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ASYMPTOTIC BEHAVIOR OF THE SPECTRA OF RF PULSES WITH LINEAR FREQUENCY MODULATION

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Asymptotic Behavior of the Spectra of RF Pulses with Linear Frequency Modulation

Abstract

The asymptotic spectra of wide-band radio frequency pulses with linear frequency modulation are evaluated. It is specifically shown that at frequencies (measured from the carrier frequency) much larger than the maximum FM deviation the effect of the FM is small. In particular, at a frequency 50 mc higher than the carrier frequency, the envelope of the spectrum of a 10 μsec rectangular pulse with a 10 mc FM deviation is less than 1% above the envelope of the spectrum of a pulse with no FM. The spectrum of a cosine pulse with these same characteristics is increased by less than 3%.

Introduction

It may sometimes be necessary to operate two or more closely spaced pulse compression radar transmitters simultaneously. To avoid interference each radar transmits at carrier frequencies sufficiently far from all other receiver bands to insure a minimum noise level for each receiver. The optimum design depends on a knowledge of the asymptotic spectral density of each signal transmitted.

Spectral densities of simple RF pulse shapes are not difficult to calculate. However, introduction of a wide-band linear frequency modulation makes these calculations considerably more difficult. It is often assumed that corrections to the spectrum are negligible at the frequency deviations of interest, i. e., 50 to 100 mc from the carrier frequency.

An evaluation of spectral densities including the effects of wide-band linear FM is presented here. The method used is first explained for the relatively simple case of a rectangular pulse envelope. By analytically extending the signal into the complex plane, the contour $-\frac{T}{2} \leq t \leq +\frac{T}{2}$ on the real time-axis (where T is the pulse duration) can be deformed to a new contour for which the Fourier transform is easily bounded. An upper bound for the spectrum is then readily obtained. It is further shown that the true spectrum is very close to this upper bound.

Following the treatment of the rectangular pulse, the method is extended to the case of a cosine pulse of the same time duration. In the absence of linear FM, a cosine pulse more closely approximates a realistic spectrum (asymptotic $\frac{1}{(\Delta f)^2}$ dependence) since there are no zero rise-times. A general continuous, piecewise linear pulse shape is treated in Appendix A. Here one again determines an upper bound for the spectrum which in the absence of FM reduces to the correct result. The result has been applied to the case of a symmetric trapezoidal pulse envelope.

In each example considered, it has been found that the linear FM increases the spectral density by only a few per cent at a frequency five FM deviations from the carrier frequency; the effect diminishes as $\frac{1}{(\Delta f)^2}$ for a further increase in Δf. One may thus conclude that under these circumstances neglecting linear FM in a calculation of the spectral density at these frequencies is a good approximation.

<u>Rectangular Pulse Envelope</u>

The spectrum (one-sided) of a rectangular pulse of unit height and duration T, centered at $t = 0$, with complex RF carrier f_o is

$$G(\Delta f) = \frac{\sin \pi \Delta f T}{\pi \Delta f} , \tag{1}$$

where $\Delta f = f - f_o$. The spectrum of the same rectangular pulse with linear FM is given by

$$G(\Delta f, \mu) = \int_{-T/2}^{T/2} e^{i\pi\mu t^2} e^{-i2\pi\Delta f t}\, dt$$

$$= T \int_{-1/2}^{1/2} e^{-i2\pi\left[\Delta f T x - \frac{\mu T^2}{2} x^2\right]} dx = T \int_{-1/2}^{1/2} e^{-i2\pi\varphi(x)}\, dx \quad (2)$$

where μ is the rate of change of frequency in cycles/(sec)^2. Primary interest is in the behavior of $G(\Delta f, \mu)$ at frequencies large enough so that

$$\Delta f >> \mu T >> \frac{1}{T} \quad ;$$

i. e., Δf is much larger than the bandwidth of the pulse with linear FM, which is in turn much larger than the bandwidth of the pulse without FM. Note that μT is the total frequency deviation due to FM during the pulse.

By completing the square in the exponent, the above integral can be evaluated in terms of the difference of two Fresnel integrals of large argument. Accurate tabulation of the Fresnel integrals at the large arguments required are not available; furthermore, generalization to other pulse shapes is not immediately obvious.* The integral will here be evaluated by the method of steepest descent.+ Generalization to other pulse shapes is then readily available. Let z be a complex variable; the analytic continuation of $\varphi(x)$ is

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*A geometrical evaluation of the Fresnel integral for large argument has been performed by E. B. Temple, M. I. T. Lincoln Laboratory, Personal Communication.

+See, for example, A. Erdelyi, Asymptotic Expansions, Dover, 1956.

$$\varphi(z) = \Delta f T z - \frac{1}{2}\mu T^2 z^2 \qquad (3)$$

One must then find contours in the z-plane on which the phase of the integrand is a slowly varying function. For $z = x + iy$ in (3),

$$\mathrm{Re}\{\varphi\} = \Delta f T x - \frac{1}{2}\mu T^2 (x^2 - y^2)$$

$$\mathrm{Im}\{\varphi\} = \Delta f T y - \mu T^2 xy$$

Contours for which $\mathrm{Im}\{\varphi\} = \text{const.}$ satisfy the equation

$$\Delta f T x - \frac{1}{2}\mu T^2 (x^2 - y^2) = \text{const.} \quad ,$$

or

$$y^2 - (x - \frac{\Delta f}{\mu T})^2 = \text{const.} \quad ,$$

and are therefore hyperbolas; the two asymptotes have slopes ± 1 and intersect the x-axis at $x = \frac{\Delta f}{\mu T} > 1$

Since the integrand is an analytic function of z, the contour may be deformed by integrating to infinity along the hyperbola passing through $z = -1/2$, and then back along the hyperbola passing through $z = 1/2$. (See Fig. 1. The contour shown is for $\Delta f > 0$.)

On C_o, (letting $\gamma = \frac{\mu T}{\Delta f}$)

$$x = \frac{1}{\gamma}(1 - \sqrt{(1 + \frac{\gamma}{2})^2 + \gamma^2 y^2}) \qquad \frac{dx}{dy} = - \frac{\gamma y}{\sqrt{(1 + \frac{\gamma}{2})^2 + \gamma^2 y^2}}$$

$$\mathrm{Im}\,\varphi = \Delta f T y \sqrt{(1 + \frac{\gamma}{2})^2 + \gamma^2 y^2} \quad ,$$

$$\mathrm{Re}\,\varphi = \varphi(-\frac{1}{2}) = -\frac{\Delta f T}{2}(1 + \frac{\gamma}{4})$$

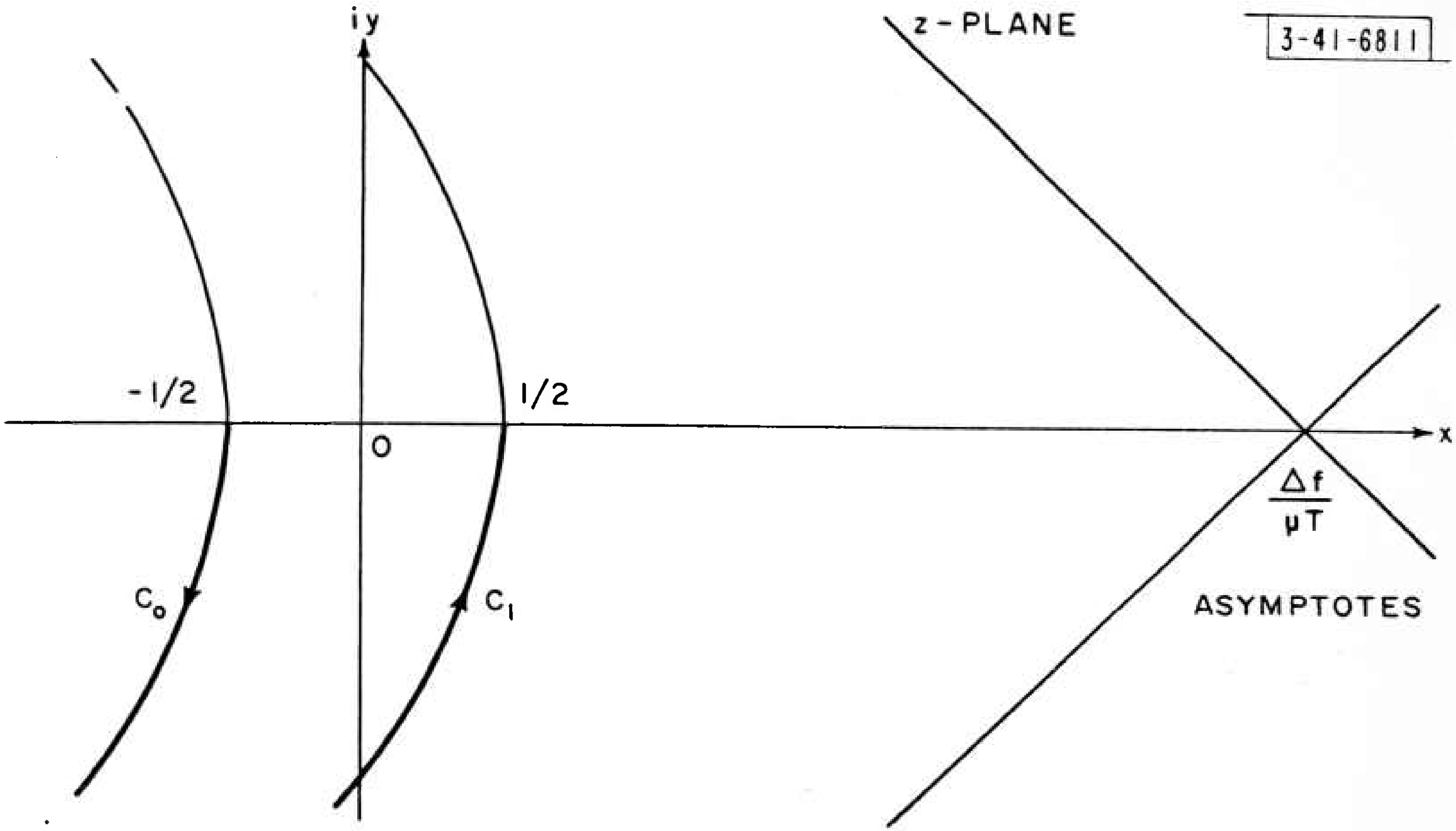


Fig. 1 Deformation of the Contour

Thus the contribution to the integral of the contour C_o is

$$\int_{C_o} e^{-i2\pi(\mathrm{Re}\,\varphi + i\,\mathrm{Im}\,\varphi)}\, dz = e^{-i2\pi\varphi(-\frac{1}{2})} \int_o^{-\infty} e^{2\pi \Delta f T y \sqrt{(1+\frac{\gamma}{2})^2 + \gamma^2 y^2}}$$

$$\left(i - \frac{\gamma y}{\sqrt{(1+\frac{\gamma}{2})^2 + \gamma^2 y^2}}\right) dy = -e^{-2\pi\varphi(-\frac{1}{2})} \int_o^{\infty} e^{-2\pi \Delta f T y \sqrt{(1+\frac{\gamma}{2})^2 + \gamma^2 y^2}}$$

$$\left(i + \frac{\gamma y}{\sqrt{(1+\frac{\gamma}{2})^2 + \gamma^2 y^2}}\right) dy \quad .$$

Similarly, on C_1

$$x = \frac{1}{\gamma}\left(1 - \sqrt{(1-\frac{\gamma}{2})^2 + \gamma^2 y^2}\right), \quad \frac{dx}{dy} = -\frac{\gamma y}{\sqrt{(1-\frac{\gamma}{2})^2 + \gamma^2 y^2}}$$

$$\mathrm{Im}\,\varphi = \Delta f T y \sqrt{(1-\frac{\gamma}{2})^2 + \gamma^2 y^2} \quad ,$$

$$\mathrm{Re}\,\varphi = \varphi(\tfrac{1}{2}) = \frac{\Delta f T}{2}\left(1 - \frac{\gamma}{4}\right) \quad .$$

The contribution from C_1 is

$$\int_{C_1} e^{-i2\pi(\mathrm{Re}\,\varphi + i\,\mathrm{Im}\,\varphi)}\, dz = e^{-i2\pi\varphi(\frac{1}{2})} \int_{-\infty}^{o} e^{2\pi \Delta f T y \sqrt{(1-\frac{\gamma}{2})^2 + \gamma^2 y^2}}$$

$$\left(i - \frac{\gamma y}{\sqrt{(1-\frac{\gamma}{2})^2 + \gamma^2 y^2}}\right) dy = e^{-i2\pi\varphi(\frac{1}{2})} \int_o^{\infty} e^{-2\pi \Delta f T y \sqrt{(1-\frac{\gamma}{2})^2 + \gamma^2 y^2}}$$

$$\left(i + \frac{\gamma y}{\sqrt{(1-\frac{\gamma}{2})^2 + \gamma^2 y^2}}\right) dy \quad .$$

Thus

$$G(\Delta f, \mu) = G_o(\Delta f, \mu) + G_1(\Delta f, \mu)$$

where

$$G_o(\Delta f, \mu) = iTe^{-i2\pi\varphi(\frac{1}{2})} \int_o^{\infty} \left(e^{-2\pi\Delta fTy\sqrt{(1-\frac{\gamma}{2})^2 + \gamma^2 y^2}} - e^{i2\pi\Delta fT} \right.$$

$$\left. e^{-2\pi\Delta fTy\sqrt{(1+\frac{\gamma}{2})^2 + \gamma^2 y^2}} \right) dy$$

and

$$G_1(\Delta f, \mu) = Te^{-i2\pi\varphi(\frac{1}{2})} \int_o^{\infty} \left(\frac{e^{-2\pi\Delta fTy\sqrt{(1-\frac{\gamma}{2})^2 + \gamma^2 y^2}}}{\sqrt{(1-\frac{\gamma}{2})^2 + \gamma^2 y^2}} - e^{i2\pi\Delta fT} \right.$$

$$\left. \frac{e^{-2\pi\Delta fTy\sqrt{(1+\frac{\gamma}{2})^2 + \gamma^2 y^2}}}{\sqrt{(1+\frac{\gamma}{2})^2 + \gamma^2 y^2}} \right) \gamma y \, dy \quad .$$

Both $|G_o(\Delta f, \mu)|$ and $|G_1(\Delta f, \mu)|$ attain their maximum values at the frequencies $\Delta f_n = \frac{2n+1}{2T}$ for $n = 0, 1, 2, \ldots, \infty$. Thus the envelopes of each are:

$$|G_o(\Delta f, \mu)|_{env.} = T \int_o^{\infty} \left(e^{-2\pi\Delta fTy\sqrt{(1-\frac{\gamma}{2})^2 + \gamma^2 y^2}} \right.$$

$$\left. + e^{-2\pi\Delta fTy\sqrt{(1+\frac{\gamma}{2})^2 + \gamma^2 y^2}} \right) dy \; ;$$

$$|G_1(\Delta f, \mu)|_{env.} = T \int_0^{\infty} \left(\frac{e^{-2\pi\Delta f T y \sqrt{(1-\frac{\gamma}{2})^2 + \gamma^2 y^2}}}{\sqrt{(1-\frac{\gamma}{2})^2 + \gamma^2 y^2}} \right.$$

$$\left. + \frac{e^{-2\pi\Delta f T y \sqrt{(1+\frac{\gamma}{2})^2 + \gamma^2 y^2}}}{\sqrt{(1+\frac{\gamma}{2})^2 + \gamma^2 y^2}} \right) \gamma y \, dy \quad .$$

Since envelopes for $G_o(\Delta f, \mu)$ and $G_1(\Delta f, \mu)$ are attained simultaneously for the same Δf_n (and $|G_1(\Delta f, \mu)|_{env.} < |G_o(\Delta f, \mu)|_{env.}$ for $\gamma < 1$), it immediately follows that (lower bound = l. b. , upper bound = u. b.)

$$|G_o(\Delta f, \mu)|_{l.\,b.\,env.} - |G_1(\Delta f, \mu)|_{u.\,b.\,env.} \leq |G(\Delta f, \mu)|_{env.}$$

$$\leq |G_o(\Delta f, \mu)|_{u.\,b.\,env.} + |G_1(\Delta f, \mu)|_{u.\,b.\,env.} \quad . \tag{4}$$

Using inequalities (B-1), (B-2), and (B-4) of Appendix B, one obtains

$$|G_o(\Delta f, \mu)|_{env.} \leq T \int_0^{\infty} \left(e^{-2\pi\Delta f T y(1-\frac{\gamma}{2})} + e^{-2\pi\Delta f T y(1+\frac{\gamma}{2})} \right) dy$$

$$= \frac{1}{2\pi\Delta f} \left(\frac{1}{1-\frac{\gamma}{2}} + \frac{1}{1+\frac{\gamma}{2}} \right)$$

and

$$|G_o(\Delta f, \mu)|_{env.} \geq 2T \int_0^{\infty} e^{-2\pi\Delta f T y \sqrt{1+\gamma^2 y^2}} \, dy \geq \frac{1}{\pi\Delta f} \left(1 - \frac{\gamma^3}{\pi\Delta f T}\right) \quad .$$

Using inequality (B-1) and

$$\frac{1}{\sqrt{(1 \pm \frac{\gamma}{2})^2 + \gamma^2 y^2}} \leq \frac{1}{1 \pm \frac{\gamma}{2}} ,$$

one further obtains

$$|G_1(\Delta f, \mu)|_{env.} \leq \frac{1}{\pi\Delta f} \left(\frac{\gamma}{2\pi\Delta fT}\right) \left(\frac{1 + \frac{3}{4}\gamma^2}{(1 - \frac{\gamma^2}{4})^3}\right) .$$

Thus

$$\frac{1}{\pi\Delta f} \left[1 - \frac{\gamma^3}{\pi\Delta fT} - \frac{\gamma(1 + \frac{3}{4}\gamma^2)}{2\pi\Delta fT(1 - \frac{\gamma^2}{4})^3}\right] \leq |G(\Delta f, \mu)|_{env.}$$

$$\leq \frac{1}{\pi\Delta f} \left[\frac{1}{1 - \frac{\gamma^2}{4}} + \frac{\gamma(1 + \frac{3}{4}\gamma^2)}{2\pi\Delta fT(1 - \frac{\gamma^2}{4})^3}\right] \qquad (5)$$

The exact spectrum envelope (see Eq. (1)) of a rectangular pulse with no FM is

$$|G(\Delta f, 0)|_{env.} = \frac{1}{\pi\Delta f} .$$

For frequencies in the asymptotic region ($\Delta f >> \mu T >> \frac{1}{T}$), the bounds in (5) are very tight. For example, if $T = 10\ \mu sec$, $\Delta f = 50$ Mc, and $\mu T = 10$ Mc, Eq. (5) becomes

$$\frac{1 - 10^{-4}}{\pi\Delta f} \leq |G(\Delta f, \mu)|_{env.} \leq \frac{1.01}{\pi\Delta f} .$$

Thus the effect of the FM on the envelope of the spectrum is quite small in the asymptotic region.

Note that a bound on the spectrum of any signal $S(t)$ which is the result of a linear operation on $g(t)$ is easily obtained from the relation

$$S(f) = H(f)\,G(f) \quad .$$

This method cannot be used, however, to extend the results of this section to other pulse shapes. For example, passing a rectangular pulse with linear FM through a filter matched to the rectangular envelope does not give a triangular pulse with linear FM.

<u>Cosine Pulse</u>

In contrast to the rectangular pulse, the cosine pulse has no zero rise-time and is therefore a more realistic model The envelope of the spectrum of the cosine pulse (in the absence of FM) is found to be

$$|G(\Delta f)|_{\text{env}} = \frac{2T}{\pi\left[(2\Delta f T)^2 - 1\right]} \quad ,$$

which has a characteristic $\frac{1}{T(\Delta f)^2}$ dependence for large Δf. If one includes linear FM, the spectral density is then

$$G(\Delta f\,\mu) = \int_{-T/2}^{T/2} \cos\frac{\pi t}{T}\, e^{-i2\pi(\Delta f t - \frac{1}{2}\mu t^2)}\, dt$$

$$= \frac{T}{2}\int_{-1/2}^{1/2} \left[e^{-i2\pi\left[\varphi(x) - \frac{x}{2}\right]} + e^{-i2\pi\left[\varphi(x) + \frac{x}{2}\right]} dx \right]$$

where $\varphi(x) = \Delta f T x - \frac{1}{2}\mu T^2 x^2$

Each of the above integrals is similar to the one encountered in the treatment of the rectangular pulse Because of the $\pm\frac{x}{2}$ term in the exponents, the two integrals are evaluated along slightly different hyperbolic contours passing through $z = -\frac{1}{2}$ and $z = \frac{1}{2}$ in the complex z-plane. The result is

$$G(\Delta f, \mu) = G_o(\Delta f, \mu) + G_1(\Delta f, \mu) \quad , \tag{6}$$

where

$$G_o(\Delta f, \mu) = -\frac{T}{2}\, e^{i2\pi\varphi(\frac{1}{2})} \Bigg[\int_o^\infty \Big(e^{-2\pi\Delta f T y \sqrt{(1 - \frac{\gamma}{2} - \frac{1}{2\Delta f T})^2 + \gamma^2 y^2}}$$

$$- e^{-2\pi\Delta f T y \sqrt{(1 - \frac{\gamma}{2} + \frac{1}{2\Delta f T})^2 + \gamma^2 y^2}} \Big) dy + e^{i2\pi\Delta f T} \int_o^\infty$$

$$\Big(e^{-2\pi\Delta f T y \sqrt{(1 + \frac{\gamma}{2} - \frac{1}{2\Delta f T})^2 + \gamma^2 y^2}} - e^{-2\pi\Delta f T y \sqrt{(1 + \frac{\gamma}{2} + \frac{1}{2\Delta f T})^2 + \gamma^2 y^2}} \Big) dy \Bigg]$$

and

$$G_1(\Delta f, \mu) = i\frac{T}{2}\, e^{i2\pi\varphi(\frac{1}{2})} \Bigg[\int_o^\infty \Bigg(\frac{e^{-2\pi\Delta f T y \sqrt{(1 - \frac{\gamma}{2} - \frac{1}{2\Delta f T})^2 + \gamma^2 y^2}}}{\sqrt{(1 - \frac{\gamma}{2} - \frac{1}{2\Delta f T})^2 + \gamma^2 y^2}}$$

$$- \frac{e^{-2\pi\Delta f T y \sqrt{(1 - \frac{\gamma}{2} + \frac{1}{2\Delta f T})^2 + \,^2 y^2}}}{\sqrt{(1 - \frac{\gamma}{2} + \frac{1}{2\Delta f T})^2 + \gamma^2 y^2}} \Bigg) \gamma y \, dy + e^{i2\pi\Delta f T} \int_o^\infty$$

$$\Bigg(\frac{e^{-2\pi\Delta f T y \sqrt{(1 + \frac{\gamma}{2} - \frac{1}{2\Delta f T})^2 + \gamma^2 y^2}}}{\sqrt{(1 + \frac{\gamma}{2} - \frac{1}{2\Delta f T})^2 + \gamma^2 y^2}} - \frac{e^{-2\pi\Delta f T y \sqrt{(1 + \frac{\gamma}{2} + \frac{1}{2\Delta f T})^2 + \gamma^2 y^2}}}{\sqrt{(1 + \frac{\gamma}{2} + \frac{1}{2\Delta f T})^2 + \gamma^2 y^2}} \Bigg)$$

$$\gamma \, y \, dy \Bigg] .$$

Each integrand for $G_o(\Delta f, \mu)$ and $G_1(\Delta f, \mu)$ is positive; thus, envelopes are obtained simultaneously at $\Delta f_n = \frac{n}{T}$ for $n = 0, 1, 2, \ldots, \infty$. Since $|G_1(\Delta f, \mu)|_{env.} < |G_o(\Delta f, \mu)|_{env.}$ for $\gamma < 1$,

$$|G_o(\Delta f, \mu)|_{l.b.\ env.} - |G_1(\Delta f, \mu)|_{u.b.\ env.} \le |G(\Delta f, \mu)|_{env.}$$

$$\le |G_o(\Delta f, \mu)|_{u.b.\ env.} + |G_1(\Delta f, \mu)|_{u.b.\ env.} \qquad (7)$$

$$|G_o(\Delta f, \mu)|_{env.} \le \frac{T}{2}\left[\int_o^\infty 2\pi y\left(e^{-2\pi\Delta f T\left(1 - \frac{\gamma}{2} - \frac{1}{2\Delta f T}\right)y} + e^{-2\pi\Delta f T\left(1 + \frac{\gamma}{2} - \frac{1}{2\Delta f T}\right)y}\right) dy\right]$$

or

$$|G_o(\Delta f, \mu)|_{env.} \le \frac{2T}{\pi(2\Delta f T - 1)^2}\left[\frac{1}{2\left(1 - \frac{\gamma}{2 - \frac{1}{\Delta f T}}\right)^2} + \frac{1}{2\left(1 + \frac{\gamma}{2 - \frac{1}{\Delta f T}}\right)^2}\right]$$

Using inequalities (B-1) and (B-4), it is possible to determine a lower bound for the envelope in the same manner as presented for the rectangular pulse. One finds the result

$$|G_o(\Delta f, \mu)|_{env.} \ge |G_o(\Delta f, \mu)|_{u.b.}\left\{1 - \frac{1}{\Delta f T\left(1 - \frac{\gamma}{2} + \frac{1}{2\Delta f T}\right)} - \frac{\gamma^3}{\pi\left(1 - \frac{\gamma}{2} - \frac{1}{2\Delta f T}\right)^2}\right\} \qquad (8)$$

To determine an upper bound for $|G_1(\Delta f, \mu)|$, let

$$\rho = 2\pi \Delta f T ,$$

and write $|G_1(\Delta f, \mu)|$ as a double integral:

$$|G_1(\Delta f, \mu)|_{env} = \frac{T}{2} \left[\int_0^\infty dy\, \gamma y \int_\rho^\infty d\rho'\, y \left(e^{-\rho' y \sqrt{\left(1 - \frac{\gamma}{2} - \frac{1}{2\Delta f T}\right)^2 + \gamma^2 y^2}} - e^{-\rho' y \sqrt{\left(1 - \frac{\gamma}{2} + \frac{1}{2\Delta f T}\right)^2 + \gamma^2 y^2}} \right) + \int_0^\infty dy\, \gamma y \int_\rho^\infty d\rho'\, y \left(e^{-\rho' y \sqrt{\left(1 + \frac{\gamma}{2} - \frac{1}{2\Delta f T}\right)^2 + \gamma^2 y^2}} - e^{-\rho' y \sqrt{\left(1 + \frac{\gamma}{2} + \frac{1}{2\Delta f T}\right)^2 + \gamma^2 y^2}} \right) \right]$$

Exchanging the order of integration and using (B-3) and (B-1)

$$|G_1(\Delta f, \mu)| \leq \frac{\gamma T}{2} \int_\rho^\infty d\rho' \int_0^\infty dy \frac{\rho'}{\Delta f T}\, y^3 \left(e^{-\rho' y \left(1 - \frac{\gamma}{2} - \frac{1}{2\Delta f T}\right)} + e^{-\rho' y \left(1 + \frac{\gamma}{2} - \frac{1}{2\Delta f T}\right)} \right)$$

$$= \frac{6\pi \gamma T}{(2\pi \Delta f T)} \int_\rho^\infty d\rho'\, \rho' \left(\frac{1}{\left[\rho'\left(1 - \frac{\gamma}{2} - \frac{1}{2\Delta f T}\right)\right]^4} + \frac{1}{\left[\rho'\left(1 + \frac{\gamma}{2} - \frac{1}{2\Delta f T}\right)\right]^4} \right)$$

$$|G_1(\Delta f, \mu)|_{env} \leq \frac{2T}{\pi (2\Delta f T)^2} \left(\frac{3\gamma}{2\pi \Delta f T} \right) \left[\frac{1}{2\left(1 - \frac{\gamma}{2} - \frac{1}{2\Delta f T}\right)^4} + \frac{1}{2\left(1 + \frac{\gamma}{2} - \frac{1}{2\Delta f T}\right)^4} \right]$$

For the case $T = 10\ \mu\text{sec}$, $\Delta f = 50$ Mc, and $\mu T = 10$ Mc, the bounds in inequality (7) are

$$1.025 |G(\Delta f, 0)|_{env} \leq |G(\Delta f, \mu)|_{env} \leq 1.03 |G(\Delta f, 0)|_{env} .$$

Thus the FM contribution to the spectral density is again quite small for $\gamma = 1/5$; this contribution continues to decrease as $\frac{1}{(\Delta f)^2}$ for a constant FM deviation.

Trapezoidal Pulse

The results obtained for a general continuous piecewise linear pulse shape (see Appendix A) are here specialized to the case of a symmetric trapezoidal pulse shape (Fig. 2).

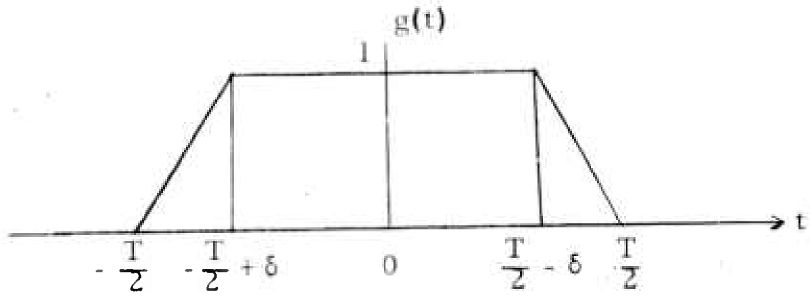


Fig. 2 Symmetric Trapezoidal Pulse

The general result (Eq. (A-3)) is

$$G(\Delta f, \mu) = T^2 \sum_k (s_k - s_{k-1}) e^{-i2\pi\varphi(x_k)} (A_k + iB_k)$$

where

$$A_k = -\int_0^\infty \frac{y}{\sqrt{1+\beta_k^2 y^2}} e^{-\alpha_k y\sqrt{1+\beta_k^2 y^2}} dy ,$$

$$B_k = \int_0^\infty \left[\frac{1}{\beta_k}\left(\sqrt{1+\beta_k^2 y^2} - 1\right) + \frac{\beta_k y^2}{\sqrt{1+\beta_k^2 y^2}}\right] e^{-\alpha_k y\sqrt{1+\beta_k^2 y^2}} dy ,$$

$$\varphi(x_k) = (\Delta f) T x_k - \frac{1}{2}\mu T^2 x_k^2 ,$$

s_k = slope between x_{k-1} and x_k ,

$$\beta_k = \frac{1}{\frac{1}{\gamma} - x_k}, \quad \alpha_k = 2\pi \Delta f T(1 - \gamma x_k)$$

and the breakpoints are at times $x_k T$. The upper bound is given by

$$|G(\Delta f, \mu)| \leq \frac{T^2}{\delta} \sum_k |A_k| + |B_k| \quad .$$

It can further be shown that

$$|A_k| \leq \frac{1}{(2\pi \Delta f T)^2 (1 - \gamma x_k)^2},$$

$$|B_k| \leq \frac{4\gamma}{(2\pi \Delta f T)^3 (1 - \gamma x_k)^4} .$$

Thus

$$|G(\Delta f, \mu)| \leq \frac{1}{(2\pi \Delta f)^2 \delta} \left[\frac{1}{(1 - \frac{\gamma}{2})^2} + \frac{1}{(1 - \frac{\gamma}{2} + \frac{\gamma \delta}{T})^2} + \frac{1}{(1 + \frac{\gamma}{2} - \frac{\gamma \delta}{T})^2} \right.$$

$$\left. + \frac{1}{(1 + \frac{\gamma}{2})^2} \right] + \frac{1}{(2\pi \Delta f)^2 \delta} \frac{2\gamma}{\pi \Delta f T} \left[\frac{1}{(1 - \frac{\gamma}{2})^4} + \frac{1}{(1 - \frac{\gamma}{2} + \frac{\gamma \delta}{T})^4} \right.$$

$$\left. + \frac{1}{(1 + \frac{\gamma}{2} - \frac{\gamma \delta}{T})^4} + \frac{1}{(1 + \frac{\gamma}{2})^4} \right] \leq \frac{1}{(\pi \Delta f)^2 \delta}$$

$$\left[\left(\frac{1}{2(1 - \frac{\gamma}{2})^2} + \frac{1}{2(1 + \frac{\gamma}{2})^2} \right) + \frac{2\gamma}{\pi \Delta f T} \left(\frac{1}{2(1 - \frac{\gamma}{2})^4} + \frac{1}{2(1 + \frac{\gamma}{2})^4} \right) \right] . \quad (9)$$

In the absence of FM, the spectrum is

$$G(\Delta f, 0) = \frac{1}{(\pi \Delta f)^2 \delta} \sin \pi \Delta f \delta \ \sin \pi \Delta f (T - \delta)$$

which is bounded by

$$|G(\Delta f, 0)| \leq \frac{1}{(\pi \Delta f)^2 \delta} .$$

For T = 10 μsec, Δf = 50 Mc, and μT = 10 Mc, the upper bound in Eq. (9) becomes

$$|G(\Delta f, \mu)| \leq \frac{1.032}{(\pi \Delta f)^2 \delta}$$

which is only 3.2 per cent higher than the upper bound in the absence of FM.

Conclusions

The results summarized below are valid in the asymptotic region of the spectrum defined by

$$\Delta f >> \mu T >> \frac{1}{T}$$

where

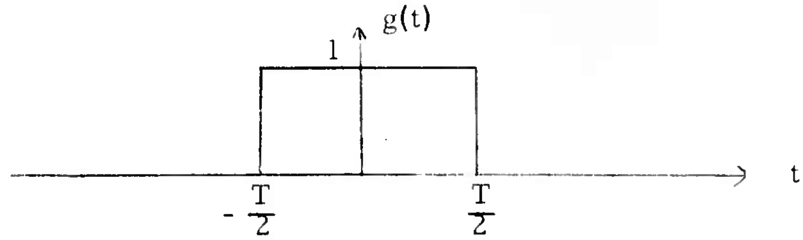
Δf = deviation from carrier frequency,

T = pulse length,

μ = frequency sweep rate,

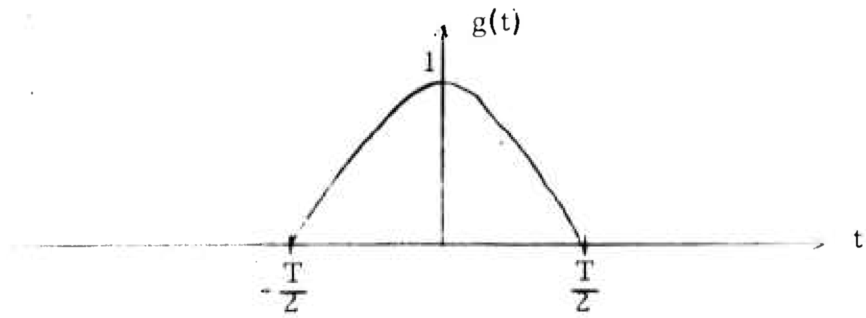
and $G(\Delta f, \mu)$ denotes the spectrum of the pulse in question for a given value of μ.

Rectangular Pulse



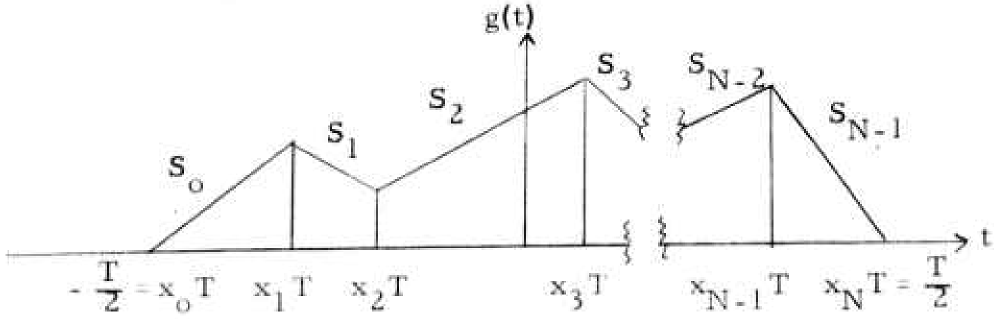
$$\left| G(\Delta f, \mu) \right|_{env} \cong \frac{1}{\pi \Delta f} \; \frac{1}{1 - \left(\frac{\gamma}{2}\right)^2}$$

Cosine Pulse



$$\left| G(\Delta f, \mu) \right|_{env} \cong \frac{T}{\pi (2\Delta f T - 1)^2} \left[\frac{1}{\left(1 - \frac{\mu T^2}{2\Delta f T - 1}\right)^2} + \frac{1}{\left(1 + \frac{\mu T^2}{2\Delta f T - 1}\right)^2} \right]$$

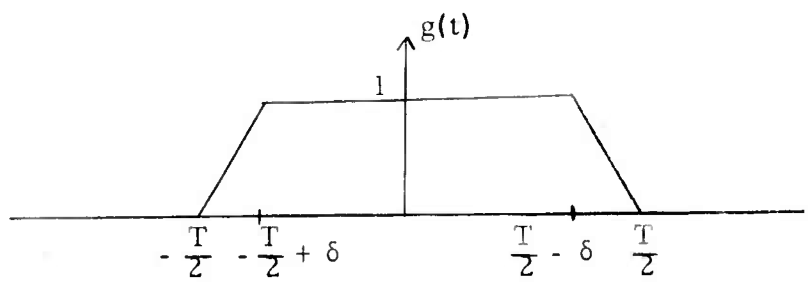
Continuous Piecewise Linear Pulse



The s_k are the slopes of the linear segments. The magnitude of the spectrum has the following bound

$$|G(\Delta f, \mu)| \le \frac{1}{(2\pi\Delta f)^2} \frac{1}{(1 - \frac{\gamma}{2})^2} \left[1 + \frac{2\gamma}{\pi \Delta f T (1 - \frac{\gamma}{2})^2} \right] \sum_k |s_k - s_{k-1}|$$

Trapezoidal Pulse (special case of a continuous, piecewise linear pulse)



$$|G(\Delta f, \mu)|_{env} \le \frac{1}{(\pi \Delta f)^2 \delta} \left[\frac{1}{2(1 - \frac{\gamma}{2})^2} + \frac{1}{2(1 + \frac{\gamma}{2})^2} + o(\frac{\gamma}{\Delta f T}) \right]$$

APPENDIX A

Continuous, Piecewise Linear Pulse

The continuous, piecewise linear pulse, $g(t)$, can be written

$$g(t) = g_k + s_k(t - t_k) , \qquad \text{for} \qquad t_k \leq t \leq t_{k+1} ,$$

where t_k, $k = 0, \ldots N$ are the breakpoints and s_k is the slope of the linear segment between t_k and t_{k+1}. Outside the interval $(-T/2,\ T/2)$, $g(t) = 0$. The asymptotic region for the spectral density is defined by

$$\Delta f >> \mu t >> \frac{1}{T} .$$

The spectrum of such a pulse with linear FM is

$$G(\Delta f, \mu) = T \sum_{k=0}^{N-1} \int_{x_k}^{x_{k+1}} \left[(g_k - T\, s_k\, x_k) + T\, s_k\, x \right] e^{-i2\pi(\Delta f T x - \frac{1}{2}\mu T^2 x^2)}\, dx$$

where $x = t/T$ and all the x_k now lie between $-1/2$ and $+1/2$. Corresponding to $g(t) = 0$ for t outside the interval $(-T/2,\ T/2)$, one may define $g_k = 0$ for $k \leq 0$ and $k \geq N$, $s_k = 0$ for $k < 0$ and $k \geq N$ and allow the summation to extend from $-\infty$ to ∞. Let $h_k = g_k - T\, s_k\, x_k$; then

$$G(\Delta f, \mu) = T \sum_k h_k \int_{x_k}^{x_{k+1}} e^{-i2\pi(\Delta f T x - \frac{1}{2}\mu T^2 x^2)}\, dx$$

$$- T^2 \sum_k s_k \int_{x_k}^{x_{k+1}} x\, e^{-i2\pi(\Delta f T x - \frac{1}{2}\mu T^2 x^2)}\, dx .$$

Summation by parts yields

$$G(\Delta f, \mu) = T^2 \sum_k (s_k - s_{k-1}) \int_{-1/2}^{x_k} (x_k - x)\, e^{-i2\pi(\Delta f T x - \frac{1}{2}\mu T^2 x^2)}\, dx .$$

After deforming the contour to that consisting of a hyperbola through $z = -1/2$ and a hyperbola through $z = x_k$,* one finds

$$G(\Delta f, \mu) = T^2 e^{-i2\pi\varphi(-\frac{1}{2})} \Sigma(s_k - s_{k-1})$$

$$\times \int_0^{-\infty} \left[x_k - \frac{1}{\gamma}\left(1 - \sqrt{(1 + \tfrac{\gamma}{2})^2 + \gamma^2 y^2}\right) - iy\right] \left[\frac{-\gamma y}{\sqrt{(1 + \frac{\gamma}{2})^2 + \gamma^2 y^2}} + i\right]$$

$$\times\, e^{2\pi \Delta f T y \sqrt{(1 + \frac{\gamma}{2})^2 + \gamma^2 y^2}}\, dy$$

$$+ T^2 \Sigma(s_k - s_{k-1})\, e^{-i2\pi\varphi(x_k)} \int_{-\infty}^{0} \left[x_k - \frac{1}{\gamma}\left(1 - \sqrt{(1 - \gamma x_k)^2 + \gamma^2 y^2}\right) - iy\right]$$

$$\times \left[\frac{-\gamma y}{\sqrt{(1 - \gamma x_k)^2 + \gamma^2 y^2}} + i\right] e^{2\pi \Delta f T y \sqrt{(1 - \gamma x_k)^2 + \gamma^2 y^2}}\, dy .$$

where

$$\gamma = \frac{\mu T}{\Delta f}$$

The first summation in this expression is zero since

$$\sum_k (s_k - s_{k-1}) = 0$$

---

*The contour of constant phase of the integrand is difficult to obtain. The phase of the integrand varies slowly on the hyperbolic contours.

and

$$\sum_k (s_k - s_{k-1})x_k = -\sum_k (x_{k+1} - x_k)s_k = -\frac{1}{T}\, g(T) = 0 \ .$$

Thus, letting $\beta_k = (\frac{1}{\gamma} - x_k)^{-1}$, $\alpha_k = 2\pi\Delta f T(1 - \gamma x_k)$

$$G(\Delta f, \mu) = T^2\, \Sigma(s_k - s_{k-1})\, e^{-i2\pi\varphi(x_k)} \tag{A-1}$$

$$\int_0^\infty \left[\frac{-y}{\sqrt{1+\beta_k^2 y^2}} + i\left(\frac{\beta_k y^2}{\sqrt{1+\beta_k^2 y^2}} + \frac{1}{\beta_k}\left(\sqrt{1+\beta_k^2 y^2} - 1\right)\right)\right] e^{-\alpha_k y\sqrt{1+\beta_k^2 y^2}}\, dy$$

and an upper bound is obtained by adding magnitudes

$$|G(\Delta f, \mu)| \le T^2 \sum_k |s_k - s_{k-1}| \int_0^\infty \Bigg| -\frac{y}{\sqrt{1+\beta_k^2 y^2}}$$

$$+ i\left(\frac{\beta_k y^2}{\sqrt{1+\beta_k^2 y^2}} + \frac{1}{\beta_k}\left(\sqrt{1+\beta_k^2 y^2} - 1\right)\right) \Bigg| e^{-\alpha_k y\sqrt{1+\beta_k^2 y^2}}\, dy \ . \tag{A-2}$$

Using the inequalities $\frac{1}{\sqrt{1+\beta_k^2 y^2}} \le 1$,

$\sqrt{1+\beta_k^2 y^2} - 1 \le \beta_k^2 y^2$, and $e^{-\alpha_k \sqrt{1+\beta_k^2 y^2}} \le e^{-\alpha_k}$ (for $\alpha_k > 0$),

the integrals may be evaluated with the result

$$|G(\Delta f, \mu)| \le \frac{1}{(2\pi\Delta f)^2} \sum_k |s_k - s_{k-1}| \left[\frac{1}{(1-\gamma x_k)^2} + \frac{2\gamma}{\pi\Delta f T(1-\gamma x_k)^4}\right]$$

$$\le \frac{1}{(2\pi\Delta f)^2}\, \frac{1}{(1-\frac{\gamma}{2})^2} \left[1 + \frac{2\gamma}{\pi\Delta f T(1-\frac{\gamma}{2})^2}\right] \sum_k |s_k - s_{k-1}| \ . \tag{A-3}$$

The last inequality is obtained using the fact that $x_k \leq 1/2$ and $0 \leq \gamma << 1$.

It is difficult to determine whether or not the bound (A-2) will ever be attained in a particular example. However, one can construct a continuous piecewise linear pulse shape with a spectrum that attains this bound at any given frequency Δf in the asymptotic region by putting the breakpoints x_k at points where the RF phase

$$\phi(x_k) = \Delta f T x_k - \frac{1}{2} \mu T^2 x_k^2$$

has the proper value to make the complex terms in (A-1) add in phase. Thus, while one cannot say that (A-2) is the envelope of $G(\Delta f, \mu)$ for an arbitrary continuous, piecewise linear pulse, it is the least upper bound for the class of pulses under consideration.

APPENDIX B

Useful Inequalities

In order to calculate the spectrum of a pulse with linear FM, the contour $-T/2 \leq t \leq T/2$ is deformed into the complex plane to a sum of contributions from hyperbolic contours. Each integrand contains the function

$$f(x, y) = e^{-\alpha y \sqrt{x^2 + \gamma^2 y^2}} \qquad x, y \geq 0 \quad ,$$

where α and γ are positive real numbers. It will here be shown that for a fixed value of $y > 0$, $f(x, y)$ is a convex function of x for $x > \frac{1}{\sqrt{\alpha}}$. It is precisely this region (actually $x >> \frac{1}{\sqrt{\alpha}}$) for which several important inequalities are necessary to determine an upper bound for the spectrum under consideration.

The first partial derivative of $f(x, y)$ is

$$\frac{\partial f(x, y)}{\partial x} = - \frac{\alpha y x}{\sqrt{x^2 + \gamma^2 y^2}} e^{-\alpha y \sqrt{x^2 + \gamma^2 y^2}} \quad ,$$

which is everywhere negative in the region of interest. The second partial derivative is

$$\frac{\partial^2 f(x, y)}{\partial x^2} = \frac{\alpha y \, e^{-\alpha y \sqrt{x^2 + \gamma^2 y^2}}}{\sqrt{x^2 + \gamma^2 y^2}} \left[\frac{\alpha y x^2}{\sqrt{x^2 + \gamma^2 y^2}} + \frac{x^2}{x^2 + \gamma^2 y^2} - 1 \right]$$

Inflection points of $f(x, y)$ as a function of x for fixed y occur at the zeros of $\frac{\partial^2 f}{\partial x^2}$. For $x > 0$ there is only one such point $x^*(y)$; as y increases from zero to infinity, $x^*(y)$ moves from zero to $\frac{1}{\sqrt{\alpha}}$. For $x > \frac{1}{\sqrt{\alpha}}$, $y \geq 0$, $\frac{\partial^2 f}{\partial x^2}$ is positive, and thus $f(x, y)$ is a convex function of x. The important features of $f(x, y)$ are indicated in Fig. B-1.

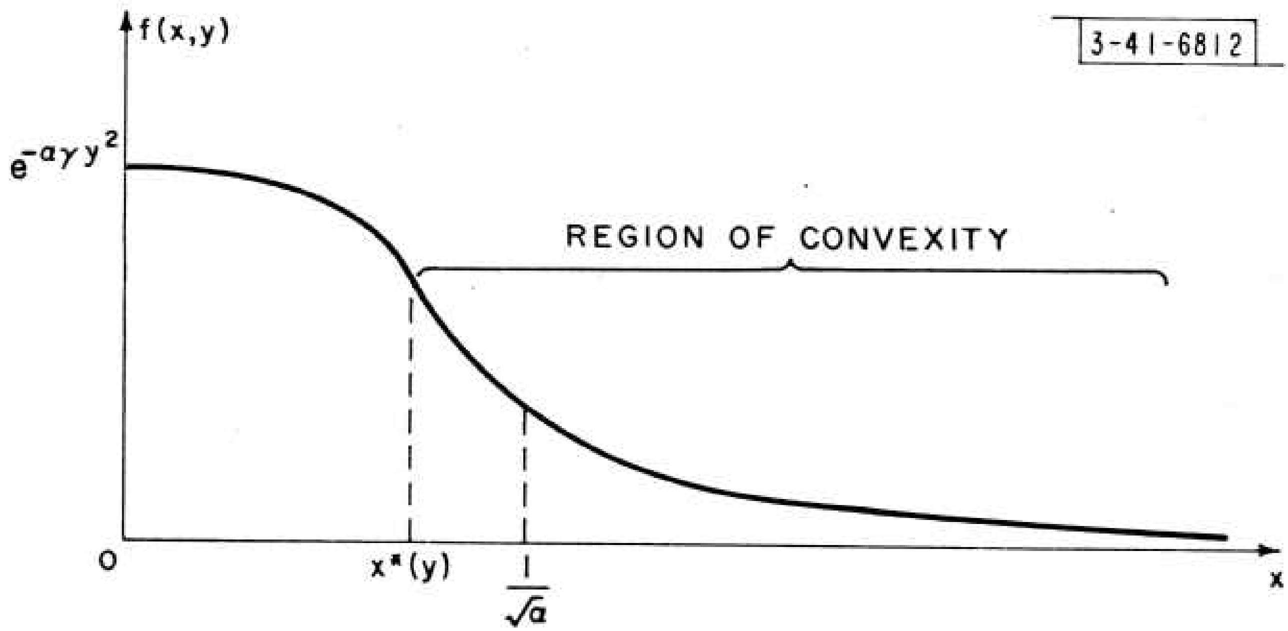


Fig. B-1 Graph of f(x, y)

First Inequality

The first inequality is

$$e^{-\alpha y\sqrt{x^2+\gamma^2 y^2}} \leq e^{-\alpha y x} \qquad x, y, \alpha, \gamma \geq 0 \quad . \tag{B-1}$$

This relation is merely a consequence of the fact that the exponent is always negative (or zero), and neglecting the $(\gamma y)^2$ term increases the function.

Second Inequality

The second inequality is

$$e^{-\alpha y\sqrt{x_1^2+\gamma^2 y^2}} + e^{-\alpha y\sqrt{x_2^2+\gamma^2 y^2}} \geq 2e^{-\alpha y\sqrt{\left(\frac{x_1+x_2}{2}\right)^2+\gamma^2 y^2}} \tag{B-2}$$

where (x_1, x_2) is an interval in the convex region, $x > \frac{1}{\sqrt{\alpha}}$, and $y \geq 0$. In the region of convexity of $f(x, y)$, the maximum value of $\left|\frac{\partial f(x, y)}{\partial x}\right|$ in any interval (x_1, x_2) occurs at x_1, and the minimum value of $\left|\frac{\partial f(x, y)}{\partial x}\right|$ occurs at x_2. Let $x_c = \frac{1}{2}(x_1 + x_2)$ denote the center of the interval. Then, by the mean value theorem, there are two points, p_1, p_2, in the intervals (x_1, x_c), (x_c, x_2) respectively, such that

$$f(x_1, y) = f(x_c, y) + \left.\frac{\partial f(x, y)}{\partial x}\right|_{x = p_1} (x_1 - x_c)$$

$$f(x_2, y) = f(x_c, y) + \left.\frac{\partial f(x, y)}{\partial x}\right|_{x = p_2} (x_2 - x_c)$$

Since $\frac{\partial f(x, y)}{\partial x} < 0$, $\frac{\partial^2 f(x, y)}{\partial x^2} > 0$,

$$\left|\frac{\partial f(x,y)}{\partial x}\right|_{x=p_1} \geq \left|\frac{\partial f(x,y)}{\partial x}\right|_{x=x_c} ,$$

and

$$\left|\frac{\partial f(x,y)}{\partial x}\right|_{x=x_c} \geq \left|\frac{\partial f(x,y)}{\partial x}\right|_{x=p_2} .$$

One then obtains

$$f(x_1, y) \geq f(x_c, y) + \left|\frac{\partial f(x,y)}{\partial x}\right|_{x=x_c} (x_c - x_1) ,$$

$$f(x_2, y) \geq f(x_c, y) - \left|\frac{\partial f(x,y)}{\partial x}\right|_{x=x_c} (x_2 - x_c) ,$$

and adding these two results yields

$$f(x_1, y) + f(x_2, y) \geq 2f(x_c, y) .$$

Third Inequality

The third inequality involves the difference of $f(x, y)$ evaluated at the ends of an interval $(x, x + \delta)$:

$$e^{-\alpha y \sqrt{x^2 + \gamma^2 y^2}} - e^{-\alpha y \sqrt{(x+\delta)^2 + \gamma^2 y^2}} \leq \alpha \delta y \, e^{-\alpha y \sqrt{x^2 + \gamma^2 y^2}} \tag{B-3}$$

According to the use of the mean value theorem in the preceding proof,

$$f(x_1, y) = f(x_c, y) + \left.\frac{\partial f(x, y)}{\partial x}\right|_{x = p_1} (x_1 - x_c)$$

$$= f(x_c, y) + \left.\frac{\partial f(x, y)}{\partial x}\right|_{x = p_1} (x_c - x_1) \quad ,$$

where $p_1 \epsilon (x_1, x_c)$. Since $\left|\left.\frac{\partial f(x, y)}{\partial x}\right|_{x = p_1}\right| \leq \left|\left.\frac{\partial f(x, y)}{\partial x}\right|_{x = x_1}\right|$,

$$f(x_1, y) - f(x_c, y) \leq \left|\left.\frac{\partial f(x, y)}{\partial x}\right|_{x = x_1}\right| (x_c \quad x_1) \quad .$$

Taking $x_1 = x$ and $x_c = x + \delta$, and using the fact that

$$\frac{x}{\sqrt{x^2 + \gamma^2 y^2}} \leq 1 \quad ,$$

one finds

$$e^{-\alpha y \sqrt{x^2 + \gamma^2 y^2}} - e^{-\alpha y \sqrt{(x + \delta)^2 + \gamma^2 y^2}} \leq \alpha \delta y e^{-\alpha y \sqrt{x^2 + \gamma^2 y^2}} \quad .$$

Fourth Inequality

The final inequality involves an integral for $\alpha > 0$:

$$I(\gamma) = \int_0^\infty y^k e^{-\alpha y \sqrt{1 + \gamma^2 y^2}} dy \geq \frac{k!}{\alpha^{k+1}} \left(1 - (k+1)(k+2) \frac{\gamma^3}{\alpha}\right) . \qquad \text{(B-4)}$$

The first derivative of $I(\gamma)$ is

$$I'(\gamma) = -\int_0^\infty \frac{\alpha\gamma^2 y^{k+2}}{\sqrt{1+\gamma^2 y^2}} e^{-\alpha y\sqrt{1+\gamma^2 y^2}} dy$$

$$\geq -\int_0^\infty \alpha\gamma^2 y^{k+2} e^{-\alpha y} dy = -\frac{\gamma^2 (k+2)!}{\alpha^{k+2}}$$

using Eq. (B-1). By the mean value theorem, there is a $\tilde{\gamma}$ in the interval $(0, \gamma)$ such that

$$I(\gamma) = I(0) + \gamma\, I'(\tilde{\gamma}) .$$

Since $I'(\tilde{\gamma}) \geq -\frac{\gamma^2 (k+2)!}{\alpha^{k+2}}$ throughout the interval $(0, \gamma)$, we find

$$I(\gamma) \geq I(0) - \frac{\gamma^3 (k+2)!}{\alpha^{k+2}} .$$

Finally, since $I(0) = \frac{k!}{\alpha^{k+1}}$, the result (B-4) is obtained.

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