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Properties of Dielectrics from Reflection Coefficients in One Dimension

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MASSACHUSETTS INSTITUTE OF TECHNOLOGY
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PROPERTIES OF DIELECTRICS FROM REFLECTION COEFFICIENTS IN ONE DIMENSION

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Group 37

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ABSTRACT

This report presents mathematical techniques for calculating dielectrics which give rise to prescribed reflection coefficients in certain problems of one-dimensional electromagnetic propagation. The techniques are, in principle at least, exact. They are based on the use of an equation of the Gel'fand-Levitan type for the one-dimensional Schrödinger equation. Although no practical applications are given, it is hoped that this report will encourage the use of newer techniques in synthesis problems.

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PROPERTIES OF DIELECTRICS FROM REFLECTION COEFFICIENTS IN ONE DIMENSION

I. INTRODUCTION

This report is based on a series of lectures by the senior author and has been only slightly revised for this written presentation. Hence, the report is more informally organized and is probably much longer than would be the case if a thorough revision had been made. Perhaps, however, the great detail will enable a novice in the inverse problem to see through the mathematics more clearly.

The report is concerned with the mathematical problem of determining the character of a dielectric scatterer from the reflection coefficient in one-dimensional electromagnetic scattering. Closely connected with this problem is the "synthesis problem" in which one wishes to construct scatterers with prescribed reflection or transmission characteristics. It is not generally known that powerful techniques are available. The objective of this report is to present these techniques in a comprehensive fashion for the benefit of those who wish such information.

The technique which we use is to map the electromagnetic equations into the one-dimensional Schrödinger equation. We then use the Gel'fand-Levitan algorithm to compute the potential and the wave function from the scattering coefficient. The results for the electromagnetic problem are then obtained from the mapping.

The usual procedure for finding dielectrics with prescribed scattering properties is to consider a family of dielectrics for which the electromagnetic equations can be solved. The parameters of this family are then adjusted until the scattering coefficients of the exactly soluble problem are as close as possible to the prescribed scattering coefficients. This procedure is always approximate. The degree of success, moreover, depends upon one's cleverness in choosing the family of dielectrics.

By contrast, the procedure for finding the dielectrics in the present report is exact, at least in principle. If computing machines are available, one can approximate the dielectrics as closely as one wishes even when one cannot solve the Gel'fand-Levitan equation exactly.

This report contains no very practical solutions because the technique which we propose is very new (it was developed within the past ten years), and most of the work in the past has been concerned with more mathematical aspects of the problem. It is hoped that this report will stimulate research for electromagnetic applications. Indeed, its sole purpose is to stimulate, if possible, research which leads to practical applications of the theory.

II. SCATTERING AND THE ONE-DIMENSIONAL SCHRÖDINGER EQUATION

The time-dependent one-dimensional Schrödinger equation is

$$-\frac{1}{i}\frac{\partial}{\partial t}\psi(x,t) = \left[-\frac{\partial^2}{\partial x^2} + V(x)\right]\psi(x,t) \quad . \tag{1}$$

We require that the solution $\psi(x,t)$ be in Hilbert space and be quadratically integrable:

$$\int_{-\infty}^{\infty} |\psi(x,t)|^2 \, dx < \infty \quad . \tag{2}$$

First, let us assume that the potential function $V(x)$ is bounded everywhere and $V(x) \to 0$ as $|x| \to \infty$. Later, we shall make $V(x) = 0$ for $|x| > a$. We proceed to solve the Schrödinger equation in the usual manner by separation of variables.

Let the general solution be

$$\psi(x,t) = \int_{-\infty}^{\infty} \chi(x|p)\, g(p)\, e^{-ip^2 t}\, dp + \sum_i \chi(x|E_i)\, g_i\, e^{-iE_i t} \tag{3}$$

and choose $\chi(x|p)$ and $\chi(x|E_i)$ so that they satisfy the following equations, respectively.

$$\left[-\frac{d^2}{dx^2} + V(x)\right]\chi(x|p) = p^2\,\chi(x|p) \qquad \text{for } p^2 > 0 \quad , \tag{4}$$

$$\left[-\frac{d^2}{dx^2} + V(x)\right]\chi(x|E_i) = E_i\,\chi(x|E_i) \qquad \text{for } E_i < 0 \quad . \tag{5}$$

In general, the solutions of these second-order differential equations are not unique unless boundary conditions are imposed.

The discrete values E_i are called "point eigenvalues," and the theory of Hilbert space requires that they be chosen so that

$$\int_{-\infty}^{\infty} \chi^*(x|E_i)\,\chi(x|E_j) = A_i \delta_{ij} \quad , \tag{6}$$

where $A_i > 0$.

With this "boundary condition," the $\chi(x|E_i)$ become unique except for a normalization constant. When the point eigenvalues E_i are chosen so that (6) is satisfied, the corresponding $\chi(x|E_i)$ are called "proper eigenfunctions." Next we need boundary conditions which will determine the $\chi(x|p)$ uniquely. Since we are interested in solving scattering problems, we wish to impose boundary conditions such that the resulting set of $\chi(x|p)$ will give a "simple" description of scattering problems.

In what follows, let us assume that $V(x)$ is bounded everywhere and that $V(x) = 0$ for $|x| > a$. Let us take the following boundary conditions:

For $p > 0$,

$$\left.\begin{aligned} \chi(x|p) &= \frac{e^{ipx}}{\sqrt{2\pi}} + \frac{b(p)}{\sqrt{2\pi}} e^{-ipx} && \text{for } x < -a \\ \chi(x|p) &= \frac{t(p)}{\sqrt{2\pi}} e^{ipx} && \text{for } x > a \end{aligned}\right\} . \quad (7)$$

For $p < 0$,

$$\left.\begin{aligned} \chi(x|p) &= \frac{e^{ipx}}{\sqrt{2\pi}} + \frac{r(p)}{\sqrt{2\pi}} e^{-ipx} && \text{for } x > a \\ \chi(x|p) &= \frac{s(p)}{\sqrt{2\pi}} e^{ipx} && \text{for } x < -a \end{aligned}\right\} . \quad (8)$$

Let us solve the specific initial value problem for which

$$g_i = 0 \quad \text{for all } i \quad ,$$

$$g(p) = 0 \quad \text{for all } p < 0 \quad .$$

Then from (7) and (3) we obtain a particular solution:

$$\left.\begin{aligned} \psi_p(x,t) &= \int_0^\infty \frac{e^{ipx}}{\sqrt{2\pi}} g(p)\, e^{-ip^2 t}\, dp + \int_0^\infty \frac{e^{-ipx}}{\sqrt{2\pi}} b(p)\, g(p)\, e^{-ip^2 t}\, dp && \text{for } x < -a \\ \psi_p(x,t) &= \int_0^\infty \frac{e^{ipx}}{\sqrt{2\pi}} t(p)\, g(p)\, e^{-ip^2 t}\, dp && \text{for } x > a \end{aligned}\right\} . \quad (9)$$

For any fixed p, the first integral in the expression for $\psi_p(x,t)$ for $x < -a$ and the integral in the expression for $\psi_p(x,t)$ for $x > a$ reduce to expressions representing waves moving in the positive x-direction. The second integral in the expression for $\psi_p(x,t)$ for $x < -a$ reduces to an expression representing a wave moving in the negative x-direction. We may think of the integrals as superpositions of such waves. With this interpretation in mind, the $b(p)$ are called the reflection coefficients for $p > 0$, and the $t(p)$ are the transmission coefficients for $p > 0$. Hence, given the boundary conditions (7) we now have defined $\chi(x|p)$ uniquely for $p > 0$.

In a similar manner, we can solve the initial value problem for which

$$g_i = 0 \quad \text{for all } i \quad ,$$

$$g(p) = 0 \quad \text{for } p > 0 \quad ,$$

and we define $\chi(x|p)$ uniquely for $p < 0$ with the help of the boundary conditions (8). The $r(p)$ and $s(p)$ are called the reflection and transmission coefficients, respectively, for $p < 0$.

Let us call $\chi(x|p)$ defined for $p > 0$, $\chi_+(x|p)$ and $\chi(x|p)$ defined for $p < 0$, $\chi_-(x|p)$. By analytic continuation we may define $\chi_+(x|p)$ for $p < 0$ and $\chi_-(x|p)$ for $p > 0$. It can then be shown that

$$\chi_+(x|-p) = \chi_+^*(x|p) \quad ,$$

$$\chi_-(x|-p) = \chi_-^*(x|p) \quad ,$$

but

$$\chi_+(x|-p) \neq \chi_-(x|p) \quad .$$

Also, by analytic continuation it can be shown that

$$\left.\begin{aligned} b(-p) &= b^*(p) \\ t(-p) &= t^*(p) \end{aligned}\right\} \quad \text{for } p > 0 \quad ,$$

$$\left.\begin{aligned} r(-p) &= r^*(p) \\ s(-p) &= s^*(p) \end{aligned}\right\} \quad \text{for } p < 0 \quad ,$$

and

$$s(p) = t(p) \qquad \text{for all } p \quad .$$

Two other relations satisfied by the reflection and transmission coefficients are

$$|b(p)|^2 + |t(p)|^2 = 1 \quad \text{for } p > 0 \quad ,$$

$$|r(p)|^2 + |s(p)|^2 = 1 \quad \text{for } p < 0 \quad .$$

These seem to imply conservation of energy. Also,

$$\lim_{|p|\to\infty} b(p) = 0 \quad ,$$

$$\lim_{|p|\to\infty} t(p) = 1 \quad ,$$

in the upper half plane. This is always true. When V(x) dies down very rapidly, it can be shown in addition that b(p) has poles at the points $p = i\sqrt{-E_i}$; that is, b(p) has poles corresponding to every point eigenvalue E_i. This, however, is true only when V(x) dies down sufficiently rapidly. Examples to illustrate this will be given later.

Once the $\chi(x|E_i)$ and $\chi(x|p)$ have been defined with the help of the boundary conditions (6), (7) and (8), we find that they satisfy the completeness and orthonormality conditions. If chosen differently, the $\chi(x|E_i)$ and $\chi(x|p)$ would not satisfy these conditions.

The orthonormality conditions are

$$\left.\begin{aligned} &\int_{-\infty}^{\infty} \chi^*(x|p)\,\chi(x|p')\,dx = \delta(p - p') \\ &\int_{-\infty}^{\infty} \chi^*(x|E_i)\,\chi(x|E_j)\,dx = A_i\delta_{ij} \\ &\int_{-\infty}^{\infty} \chi^*(x|p)\,\chi(x|E_i)\,dx = 0 \end{aligned}\right\} \quad . \tag{10}$$

The completeness relationship is

$$\int_{-\infty}^{\infty} \chi^*(x|p)\,\chi(x'|p)\,dp + \sum_i \frac{\chi^*(x|E_i)\,\chi(x'|E_i)}{A_i} = \delta(x - x') \quad . \tag{11}$$

The satisfaction of these conditions leads to a complete analogue with Fourier transform theory.

Suppose a function f(x) is in L^2. Then as a result of the completeness theorem we may expand f(x) as

$$f(x) = \int_{-\infty}^{\infty} \chi(x|p)\, g(p)\, dp + \sum \chi(x|E_i)\, g_i \quad , \tag{12}$$

where

$$g(p) = \int_{-\infty}^{\infty} \chi^*(x|p)\, f(x)\, dx \quad ,$$

$$A_i g_i = \int_{-\infty}^{\infty} \chi^*(x|E_i)\, f(x)\, dx \quad .$$

Hence we may write

$$f(x) \Longleftrightarrow g(p);\ g_i \quad .$$

Also suppose that

$$f^1(x) \Longleftrightarrow g^1(p);\ g_i^1 \quad ,$$

$$f^2(x) \Longleftrightarrow g^2(p);\ g_i^2 \quad .$$

Then

$$\int_{-\infty}^{\infty} f^{1*}(x)\, f^2(x)\, dx = \int_{-\infty}^{\infty} g^{1*}(p)\, g^2(p)\, dp + \sum_i g_i^{1*} g_i^2 A_i \quad . \tag{13}$$

But this is simply Parseval's theorem.

In the solution of scattering problems we are interested in the reflection and transmission coefficients. In particular, for $p > 0$, $|b(p)|^2$ and for $p < 0$, $|r(p)|^2$ give us the relative probability that a particle of momentum p will be reflected. Similarly, for $p > 0$, $|t(p)|^2$ and for $p < 0$, $|s(p)|^2$ give us the relative probability that a particle of momentum p will be transmitted. The relations

$$|b(p)|^2 + |t(p)|^2 = 1 \quad ,$$

$$|r(p)|^2 + |s(p)|^2 = 1 \quad ,$$

state that one event or the other will occur. To determine the reflection and transmission coefficients, Eq. (4) must be solved. This can be done in the usual manner by matching boundary conditions where $V(x)$ vanishes.

For an alternate integral equation technique, we write

$$\chi(x|p) = \frac{e^{ipx}}{\sqrt{2\pi}} - \frac{i}{2|p|} \int_{-\infty}^{\infty} e^{i|p|\,|x-x'|}\, V(x')\, \chi(x'|p)\, dx' \quad . \tag{14}$$

This expression for $\chi(x|p)$ satisfies the wave equation and all boundary conditions.

For example, take $p > 0$. Then (14) gives

$$\left.\begin{aligned} \chi(x|p) &= \frac{e^{ipx}}{\sqrt{2\pi}} - \frac{ie^{-ipx}}{2p}\int_{-\infty}^{\infty} e^{ipx'}\, V(x')\,\chi(x'|p)\,dx' && \text{for } x < -a \\ \chi(x|p) &= \frac{e^{ipx}}{\sqrt{2\pi}}\left\{1 - \sqrt{\frac{\pi}{2}}\,\frac{i}{p}\int_{-\infty}^{\infty} e^{-ipx'}\, V(x')\,\chi(x'|p)\,dx'\right\} && \text{for } x > a \end{aligned}\right\} . \qquad (15)$$

The expressions (15) may be written in the form (7) if we choose

$$\left.\begin{aligned} b(p) &= -\sqrt{\frac{\pi}{2}}\,\frac{i}{p}\int_{-\infty}^{\infty} e^{ipx'}\, V(x')\,\chi(x'|p)\,dx' \\ t(p) &= 1 - \sqrt{\frac{\pi}{2}}\,\frac{i}{p}\int_{-\infty}^{\infty} e^{-ipx'}\, V(x')\,\chi(x'|p)\,dx' \end{aligned}\right\} \text{for } p > 0 . \qquad (16)$$

By analytic continuation, $b(p)$ and $t(p)$ can be defined for $p < 0$.

Similarly, take $p < 0$. Then (14) gives

$$\left.\begin{aligned} \chi(x|p) &= \frac{e^{ipx}}{\sqrt{2\pi}} + \frac{i}{2p}\, e^{-ipx}\int_{-\infty}^{\infty} e^{ipx'}\, V(x')\,\chi(x'|p)\,dx' && \text{for } x > a \\ \chi(x|p) &= \frac{e^{ipx}}{\sqrt{2\pi}}\left[1 + \sqrt{\frac{\pi}{2}}\,\frac{i}{p}\int_{-\infty}^{\infty} e^{-ipx'}\, V(x')\,\chi(x'|p)\,dx' \quad \text{for } x < -a\right] \end{aligned}\right\} . \qquad (17)$$

The expression (17) may be written in the form (8) if we choose

$$\left.\begin{aligned} r(p) &= \sqrt{\frac{\pi}{2}}\,\frac{i}{p}\int_{-\infty}^{\infty} e^{ipx'}\, V(x')\,\chi(x'|p)\,dx' \\ s(p) &= 1 + \sqrt{\frac{\pi}{2}}\,\frac{i}{p}\int_{-\infty}^{\infty} e^{-ipx'}\, V(x')\,\chi(x'|p)\,dx' \end{aligned}\right\} \text{for } p < 0 . \qquad (18)$$

By analytic continuation, $r(p)$ and $s(p)$ can be defined for $p > 0$.

As an example of the solution of the direct problem of the one-dimensional Schrödinger equation, let us consider the delta function potential $V(x) = 2B\delta(x)$, and let us take $B > 0$ so that there are no point eigenvalues. Hence $g_i = 0$ for all i. From (16),

$$\left.\begin{aligned} b(p) &= -\sqrt{\frac{\pi}{2}}\,\frac{i}{p}\, 2B\chi(0|p) \\ t(p) &= 1 - \sqrt{\frac{\pi}{2}}\,\frac{i}{p}\, 2B\chi(0|p) \end{aligned}\right\} \text{for } p > 0 .$$

From (18),

$$\left.\begin{aligned} r(p) &= \sqrt{\frac{\pi}{2}}\,\frac{i}{p}\, 2B\chi(0|p) \\ s(p) &= 1 + \sqrt{\frac{\pi}{2}}\,\frac{i}{p}\, 2B\chi(0|p) \end{aligned}\right\} \text{for } p < 0 .$$

From (14),

$$\chi(0|p) = \frac{p}{\sqrt{2\pi}\,(p + iB)} \qquad \text{for } p > 0 \quad ,$$

$$\chi(0|p) = \frac{p}{\sqrt{2\pi}\,(p - iB)} \qquad \text{for } p < 0 \quad .$$

Hence,

$$\left.\begin{aligned} b(p) &= \frac{-iB}{p + iB} \\ t(p) &= \frac{p}{p + iB} \end{aligned}\right\} \qquad \text{for } p > 0 \quad .$$

$$\left.\begin{aligned} r(p) &= \frac{iB}{p - iB} \\ s(p) &= \frac{p}{p - iB} \end{aligned}\right\} \qquad \text{for } p < 0 \quad .$$

By analytic continuation,

$$\left.\begin{aligned} b(-p) &= \frac{-iB}{-p + iB} = \frac{iB}{p - iB} = b^*(p) \\ t(-p) &= \frac{-p}{-p + iB} = \frac{p}{p - iB} = t^*(p) \end{aligned}\right\} \qquad \text{for } p > 0 \quad ,$$

$$\left.\begin{aligned} r(-p) &= \frac{iB}{-p - iB} = \frac{-iB}{p + iB} = r^*(p) \\ s(-p) &= \frac{-p}{-p - iB} = \frac{p}{p + iB} = s^*(p) \end{aligned}\right\} \qquad \text{for } p < 0 \quad .$$

Hence $s(p) = t(p)$ for all p and $r(p) = b(p)$ for all p, the latter being so only because of the symmetry of the delta function potential. Also,

$$|b(p)|^2 + |t(p)|^2 = \frac{B^2}{p^2 + B^2} + \frac{p^2}{p^2 + B^2} = 1 \qquad \text{for } p > 0 \quad ,$$

$$|r(p)|^2 + |s(p)|^2 = \frac{B^2}{p^2 + B^2} + \frac{p^2}{p^2 + B^2} = 1 \qquad \text{for } p < 0 \quad .$$

III. EXAMPLES OF ELECTROMAGNETIC PROBLEMS WHICH CAN BE MAPPED INTO A ONE-DIMENSIONAL SCHRÖDINGER EQUATION

We shall next give three examples of one-dimensional electromagnetic theory. It will be shown that with an appropriate mapping each example can be reduced to a one-dimensional Schrödinger equation.

A. Reflection and Transmission of Light of a Fixed Frequency at Varying Angles of Incidence by a Dielectric Slab

Consider a medium in which the dielectric constant $\epsilon = \epsilon(x)$ is a function of one variable x only and is independent of time. In order to make the discussion concrete, we assume that $\epsilon(x) = 1$ for $x < a$ and $x > b$. Many of the results will still hold if $\epsilon(x)$ dies down sufficiently rapidly outside the range $a < x < b$.

Consider a ray of light impinging on this material, the angle of incidence being α. The plane of incidence is the xy-plane; the z-axis projects out of the paper (Fig. 1). We are interested primarily in the amount of energy reflected and transmitted by the material. It will be shown that, for a ray polarized so that the electric field is parallel to the z-axis, the electromagnetic equation

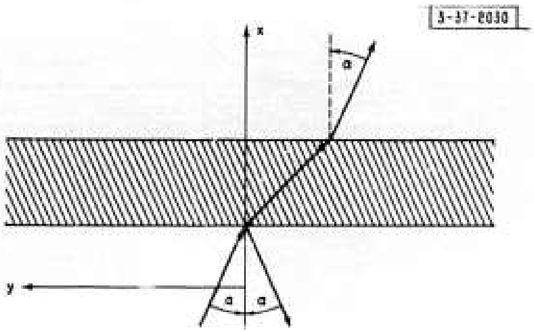

Fig. 1. Roy picture showing reflection and transmission of light by a dielectric slob.

can be mapped into the Schrödinger equation and the b(p) and t(p) of this associated Schrödinger equation will turn out to be essentially the same as the electromagnetic reflection and transmission coefficients.

Using Gaussian units, Maxwell's equations in a source-free region are

$$\underline{\nabla} \cdot \underline{D} = 0 \quad ,$$

$$\underline{\nabla} \cdot \underline{B} = 0 \quad ,$$

$$\underline{\nabla} \times \underline{E} = -\frac{1}{c}\frac{\partial \underline{B}}{\partial t} \quad ,$$

$$\underline{\nabla} \times \underline{H} = \frac{1}{c}\frac{\partial \underline{D}}{\partial t} \quad , \tag{19}$$

where c is the free-space velocity of light. We also have the constitutive equations

$$\underline{B} = \mu \underline{H} \quad ,$$

$$\underline{D} = \epsilon(x)\, \underline{E} \quad . \tag{20}$$

We assume that $\mu = \mu_o$, where μ_o is the permeability of free space and, to eliminate time dependence, write

$$\underline{E}(\underline{x}, t) = \underline{E}(\underline{x})\, e^{-i\omega t} \quad ,$$

$$\underline{H}(\underline{x}, t) = \underline{H}(\underline{x})\, e^{-i\omega t} \quad . \tag{21}$$

The bar under x indicates dependence on both x and y, while x without a bar, such as in $\epsilon(x)$, indicates dependence on the one variable x alone. Substituting in (19) from (20) and (21), we obtain

$$\underline{\nabla} \cdot \epsilon(x)\, \underline{E}(\underline{x}) = 0 \quad ,$$

$$\underline{\nabla} \cdot \underline{H}(\underline{x}) = 0 \quad ,$$

$$\underline{\nabla} \times \underline{E}(\underline{x}) = \frac{i\mu_o \omega}{c}\, \underline{H}(\underline{x}) \quad ,$$

$$\underline{\nabla} \times \underline{H}(\underline{x}) = \frac{-i\epsilon(x)\,\omega}{c}\, \underline{E}(\underline{x}) \quad .$$

Proceeding in the usual manner,

$$\underline{\nabla} \times \underline{\nabla} \times \underline{E}(\underline{x}) = \underline{\nabla}\underline{\nabla} \cdot \underline{E}(\underline{x}) - \underline{\nabla}^2 \underline{E}(\underline{x})$$

$$= \frac{i\mu_o \omega}{c}\, \underline{\nabla} \times \underline{H}(\underline{x}) = \frac{\mu_o \epsilon(x)\,\omega^2}{c^2}\, \underline{E}(\underline{x}) \quad .$$

But

$$\underline{\nabla} \cdot \epsilon(x)\, \underline{E}(\underline{x}) = \underline{E}(\underline{x}) \cdot \underline{\nabla}\epsilon(x) + \epsilon(x)\, \underline{\nabla} \cdot \underline{E}(x) \quad .$$

Hence,

$$\underline{\nabla} \cdot \underline{E}(\underline{x}) = \frac{-\underline{E}(\underline{x}) \cdot \underline{\nabla}\epsilon(x)}{\epsilon(x)} \quad ,$$

and

$$\underline{\nabla}^2 \underline{E}(\underline{x}) + \frac{\mu_o \epsilon(x)\, \omega^2}{c^2} \underline{E}(\underline{x}) = -\underline{\nabla}\left[\frac{\underline{E}(\underline{x}) \cdot \underline{\nabla}\epsilon(x)}{\epsilon(x)}\right] \quad .$$

Let us write

$$\underline{E}(\underline{x}) = \underline{E}_n(\underline{x}) + \underline{E}_p(\underline{x}) \quad ,$$

where

$$\underline{E}_n(x) = [0, 0, E_z(\underline{x})]$$

is the component of the electric field normal to the plane of incidence, and

$$\underline{E}_p(\underline{x}) = [E_x(\underline{x}), E_y(\underline{x}), 0]$$

is the component of the electric field parallel to the plane of incidence. Then $E_z(\underline{x})$ satisfies the equation

$$\underline{\nabla}^2 E_z(\underline{x}) + n^2(x)\, k^2 E_z(\underline{x}) = 0 \tag{22}$$

where

$$k = \frac{\omega}{c} \quad ,$$

$$n(x) = \sqrt{\mu_o \epsilon(x)} \quad .$$

The boundary conditions we wish to impose on $E_z(\underline{x})$ are

$$\left.\begin{aligned} E_z(\underline{x}) &= \frac{e^{i\underline{k}\cdot\underline{x}}}{\sqrt{2\pi}} + \frac{B\, e^{i\underline{k}'\cdot x}}{\sqrt{2\pi}} && \text{for } x < a \\ E_z(\underline{x}) &= \frac{T\, e^{i\underline{k}\cdot\underline{x}}}{\sqrt{2\pi}} && \text{for } x > b \end{aligned}\right\} \quad , \tag{23}$$

where $\underline{k}$ and $\underline{k}'$ are vectors given in terms of their components by

$$\left.\begin{aligned} \underline{k} &= (k \cos\alpha,\ k \sin\alpha, 0) \\ \underline{k}' &= (-k \cos\alpha,\ k \sin\alpha, 0) \end{aligned}\right\} \quad . \tag{24}$$

Then the first term in the expression for $E_z(\underline{x})$ for $x > a$ represents a wave moving in the direction $(\cos\alpha,\ \sin\alpha, 0)$. We may consider this to be an incident wave. The second term represents a wave moving in the direction $(-\cos\alpha,\ \sin\alpha, 0)$, i.e., in the direction of specular reflection. It represents a reflected wave. B is the reflection coefficient, and $|B|^2$ gives the relative proportion of energy reflected at the frequency ω. Similarly, the expression for $E_z(\underline{x})$ for $x > b$

represents a wave moving in the direction $(\cos\alpha, \sin\alpha, 0)$. It represents a wave transmitted through the material. T is the transmission coefficient, and $|T|^2$ gives the relative proportion of energy transmitted at the frequency ω.

We shall now show that with the boundary conditions (23) on $E_z(\underline{x})$ we can reduce equation (22) to the Schrödinger equation by an easy mapping and identify B and T with b(p) and t(p), respectively.

Let us consider a fixed frequency $\omega > 0$. To separate spatial variables we write

$$E_z(\underline{x}) = u(x)\, e^{ik\sin\alpha\, y} \quad . \tag{25}$$

We also define

$$\left.\begin{aligned} p &= k\cos\alpha \\ V(x) &= -k^2(n^2-1) = -k^2[\epsilon(x)\,\mu_o - 1] \end{aligned}\right\} \quad . \tag{26}$$

Since ω is fixed, p is a function of the angle α. Usually, we will be dealing with $\epsilon(x) \geqslant 1$, and hence $V(x) \leqslant 0$.

Substituting in Eq. (22), we obtain

$$\left[-\frac{d^2}{dx^2} + V(x)\right] u(x) = p^2 u(x) \quad , \tag{27}$$

where $p^2 > 0$, but this is just the time-independent Schrödinger equation.

Suppose the boundary conditions we impose on u(x) are

$$\begin{aligned} u(x) &= \frac{e^{ipx}}{\sqrt{2\pi}} + \frac{b(p)\, e^{-ipx}}{\sqrt{2\pi}} \qquad \text{for } x < a \quad , \\ u(x) &= \frac{t(p)\, e^{ipx}}{\sqrt{2\pi}} \qquad \text{for } x > b \quad . \end{aligned} \tag{28}$$

Then from (25) and (26) we obtain

$$\begin{aligned} E_z(\underline{x}) &= \frac{e^{i(k\cos\alpha x + k\sin\alpha y)}}{\sqrt{2\pi}} + \frac{b(p)\, e^{i(-k\cos\alpha x + k\sin\alpha y)}}{\sqrt{2\pi}} \\ &= \frac{e^{i\underline{k}\cdot\underline{x}}}{\sqrt{2\pi}} + \frac{b(p)\, e^{\underline{k}'\cdot\underline{x}}}{\sqrt{2\pi}} \qquad \text{for } x < a \quad , \\ E_z(\underline{x}) &= \frac{t(p)\, e^{i(k\cos\alpha x + k\sin\alpha y)}}{\sqrt{2\pi}} \\ &= \frac{t(p)\, e^{i\underline{k}\cdot\underline{x}}}{\sqrt{2\pi}} \qquad \text{for } x > b \quad . \end{aligned}$$

If we identify B with b(p) and T with t(p), these are just the boundary conditions we wish to impose on $E_z(\underline{x})$. Hence, we can solve this particular electromagnetic problem by using the mapping (26) to find the associated Schrödinger equation and identifying its reflection and transmission coefficients with the reflection and transmission coefficients of the electromagnetic problem.

We might note again that in quantum mechanics p represents momentum and $-\infty < p < \infty$. However, p in Eq. (27) as defined by the mapping (26) is a function of the optical angle of incidence α, and this imposes restrictions on the range of p. The technique is valid for complex α and might therefore be used to handle more complicated wave fronts.

B. A Transmission Line Problem

The second example we shall give is a transmission line problem. It will differ from the above example which dealt with a fixed frequency and varying angles of incidence in that now we shall consider a fixed angle for varying frequencies.

Consider a transmission line with distributed impedances L(z) and capacitances C(z). The transmission line equations are

$$\left.\begin{aligned} \frac{\partial V(z,t)}{\partial z} &= -L(z)\,\frac{\partial I(z,t)}{\partial t} \\ \frac{\partial I(z,t)}{\partial z} &= -C(z)\,\frac{\partial V(z,t)}{\partial t} \end{aligned}\right\} , \qquad (29)$$

where V(z,t) and I(z,t) are the potential and the current, respectively. For the special case

$$L = L_o \quad ,$$

$$C = C_o \quad ,$$

we obtain at once

$$\frac{\partial^2 V(z,t)}{\partial z^2} = L_o C_o \frac{\partial^2 V(z,t)}{\partial t^2} \quad .$$

This has a solution of the form

$$V(z,t) = F(z - vt) + g(z + vt) \quad ,$$

where

$$v = \frac{1}{\sqrt{L_o C_o}} \quad .$$

Apparently, we have again a wave propagation problem in which reflection and transmission coefficients are of importance. We propose to show that this problem also can be reduced to the Schrödinger equation.

Let us write

$$V(z,t) = V(z)\, e^{-i\omega t} \quad ,$$

$$I(z,t) = I(z)\, e^{-i\omega t} \quad .$$

As above, we take L to be a constant, i.e., $L = L_o$. However, for $a < z < b$ we take C to be a function of z. For $z < a$ and $z > b$ we take C to be a constant $C = C_o$. Then

$$\frac{V(z)}{\partial z} = i\omega L_o I(z) \quad ,$$

$$\frac{\partial I(z)}{\partial z} = i\omega C(z)\, V(z) \quad ,$$

and hence,

$$\frac{\partial^2 V(z)}{\partial z^2} + \omega^2 L_o C(z)\ V(z) = 0 \quad . \tag{30}$$

We define $k^2 = \omega^2 L_o C_o$ and impose the following boundary conditions on the solution

$$V(z) = \frac{e^{ikz}}{\sqrt{2\pi}} + \frac{B(k)\ e^{-ikz}}{\sqrt{2\pi}} \qquad \text{for } z < a \quad ,$$

$$V(z) = \frac{T(k)\ e^{ikz}}{\sqrt{2\pi}} \qquad \text{for } z > b \quad . \tag{31}$$

We shall show that if we can solve a certain Schrödinger equation then we can solve (30) subject to the boundary conditions (31). For convenience, let us rewrite (30) in the form

$$\frac{\partial^2 V(z)}{\partial z^2} + k^2 \epsilon(z)\ V(z) = 0 \quad , \tag{32}$$

where

$$\epsilon(z) = \frac{C(z)}{C_o} \quad .$$

Next let

$$\eta(z) = \epsilon^{1/4}(z) \tag{33}$$

and introduce a new independent variable x related to z by

$$\frac{dz}{dx} = \eta^{-2}(z)$$

or

$$\eta^2(z) = \frac{dx}{dz} \quad . \tag{34}$$

We also introduce a change of dependent variable,

$$u = \eta V \quad . \tag{35}$$

Then

$$\frac{dV}{dz} = \frac{1}{\eta}\frac{du}{dz} - \frac{u}{\eta^2}\frac{d\eta}{dz} \quad .$$

$$\frac{d^2 V}{dz^2} = \frac{1}{\eta}\frac{d^2 u}{dz^2} - \frac{2}{\eta^2}\frac{d\eta}{dz}\frac{du}{dz} - \frac{u}{\eta^2}\frac{d^2\eta}{dz^2} + \frac{2u}{\eta^3}\left(\frac{d\eta}{dz}\right)^2 \quad .$$

Also,

$$\frac{du}{dz} = \eta^2 \frac{du}{dx} \quad ,$$

$$\frac{d^2 u}{dz^2} = \eta^4 \frac{d^2 u}{dx^2} + 2\eta^3 \frac{du}{dx}\frac{d\eta}{dx} \quad ,$$

$$\frac{d\eta}{dz} = \eta^2 \frac{d\eta}{dx} \quad ,$$

$$\frac{d^2\eta}{dz^2} = \eta^4 \frac{d^2\eta}{dx^2} + 2\eta^3 \left(\frac{d\eta}{dx}\right)^2 \quad .$$

Substituting in (32), we obtain

$$\frac{1}{\eta}\left[\eta^4 \frac{d^2u}{dx^2} + 2\eta^3 \frac{du}{dx}\frac{d\eta}{dx}\right] - \frac{2}{\eta^2}\left(\eta^2 \frac{d\eta}{dx}\right)\left(\eta^2 \frac{du}{dx}\right)$$

$$- \frac{u}{\eta^2}\left[\eta^4 \frac{d^2\eta}{dx^2} + 2\eta^3 \left(\frac{d\eta}{dx}\right)^2\right] + \frac{2u}{\eta^3}\left(\eta^2 \frac{d\eta}{dx}\right)^2 + k^2\eta^4 \frac{u}{\eta} = 0 \quad .$$

From this we have

$$\frac{d^2u}{dx^2} - \frac{u}{\eta}\frac{d^2\eta}{dx^2} + k^2u = 0 \quad .$$

We write this as

$$\left[-\frac{d^2}{dx^2} + q(x)\right] u(x) = k^2u \quad , \tag{36}$$

where

$$q(x) = \frac{1}{\eta}\frac{d^2\eta}{dx^2} \quad . \tag{37}$$

Equation (36) is the Schrödinger equation corresponding to Eq. (30) or (32). Note that q(x) is independent of frequency. Incidentally, the mapping is possible only if $\epsilon(z)$ is continuous.

We shall now turn our attention to an investigation of the mapping and the shape that q(x) will have for a realistically chosen $\epsilon(z)$.

A first observation is that x is a monotonic function of z. Also, since the first-order differential equation relating x to z allows for one constant of integration, we may put x = A when z = a; then z = b will correspond to some value of x – say, x = B. The mapping is not unique since A can be chosen arbitrarily, but a change in A merely implies a shift in x.

Consider a particular $\epsilon(z)$ such that $\epsilon(z)$ is some function of z for $a < z < b$, and $\epsilon(z) = 1$ for $z < a$ and $z > b$. Then we want to determine first how x depends on z for $z < a$ and $z > b$. Since x is a monotonic function of z,

$z < a$ corresponds to $x < A$,

$z > b$ corresponds to $x > B$.

From (34), we have at once that

$x - A = z - a \quad$ for $z < a$, i.e., $x < A \quad ,$

$x - B = z - b \quad$ for $z > b$, i.e., $x > B \quad .$

Suppose Fig. 2(a) represents a typical $\epsilon(z)$. It is essential that $\epsilon(z)$ be continuous for the mapping determined by (33), (34) and (35) to be possible. However, it is not necessary that the derivatives of $\epsilon(z)$ be continuous. Given an $\epsilon(z)$ as shown in Fig. 2(a), $\eta(x)$ would have a shape as shown in

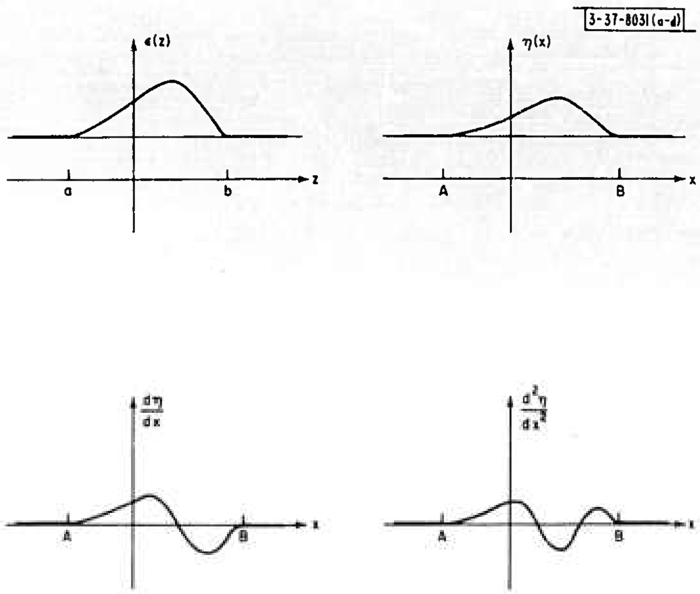

Fig. 2. $\epsilon(z)$ as a function of z and, corresponding to this $\epsilon(z)$, the general shapes taken by $\eta(x)$, $d\eta(x)/dx$ and $d^2\eta(x)/dx^2$ as functions of x.

Fig. 2(b). Furthermore, $d\eta/dx$ and $d^2\eta/dx^2$ would have shapes as shown in Figs. 2(c) and (d), respectively. Note that Figs. 2(b), (c) and (d) are not the exact representations of $\eta(x)$, $d\eta/dx$, and $d^2\eta/dx^2$ for $\epsilon(z)$ as shown in Fig. 2(a), but merely are intended to show the general shape of these functions for the given $\epsilon(z)$.

$q(x) = (1/\eta(x))\,(d^2\eta/dx^2)$ will have a shape very similar to that of $d^2\eta/dx^2$, the important thing being that it will have two maxima and one minimum. Hence, if we wish to solve our transmission line problem through its associated Schrödinger equation for a realistic $\epsilon(z)$ as shown in Fig. 2(a), we must solve the Schrödinger equation for a q(x) such as shown in Fig. 3. We might note that, if the bumps in Fig. 3 were replaced by delta function like discontinuities such a q(x) would still correspond to a realistic $\epsilon(z)$. Since $q(x) = 0$ for $x < A$ and $x > B$, we again have a scattering problem and shall look for solutions of the Schrödinger equation which describe scattering. We shall show that if we can solve this scattering problem, we will also have solved the transmission line problem and will be able to relate the coefficients B(k) and T(k) to the transmission and reflection coefficients of the Schrödinger equation. Let us take as solutions of the Schrödinger equation

$$\left.\begin{aligned} u(x) &= \frac{e^{ikx}}{\sqrt{2\pi}} + \frac{b(k)\,e^{-ikx}}{\sqrt{2\pi}} \qquad \text{for } x < A \\ u(x) &= \frac{t(k)\,e^{ikx}}{\sqrt{2\pi}} \qquad \text{for } x > B \end{aligned}\right\} . \tag{38}$$

But when $x < A$ we have $z < a$; $\eta = 1$ and therefore $V(z) = u(x)$. Also, $x - A = z - a$. Hence

$$V(z) = \frac{e^{ikz}\, e^{-ik(a-A)}}{\sqrt{2\pi}} + \frac{b(k)\, e^{-ikz}}{\sqrt{2\pi}}\, e^{ik(a-A)}$$

$$= e^{-ik(a-A)}\left[\frac{e^{ikz}}{\sqrt{2\pi}} + \frac{b(k)\, e^{-ikz}}{\sqrt{2\pi}}\, e^{2ik(a-A)}\right]$$

$$= e^{-ik(a-A)}\left[\frac{e^{ikz}}{\sqrt{2\pi}} + \frac{B(k)\, e^{-ikz}}{\sqrt{2\pi}}\right] ,$$

where

$$B(k) = b(k)\, e^{2ik(a-A)} .$$

Similarly, when $x > B$ we have $z > b$; $\eta = 1$ and therefore $V(z) = u(x)$. Also, now $x - B = z - b$. Hence,

$$V(z) = \frac{t(k)\, e^{ikz}}{\sqrt{2\pi}}\, e^{-ik(b-B)}$$

$$= \frac{t(k)\, e^{ikz}}{\sqrt{2\pi}}\, e^{-ik(a-A)}\, e^{-ik(b-B)+ik(a-A)}$$

$$= \frac{T(k)\, e^{ikz}}{\sqrt{2\pi}}\, e^{-ik(a-A)} ,$$

where

$$T(k) = t(k)\, e^{ik(a-b+B-A)}$$

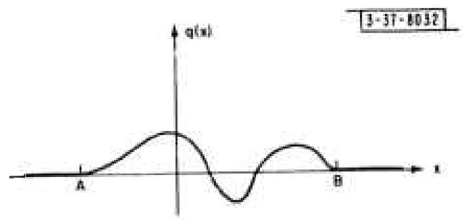

Fig. 3. q(x) as a function of x.

The factor $e^{-ik(a-A)}$ merely changes the incident wave. Hence, we can obtain a solution of our transmission line equation (32) subject to the boundary conditions (31) with the help of the mapping given by (33), (34) and (35). We solve the Schrödinger equation (36) subject to the boundary conditions (38) and obtain the B(k) and T(k) of the transmission line problem from the reflection coefficients b(k) and the transmission coefficients t(k) of the solution of the Schrödinger equation by using the relationships

$$\left.\begin{aligned} B(k) &= b(k)\, e^{2ik(a-A)} \\ T(k) &= t(k)\, e^{ik(a-b+B-A)} \end{aligned}\right\} . \qquad (39)$$

In practice, reducing the transmission line equation to the Schrödinger equation involves the following steps. Given $\epsilon(z)$, we use (33) and (34) to find x as a function of z. The choice of the

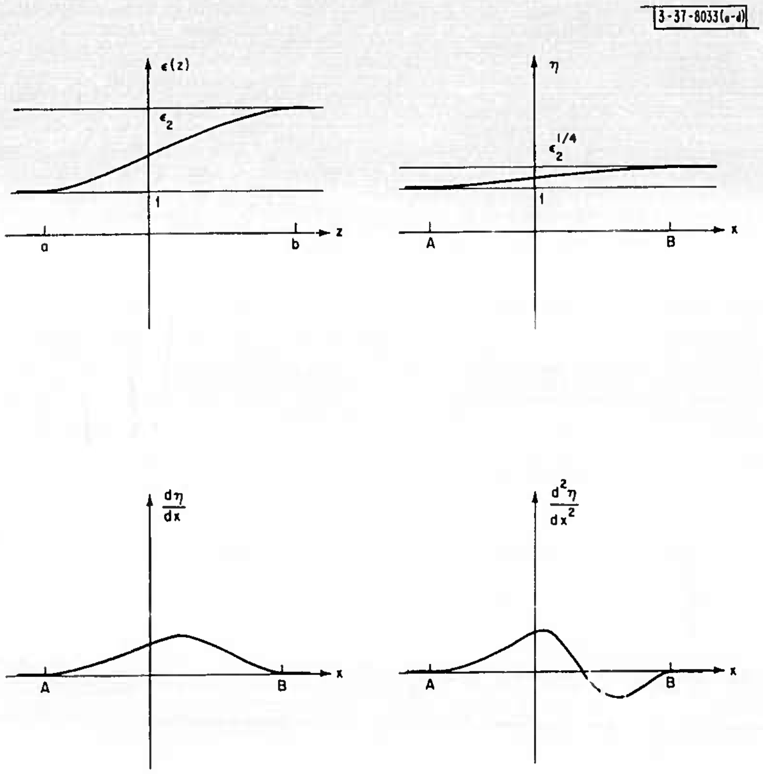

Fig. 4. $\epsilon(z)$ as a function of z, and the corresponding general shapes of $\eta(x)$, $d\eta(x)/dx$ and $d^2\eta(x)/dx^2$ as functions of x.

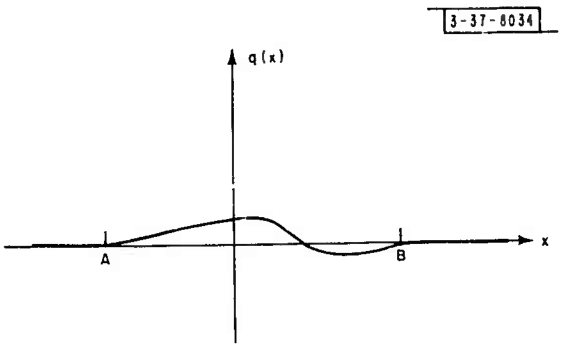

Fig. 5. $q(x)$ as a function of x.

constant of integration A is arbitrary but is usually determined by symmetry considerations. We now have ϵ and η as functions of x and can use (37) to compute q(x).

Next, we shall generalize the above problem slightly. Suppose we again take the transmission line equation (32) and let $\epsilon(z) = 1$ for $z < a$; $\epsilon(z)$ = some function of z for $a < z < b$; but $\epsilon(z) = \epsilon_2 = \text{constant} \neq 1$ for $z > b$.

We shall require that V(z) satisfy the following boundary conditions:

$$\left.\begin{aligned} V(z) &= \frac{e^{ikz}}{\sqrt{2\pi}} + \frac{B(k)\, e^{ikz}}{\sqrt{2\pi}} && \text{for } z < a \\ V(z) &= \frac{T(k)\, e^{ik\sqrt{\epsilon_2}z}}{\sqrt{2\pi}} && \text{for } z > b \end{aligned}\right\} \quad . \tag{40}$$

The mapping is independent of the boundary conditions, and we shall use the same mapping as above. Again, we choose z = a to correspond to x = A, and since x is a monotonic function of z we have $z < a$ corresponds to $x < A$. Also, since for $z < a$ we choose $\epsilon(z) = 1$, we have as before $x - A = z - a$ for $z < a$ and $x < A$. z = b will correspond to some value of x – say, x = B – and $z > b$ will correspond to $x > B$. However, for $z > b$ we now have $\epsilon(z) = \epsilon_2$ and hence $\eta = \epsilon_2^{1/4}$. Therefore,

$$\frac{dx}{dz} = \sqrt{\epsilon_2} \quad ,$$

and

$$x - B = \sqrt{\epsilon_2}\,(z - b) \quad .$$

Again let us assume a simple realistic shape for $\epsilon(z)$ such as shown in Fig. 4(a). The approximate shapes for η, $d\eta/dx$ and $d^2\eta/dx^2$ are shown in Figs. 4(b), (c) and (d). q(x) will of course again have a shape similar to that of $d^2\eta/dx^2$. Therefore, in this case we shall be dealing with a q(x) having a shape like that shown in Fig. 5.

We solve the Schrödinger equation for this q(x) subject to the boundary conditions (38). For $x < A$ we have $z < a$; $\eta = 1$; $V(z) = u(x)$; and $x - A = z - a$ as before. The boundary conditions on V(z) for $z < a$ are also the same as above. Hence, again we obtain

$$B(k) = b(k)\, e^{2ik(a-A)} \quad .$$

For $x > B$ we have $z > b$; $\eta = \epsilon_2^{1/4}$; $V(z) = u(x)/\epsilon_2^{1/4}$; and $x - B = \sqrt{\epsilon_2}\,(z - b)$. Hence,

$$V(z) = \frac{t(k)\, e^{ik\sqrt{\epsilon_2}z}}{\epsilon_2^{1/4}\sqrt{2\pi}}\, e^{ik(B-\sqrt{\epsilon_2}b)}$$

$$= \frac{t(k)\, e^{ik\sqrt{\epsilon_2}z}}{\epsilon_2^{1/4}\sqrt{2\pi}}\, e^{-ik(a-A)}\, e^{ik(a-\sqrt{\epsilon_2}b+B-A)} \quad .$$

In this case, the transmission coefficient t(k) of the associated Schrödinger equation and the T(k) of the transmission line equation are related in the following manner;

$$T(k) = \frac{t(k)}{\epsilon_2^{1/4}} e^{ik(a-\sqrt{\epsilon_2}b+B-A)} \quad ,$$

and again we are able to solve the transmission line problem by mapping it into the Schrödinger equation.

C. Reflection and Transmission, for Varying Frequencies, of a Plane Wave Normally Incident on a Dielectric Slab

Finally, we shall show that Eq. (32), the transmission line equation, also occurs in a simple electromagnetic problem. In this case, k and ϵ are defined differently, of course.

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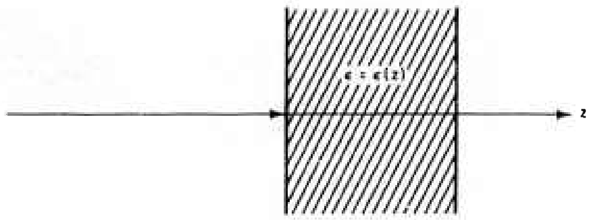

Fig. 6. Plane wave incident normally on a slab of dielectric constant, $\epsilon(z)$.

Consider normal incidence of a plane wave traveling in the z-direction on a slab of material whose dielectric constant $\epsilon = \epsilon(z)$ varies in the z-direction (Fig. 6). It was shown previously that, assuming a time dependence $e^{-i\omega t}$ and using Maxwell's equations and the constitutive equations, we find that the electric field $\underline{E}(\underline{x})$ must satisfy the equation

$$\underline{\nabla}^2 \underline{E}(\underline{x}) + \frac{\mu_o \epsilon(z)\, \omega^2}{c^2} \underline{E}(\underline{x}) = -\underline{\nabla} \left[\frac{\underline{E}(\underline{x}) \cdot \underline{\nabla}\epsilon(z)}{\epsilon(z)}\right] \quad . \tag{41}$$

Since we are now assuming that $\epsilon = \epsilon(z)$ varies in the direction of propagation and since $\underline{E}(\underline{x})$ must be normal to this direction, we see at once that the right-hand side of (41) vanishes.

Let us assume further that

$$\underline{E}(\underline{x}) = [E_x(z), 0, 0] \quad .$$

Then we have at once

$$\frac{\partial^2 E_x(z)}{\partial z^2} + \frac{\mu_o \omega^2}{c^2} \epsilon(z)\, E_x(z) = 0 \quad .$$

Similarly, if we had assumed that

$$\underline{E}(\underline{x}) = [0, E_y(z), 0] \quad ,$$

we would have obtained the same equation for $E_y(z)$, namely,

$$\frac{\partial^2 E_y(z)}{\partial z^2} + \frac{\mu_o \omega^2}{c^2} \epsilon(z)\, E_y(z) = 0 \quad .$$

We can write either of these in the form of the transmission line equation (32):

$$\frac{\partial^2 E_v(z)}{\partial z^2} + k^2 \hat{\epsilon}(z)\, E_v(z) = 0 \quad , \tag{42}$$

where now v = x or y,

$$k^2 = \frac{\omega^2 \mu_o \epsilon_o}{c^2} \quad ,$$

$$\hat{\epsilon}(z) = \frac{\epsilon(z)}{\epsilon_o} \quad ,$$

and ϵ_o is the dielectric constant of free space.

IV. THE INVERSE PROBLEM FOR THE SCHRÖDINGER EQUATION – THE GEL'FAND-LEVITAN ALGORITHM AND ITS PROOF

The direct problem of solving the two time-independent equations arising from the Schrödinger equation has been treated earlier in this report.

To recapitulate briefly in this case specifying

(a) V(x),

(b) A_i the normalization of $\chi(x|E_i)$:

$$A_i = \int_{-\infty}^{\infty} |\chi(x|E_i)|^2\, dx \quad .$$

(c) The boundary conditions on $\chi(x|p)$:

$$\left.\begin{aligned} \lim_{x\to-\infty} \chi(x|p) &= \frac{e^{ipx}}{\sqrt{2\pi}} + \frac{b(p)\, e^{-ipx}}{\sqrt{2\pi}} \\ \lim_{x\to\infty} \chi(x|p) &= \frac{t(p)\, e^{ipx}}{\sqrt{2\pi}} \end{aligned}\right\} \quad \text{for } p > 0 \quad ,$$

$$\left.\begin{aligned} \lim_{x\to\infty} \chi(x|p) &= \frac{e^{ipx}}{\sqrt{2\pi}} + \frac{r(p)\, e^{-ipx}}{\sqrt{2\pi}} \\ \lim_{x\to-\infty} \chi(x|p) &= \frac{s(p)\, e^{ipx}}{\sqrt{2\pi}} \end{aligned}\right\} \quad \text{for } p < 0 \quad ,$$

we solve the two equations

$$\left[-\frac{d^2}{dx^2} + V(x)\right] \chi(x|p) = p^2 \chi(x|p) \qquad \text{for } p^2 > 0 \quad ,$$

$$\left[-\frac{d^2}{dx^2} + V(x)\right] \chi(x|E_i) = E_i \chi(x|E_i) \qquad \text{for } E_i < 0 \quad ,$$

and determine E_i, b(p), t(p), r(p), s(p), $\chi(x|p)$ and $\chi(x|E_i)$. It was also shown that the $\chi(x|p)$ and $\chi(x|E_i)$ thus determined satisfy the completeness and orthonormality conditions, and hence, that any function f(x) can be expanded in the following manner:

$$f(x) = \int_{-\infty}^{\infty} \chi(x|p)\, g(p)\, dp + \sum_i \chi(x|E_i)\, g_i \quad ,$$

where

$$g(p) = \int_{-\infty}^{\infty} \chi^*(x|p)\, f(x)\, dx \quad ,$$

$$A_i g_i = \int_{-\infty}^{\infty} \chi^*(x|E_i)\, f(x)\, dx \quad .$$

We shall next discuss the inverse problem, namely, the problem of determining $V(x)$ from a knowledge of A_i, E_i, $b(p)$ and the boundary conditions on $\chi(x|p)$. This problem was first treated by Gel'fand and Levitan. However, they dealt with the radial equation for $\ell = 0$, and we shall be dealing with the one-dimensional equation for the range $-\infty < x < \infty$.

We shall first give the Gel'fand-Levitan algorithm and then proceed to the proof. The proof will be given in two parts.

Gel'fand-Levitan Algorithm

If, given $b(p)$, E_i, and A_i, we define

$$R(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} b(p)\, e^{-ipx}\, dp + \sum_i \frac{e^{\sqrt{-E_i}\,x}}{A_i} \quad , \tag{43}$$

and assume that the Gel'fand-Levitan equation for $x \geqslant y$,

$$K(x,y) = -R(x+y) - \int_{-\infty}^{x} K(x,z)\, R(z+y)\, dz \quad , \tag{44}$$

has a unique solution for $K(x,y)$, then the potential $V(x)$ is given by

$$V(x) = 2\, \frac{d}{dx}\, [K(x,x)] \quad , \tag{45}$$

and for $p > 0$,

$$\left.\begin{aligned} \chi(x|p) &= \frac{e^{ipx}}{\sqrt{2\pi}} + \frac{b(p)\, e^{-ipx}}{\sqrt{2\pi}} + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} K(x,y)\, [e^{ipy} + b(p)\, e^{-ipy}]\, dy \\ \chi(x|E_i) &= e^{\sqrt{-E_i}\,x} + \int_{-\infty}^{x} K(x,y)\, e^{\sqrt{-E_i}\,y}\, dy \end{aligned}\right\} \tag{46}$$

In principle, the inverse problem which requires the solution of a linear integral equation – the Gel'fand-Levitan equation – is no more difficult than the direct problem which requires the solution of a linear differential equation. We note that for fixed x the Gel'fand-Levitan equation is a Fredholm equation and hence is of a well understood standard form.

Also, as $x \to -\infty$ we have $R(x) \to 0$. Each of the terms of the sum $\sum_i e^{\sqrt{-E_i}\,x}/A_i$ certainly tends to zero as $x \to -\infty$, and with the help of the Riemann-Lebesque lemma it can be shown that the integral in the expression for $R(x)$ vanishes as $x \to -\infty$.† The given $b(p)$ will of course satisfy the relations $b(-p) = b^*(p)$, $\lim_{|p| \to \infty} b(p) = 0$ in the upper half plane. We might point out that as a consequence of the first of these two relations $R(x)$ is real and this in turn implies that $K(x,y)$ is real.

In general, it is necessary that $b(p)$, E_i, and A_i are specified before the Gel'fand-Levitan algorithm can be applied. However, for very short range potentials $b(p)$ and E_i are sufficient and A_i need not be given.

Let us specify two functions $\psi(x|p)$ and $\psi(x|E_i)$ which are solutions of the equations

$$\left.\begin{aligned} \left[-\frac{d^2}{dx^2} + V(x)\right] \psi(x|p) &= p^2\psi(x|p) \qquad \text{for } p^2 > 0 \\ \left[-\frac{d^2}{dx^2} + V(x)\right] \psi(x|E_i) &= E_i\psi(x|E_i) \qquad \text{for } E_i < 0 \end{aligned}\right\} . \tag{47}$$

The $\psi(x|E_i)$ are to be identical with the $\chi(x|E_i)$ and hence are determined by imposing the same boundary condition on them as was imposed on the $\chi(x|E_i)$, namely,

$$\int_{-\infty}^{\infty} \psi^*(x|E_i)\, \psi(x|E_j)\, dx = A_i\delta_{ij}$$

However, the boundary condition

$$\lim_{x \to -\infty} \psi(x|p) = \frac{e^{ipx}}{\sqrt{2\pi}} \qquad \text{for all } p \tag{48}$$

which we impose on the $\psi(x|p)$ is different from the condition imposed on $\chi(x|p)$. One reason for the choice of this boundary condition is convenience. Also, with this condition the $\psi(x|p)$ are analytic functions for all p. The two independent solutions $\psi(x|p)$ and $\psi(x|-p)$ may now be superimposed and we write

$$\left.\begin{aligned} \chi(x|p) &= \psi(x|p) + b(p)\, \psi(x|-p) \qquad \text{for } p > 0 \\ \chi(x|p) &= \psi(x|p) + r(p)\, \psi(x|-p) \qquad \text{for } p < 0 \end{aligned}\right\} . \tag{49}$$

Next we proceed to Part I of the proof of the Gel'fand-Levitan algorithm. In this part of the proof we shall show that

$$\left.\begin{aligned} \psi(x|p) &= \frac{e^{ipx}}{\sqrt{2\pi}} + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} K(x,y)\, e^{ipy}\, dy \\ \psi(x|E_i) &= e^{\sqrt{-E_i}x} + \int_{-\infty}^{x} K(x,y)\, e^{\sqrt{-E_i}y}\, dy \end{aligned}\right\} \tag{50}$$

satisfy the equations (47) when (45) is satisfied. It is immediately apparent that $\psi(x|p)$ as given by (50) satisfies the boundary condition(48).

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† The Riemonn-Lebesque theorem (Lighthill, Introduction to Fourier Analysis ond Generalized Functions, p. 46) stotes thot: If $b(p)$ is on ordinary function obsalutely integroble from $-\infty$ to ∞, then its Fourier transform $\to 0$ os $|x| \to \infty$. However, $b(p)$ in this cose consists of a given set of values ond is therefore certoinly obsolutely integroble over $(-\infty,\infty)$.

Next we calculate $(d^2/dx^2)\,\psi(x|p) + p^2\psi(x|p)$.

$$\frac{d}{dx}\,\psi(x|p) = i\,\frac{pe^{ipx}}{\sqrt{2\pi}} + \frac{K(x,x)}{\sqrt{2\pi}}\,e^{ipx} + \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} K_x(x,y)\,e^{ipy}\,dy \quad ,$$

$$\frac{d^2}{dx^2}\,\psi(x|p) = -p^2\,\frac{e^{ipx}}{\sqrt{2\pi}} + \frac{ipK(x,x)}{\sqrt{2\pi}}\,e^{ipx} + \frac{e^{ipx}}{\sqrt{2\pi}}\,\frac{d}{dx}\,K(x,x)$$

$$+ \frac{K_x(x,x)}{\sqrt{2\pi}}\,e^{ipx} + \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} K_{xx}(x,y)\,e^{ipy}\,dy \quad ,$$

where

$$K_x(x,x) = \frac{d}{dx}\,[K(x,y)]\big|_{y=x} \quad ,$$

$$K_y(x,x) = \frac{d}{dy}\,[K(x,y)]\big|_{y=x} \quad . \tag{51}$$

Integrating by parts and making use of the fact that as $x \to -\infty$, $R(x) \to 0$, and therefore also $K(x,y) \to 0$, we have

$$p^2\psi(x|p) = \frac{p^2\,e^{ipx}}{\sqrt{2\pi}} + \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} K(x,y)\,p^2\,e^{ipy}\,dy$$

$$= \frac{p^2\,e^{ipx}}{\sqrt{2\pi}} - \frac{ipK(x,x)}{\sqrt{2\pi}}\,e^{ipx} + \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} K_y(x,y)\,ip\,e^{ipy}\,dy$$

$$= \frac{p^2\,e^{ipx}}{\sqrt{2\pi}} - \frac{ipK(x,x)}{\sqrt{2\pi}}\,e^{ipx} + \frac{K_y(x,x)}{\sqrt{2\pi}}\,e^{ipx} - \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} K_{yy}(x,y)\,e^{ipy}\,dy \quad .$$

Hence,

$$\frac{d^2\psi(x|p)}{dx^2} + p^2\psi(x|p) = \frac{e^{ipx}}{\sqrt{2\pi}}\,\left\{\frac{d}{dx}\,[K(x,x)] + K_x(x,x) + K_y(x,x)\right\}$$

$$+ \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} [K_{xx}(x,y) - K_{yy}(x,y)]\,e^{ipy}\,dy$$

$$= \frac{2\,e^{ipx}}{\sqrt{2\pi}}\,\frac{d}{dx}\,[K(x,x)] + \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} [K_{xx}(x,y) - K_{yy}(x,y)]\,e^{ipy}\,dy \quad . \tag{52}$$

Since

$$K_x(x,x) + K_y(x,x) = \frac{d}{dx}\,[K(x,x)] \quad , \tag{53}$$

we proceed to find an expression for $K_{xx}(x,y) - K_{yy}(x,y)$. We have

$$K(x,y) = -R(x+y) - \int_{-\infty}^{x} K(x,z)\,R(z+y)\,dz \quad .$$

Henee,

$$K_x(x,y) = -R'(x+y) - K(x,x)\,R(x+y) - \int_{-\infty}^{x} K_x(x,z)\,R(z+y)\,dz \quad ,$$

$$K_{xx}(x,y) = -R''(x+y) - K(x,x)\,R'(x+y) - \frac{d}{dx}[K(x,x)]\,R(x+y)$$

$$-K_x(x,x)\,R(x+y) - \int_{-\infty}^{x} K_{xx}(x,z)\,R(z+y)\,dz \quad ,$$

$$K_y(x,y) = -R'(x+y) - \int_{-\infty}^{x} K(x,z)\,R'(z+y)\,dz$$

$$= -R'(x+y) - K(x,x)\,R(x+y) + \int_{-\infty}^{x} K_z(x,z)\,R(z+y)\,dz \quad ,$$

$$K_{yy}(x,y) = -R''(x+y) - K(x,x)\,R'(x+y) + \int_{-\infty}^{x} K_z(x,z)\,R'(z+y)\,dz$$

$$= -R''(x+y) - K(x,x)\,R'(x+y) + K_y(x,x)\,R(x+y) - \int_{-\infty}^{x} K_{zz}(x,z)\,R(z+y)\,dz \quad .$$

Then

$$K_{xx}(x,y) - K_{yy}(x,y) = -\left\{\frac{d}{dx}[K(x,x)] + K_x(x,x) + K_y(x,x)\right\} R(x+y)$$

$$-\int_{-\infty}^{x} [K_{xx}(x,z) - K_{zz}(x,z)]\,R(z+y)\,dz \quad ,$$

or using (53),

$$K_{xx}(x,y) - K_{yy}(x,y) = -2\,\frac{d}{dx}[K(x,x)]\,R(x+y)$$

$$-\int_{-\infty}^{x} [K_{xx}(x,z) - K_{zz}(x,z)]\,R(z+y)\,dz \quad . \tag{54}$$

This is an integral equation for $K_{xx}(x,y) - K_{yy}(x,y)$. Let us try to find a solution in the form

$$K_{xx}(x,y) - K_{yy}(x,y) = L(x,y)\,2\,\frac{d}{dx}[K(x,x)] \quad . \tag{55}$$

Substituting (55) in (54), we obtain

$$L(x,y) = -R(x+y) - \int_{-\infty}^{x} L(x,z)\,R(z+y)\,dz \quad . \tag{56}$$

Hence, $L(x,y)$ satisfies the same equation as $K(x,y)$. If we assume that this equation has a unique solution, then $L(x,y) = K(x,y)$ and

$$K_{xx}(x,y) - K_{yy}(x,y) = 2K(x,y)\,\frac{d}{dx}K[x,x] \quad . \tag{57}$$

Using (57) in expression (52), we obtain

$$\frac{d^2\psi(x|p)}{dx^2} + p^2\psi(x|p) = \frac{2\,e^{ipx}}{\sqrt{2\pi}} \frac{d}{dx}[K(x,x)]$$

$$+ \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} 2K(x,y) \frac{d}{dx}[K(x,x)]\, e^{ipy}\, dy$$

$$= \frac{2d[K(x,x)]}{dx} \left[\frac{e^{ipx}}{\sqrt{2\pi}} + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} K(x,y)\, e^{ipy}\, dy\right]$$

$$= 2 \frac{d}{dx}[K(x,x)]\, \psi(x|p) \quad .$$

Hence, if

$$V(x) = 2 \frac{d}{dx}[K(x,x)] \quad ,$$

then $\psi(x|p)$ satisfies Eq. (47). The proof for $\psi(x|E_i)$ is identical. This completes Part I of the proof.

In Part I of the proof of the Gel'fand-Levitan algorithm we have shown that the $\psi(x|p)$ and $\psi(x|E_i)$ as given by (50) satisfy the equations (47). Also, $\psi(x|p)$ obviously obeys the boundary condition (48). In Part II of the proof we shall prove the following completeness relationship:

$$\int_{-\infty}^{\infty} \psi(x|p)\, \psi^*(x'|p)\, dp + \int_{-\infty}^{\infty} \psi(x|p)\, b(-p)\, \psi^*(x'|-p)\, dp$$

$$+ \sum_i \frac{\psi(x|E_i)\, \psi(x'|E_i)}{A_i} = \delta(x - x') \quad . \tag{58}$$

We shall then use this completeness relationship to show that the $\psi(x|p)$ are linearly independent, that the $\psi(x|E_i)$ are quadratically integrable, have the normalization A_i, and are therefore the point eigenvalues of the Schrödinger equation (47), and finally, that the b(p) as used in (43) are indeed the reflection coefficients of the Schrödinger equation.

Let us first prove (58). For $x > y$ we have

$$K(x,y) = -R(x+y) - \int_{-\infty}^{x} K(x,z)\, R(z+y)\, dz$$

$$= -R(x+y) - \frac{1}{2\pi} \int_{-\infty}^{x} dz \int_{-\infty}^{\infty} K(x,z)\, b(p)\, e^{-ip(z+y)}\, dp - \int_{-\infty}^{x} \sum_i \frac{K(x,z)\, e^{\sqrt{-E_i}(z+y)}}{A_i}\, dz$$

$$= -R(x+y) - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[\psi(x|-p) - \frac{e^{-ipx}}{\sqrt{2\pi}}\right] b(p)\, e^{-ipy}\, dp$$

$$- \sum_i \frac{\left[\psi(x|E_i) - e^{\sqrt{-E_i}x}\right] e^{\sqrt{-E_i}y}}{A_i} \quad .$$

Therefore,

$$K(x,y) = -\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \psi(x|-p)\, b(p)\, e^{-ipy}\, dp - \sum_i \frac{\psi(x|E_i)\, e^{\sqrt{-E_i}y}}{A_i} \quad . \tag{59}$$

Also,

$$K(x,y) = -\int_{-\infty}^{\infty} \psi(x|-p)\, b(p) \left[\psi(y|-p) - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y} K(y,z)\, e^{-ipz}\, dz\right] dp$$

$$- \sum_i \frac{\psi(x|E_i)}{A_i} \left[\psi(y|E_i) - \int_{-\infty}^{y} K(y,z)\, e^{\sqrt{-E_i}z}\, dz\right]$$

$$= -\int_{-\infty}^{\infty} \psi(x|-p)\, b(p)\, \psi(y|-p)\, dp - \sum_i \frac{\psi(x|E_i)\, \psi(y|E_i)}{A_i}$$

$$+ \int_{-\infty}^{y} K(y,z) \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \psi(x|-p)\, b(p)\, e^{-ipz}\, dp + \sum_i \frac{\psi(x|E_i)\, e^{\sqrt{-E_i}z}}{A_i}\right] dz \quad .$$

But, according to (59), the expression in brackets in the last integral is just $-K(x,z)$. Hence, rearranging terms we now have

$$\int_{-\infty}^{\infty} \psi(x|-p)\, b(p)\, \psi(y|-p)\, dp + \sum_i \frac{\psi(x|E_i)\, \psi(y|E_i)}{A_i} = -K(x,y)$$

$$- \int_{-\infty}^{y} K(y,z)\, K(x,z)\, dz \quad . \tag{60}$$

This is true for $x > y$. However, the left-hand side of (60) is symmetric in x and y so that for $x < y$ we have

$$\int_{-\infty}^{\infty} \psi(y|-p)\, b(p)\, \psi(x|-p)\, dp + \sum_i \frac{\psi(y|E_i)\, \psi(x|E_i)}{A_i} = -K(y,x)$$

$$- \int_{-\infty}^{x} K(x,z)\, K(y,z)\, dz \quad .$$

Introducing the Heaviside step function,

$$\eta(x) = \begin{cases} 1 & \text{when } x > 0 \quad , \\ 0 & \text{when } x < 0 \quad , \end{cases} \tag{61}$$

and replacing p by $-p$ in the integral on the left-hand side, we finally have

$$\int_{-\infty}^{\infty} \psi(x|p)\, b(-p)\, \psi^*(y|-p)\, dp + \sum_i \frac{\psi(x|E_i)\, \psi(y|E_i)}{A_i} = -\eta(x-y)\, K(x,y) - \eta(y-x)\, K(y,x)$$

$$- \eta(x-y) \int_{-\infty}^{y} K(y,z)\, K(x,z)\, dz - \eta(y-x) \int_{-\infty}^{x} K(x,z)\, K(y,z)\, dz \quad . \tag{62}$$

Next, using (50), we write

$$\int_{-\infty}^{\infty} \psi(x|p)\, \psi^*(y|p)\, dp = \int_{-\infty}^{\infty} \left[\frac{e^{ipx}}{\sqrt{2\pi}} + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} K(x,z)\, e^{ipz}\, dz\right]$$

$$\times \left[\frac{e^{-ipy}}{\sqrt{2\pi}} + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y} K(y,v)\, e^{-ipv}\, dv\right] dp$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ip(x-y)}\, dp + \frac{1}{2\pi} \int_{-\infty}^{y} K(y,v)\, dv \int_{-\infty}^{\infty} e^{ip(x-v)}\, dp$$

$$+ \frac{1}{2\pi} \int_{-\infty}^{x} K(x,z)\, dz \int_{-\infty}^{\infty} e^{ip(z-y)}\, dp$$

$$+ \frac{1}{2\pi} \int_{-\infty}^{x} K(x,z)\, dz \int_{-\infty}^{y} K(y,v)\, dv \int_{-\infty}^{\infty} e^{ip(z-v)}\, dp \quad .$$

But

$$\int_{-\infty}^{\infty} e^{ip(x-y)}\, dp = 2\pi\delta(x-y) \quad .$$

Also,

$$\frac{1}{2\pi} \int_{-\infty}^{y} K(y,v)\, dv \int_{-\infty}^{\infty} e^{ip(x-v)}\, dp = \int_{-\infty}^{y} K(y,v)\, \delta(x-v)\, dv$$

$$= \begin{cases} K(y,x) & \text{when } x < y \\ 0 & \text{when } x > y \end{cases}$$

$$= \eta(y-x)\, K(y,x) \quad ,$$

$$\frac{1}{2\pi} \int_{-\infty}^{x} K(x,z)\, dz \int_{-\infty}^{\infty} e^{ip(z-y)}\, dp = \int_{-\infty}^{x} K(x,z)\, \delta(z-y)\, dz$$

$$= \begin{cases} K(x,y) & \text{when } y < x \\ 0 & \text{when } y > x \end{cases}$$

$$= \eta(x-y)\, K(x,y) \quad ,$$

$$\frac{1}{2\pi} \int_{-\infty}^{x} K(x,z)\, dz \int_{-\infty}^{y} K(y,v)\, dv \int_{-\infty}^{\infty} e^{ip(z-v)}\, dp = \int_{-\infty}^{x} K(x,z)\, dz \int_{-\infty}^{y} K(y,v)\, dv\delta(v-z)$$

$$= \int_{-\infty}^{y} K(x,z)\, K(y,z)\, dz \quad \text{when } x > y$$

$$= \int_{-\infty}^{x} K(x,z)\, K(y,z)\, dz \quad \text{when } y > x$$

$$= \eta(x-y) \int_{-\infty}^{y} K(x,z)\, K(y,z)\, dz + \eta(y-x) \int_{-\infty}^{x} K(x,z)\, K(y,z)\, dz \quad .$$

Hence,

$$\int_{-\infty}^{\infty} \psi(x|p)\, \psi^*(y|p)\, dp = \delta(x-y) + \eta(y-x)\, K(y,x) + \eta(x-y)\, K(x,y)$$

$$+ \eta(x-y) \int_{-\infty}^{y} K(x,z)\, K(y,z)\, dz + \eta(y-x) \int_{-\infty}^{x} K(y,z)\, K(x,z)\, dz \quad . \tag{63}$$

Adding (62) and (63), we obtain

$$\int_{-\infty}^{\infty} \psi(x|p)\, \psi^*(y|p)\, dp + \int_{-\infty}^{\infty} \psi(x|p)\, b(-p)\, \psi^*(y|-p)\, dp$$

$$+ \sum_i \frac{\psi(x|E_i)\, \psi(y|E_i)}{A_i} = \delta(x-y) \quad .$$

This completes the proof of the completeness relationship (58).

A direct consequence of (58) is that any quadratically integrable function f(x) can be expanded in the following manner:

$$f(x) = \int_{-\infty}^{\infty} \psi(x|p)\, a(p)\, dp + \sum_i \psi(x|E_i)\, a_i \quad , \tag{64}$$

where

$$a(p) = \int_{-\infty}^{\infty} f(y)\, \psi^*(y|p)\, dy + \int_{-\infty}^{\infty} f(y)\, b(-p)\, \psi^*(y|-p)\, dy \quad , \tag{65}$$

$$a_i = \int_{-\infty}^{\infty} \frac{f(y)\, \psi(y|E_i)}{A_i}\, dy \quad . \tag{66}$$

Let us treat two special cases:

(a) $f(x) = \psi(x|p)$,

(b) $f(x) = \psi(x|E_i)$.

$\psi(x|p)$ is not quadratically integrable, but it is still symbolically possible to treat case (a).

(a) When $f(x) = \psi(x|p)$, we see at once that in this case we must have

$$a(p) = \delta(p - p') \quad ,$$

$$a_i = 0 \quad .$$

The first of these gives

$$\int_{-\infty}^{\infty} \psi(y|p')\, \psi^*(y|p)\, dy + \int_{-\infty}^{\infty} \psi(y|p')\, b(-p)\, \psi^*(y|-p)\, dy = \delta(p-p') \quad . \tag{67}$$

The second gives

$$\int_{-\infty}^{\infty} \frac{\psi(y|p')\, \psi(y|E_i)}{A_i}\, dy = 0 \quad . \tag{68}$$

Hence, the $\psi(x|p)$ and $\psi(x|E_i)$ are orthogonal.

(b) When $f(x) = \psi(x|E_i)$, we see at once that in this case we must have

$$a(p) = 0 \quad ,$$

$$a_i = \delta_{ij} \quad .$$

From the second of these we obtain

$$\int_{-\infty}^{\infty} \frac{\psi(y|E_j)\,\psi(y|E_i)}{A_i}\,dy = \delta_{ij} \quad ,$$

or

$$\int_{-\infty}^{\infty} \psi(y|E_j)\,\psi(y|E_i)\,dy = A_i\delta_{ij} \quad . \tag{69}$$

Hence, the $\psi(x|E_i)$ are quadratically integrable and have the proper normalization.

We shall now prove that the b(p) as used in the Gel'fand-Levitan algorithm are indeed the reflection coefficients.

Let us consider the outgoing wave, i.e., $\chi(x|p)$ for $p > 0$, and assume that the reflection coefficient is $\hat{b}(p)$ and not b(p). With the help of the completeness relationships for $\psi(x|p)$ and $\chi(x|p)$, we shall show that $\hat{b}(p) \equiv b(p)$.

Since $\chi(x|p)$ and $\psi(x|p)$ are solutions of the same Schrodinger equation for different boundary conditions, we may write $\chi(x|p)$ as a linear combination of the two independent solutions $\psi(x|p)$ and $\psi(x|-p)$. We write

$$\chi(x|p) = \psi(x|p) + \hat{b}(p)\,\psi(x|-p) \quad \text{for } p > 0 \quad . \tag{70}$$

We have chosen $\psi(x|p)$ so that it satisfies the boundary condition (48), and as a result of this $\chi(x|p)$ as given by (70) will satisfy the correct boundary condition

$$\lim_{x \to -\infty} \chi(x|p) = \frac{e^{ipx}}{\sqrt{2\pi}} + \frac{\hat{b}(p)\,e^{-ipx}}{\sqrt{2\pi}} \quad \text{for } p > 0 \quad .$$

However, the $\chi(x|p)$ satisfying this boundary condition is unique, and hence $\hat{b}(p)$ is the reflection coefficient as defined previously.

Similarly, uniqueness of a solution for a given boundary condition gives us

$$\chi(x|p) = \hat{t}(p)\,\psi(x|p) \quad \text{for } p < 0 \quad . \tag{71}$$

$\hat{t}(p)$ is the transmission coefficient, and $|\hat{b}(p)|^2 + |\hat{t}(p)|^2 = 1$ for all p.

Again we choose the proper eigenfunctions $\chi(x|E_i)$ and $\psi(x|E_i)$ to be identical and to have the same normalization A_i. We might point out that since K(x, y) is real, $\psi(x|E_i)$ and $\chi(x|E_i)$ are real.

We now have the completeness relationship (11) for the $\chi(x|p)$ and $\chi(x|E_i)$, namely,

$$\int_{-\infty}^{\infty} \chi(x|p)\,\chi^*(x'|p)\,dp + \sum_i \frac{\chi(x|E_i)\,\chi(x'|E_i)}{A_i} = \delta(x - x') \quad ,$$

and the completeness relationship (58) for the $\psi(x|p)$ and $\psi(x|E_i) = \chi(x|E_i)$, namely,

$$\int_{-\infty}^{\infty} \psi(x|p)\,\psi^*(x'|p)\,dp + \int_{-\infty}^{\infty} \psi(x|p)\,b(-p)\,\psi^*(x'|-p)\,dp + \sum_i \frac{\psi(x|E_i)\,\psi(x'|E_i)}{A_i} = \delta(x - x') \quad .$$

Let us consider (11) and substitute $\chi(x|p)$ from (70) for the range 0 to ∞, $\chi(x|p)$ from (71) for the range $-\infty$ to 0 and $\chi(x|E_i) = \psi(x|E_i)$. Then we obtain

$$\int_0^\infty \psi(x|p)\,\psi^*(x'|p)\,dp + \int_0^\infty \psi(x|p)\,\hat{b}^*(p)\,\psi^*(x'|-p)\,dp$$

$$+ \int_0^\infty \hat{b}(p)\,\psi(x|-p)\,\psi^*(x'|p)\,dp + \int_0^\infty |\hat{b}(p)|^2\,\psi(x|-p)\,\psi^*(x'|-p)\,dp$$

$$+ \int_{-\infty}^0 |\hat{t}(p)|^2\,\psi(x|p)\,\psi^*(x'|p)\,dp + \sum_i \frac{\psi(x|E_i)\,\psi(x'|E_i)}{A_i} = \delta(x - x') \qquad (72)$$

We note at once that

$$\int_0^\infty |\hat{b}(p)|^2\,\psi(x|-p)\,\psi^*(x'|-p)\,dp = \int_{-\infty}^0 |\hat{b}(p)|^2\,\psi(x|p)\,\psi^*(x'|p)\,dp \quad .$$

Also, since $|\hat{b}(p)|^2 + |\hat{t}(p)|^2 = 1$, we can add the fourth and fifth integrals of (72) and obtain as their sum

$$\int_{-\infty}^0 \psi(x|p)\,\psi^*(x'|p)\,dp$$

This integral in turn, when added to the first integral of (72), gives for their sum

$$\int_{-\infty}^\infty \psi(x|p)\,\psi^*(x'|p)\,dp \quad .$$

Finally, we note that since $\hat{b}(-p) = \hat{b}^*(p)$,

$$\int_0^\infty \psi(x|p)\,\hat{b}^*(p)\,\psi^*(x'|-p)\,dp + \int_0^\infty \hat{b}(p)\,\psi(x|-p)\,\psi^*(x'|p)\,dp$$

$$= \int_0^\infty \psi(x|p)\,\hat{b}(-p)\,\psi^*(x'|-p)\,dp + \int_{-\infty}^0 \hat{b}(-p)\,\psi(x|p)\,\psi^*(x'|-p)\,dp$$

$$= \int_{-\infty}^\infty \psi(x|p)\,\hat{b}(-p)\,\psi^*(x'|-p)\,dp \quad .$$

With these simplifications, (72) now becomes

$$\int_{-\infty}^\infty \psi(x|p)\,\psi^*(x'|p)\,dp + \int_{-\infty}^\infty \psi(x|p)\,\hat{b}(-p)\,\psi^*(x'|-p)\,dp$$

$$+ \sum_i \frac{\psi(x|E_i)\,\psi(x'|E_i)}{A_i} = \delta(x - x') \quad . \qquad (73)$$

It now appears that, as a consequence of (73), we can expand any quadratically integrable function $f(x)$ in the following manner:

$$f(x) = \int_{-\infty}^\infty \psi(x|p)\,\hat{a}(p)\,dp + \sum_i \psi(x|E_i)\,a_i \quad ,$$

where

$$\hat{a}(p) = \int_{-\infty}^{\infty} f(y)\, \psi^*(y|p)\, dy + \int_{-\infty}^{\infty} f(y)\, \hat{b}(-p)\, \psi^*(y|-p)\, dy \quad ,$$

$$a_i = \int_{-\infty}^{\infty} \frac{f(y)\, \psi(y|E_i)}{A_i}\, dy \quad .$$

However, it follows immediately that

$$\hat{a}(p) \equiv a(p) \quad ,$$

where a(p) is given by (65). Hence, for any quadratically integrable function f(x) we must have for all p

$$[\hat{b}(-p) - b(-p)] \int_{-\infty}^{\infty} f(x)\, \psi(x|p)\, dx \equiv 0 \quad .$$

This can be true only if $b(p) = \hat{b}(p)$ for all p.

V. SOME EXAMPLES OF THE SOLUTION OF INVERSE SCATTERING PROBLEMS

We shall now give some examples of the extreme potency of the Gel'fand-Levitan algorithm and show how it can be used to reconstruct the entire scattering problem from the scattering coefficients.

A. The Delta Function Potential

First let us consider the case for which there are no bound states and

$$b(p) = -i\, \frac{A}{2}\, \frac{e^{-2i\alpha p}}{p + \frac{iA}{2}} \quad . \tag{74}$$

This reflection coefficient is of the proper form, since it satisfies the relations

$$b(-p) = b^*(p) \quad ,$$

$$\lim_{|p| \to \infty} b(p) = 0 \quad \text{in the upper half plane} \quad .$$

In this case, we now have

$$R(x) = -\frac{iA}{4\pi} \int_{-\infty}^{\infty} \frac{e^{-ip(2\alpha + x)}}{p + \frac{iA}{2}}\, dp \quad . \tag{75}$$

Evaluating the integral in the usual fashion by contour integration, i.e., closing the contour in the upper half plane when $x < -2\alpha$, we obtain

$$R(x) = 0 \quad \text{for } x < -2\alpha \quad .$$

Similarly, closing the contour in the lower half plane when $x > -2\alpha$, we obtain

$$R(x) = -\frac{A}{2}\, e^{-\frac{A}{2}(x + 2\alpha)} \quad \text{when } x > -2\alpha \quad .$$

Hence,

$$R(x) = -\frac{A}{2}\,\eta(x+2\alpha)\,e^{-\frac{A}{2}(x+2\alpha)} \quad . \tag{76}$$

The Gel'fand-Levitan equation for $x > y$ is then, for the given example,

$$K(x,y) = \frac{A}{2}\,\eta(x+y+2\alpha)\,e^{-\frac{A}{2}(x+y+2\alpha)} + \frac{A}{2}\int_{-\infty}^{x} K(x,z)\,\eta(z+y+2\alpha)\,e^{-\frac{A}{2}(z+y+2\alpha)}\,dz \quad . \tag{77}$$

We note than when

$$x < -\alpha \quad ,$$

$$x + y < -2\alpha \quad ,$$

$$z + y < -2\alpha \quad ,$$

and

$$K(x,y) \equiv 0 \quad ,$$

and hence also

$$V(x) \equiv 0 \quad .$$

Equation (77) suggests that we write

$$K(x,y) = \eta(x+y+2\alpha)\,g(x,y) \quad . \tag{78}$$

Substituting this expression for $K(x,y)$, we obtain

$$\eta(x+y+2\alpha)\,g(x,y) = \frac{A}{2}\,\eta(x+y+2\alpha)\,e^{-\frac{A}{2}(x+y+2\alpha)} + \frac{A}{2}\int_{-\infty}^{x} \eta(x+z+2\alpha)\,\eta(y+z+2\alpha)\,g(x,z)\,e^{-\frac{A}{2}(y+z+2\alpha)}\,dz \quad .$$

Consider the product of two step functions. Evidently,

$$\eta(x-a)\,\eta(x-b) = \begin{cases} \eta(x-a) & \text{when } a > b \\ \eta(x-b) & \text{when } b > a \end{cases}$$

$$= \eta(a-b)\,\eta(x-a) + \eta(b-a)\,\eta(x-b) \quad .$$

Hence,

$$\eta(x+z+2\alpha)\,\eta(y+z+2\alpha) = \eta(y-x)\,\eta(x+z+2\alpha) + \eta(x-y)\,\eta(y+z+2\alpha)$$

$$= \eta(y+z+2\alpha) \quad ,$$

since $x > y$.

Using this expression for the product of the step functions, we now have

$$\eta(x + y + 2\alpha)\, g(x,y) = \frac{A}{2}\, \eta(x + y + 2\alpha)\, e^{-\frac{A}{2}(x+y+2\alpha)}$$

$$+ \frac{A}{2} \int_{-\infty}^{x} \eta(y + z + 2\alpha)\, g(x,z)\, e^{-\frac{A}{2}(y+z+2\alpha)}$$

$$= \frac{A}{2}\, \eta(x + y + 2\alpha)\, e^{-\frac{A}{2}(x+y+2\alpha)}$$

$$+ \frac{A}{2}\, \eta(x + y + 2\alpha) \int_{-(y+2\alpha)}^{x} g(x,z)\, e^{-\frac{A}{2}(y+z+2\alpha)}\, dz$$

We have already shown that, for $x + y + 2\alpha < 0$, $K(x,y) = 0$. Hence we now treat the case $x + y + 2\alpha > 0$ so that $\eta(x + y + 2\alpha) = 1$. Then

$$g(x,y) = \frac{A}{2}\, e^{-\frac{A}{2}(x+y+2\alpha)} + \frac{A}{2} \int_{-(y+2\alpha)}^{x} g(x,z)\, e^{-\frac{A}{2}(y+z+2\alpha)}\, dz$$

$$= e^{-\frac{A}{2}(y+2\alpha)}\, f(x,y) \quad , \tag{79}$$

where

$$f(x,y) = \frac{A}{2}\, e^{-\frac{A}{2}x} + \frac{A}{2}\, e^{-A\alpha} \int_{-(y+2\alpha)}^{x} e^{-Az}\, f(x,z)\, dz \quad , \tag{80}$$

and

$$\frac{df(x,y)}{dy} = \frac{A}{2}\, e^{A(y+\alpha)}\, f(x, -y - 2\alpha) \quad . \tag{81}$$

We can satisfy (81) by letting

$$f(x,y) = e^{\frac{A}{2}y}\, v(x) \quad ,$$

and substituting in (80) we obtain

$$e^{\frac{A}{2}y}\, v(x) = \frac{A}{2}\, e^{-\frac{A}{2}x} + \frac{A}{2}\, e^{-A\alpha}\, v(x) \int_{-(y+2\alpha)}^{x} e^{-\frac{A}{2}z}\, dz$$

$$= \frac{A}{2}\, e^{-\frac{A}{2}x} - e^{-A\alpha}\, v(x) \left[e^{-\frac{A}{2}x} - e^{\frac{A}{2}(y+2\alpha)} \right] \quad ,$$

which gives

$$v(x) = \frac{A}{2}\, e^{A\alpha} \quad .$$

Hence,

$$f(x,y) = \frac{A}{2}\, e^{\frac{A}{2}(y+2\alpha)} \quad ,$$

and

$$g(x,y) = \frac{A}{2} \quad .$$

Then

$$K(x,y) = \eta(x + y + 2\alpha)\,\frac{A}{2} \quad , \tag{82}$$

and

$$V(x) = 2\,\frac{d}{dx}\,[K(x,x)]$$
$$= A\delta(x + \alpha) \quad . \tag{83}$$

Having determined K(x, y), we can now also find the eigenfunctions $\psi(x|p)$:

$$\psi(x|p) = \frac{e^{ipx}}{\sqrt{2\pi}} + \frac{A}{2\sqrt{2\pi}} \int_{-\infty}^{x} \eta(y + x + 2\alpha)\, e^{ipy}\, dy$$

$$= \frac{e^{ipx}}{\sqrt{2\pi}} + \frac{A}{2\sqrt{2\pi}}\, \eta(x + \alpha) \int_{-(x+2\alpha)}^{x} e^{ipy}\, dy$$

$$= \frac{e^{ipx}}{\sqrt{2\pi}} \left[1 - \frac{Ai}{2p}\,\eta(x + \alpha)\right] + \frac{Ai}{2p}\,\eta(x + \alpha)\, e^{-2ip\alpha}\, \frac{e^{-ipx}}{\sqrt{2\pi}} \quad . \tag{84}$$

From this we see at once that $\psi(x|p)$ satisfies (48), namely,

$$\lim_{x \to -\infty} \psi(x|p) = \frac{e^{ipx}}{\sqrt{2\pi}} \quad \text{for all } p \quad .$$

Also,

$$\frac{d\psi(x|p)}{dx} = \frac{ip\, e^{ipx}}{\sqrt{2\pi}} \left[1 - \frac{Ai}{2p}\,\eta(x + \alpha)\right] - \frac{Ai}{2p}\,\delta(x + \alpha)\, \frac{e^{ipx}}{\sqrt{2\pi}}$$
$$+ \frac{Ai}{2p}\,\delta(x + \alpha)\, e^{-2ip\alpha}\, \frac{e^{-ipx}}{\sqrt{2\pi}} + \frac{A}{2}\,\eta(x + \alpha)\, e^{-2ip\alpha}\, \frac{e^{-ipx}}{\sqrt{2\pi}}$$

$$= \frac{ip\, e^{ipx}}{\sqrt{2\pi}} + \frac{Ai}{2p}\, \frac{\delta(x + \alpha)}{\sqrt{2\pi}}\, (e^{-2ip\alpha}\, e^{-ipx} - e^{ipx})$$
$$+ \frac{A}{2}\, \frac{\eta(x + \alpha)}{\sqrt{2\pi}}\, (e^{ipx} + e^{-2ip\alpha}\, e^{-ipx})$$

$$= \frac{ip\, e^{ipx}}{\sqrt{2\pi}} + \frac{A}{2}\, \frac{\eta(x + \alpha)}{\sqrt{2\pi}}\, (e^{ipx} + e^{-2ip\alpha}\, e^{-ipx}) \quad ,$$

$$\frac{d^2\psi(x|p)}{dx^2} = -p^2\, \frac{e^{ipx}}{\sqrt{2\pi}} + \frac{A}{2}\, \frac{\delta(x + \alpha)}{\sqrt{2\pi}}\, (e^{ipx} + e^{-2ip\alpha}\, e^{-ipx})$$
$$+ \frac{ipA}{2}\, \frac{\eta(x + \alpha)}{\sqrt{2\pi}}\, (e^{ipx} - e^{-2ip\alpha}\, e^{-ipx})$$

$$\frac{d^2\psi(x|p)}{dx^2} = -p^2 \frac{e^{ipx}}{\sqrt{2\pi}} + \frac{ipA}{2} \frac{\eta(x+\alpha)}{\sqrt{2\pi}} (e^{ipx} - e^{-2ip\alpha} e^{-ipx}) + \frac{A}{\sqrt{2\pi}} e^{-ip\alpha} .$$

Hence,

$$-\frac{d^2\psi(x|p)}{dx^2} + V(x)\,\psi(x|p) = p^2 \left\{ \frac{e^{ipx}}{\sqrt{2\pi}} [1 - \frac{Ai}{2p} \eta(x+\alpha)] + \frac{Ai}{2p} \eta(x+\alpha)\, e^{-2ip\alpha} \frac{e^{-ipx}}{\sqrt{2\pi}} \right\} - \frac{A}{\sqrt{2\pi}} e^{-ip\alpha}$$

$$+ A\delta(x+\alpha) \left\{ \frac{e^{ipx}}{\sqrt{2\pi}} [1 - \frac{Ai}{2p} \eta(x+\alpha)] + \frac{Ai}{2p} \eta(x+\alpha)\, e^{-2ip\alpha} \frac{e^{-ipx}}{\sqrt{2\pi}} \right\} = p^2 \psi(x|p) .$$

$\psi(x|p)$ satisfies the Schrödinger equation:

$$\left[-\frac{d^2}{dx^2} + A\delta(x+\alpha) \right] \psi(x|p) = p^2 \psi(x|p) , \qquad (85)$$

subject to the boundary condition

$$\lim_{x\to-\infty} \psi(x|p) = \frac{e^{ipx}}{\sqrt{2\pi}} \quad \text{for all } p ,$$

and hence for $p > 0$,

$$\chi(x|p) = \psi(x|p) + b(p)\,\psi(x|-p) \qquad (86)$$

is the unique solition which satisfies the same Schrödinger equation subject to the different boundary condition

$$\lim_{x\to-\infty} \chi(x|p) = \frac{e^{ipx}}{\sqrt{2\pi}} + \frac{b(p)\, e^{-ipx}}{\sqrt{2\pi}} .$$

Substituting in (86) from (74) and (84), we have

$$\chi(x|p) = \psi(x|p) + b(p)\,\psi(x|-p)$$

$$= \frac{e^{ipx}}{\sqrt{2\pi}} [1 - \frac{Ai}{2p} \eta(x+\alpha)] + \frac{Ai}{2p} \eta(x+\alpha)\, e^{-2ip\alpha} \frac{e^{-ipx}}{\sqrt{2\pi}}$$

$$- \frac{Ai}{2} \frac{e^{-2ip\alpha}}{(p + \frac{iA}{2})} \frac{e^{-ipx}}{\sqrt{2\pi}} [1 + \frac{Ai}{2p} \eta(x+\alpha)] - \frac{A^2 \eta(x+\alpha)\, e^{ipx}}{4p(p + i\frac{A}{2})\sqrt{2\pi}}$$

$$= \frac{e^{ipx}}{\sqrt{2\pi}} \left[1 - \frac{Ai\eta(x+\alpha)}{2p} - \frac{A^2 \eta(x+\alpha)}{4p(p + i\frac{A}{2})} \right]$$

$$+ \frac{e^{-ipx}}{\sqrt{2\pi}} \left\{ \frac{Ai}{2p} \eta(x+\alpha) - \frac{Ai}{2} \frac{[1 + \frac{Ai}{2p} \eta(x+\alpha)]}{(p + \frac{iA}{2})} \right\} e^{-2i\alpha p} .$$

Letting $x \to \infty$, we can now also find the transmission coefficient $t(p)$:

$$\lim_{x\to\infty} \chi(x|p) = \frac{e^{ipx}}{\sqrt{2\pi}} \left[1 - \frac{Ai}{2p} - \frac{A^2}{4p(p + \frac{iA}{2})}\right] + \frac{e^{-ipx}}{\sqrt{2\pi}} \left[\frac{Ai}{2p} - \frac{Ai}{2p} \frac{(1 + \frac{Ai}{2p})}{(1 + \frac{Ai}{2p})}\right] e^{-2i\alpha p} ,$$

or

$$\lim_{x\to\infty} \chi(x|p) = t(p) \frac{e^{ipx}}{\sqrt{2\pi}} ,$$

where

$$t(p) = 1 - \frac{Ai}{2p} - \frac{A^2}{2p(2p + iA)} ,$$

or

$$t(p) = \frac{2p}{2p + iA} . \tag{87}$$

Also, as has been shown to be generally true,

$$t(-p) = \frac{-2p}{-2p + iA} = \frac{2p}{2p - iA} = t^*(p) ,$$

and

$$|b(p)|^2 + |t(p)|^2 = \frac{A^2}{4p^2 + A^2} + \frac{4p^2}{4p^2 + A^2} = 1 .$$

We can also find r(p), the reflection coefficient, from the other side by proceeding in the following manner.

For $p < 0$,

$$\lim_{x\to-\infty} \chi(x|p) = s(p) \frac{e^{ipx}}{\sqrt{2\pi}} .$$

It was shown earlier in this report that

$$s(p) = t(p) \quad \text{for all } p .$$

Hence, we write for $p < 0$:

$$\chi(x|p) = s(p)\, \psi(x|p)$$

$$= \frac{2p}{2p - iA} \left\{ \frac{e^{ipx}}{\sqrt{2\pi}} [1 - \frac{Ai}{2p} \eta(x + \alpha)] + \frac{Ai}{2p} \eta(x + \alpha)\, e^{-2ip\alpha} \frac{e^{-ipx}}{\sqrt{2\pi}} \right\} .$$

In order to find r(p), we now let $x \to +\infty$ and find

$$\lim_{x\to\infty} \chi(x|p) = \frac{2p}{2p - iA} \left[\frac{e^{ipx}}{\sqrt{2\pi}} (1 - \frac{Ai}{2p}) + \frac{Ai}{2p} e^{-2ip\alpha} \frac{e^{ipx}}{\sqrt{2\pi}}\right]$$

$$= \frac{e^{ipx}}{\sqrt{2\pi}} + \frac{Ai}{2p - Ai} e^{-2ip\alpha} \frac{e^{-ipx}}{\sqrt{2\pi}} .$$

Hence,

$$r(p) = \frac{Ai}{2p - Ai} e^{-2ip\alpha} .$$

We might note that only for $\alpha = 0$ do we have $r(p) = b(p)$ for all p.

Before proceeding to further examples, we emphasize again the extreme potency of the Gel'fand-Levitan algorithm. In the previous example, we have reconstructed the entire scattering problem from a knowledge of the reflection coefficient on one side. We have found the potential function producing this reflection coefficient, the corresponding eigenfunctions of the Schrödinger equation, the transmission coefficient, and the reflection coefficient on the other side.

B. Reflectionless Potential

In this example, we shall use the Gel'fand-Levitan algorithm to construct a reflectionless potential.

We put $b(p) = 0$, and let us also assume that there is but one bound state, E_1, having a normalization A_1. Then

$$R(x) = \frac{e^{\sqrt{-E_1}x}}{A_1} , \tag{88}$$

$$K(x,y) = -\frac{e^{\sqrt{-E_1}(x+y)}}{A_1} - \int_{-\infty}^{x} K(x,z) \frac{e^{\sqrt{-E_1}(z+y)}}{A_1} dz$$

$$= -\frac{e^{\sqrt{-E_1}y}}{A_1} \left\{ e^{\sqrt{-E_1}x} + \int_{-\infty}^{x} K(x,z)\, e^{\sqrt{-E_1}z} \right\} dz . \tag{89}$$

Equation (89) suggests that we write

$$K(x,y) = -\frac{e^{\sqrt{-E_1}y}}{A_1} f(x) , \tag{90}$$

where

$$f(x) = e^{\sqrt{-E_1}x} - \frac{f(x)}{A_1} \int_{-\infty}^{x} e^{2\sqrt{-E_1}z} dz$$

$$= e^{\sqrt{-E_1}x} - \frac{f(x)\, e^{2\sqrt{-E_1}x}}{2A_1\sqrt{-E_1}} , \tag{91}$$

or

$$f(x) = \frac{2A_1\sqrt{-E_1}}{2A\sqrt{-E_1}\, e^{-\sqrt{-E_1}x} + e^{\sqrt{-E_1}x}} , \tag{92}$$

and

$$K(x,y) = \frac{-2\sqrt{-E_1}\, e^{\sqrt{-E_1}y}}{e^{\sqrt{-E_1}x} + 2A\sqrt{-E_1}\, e^{-\sqrt{-E_1}x}}$$

$$= -\sqrt{-E_1}\, e^{\sqrt{-E_1}(y-x_o)} \operatorname{sech} \sqrt{-E_1}\,(x - x_o) \quad , \tag{93}$$

where

$$x_o = \frac{1}{2\sqrt{-E_1}} \ln 2A_1\sqrt{-E_1} \quad . \tag{94}$$

Hence, we obtain a reflectionless potential $V(x)$ given by

$$V(x) = 2 \frac{d}{dx} [K(x,x)] = 2E_1 \operatorname{sech}^2\sqrt{-E_1}\,(x - x_o) \quad . \tag{95}$$

Also,

$$\psi(x|p) = \frac{e^{ipx}}{\sqrt{2\pi}} - \frac{\sqrt{-E_1}}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{\sqrt{-E_1}(y-x_o)} \operatorname{sech} \sqrt{-E_1}\,(x - x_o)\, e^{ipy}\, dy$$

$$= \frac{e^{ipx}}{\sqrt{2\pi}} - \sqrt{-E_1}\, \frac{\operatorname{sech}\sqrt{-E_1}\,(x - x_o)\, e^{\sqrt{-E_1}(x-x_o)}\, e^{ipx}}{(\sqrt{-E_1} + ip)\sqrt{2\pi}}$$

$$= \frac{e^{ipx}}{\sqrt{2\pi}} \left[1 - \frac{\sqrt{-E_1}\, \operatorname{sech}\sqrt{-E_1}\,(x - x_o)\, e^{\sqrt{-E_1}(x-x_o)}}{(\sqrt{-E_1} + ip)}\right] \quad . \tag{96}$$

Since $b(p) = 0$,

$$\chi(x|p) = \frac{e^{ipx}}{\sqrt{2\pi}} \left[1 - \frac{\sqrt{-E_1}\, \operatorname{sech}\sqrt{-E_1}\,(x - x_o)\, e^{\sqrt{-E_1}(x-x_o)}}{\sqrt{-E_1} + ip}\right] \tag{97}$$

In order to determine $t(p)$, we consider

$$\lim_{x\to\infty} \chi(x|p) = \frac{e^{ipx}}{\sqrt{2\pi}} \left[1 - \frac{2\sqrt{-E_1}}{\sqrt{-E_1} + ip}\right] \quad .$$

Hence,

$$t(p) = \frac{p + i\sqrt{-E_1}}{p - i\sqrt{-E_1}} \quad . \tag{98}$$

We may now write for $p < 0$,

$$\chi(x|p) = \frac{e^{ipx}}{\sqrt{2\pi}} \frac{(p - i\sqrt{-E_1})}{(p + i\sqrt{-E_1})} \left[1 - \frac{\sqrt{-E_1}\, \operatorname{sech}\sqrt{-E_1}\,(x - x_o)\, e^{\sqrt{-E_1}(x-x_o)}}{\sqrt{-E_1} + ip}\right] \quad . \tag{99}$$

Then

$$\lim_{x\to\infty} \chi(x|p) = \frac{e^{ipx}}{\sqrt{2\pi}} \quad ,$$

and hence r(p), the reflection coefficient on the other side, is also zero. That this must be so is apparent at once from the relations

$$|b(p)|^2 + |t(p)|^2 = 1 \qquad \text{for } p > 0 \quad ,$$

$$|r(p)|^2 + |s(p)|^2 = 1 \qquad \text{for } p < 0 \quad ,$$

$$s(p) = t(p) \qquad \text{for all } p \quad .$$

Finally, we might mention that in this case

$$\chi(x|E_1) = \psi(x|E_1) = e^{\sqrt{-E_1}x} - \sqrt{-E_1}\int_{-\infty}^{x} e^{\sqrt{-E_1}(2y-x_0)} \operatorname{sech}\sqrt{-E_1}\,(x-x_0)\,dy$$

$$= e^{\sqrt{-E_1}x} - \frac{\sqrt{-E_1}\operatorname{sech}\sqrt{-E_1}\,(x-x_0)\, e^{\sqrt{-E_1}(2x-x_0)}}{2\sqrt{-E_1}}$$

$$= \frac{e^{\sqrt{-E_1}x}}{2}\operatorname{sech}\sqrt{-E_1}\,(x-x_0) \quad . \qquad (100)$$

Again the entire scattering problem has been reconstructed with the help of the Gel'fand-Levitan algorithm.

VI. REFLECTION COEFFICIENTS FOR POTENTIALS WHICH VANISH IDENTICALLY FOR $x < -\alpha$

Unfortunately, the potentials we have been obtaining so far are not of practical interest, since they do extend to infinity and are therefore impossible to construct in actuality.

We shall next give conditions on the reflection coefficient b(p) which will result in potentials which will be identically zero on one side at least.

We first prove the following theorem:

Theorem

If $R(x) \equiv 0 \quad \text{for } x < -2\alpha$,

Then $V(x) \equiv 0 \quad \text{for } x < -\alpha$.

We hope to be able to choose b(p) and the point eigenvalues E_i so that $R(x) \equiv 0$ when $x < -2\alpha$ and thus obtain a corresponding potential V(x) which is identically zero for $x < -\alpha$. To prove the theorem, we use the Gel'fand-Levitan equation for $x > y$:

$$K(x,y) = -R(x+y) - \int_{-\infty}^{x} K(x,z)\,R(y+z)\,dz \quad .$$

If

$$x < -\alpha \quad ,$$

then

$$y < -\alpha \quad ,$$

and

$$x + y < -2\alpha \quad ,$$

hence,

$$R(x + y) \equiv 0 \quad .$$

Also,

$$z < x < -\alpha \quad ,$$

and hence

$$y + z < -2\alpha \quad ,$$

and

$$R(z + y) = 0 \quad .$$

Therefore,

$$K(x, y) \equiv 0 \quad \text{when } x < -\alpha \quad ,$$

and

$$V(x) = 2 \frac{d}{dx} [K(x, x)] \equiv 0 \quad \text{when } x < -\alpha \quad .$$

This completes the proof of the theorem.

Let us now consider b(p) in the complex plane. Suppose b(p) has poles at $p = i\tau_j$ where $\tau_j > 0$. We shall show that, if $E_j = -\tau_j^2$ and A_j is chosen properly, we will be able to make $R(x) \equiv 0$ for $x < -2\alpha$; hence, b(p) will be the reflection coefficient for a potential V(x) which is identically zero for $x < -\alpha$.

Assume that b(p) has the form

$$b(p) = g(p)\, e^{-2i\alpha p} \quad , \tag{101}$$

where

$$\left.\begin{array}{ll} g(-p) = g^*(p) & \text{on the real axis} \\ g(p) = 0(p^{-K+1}) & \text{where } K \geqslant 1 \end{array}\right\} \quad , \tag{102}$$

and g(p) has poles of residue r_j at the points

$$p = i\tau_j = i\sqrt{-E_j} > 0 \quad . \tag{103}$$

Then

$$R(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} g(p)\, e^{-ip(x+2\alpha)}\, dp + \sum_j \frac{e^{\sqrt{-E_j}x}}{A_j} \quad . \tag{104}$$

When $x < -2\alpha$,

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} g(p)\, e^{-ip(x+2\alpha)}\, dp = i \sum_j r_j\, e^{2\alpha\tau_j + \tau_j x}$$

$$= \sum_j i r_j\, e^{2\alpha\tau_j}\, e^{\sqrt{-E_j}\,x} \quad .$$

Hence, if we choose A_j so that

$$\frac{1}{A_j} = -i r_j\, e^{2\alpha\tau_j} \quad , \tag{105}$$

then $R(x) \equiv 0$ for $x < -2\alpha$, and hence $V(x) \equiv 0$ for $x < -\alpha$.

As an example, consider the case where

$$\left.\begin{aligned} b(p) &= \frac{iB}{2}\,\frac{e^{-2i\alpha p}}{p - \frac{iB}{2}} \\ E_1 &= -\frac{B^2}{4} \end{aligned}\right\} \quad . \tag{106}$$

In this case,

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} b(p)\, e^{-ipx}\, dp = \frac{iB}{4\pi}\int_{-\infty}^{\infty} \frac{e^{-ip(x+2\alpha)}}{p - \frac{iB}{2}}\, dp$$

$$= -\frac{B}{2}\, e^{B\alpha}\, e^{\frac{B}{2}x} \qquad \text{for } x < -2\alpha \quad .$$

Hence,

$$R(x) = -\frac{B}{2}\, e^{B\alpha}\, e^{\frac{B}{2}x} + \frac{e^{\frac{B}{2}x}}{A_1} \qquad \text{for } x < -2\alpha \quad . \tag{107}$$

If we choose

$$\frac{1}{A_1} = \frac{B}{2}\, e^{B\alpha} \quad , \tag{108}$$

then

$$R(x) \equiv 0 \qquad \text{for } x < -2\alpha \quad .$$

When $x > -2\alpha$, we close the contour used to evaluate the integral in the expression for $R(x)$ in the lower half plane. This contour contains no poles of the integrand, and hence in this case the value of the integral is zero. Then

$$R(x) = \frac{e^{\frac{B}{2}x}}{A_1} = \frac{B}{2}\, e^{\frac{B}{2}(x+2\alpha)} \qquad \text{for } x > -2\alpha \quad . \tag{109}$$

We now have

$$R(x) = \frac{B}{2}\,\eta(x + 2\alpha)\, e^{\frac{B}{2}(x+2\alpha)} \quad , \tag{110}$$

and for $x > y$,

$$K(x,y) = -\frac{B}{2}\,\eta(x + y + 2\alpha)\, e^{\frac{B}{2}(x+y+2\alpha)}$$

$$-\frac{B}{2}\int_{-\infty}^{x} K(x,z)\,\eta(y + z + 2\alpha)\, e^{\frac{B}{2}(y+z+2\alpha)}\, dz \quad , \tag{111}$$

when

$$x < -\alpha \quad ,$$

$$x + y < -2\alpha \quad ,$$

$$z + y < -2\alpha \quad ,$$

and

$$K(x,y) \equiv 0 \quad ,$$

and hence,

$$V(x) \equiv 0 \quad .$$

Equation (111) suggests that we write

$$K(x,y) = \eta(x + y + 2\alpha)\, g(x,y) \quad , \tag{112}$$

and, proceeding as in a previous example given in this report, we obtain

$$g(x,y) = -\frac{B}{2} \quad , \tag{113}$$

$$K(x,y) = -\frac{B}{2}\,\eta(x + y + 2\alpha) \quad , \tag{114}$$

$$V(x) = -B\delta(x + \alpha) \quad . \tag{115}$$

As we expected, this potential is identically zero for $x < -\alpha$. Suppose now that

$$b(p) = \frac{iB}{2}\,\frac{1}{p - \frac{iB}{2}} \tag{116}$$

and that there are no point eigenvalues. We will now not be able to make $R(x) \equiv 0$ for $x < -2\alpha$. Let us see what the corresponding potential will be in this case.

We have

$$R(x) = \begin{cases} -\frac{B}{2}\, e^{\frac{B}{2}x} & \text{for } x < 0 \quad , \\ 0 & \text{for } x > 0 \quad , \end{cases}$$

or

$$R(x) = -\frac{B}{2}\,\eta(-x)\, e^{\frac{B}{2}x} \quad . \tag{117}$$

Hence, for $x > y$,

$$K(x,y) = \frac{B}{2}\,\eta(-x-y)\,e^{\frac{B}{2}(x+y)} + \frac{B}{2}\int_{-\infty}^{x} K(x,z)\,\eta(-y-z)\,e^{\frac{B}{2}(y+z)}\,dz \quad . \tag{118}$$

The solution of this equation may be shown to be

$$K(x,y) = \frac{B\eta(-x)\,e^{\frac{B}{2}(x+y)}}{2-e^{Bx}} + \frac{B}{2}\,\eta(x)\,\eta(x+y) + B\eta(x)\,\eta(-x-y)\,e^{\frac{B}{2}(x+y)} \quad . \tag{119}$$

Then

$$K(x,x) = \frac{B\eta(-x)\,e^{Bx}}{2-e^{Bx}} + \frac{B}{2}\,\eta(x) \quad , \tag{120}$$

and

$$V(x) = 2\,\frac{d}{dx}\,[K(x,x)] = \frac{4B^2\eta(-x)\,e^{Bx}}{(2-e^{Bx})^2} + B\delta(x)\left(\frac{2-3\,e^{Bx}}{2-e^{Bx}}\right)$$

$$= \frac{4B^2\eta(-x)\,e^{Bx}}{(2-e^{Bx})^2} - B\delta(x) \quad . \tag{121}$$

Hence, in this case the potential has an exponential tail for $x < 0$.

Finally, let us consider as an example the case for which

$$\left.\begin{aligned} b(p) &= \frac{-e^{-2i\alpha p}}{(p+i)\,(p-i)} \\ E_1 &= -1 \end{aligned}\right\} \quad . \tag{122}$$

In this case,

$$R(x) = -\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{-ip(x+2\alpha)}}{(p+i)\,(p-i)}\,dp + \frac{e^{x}}{A_1} \quad . \tag{123}$$

When $x < -2\alpha$, the exponent in the integral is positive, and we close the contour in the upper half plane. Evaluating the integral thus, we obtain for $x < -2\alpha$:

$$-\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{-ip(x+2\alpha)}}{(p+i)\,(p-i)}\,dp = -i\,\frac{e^{x+2\alpha}}{2i} = -e^{x}\,\frac{e^{2\alpha}}{2} \quad .$$

If we choose

$$\frac{1}{A_1} = \frac{e^{2\alpha}}{2} \quad , \tag{124}$$

then $R(x) \equiv 0$ for $x < -2\alpha$, and from the theorem we expect $V(x) \equiv 0$ for $x < -\alpha$. When $x > -2\alpha$, the exponent in the integral is negative, and we close the contour in the lower half plane. Thus we obtain for $x > -2\alpha$:

$$-\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{-ip(x+2\alpha)}}{(p+i)\,(p-i)}\,dp = \frac{2\pi i}{2\pi}\,\frac{e^{-x-2\alpha}}{(-2i)} = -e^{-x}\,\frac{e^{-2\alpha}}{2} \quad .$$

Hence,

$$R(x) = -e^{-x}\frac{e^{-2\alpha}}{2} + e^{x}\frac{e^{2\alpha}}{2} = \sinh(x + 2\alpha) \quad \text{for } x < -2\alpha \quad ,$$

or

$$R(x) = \eta(x + 2\alpha)\sinh(x + 2\alpha) \quad . \tag{125}$$

For $x > y$,

$$K(x, y) = -\eta(x + y + 2\alpha)\sinh(x + y + 2\alpha) - \int_{-\infty}^{x} K(x, z)\,\eta(y + z + 2\alpha)\sinh(y + z + 2\alpha)\,dz \quad . \tag{126}$$

We note that when

$$x < -\alpha \quad ,$$

$$x + y < -2\alpha \quad ,$$

$$z + y < -2\alpha \quad ,$$

hence

$$K(x, y) \equiv 0 \quad ,$$

and

$$V(x) \equiv 0 \quad \text{when } x < -\alpha$$

as we expected. Let us now try to solve the integral equation (126) and find $V(x)$ for $x > -\alpha$. If we write

$$K(x, y) = \eta(x + y + 2\alpha)\,g(x, y) \quad , \tag{127}$$

then

$$\eta(x + y + 2\alpha)\,g(x, y) = -\eta(x + y + 2\alpha)\sinh(x + y + 2\alpha) - \int_{-\infty}^{x} g(x, z)\,\eta(x + z + 2\alpha)\,\eta(y + z + 2\alpha)\sinh(y + z + 2\alpha)\,dz$$

$$= -\eta(x + y + 2\alpha)\sinh(x + y + 2\alpha) - \int_{-\infty}^{x} g(x, z)\,\eta(y + z + 2\alpha)\sinh(y + z + 2\alpha)\,dz$$

$$= -\eta(x + y + 2\alpha)\sinh(x + y + 2\alpha) - \eta(x + y + 2\alpha)\int_{-(y+2\alpha)}^{x} g(x, z)\sinh(y + z + 2\alpha)\,dz \quad .$$

Hence,

$$g(x, y) = -\sinh(x + y + 2\alpha) - \int_{-(y+2\alpha)}^{x} g(x, z)\sinh(y + z + 2\alpha)\,dz \quad . \tag{128}$$

Also,

$$\frac{dg(x,y)}{dy} = -\cosh(x+y+2\alpha) - \int_{-(y+2\alpha)}^{x} g(x,z)\cosh(x+y+2\alpha)\,dz \quad ,$$

$$\frac{d^2g(x,y)}{dy^2} = -\sinh(x+y+2\alpha) - \int_{-(y+2\alpha)}^{x} g(x,z)\sinh(x+y+2\alpha)\,dz - g(x,-y-2\alpha) \quad ,$$

or

$$\frac{d^2g(x,y)}{dy^2} = g(x,y) - g(x,-y-2\alpha) \quad . \tag{129}$$

This is satisfied by

$$g(x,y) = v_1(x)\,u(y) + v_2(x) \quad , \tag{130}$$

where u(y) satisfied

$$\frac{d^2u(y)}{dy^2} = u(y) - u(-y-2\alpha) \quad . \tag{131}$$

From (131),

$$u(y) = \sinh\sqrt{2}\,(y+\alpha) \quad , \tag{132}$$

and

$$g(x,y) = v_1(x)\sinh\sqrt{2}\,(y+\alpha) + v_2(x) \quad . \tag{133}$$

Substituting this expression for g(x, y) in the integral equation (128), we obtain

$$\begin{aligned}
v_1(x)\sinh\sqrt{2}\,(y+\alpha) + v_2(x) &= -\sinh(x+y+2\alpha) \\
&\quad - \int_{-(y+2\alpha)}^{x} [v_1(x)\sinh\sqrt{2}\,(z+\alpha) + v_2(x)]\,[\sinh(y+z+2\alpha)]\,dz \\
&= -\sinh(x+y+2\alpha) - v_2(x)\,[\cosh(y+z+2\alpha)]_{-(y+2\alpha)}^{x} \\
&\quad - v_1(x)\left\{\frac{1}{2(1+\sqrt{2})}\sinh[(1+\sqrt{2})\,z+y+(2+\sqrt{2})\,\alpha]\right. \\
&\quad \left. - \frac{1}{2(1-\sqrt{2})}\sinh[(1-\sqrt{2})\,z+y+(2-\sqrt{2})\,\alpha]\right\}_{-(y+2\alpha)}^{x} \\
&= -\sinh(x+y+2\alpha) - v_2(x)[\cosh(y+x+2\alpha)-1] \\
&\quad - v_1(x)\left\{\frac{1}{2(1+\sqrt{2})}\sinh[x+y+2\alpha+\sqrt{2}\,(x+\alpha)]\right. \\
&\quad + \frac{1}{2(1+\sqrt{2})}\sinh\sqrt{2}\,(y+\alpha) \\
&\quad - \frac{1}{2(1-\sqrt{2})}\sinh[x+y+2\alpha-\sqrt{2}\,(x+\alpha)] \\
&\quad \left. + \frac{1}{2(1-\sqrt{2})}\sinh\sqrt{2}\,(y+\alpha)\right\} \quad ,
\end{aligned}$$

or

$$0 = -\sinh(x + y + 2\alpha) - v_2(x)\cosh(x + y + 2\alpha)$$

$$-v_1(x)\left\{\frac{1}{2(1+\sqrt{2})}\sinh[(x + y + 2\alpha) + \sqrt{2}(x+\alpha)]\right.$$

$$\left.-\frac{1}{2(1-\sqrt{2})}\sinh[x + y + 2\alpha - \sqrt{2}(x+\alpha)]\right\} ,$$

or

$$0 = -e^{x+y+2\alpha} + e^{-(x+y+2\alpha)} - v_2(x)[e^{x+y+2\alpha} + e^{-(x+y+2\alpha)}]$$

$$-v_1(x)\left\{\frac{1}{2(1+\sqrt{2})}[e^{x+y+2\alpha+\sqrt{2}(x+\alpha)} - e^{-(x+y+2\alpha)-\sqrt{2}(x+\alpha)}]\right\}$$

$$-\frac{1}{2(1-\sqrt{2})}[e^{x+y+2\alpha-\sqrt{2}(x+\alpha)} - e^{-(x+y+2\alpha)+\sqrt{2}(x+\alpha)}]$$

$$= e^{x+y+2\alpha}\left[-1 - v_2(x) - \frac{v_1(x)\,e^{\sqrt{2}(x+\alpha)}}{2(1+\sqrt{2})} + \frac{v_1(x)\,e^{-\sqrt{2}(x+\alpha)}}{2(1-\sqrt{2})}\right]$$

$$+ e^{-(x+y+2\alpha)}\left[1 - v_2(x) + \frac{v_1(x)\,e^{-\sqrt{2}(x+\alpha)}}{2(1+\sqrt{2})} - \frac{v_1(x)\,e^{\sqrt{2}(x+\alpha)}}{2(1-\sqrt{2})}\right] .$$

Hence, $v_1(x)$ and $v_2(x)$ must satisfy

$$\left.\begin{aligned} 1 + v_2(x) + \frac{v_1(x)}{2}\left[\frac{e^{\sqrt{2}(x+\alpha)}}{1+\sqrt{2}} - \frac{e^{-\sqrt{2}(x+\alpha)}}{1-\sqrt{2}}\right] = 0 \\ 1 - v_2(x) - \frac{v_1(x)}{2}\left[\frac{e^{\sqrt{2}(x+\alpha)}}{1-\sqrt{2}} - \frac{e^{-\sqrt{2}(x+\alpha)}}{1+\sqrt{2}}\right] = 0 \end{aligned}\right\} . \tag{134}$$

Therefore,

$$2 + \frac{v_1(x)}{2}\left\{e^{\sqrt{2}(x+\alpha)}\left[\frac{1}{1+\sqrt{2}} - \frac{1}{1-\sqrt{2}}\right] + e^{-\sqrt{2}(x+\alpha)}\left[\frac{1}{1+\sqrt{2}} - \frac{1}{1-\sqrt{2}}\right]\right.$$

$$= 2 + v_1(x)\sqrt{2}\,[e^{\sqrt{2}(x+\alpha)} + e^{-\sqrt{2}(x+\alpha)}] = 0 ,$$

or

$$v_1(x) = -\frac{1}{\sqrt{2}}\,\frac{2}{[e^{\sqrt{2}(x+\alpha)} + e^{-\sqrt{2}(x+\alpha)}]} = -\frac{1}{\sqrt{2}\cosh\sqrt{2}(x+\alpha)} . \tag{135}$$

Also,

$$1 + v_2(x) + \frac{[e^{\sqrt{2}(x+\alpha)} - e^{-\sqrt{2}(x+\alpha)}] - \sqrt{2}\,[e^{\sqrt{2}(x+\alpha)} + e^{-\sqrt{2}(x+\alpha)}]}{\sqrt{2}\,[e^{\sqrt{2}(x+\alpha)} + e^{-\sqrt{2}(x+\alpha)}} = 0 ,$$

and

$$v_2(x) = -\frac{1}{\sqrt{2}} \frac{[e^{\sqrt{2}(x+\alpha)} - e^{-\sqrt{2}(x+\alpha)}]}{[e^{\sqrt{2}(x+\alpha)} + e^{-\sqrt{2}(x+\alpha)}]}$$

$$= -\frac{1}{\sqrt{2}} \frac{\sinh\sqrt{2}\,(x + \alpha)}{\cosh\sqrt{2}\,(x + \alpha)} \quad . \tag{136}$$

Hence,

$$g(x,y) = -\frac{1}{\sqrt{2}} \frac{\sinh\sqrt{2}\,(x + \alpha) + \sinh\sqrt{2}\,(y + \alpha)}{\cosh\sqrt{2}\,(x + \alpha)} \quad , \tag{137}$$

and

$$K(x,y) = -\frac{\eta(x + y + 2\alpha)}{\sqrt{2}} \frac{[\sinh\sqrt{2}(x + \alpha) + \sinh\sqrt{2}(y + \alpha)]}{\cosh\sqrt{2}(x + \alpha)} \quad , \tag{138}$$

$$K(x,x) = -\sqrt{2}\,\eta(x + \alpha)\,\tanh\sqrt{2}\,(x + \alpha) \quad , \tag{139}$$

and

$$V(x) = 2\,\frac{d}{dx}\,[K(x,x)]$$

$$= -4\eta(x + \alpha)\,\operatorname{sech}^2\sqrt{2}\,(x + \alpha) \quad . \tag{140}$$

Hence, again we have obtained a potential which is identically zero for $x < -\alpha$.

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