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ON THE LINE GRAPH OF A PROJECTIVE PLANE by A. J. Hoffman 9/18/63 Introduction

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If G is a (finite, undirected) graph, its line graph (also called the interchange graph, and the adjoint graph) is the graph G whose vertices are the edges of G, with two vertices of G adjacent if the corresponding edges of G are adjacent. Let π be a projective plane with n + 1 points on a line, and let $G(\pi)$ be the bipartite graph whose vertices are the $2(n^2 + n + 1)$ points and lines of π , with two vertices adjacent if and only if one of the vertices is a point, the other is a line, and the point is on the line. The graph we shall study is $(G(\pi))$

For any graph G, let

 $A(G) = A = (a_{ij}) = \begin{cases} l \text{ if } i \text{ and } j \text{ are adjacent vertices} \\ 0 \text{ otherwise} \end{cases}$

A is called the adjacency matrix of G, and in recent years there have been several investigations to determine to what extent a regular, coconnected graph is determined by the characteristic roots of its adjacency matrix. In the case where G is a line graph, the following

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results have been obtained:

(i) if G is the line graph of the complete bipartite graph on n + n vertices, and H is a regular connected graph on n^2 vertices such that A(H) has the same characteristic roots as A(G), then H = G unless n = 4, when there is exactly one exception [9].

(ii) if G is the line graph of the complete graph on n vertices, and H is a regular connected graph on n(n-1)/2 vertices, such that A(H) has the same characteristic roots as A(G), then H = G, unless n = 8, when there are exactly two exceptions ([1], [2], [3], [4], [5], [8]).

In this paper, we shall prove that if H is a regular connected graph on (n + 1) $(n^2 + n + 1)$ vertices such that A(H) has the same characteristic roots as A((G(π))^{*}), then H = (G(π_1))^{*}, where π_1 is some projective plane of the same order as π . Thus the characteristic roots of A((G(π))^{*}) do determine the class of graphs (G(π))^{*}, but do not distinguish between projective planes of the same order.

2. The Characteristic Roots of $A((G(\pi))^{-})$.

It is useful to first determine the characteristic roots of $A(G(\pi) \).$

Lemma 1: A regular connected graph G on $2(n^2 + n + 1)$ vertices has as the distinct characteristic roots of $A(\vec{G})$

$$(2.1) (n+1), - (n+1), \sqrt{n}, - \sqrt{n}$$

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if and only if $G = G(\pi)$, where π is a projective plane of the order n.

(2.2) Proof: By definition, if
$$G = G(\pi)$$

 $A(G) = \begin{pmatrix} O & B \\ B^T & O \end{pmatrix}$,

where B is a point-line incidence matrix of π . The characteristic roots of (2.2) are the singluar values of B and their negatives. But the singular values of B are n + 1 and \sqrt{n} ([7]).

Conversely, assume A has (2.1) as its distinct characteristic roots. Then ([6]), if A = A(G), because G is bipartite, A is of the form (2.2), where B is a (0, 1) matrix with row and column sums equal to n + 1, and BB^T has all but one characteristic root equal to n. Hence BB^T - nI is a nonnegative integral symmetric matrix of rank one with every diagonal entry equal to 1. This implies BB^T - nI = J; i.e., B is the incidence matrix of a projective plane π of order n.

Another derivation of lemma 1 is given in the thesis of R. R. Singleton [10], in which it is proved that a regular connected graph H of valence n + 1 and girth 6 has $2(n^2 + n + 1)$ vertices if and only if H = G(π).

(2.3) Lemma 2: The distinct characteristic roots of $A(G(\pi)^*)$ are (2.3) $2n, -2, n-1 - \sqrt{n}$. Proof: Let $A = A((G(\pi))^*)$, B be the adjacency matrix

for $G(\pi)$. Let K be the $2(n^2 + n + 1)$ by $(n + 1)(n^2 + n + 1)$ matrix

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whose rows correspond to the points and lines of π , and whose columns correspond to the edges of $(G(\pi))^{\ddagger}$; i.e., each colum of K contains two l's, corresponding to an incident point and line of π , the remaining entry in the column being 0. Clearly,

$$KK^{T} = (n + 1)I + B, K^{T}K = 2I + A$$
.

The distinct characteristic roots of KK^T and K^TK are the same except possibly for 0. But K^TK is singular, since its rank is at most $2(n^2 + n + 1)$, while its order is $(n + 1) (n^2 + n + 1)$; KK^T is singular, since the sum of the rows of K corresponding to points of π minus the sum of the rows of K corresponding to lines of π is the zero vector. Thus the distinct eigenvalues of KK^T and of K^TK are the same. Invoking (2.1) then proves (2.3).

3. Theorem

If G is a regular connected graph with no edges joining a vertex to itself, if G has $(n + 1) (n^2 + n + 1)$ vertices and the adjacency matrix of G has (2.3) as its distinct eigenvalues, then $G = (G(\pi))^*$, for some projective plane π of order n.

In the lemmas that follow, we assume that G satisfies the hypothesis of the theorem, A = A(G), J is the matrix every entry of which is 1.

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Lemma 3: Let (3.1) $P(x) = \frac{1}{2} (x^3 - (2n - 4) x^2 + (n^2 - 7n + 5) x + 2 (n^2 - 3n + 1))$, then P(A) = J. Proof: It has been shown [6] that the adjacency matrix of a regular connected graph of valence d on N vertices, with distinct eigenvalues d, α_1 , ... α_t , satisfies P(B) = J, where P(x) = N $\prod_i (x - \alpha_i)/\prod_i (d - \alpha_1)$. From (2.3), we then calculate (3.1). Lemma 4: If two vertices of G are adjacent, then there

are exactly n - 1 vertices of G adjacent to both. If two vertices of G are not adjacent, then there are no vertices or exactly one vertex adjacent to both.

Proof: Let i be any vertex of G. Then i has valence 2n, so there are 2n vertices j_1, \ldots, j_{2n} such that $a_{ij_t} = 1$, $t = 1, \ldots, 2n$. We first show that

(3.2) $\sum_{t} (A^{2})_{ij_{t}} = 2n(n-1)$.

This follows from (3.1); for the left side of (3.2) is $(A^3)_{ii}$, and by (3.1), $(A^3)_{ii} = 2(J)_{ii} + (2n - 4) (A^2)_{ii} - (n^2 - 7n + 5) A_{ii} - 2(n^2 - 3n + 1)$. But $J_{ii} = 1$, $(A^2)_{ii} = 2n$, $A_{ii} = 0$, and (3.2) follows.

Next, consider the matrix

(3.3)
$$B = A^2 - 2nI - (n - 1) A$$

We shall show that every entry of B is 0 or 1. Certainly every entry is an integer. Let i be any row of B. From the fact that $\sum_{j} (A^2)_{ij} = (2n)^2$, we infer that (3.4) $\sum_{j} b_{ij} = 2n^2$. We next evaluate $\sum_{j} b_{ij}^{2} = (B^{2})_{ii}$. We have from (3.3) (3.5) $B^{2} = A^{4} - 2(n-1)A^{3} + (n^{2} - 6n + 1)A^{2} + 4n(n-1)A + 4n^{2}I$. Further, $I_{ii} = 1$, $A_{ii} = 0$, $A_{ii}^{2} = 2n$, $A_{ii}^{3} = 2n(n-1)$ from (3.2). To evaluate $(A^{4})_{ii}$, we use (3.1), with P(A) = J, and obtain A P(A) = AJ =2n J. Since A P(A) is a fourth degree polynomial in A, we can evaluate $(A^{4})_{ii} = 4n - 2n(n^{2} - 7n + 5) + 2n(n-1)(2n-4)$.

Putting these expressions in (3.5), we obtain

(3.6)
$$(B^2)_{ii} = \sum_j b_{ij}^2 = 2n^2$$
.

From (3.5) and (3.6) we infer that each of the integers b_{ij} is 0 or 1. Recalling the definition of B in (3.3), this proves the second sentence of the lemma. To prove the first sentence, note from (3.2) and (3.3) that $\sum_{t} b_{ij} = 0$. Since each b_{ij} is 0 or 1, each $b_{ij} = 0$. By (3.3), this proves the first sentence of the lemma.

<u>Lemma 5:</u> G contains $2(n^2 + n + 1)$ cliques $C_1, \ldots,$

 $C_{2(n^2 + n + 1)}$ with the following properties:

(3.7) Each C, contains exactly n + 1 vertices

$$(3.8) \qquad \text{Each vertex of } G \text{ is contained in exactly two } C_i$$

(3.9) Each pair of adjacent vertices of G is contained in exactly one C_i.

Proof: The set of cliques C_i will consist of all cliques with n + 1 vertices, which establishes (3.7). To prove (3.9), let i and

j be adjacent vertices of G. Let k and l each be adjacent to both i and j. If k and l were not adjacent, we would have a violation of the second sentence of lemma 4. Hence, the n - l vertices adjacent to both i and j (by the first sentence of lemma 4) are adjacent to each other. These vertices, together with i and j are the unique clique with n + l vertices containing i and j.

Let T be the total number of (n + 1) - cliques, and let us count the number of incidences of cliques with pairs of vertices contained in the clique. This is

$$T\binom{n+1}{2} = \frac{1}{2} 2n (n+1) (n^2 + n + 1),$$

for the right hand side is the total number of pairs of adjacent vertices. This equation yields $T = 2 (n^2 + n + 1)$. Thus all that remains to be proven is (3.8). Since the valence of each vertex i is 2n, there must be at least two (n + 1) - cliques containing i. If these two cliques did not contain all vertices adjacent to i, there would have to be some vertex $j \neq i$ in both cliques, violating (3.9).

We are now ready to prove the theorem. Let G be the graph whose vertices are the (n + 1) - cliques of G. Two vertices of \tilde{G} are adjacent if the corresponding cliques of G have a common vertex. It follows from lemma 5 that \tilde{G} is a regular connected graph of valence n + 1, and that $G = \tilde{G}^*$. We will be finished if we

prove that $\tilde{G} = G(\pi)$. Let L be the vertex-edge incidence matrix of \tilde{G} , and let \tilde{A} be the adjacency matrix of \tilde{G} . Assume \tilde{A} has distinct characteristic roots n + 1, α_1 , ..., α_t . Since

$$LL^{T} = (n + 1)I + \tilde{A}, L^{T}L = 2I + A,$$

and (except possibly for 0) the distinct characteristic roots of LL^{T} and $L^{T}L$ are the same, it follows by the same reasoning as in Lemma 2 that the distinct characteristic roots (with the possible exception of -2) of A are (3.10) 2n, $n = 1 + \alpha_{\bullet}$.

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Comparing (3.10) with (2.3), we see that, if -2 is of the form $n - 1 + \alpha_{\epsilon}$, then \tilde{A} has the same distinct characteristic roots as the adjacency matrix for G(π), and (by the "only if" part of lemma 1) we are finished. Therefore, assume otherwise, so that (comparing (3.10) with (2.3)) we find that the distinct characteristic roots of \tilde{A} are

n+1, $-\sqrt{n}$.

Since \tilde{G} is regular and connected, we can, as in lemma 3, use the theorem of [6] to assert that

$$2(\tilde{A}^2 - nI) = J.$$

But since A is a (0, 1) matrix, this is absurd.

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