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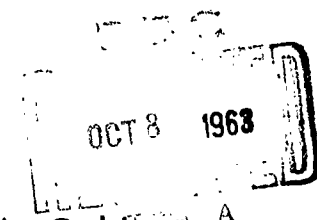
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Fuel Optimization in Orbital Rendezvous

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FUEL OPTIMIZATION IN ORBITAL RENDEZVOUS

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The linearized equations of motion for a ferry vehicle attempting rendezvous with an orbital satellite (2-dimensional Clohessy-Wiltshire equations) are:

$$
(1) \qquad
\left.
\begin{aligned}
\ddot{x} - 2\omega\dot{y} &= u_1 \\
\ddot{y} + 2\omega\dot{x} - 3\omega^2 y &= u_2
\end{aligned}
\right\}
$$

where u_1 and u_2 are the components of the rocket thrust (the mass of the ferry has been normalized to unity). The amplitude of the thrust is

$$
|u| = (u_1^2 + u_2^2)^{\frac{1}{2}}
$$

and it is assumed that besides the initial and terminal conditions for (1) the additional restraint

$$
(2) \qquad |u(t)| \leq K \qquad\qquad\qquad (0 \leq t \leq T)
$$

is imposed.

We wish a thrust function $u = (u_1, u_2)$ such that, starting with an initial position (x_0, y_0) and velocity $(\dot{x}_0, \dot{y}_0)$ at $t = 0$, we rendezvous at $t = T$ so that

$$
(3) \qquad x(T) = y(T) = \dot{x}(T) = \dot{y}(T) = 0.
$$

We ask that T be less than a full period so $\omega T < 2\pi$. If it is assumed that the rate of fuel consumption is proportional to the amplitude of the thrust then the $\underline{\text{total}}$ fuel consumption will be (proportional to)

$$
(4) \qquad f(u) = \int_0^T (u_1^2 + u_2^2)^{\frac{1}{2}} dt = \int_0^T |u(t)| dt
$$

and our problem is to determine a thrust function satisfying (2) and (3) (subject to (1)) which minimizes (4), if such exists.

We let M be the set of transfer functions (thrust functions u for which (3) holds from (1) and the given initial conditions). We employ the norm

$$(5) \qquad \|u\|_2 = [\int_0^T |u(t)|^2 dt]^{\frac{1}{2}} = [\int_0^T (u_1^2 + u_2^2) dt]^{\frac{1}{2}}$$

and consider the Hilbert space X of $u = (u_1, u_2)$ for which $\|u\|_2$ is finite. In the topology thus defined, M will be a closed linear manifold (see below). For any K, let M_K be the subset of M for which (2) holds; we assume K sufficiently large that M_K is non-empty. Then

<u>Lemma I</u>:   <u>The functional</u> $f(u)$ <u>actually achieves its minimum on</u> M_K.

<u>Proof</u>: We observe that for any u, ũ ε X

$$|f(u) - f(\tilde{u})| \leq \int_0^T \|u| - |\tilde{u}\| dt \leq \int_0^T |u - \tilde{u}| dt \leq T^{\frac{1}{2}} \|u - \tilde{u}\|_2$$

(using the E_2 triangle inequality pointwise and then the Schwartz inequality). Thus f is (strongly) continuous on X. We also observe that if $\alpha + \beta = 1$ with $\alpha \geq 0, \beta \geq 0$ then

$$f(\alpha u + \beta \tilde{u}) = \int_0^T |\alpha u + \beta \tilde{u}| dt \leq \int_0^T (\alpha|u| + \beta|\tilde{u}|) dt = \alpha f(u) + \beta f(\tilde{u})$$

(using, again, the E_2 triangle inequality pointwise) so f is convex. Since f is convex and (strongly) continuous, the sets

$$S_\alpha = \{u : f(u) \leq \alpha\}$$

are convex and strongly closed in **X**. Hence they are also weakly closed

(c.f., [1] p. 422) and (c.f.,[2] p. 61) f is weakly lower semi-continuous.

In order that u be a transfer function (u ε M) we must have

$x = y = \dot{x} = \dot{y} = 0$ at $t = T$ given $x_0, y_0, \dot{x}_0, \dot{y}_0$ and the differential

equations (1). The solution of (1) is given by

$$x(t) = a_1(t) + \int_0^t f^1(t - \tau)u(\tau)d\tau,$$

$$\dot{x}(t) = a_2(t) + \int_0^t f^2(t - \tau)u(\tau)d\tau,$$

(6)

$$y(t) = a_3(t) + \int_0^t f^3(t - \tau)u(\tau)d\tau,$$

$$\dot{y}(t) = a_4(t) + \int_0^t f^4(t - \tau)u(\tau)d\tau,$$

where

$$a_1(t) = x_0 + \dot{x}_0(\frac{4}{\omega} \sin \omega t - 3t) + 6y_0(\omega t - \sin \omega t) + \frac{2}{\omega} \dot{y}_0(1 - \cos \omega t),$$

$$a_2(t) = \dot{x}_0(4 \cos \omega t - 3) + 6\omega y_0(1 - \cos \omega t) + 2\dot{y}_0 \sin \omega t,$$

$$a_3(t) = \frac{2}{\omega} \dot{x}_0(\cos \omega t - 1) + y_0(4 - 3\cos \omega t) + \frac{1}{\omega} \dot{y}_0 \sin \omega t,$$

$$a_4(t) = - 2 \dot{x}_0 \sin \omega t + 3\omega y_0 \sin \omega t + \dot{y}_0 \cos \omega t,$$

and

$$f^1(s) = (\frac{4}{\omega} \sin \omega s - 3s, \frac{2}{\omega}[1 - \cos \omega s]),$$

$$f^2(s) = (4 \cos \omega s - 3 , 2 \sin \omega s),$$

$$f^3(s) = (\frac{2}{\omega}[\cos \omega s - 1], \frac{1}{\omega} \sin \omega s),$$

$$f^4(s) = (- 2 \sin \omega s , \cos \omega s).$$

Thus, the conditions (3) (which define M) are equivalent to

(3') $\langle u, F^i \rangle = A_i$ $(i = 1,2,3,4)$

where $F^i = F^i(t) = f^i(T - t)$ and $A_i = - a_i(T)$ for $i = 1,2,3,4$.

(We have used $\langle \alpha, \beta \rangle$ to denote the scalar product in X,

$$\langle \alpha, \beta \rangle = \int_0^T [\alpha_1(t)\beta_1(t) + \alpha_2(t)\beta_2(t)]dt).$$

The condition (2) (holding for almost all $t \in [0,T]$) is equivalent to

(2') $\langle u, g \rangle \leq K \int_0^T |g(t)|dt$ (all $g \in X$).

Certainly (2) implies (2') as

$$\langle u, g \rangle = \int_0^T u(t)g(t)dt \leq \int_0^T |u(t)||g(t)|dt \leq K \int_0^T |g(t)|dt.$$

Conversely, if (2) were false on a set of positive measure there would be

a $K' > K$ and a set $S \subset [0,T]$ of positive measure such that

$$|u(t)| \geq K' > K \qquad\qquad (t \in S).$$

If one now took

$$g(t) = \begin{cases} \dfrac{u(t)}{|u(t)|} & t \in S, \\ 0 & t \notin S, \end{cases}$$

then $g \in X$ and

$$\langle u, g \rangle = \int_0^T u(t)g(t)dt = \int_S u(t) \frac{u(t)}{|u(t)|} dt = \int_S |u(t)|dt \geq K' \text{ meas}(S)$$

while

$$K \int_0^T |g|dt = K \int_S |\frac{u}{|u|}|dt = K \int_S 1\,dt = K \text{ meas}(S) < K' \text{ meas}(S)$$

which would contradict (2').

It follows that $u \in M_K$ is characterized (except on subsets of measure zero) by the conditions (2') and (3'). Then, if $\{u_\alpha\}$ is a net of elements of M_K weakly convergent to u, one has

$$\langle u, f^i \rangle = \lim_\alpha \langle u_\alpha, f^i \rangle = \lim_\alpha a^i = a^i \qquad (i = 1,2,3,4)$$

$$\langle u, g \rangle = \lim_\alpha \langle u_\alpha, g \rangle \leq \lim_\alpha K \int_0^T |g| dt = K \int_0^T |g| dt \quad (g \in X)$$

so u is also in M_K and M_K is weakly closed. Also (2) implies, for $u \in M_K$,

$$\|u\|_2 = [\int_0^T |u|^2 dt]^{\frac{1}{2}} \leq [\int_0^T K^2 dt]^{\frac{1}{2}} = KT^{\frac{1}{2}}$$

so M_K is strongly bounded and, hence, weakly compact (c.f.,[1] p. 425).

Thus, in the weak topology, we have a lower semi-continuous function f on a compact set M_K and (c.f.,[2] p. 65) f must achieve its minimum on M_K. \hfill QED

We have not shown that the $u \in M_K$ which minimizes f is unique but will call any transfer function which minimizes f (subject to (2)) an optimal (or, more precisely, K-optimal) transfer function. Now that we know such functions exist it is of interest to characterize them. There is a theorem, the so-called "Maximum Principle", due to Pontryagin (c.f.,[3], pp. 66-68) which provides a characterization. Introduce the functions $(\psi_0, \ldots, \psi_5)$ satisfying the linear system

(7) $$\dot{\psi}_k = - \Sigma_j [\frac{\partial}{\partial x_k} R_j(\bar{x}, u)] \psi_j, \qquad \psi_0(T) < 0$$

where the original equation (1) has been written as a first order

system for $\bar{x} = (x_0,\ldots,x_5)$,

$$(1') \qquad \dot{x}_k = R_k(\bar{x},u) \qquad\qquad (k = 0,\ldots,5)$$

or, in detail,

$$(1'') \quad \begin{aligned} \dot{x}_0 &= |u| = (u_1^2 + u_2^2)^{\frac{1}{2}} &&= R_0(\bar{x},u) \\ \dot{x}_1 &= x_2 &&= R_1(\bar{x},u) \\ \dot{x}_2 &= 2\omega x_4 + u_1 &&= R_2(\bar{x},u) \\ \dot{x}_3 &= x_4 &&= R_3(\bar{x},u) \\ \dot{x}_4 &= -2\omega x_2 + 3\omega^2 x_3 + u_2 &&= R_4(\bar{x},u) \\ \dot{x}_5 &= 1 &&= R_5(\bar{x},u) \end{aligned}$$

(the original coordinate functions, x and y, are now x_1 and x_3) and

$$(7') \quad \begin{aligned} \dot{\psi}_0 &= 0 &&(\psi_0(t_1) \le 0) \\ \dot{\psi}_1 &= 0 \\ \dot{\psi}_2 &= -\psi_1 + 2\omega\psi_4 \\ \dot{\psi}_3 &= -3\omega^2\psi_4 \\ \dot{\psi}_4 &= -2\omega\psi_2 - \psi_3 \\ \dot{\psi}_5 &= 0. \end{aligned}$$

Construct the function

$$H = H(\psi,\bar{x},u) = \Sigma_k \psi_k R_k(\bar{x},u)$$

$$(8) \quad = \psi_0|u| + \psi_1 x_2 + \psi_2(2\omega x_4 + u_1) + \psi_3 x_4 + \psi_4(-2\omega x_2 + 3\omega^2 x_3 + u_2) + \psi_5$$

$$= \psi_0|u| + (\varphi,u) + [\psi_1 x_2 + 2\omega\psi_2 x_4 + \psi_3 x_4 + \psi_4(-2\omega x_2 + 3\omega^2 x_3) + \psi_5]$$

where $\varphi = (\varphi_1, \varphi_2) = (\psi_2, \psi_4)$.

The Maximum Principle says that a necessary condition that $\bar{u}$ be K-optimal (since f is convex and M_K is a convex set this is also sufficient) is that there exists a solution of $(7')$ such that, for almost every $t \varepsilon [0,T]$,

$$(9) \qquad H(\psi, \bar{x}, \bar{u}) = \max_{|u| \leq K} H(\psi, \bar{x}, u).$$

We observe that the maximization in (9) involves only $H_0 = \psi_0 |u| + (\varphi, u)$. This clearly attains its maximum, subject to (2), if u is equal to a multiple of $\varphi/|\varphi|$, so that $(\varphi, u) = |\varphi||u|$. Thus the function $H_0 = w|u|$, where $w = \psi_0 + |\varphi|$, is to be maximized with $u = \alpha \varphi/|\varphi|$ and $0 \leq |u| \leq K$. Hence:

$$\alpha = \begin{cases} K & \text{if} \quad w > 0, \\ 0 & \text{if} \quad w < 0. \end{cases}$$

It is convenient to normalize H and ψ by taking $\psi_0(T) = -1$ (it must be negative) so

$$w(t) = -1 + |\varphi| = -1 + (\psi_2^2 + \psi_4^2)^{\frac{1}{2}}.$$

As all that concerns us is the sign of w, we multiply through by the positive function $(1 + |\varphi|)$ to get

$$(10) \qquad W(t) = \psi_2^2 + \psi_4^2 - 1 = (\varphi, \varphi) - 1$$

and

$$(11) \qquad u(t) = \begin{cases} K \varphi/|\varphi| & W > 0 \\ 0 & W < 0. \end{cases}$$

Since $W = 0$ only at isolated points (unless $W \equiv 0$, which is impossible if u is to be K-optimal) we do not care particularly how u is defined there.

We note from (11) that, for a K-optimal transfer function u,

<u>One always wishes either to be firing at full thrust in some direction</u> $(W > 0)$ <u>or to be coasting</u> $(W < 0)$.

The total fuel consumption is, then, proportional to

$$(12) \qquad f(u) = \int_0^T |u| dt = K\tau(u)$$

where $\tau(u)$ is the amount of time during which the rockets are to be firing for a K-optimal transfer function u. Since, for a given K, any K-optimal transfer function u must give the same value for $f(u)$, $\tau(u)$ must depend on K alone, call it τ_K. Thus, for any K-optimal transfer function u, the total amount of "firing time" is τ_K and the total amount of "coasting time" is $(T - \tau_K)$, $f(u) = K\tau_K$.

Clearly, if u satisfies (2) for any K it also satisfies (2) for any larger value of K. Thus $M_K \subset M_{K'}$ for $K < K'$ and

$$(13) \qquad K\tau_K = \min_{u \epsilon M_K} f(u) \geq \min_{u \epsilon M_{K'}} f(u) = K'\tau_{K'} \qquad (K < K')$$

so that not only is τ_K monotonically decreasing but

$$(14) \qquad \tau_K = \theta(1/K).$$

This may be restated as

> As K (the maximum available thrust) increases the amount of
> "coasting time" increases and the amount of "firing time" goes
> to zero at least of the order of $1/K$. One expects, "eventually",
> to have "infinitely" powerful thrusts exerted "instantaneously" -
> i.e., impulses.

Of interest is the question of how many firing intervals there can be, which is equivalent to determining a bound for the number of times the rockets can "switch" from full thrust to coasting or vice-versa. From the characterization (11) of K-optimal transfer functions we see that a "switch" corresponds to a root of the function $W = W(t)$ defined in (10) from the system (7'). At this point we note that the general solution of (7') is given by

$$\psi_0 = c_0 \qquad (\text{normalize so } \psi_0 = -1)$$

$$\psi_1 = -c_1$$

(15)
$$\psi_2 = (c_4 - 3c_1 t) - 2c_2 \sin \omega t + 2c_3 \cos \omega t$$

$$\psi_3 = -2\omega(c_4 - 3c_1 t) + 3c_2 \omega \sin \omega t - 3c_3 \omega \cos \omega t$$

$$\psi_4 = -(2/\omega)c_1 - c_2 \cos \omega t - c_3 \sin \omega t$$

$$\psi_5 = c_5$$

so that $W = |\varphi|^2 - 1 = \psi_2^2 + \psi_4^2 - 1$ has the form

$$W(t) = 3\gamma^2 \sin^2(\omega t + \phi) + 4\gamma(c_4 - 3c_1 t) \sin(\omega t + \phi) - \frac{4}{\omega} c_1 \gamma \cos(\omega t + \phi)$$

(16)
$$+ 9c_1^2 t^2 - 6c_1 c_4 t + (c_4^2 + \frac{4}{\omega^2} c_1^2 + \gamma^2 - 1)$$

where $\gamma = (c_2^2 + c_3^2)^{\frac{1}{2}}$ and $\phi = -\tan^{-1}(\frac{c_3}{c_2})$.

On differentiating (16) we obtain (letting $\theta = \omega t + \phi$, for convenience)

$$\dot{W} = 6\gamma^2\omega \sin\theta \cos\theta - 8c_1\gamma \sin\theta + 4\gamma\omega(c_4 - 3c_1t) \cos\theta + 18c_1^2t - 6c_1c_4$$

and

$$W = -12\gamma^2\omega^2 \sin^2\theta - 20c_1\gamma\omega \cos\theta - 4\gamma\omega^2(c_4 - 3c_1t) \sin\theta + (18c_1^2 + 6\gamma^2\omega^2).$$

Letting $\dot{W} = G \sin\theta$ and writing G in terms of $\xi = \cos\theta$ gives

$$G = -12\gamma^2\omega^2(1 - \xi^2)^{\frac{1}{2}} - 20c_1\gamma\omega\xi(1 - \xi^2)^{-\frac{1}{2}} - 4\gamma\omega[(\omega c_4 + 3c_1\phi) - 3c_1\cos^{-1}\xi]$$

$$+ (18c_1^2 + 6\gamma^2\omega^2)(1 - \xi^2)^{-\frac{1}{2}}$$

and, differentiating with respect to ξ ($G' = dG/d\xi$),

$$G' = 12\gamma^2\omega^2\xi(1 - \xi^2)^{-\frac{1}{2}} - 20c_1\gamma\omega(1 - \xi^2)^{-\frac{1}{2}} - 20c_1\gamma\omega\xi^2(1 - \xi^2)^{-3/2}$$

$$- 12\gamma\omega c_1(1 - \xi^2)^{-\frac{1}{2}} + (18c_1^2 + 6\gamma^2\omega^2)\xi(1 - \xi^2)^{-3/2}$$

$$= 2\gamma^2\omega^2(1 - \xi^2)^{-3/2}[6\xi(1 - \xi^2) - 16\alpha(1 - \xi^2) - 10\alpha\xi^2 + 3(1 + 3\alpha^2)\xi]$$

$$= -2\gamma^2\omega^2(1 - \xi^2)^{-3/2}[6\xi^3 - 6\alpha\xi^2 - 9(1 + 1\alpha^2)\xi + 16\alpha]$$

(where we have introduced $\alpha = c_1/\gamma\omega$ to simplify the expression). Since $\dot{G} = G'\dot{\xi} = -\omega(1 - \xi^2)^{\frac{1}{2}} G'$ this gives

(17) $\qquad (1 - \xi^2)\dot{G} = 12\gamma^2\omega[\xi^3 - \alpha\xi^2 - \frac{3}{2}(1 + \alpha^2)\xi + \frac{8}{3}\alpha].$

For any value of the parameter α, the cubic polynomial on the right of (17) (and, a fortiori, $\dot{G}$) will vanish for at most three distinct real values of ξ (we are interested only in those roots which lie between -1 and $+1$, however, so these may not all be significant). Recalling

that $\omega T < 2\pi$ we see that $\xi = \cos(\omega t + \phi)$ can take any value at most twice for $0 \leq t \leq T$ and there is an upper bound of <u>six</u> on the number of roots of $\overset{\circ}{G}$ in $[0,T]$. Since there must be a root of $\overset{\circ}{G}$ between any two roots of G, it follows that G can have at most <u>seven</u> roots in $[0,T]$. Now $W = G \sin \Theta$ can vanish only when either of its factors does and $\sin \Theta$ vanishes at most twice in $[0,T]$ so $\overset{\cdot}{W}$ can have a maximum of seven plus two or <u>nine</u> roots in $[0,T]$. As above, this gives a bound of ten on the roots of $\overset{\cdot}{W}$ and of <u>eleven</u> on the possible roots of W in $[0,T]$. Each "switch" from "full thrust" to "coast" (or vice-versa) of the rocket will occur, by (11), only at a root of $\overset{\cdot}{W}$ in $[0,T]$. Thus, we have shown that

> <u>There are at most</u> 6 <u>firing intervals during the trip for a</u>
> <u>ferry making a rendezvous under an optimal thrust program.</u>

These results can be extended somewhat. Suppose one were to consider, instead of the linearized equations (1), the equations of motion:

$$
\begin{aligned}
&\dot{x}_0 = R_0 = (u_1^2 + u_2^2)^{\frac{1}{2}} && x_0(t_0) = 0 && \text{---} \\
&\dot{x}_1 = R_1 = x_2 && x_1(t_0) = a_1 && x_1(t_1) = b_1 \\
&\dot{x}_2 = R_2 = -x_1(x_1^2 + x_3^2)^{-3/2} + x_5^{-1} u_1 && x_2(t_0) = a_2 && x_2(t_1) = b_2 \\
(18)\ &\dot{x}_3 = R_3 = x_4 && x_3(t_0) = a_3 && x_3(t_1) = b_3 \\
&\dot{x}_4 = R_4 = -x_3(x_1^2 + x_3^2)^{-3/2} + x_5^{-1} u_2 && x_4(t_0) = a_4 && x_4(t_1) = b_4 \\
&\dot{x}_5 = R_5 = -\alpha(u_1^2 + u_2^2)^{\frac{1}{2}} && x_5(t_0) = m_0 && \text{---} \\
&\dot{x}_6 = R_6 = 1 && x_6(t_0) = t_0 && x_6(t_1) = T.
\end{aligned}
$$

Here (u_1, u_2) is to be chosen so as to minimize $x_0(t_1)$ subject to meeting the initial and terminal conditions specified in (18) and the

same condition (2). As above, $x_0(t_1)$ represents total fuel consumption over the trip; x_1 and x_3 are the coordinates and x_2 and x_4 the velocity components of the ferry. Also, x_5 is the mass of the ferry (the decrease in mass of which with fuel consumption is now being considered); x_6 is introduced to avoid an explicit condition on t_1.

The Maximum Principle now, as with (7) above, introduces new functions $\{\psi_0, \ldots, \psi_6\}$ satisfying the system of equations

$$
\begin{aligned}
\dot{\psi}_0 &= 0 \\
\dot{\psi}_1 &= (x_3^2 - 2x_1^2)(x_1^2 + x_3^2)^{-5/2}\psi_2 - 3x_1x_3(x_1^2 + x_3^2)^{-5/2}\psi_4 \\
\dot{\psi}_2 &= -\psi_1 \\
(19) \quad \dot{\psi}_3 &= -3x_1x_3(x_1^2 + x_3^2)^{-5/2}\psi_2 + (x_1^2 - 2x_3^2)(x_1^2 + x_2^2)^{-5/2}\psi_4 \\
\dot{\psi}_4 &= -\psi_3 \\
\dot{\psi}_5 &= -x_5^{-2}u_1\psi_2 - x_5^{-2}u_2\psi_4 \\
\dot{\psi}_6 &= 0
\end{aligned}
$$

to which must be added the terminal conditions that

$$
(20) \quad
\begin{aligned}
\psi_0(t_1) &\leq 0 \\
\psi_5(t_1) &= 0
\end{aligned}
$$

($\psi_0(t_1) = 0$ is impossible for large K and we normalize so $\psi_0(t_1) = -1$).

Then, constructing $H = \Sigma_j \psi_j R_j$ as before,

$$
H = H(\psi, \bar{x}, u) = [- |u| + x_5^{-1}(\psi_2 u_1 + \psi_4 u_2) - \alpha\psi_5|u|]
$$

$$
(21) \qquad\qquad + [x_2\psi_1 - x_1(x_1^2 + x_3^2)^{-3/2}\psi_2 + x_4\psi_3 - x_3(x_1^2 + x_3^2)^{-3/2} + \psi_6]
$$

and this will be maximized by taking

$$(22) \qquad u = \begin{cases} K\varphi/|\varphi| & \bar{w} > 0 \\ 0 & \bar{w} < 0 \end{cases}$$

where we now have

$$(23) \qquad \bar{w} = |\varphi| - m(1 + \alpha\psi_5)$$

with $\varphi = (\psi_2, \psi_4)$ as before and $m = x_5$. Note that $m = x_5$ and α are always positive and, as

$$\psi_5 = -x_5^{-2} u \cdot \varphi = -x_5^{-2}|u||\varphi| \leq 0, \qquad \psi_5(t_1) = 0,$$

we also have ψ_5 always non-negative for $t \, \epsilon \, [t_0, t_1]$. As before, we can therefore multiply $\bar{w}$ by the (<u>always positive</u>) function $[|\varphi| + m(1 + \alpha\psi_5)]$ to get

$$(24) \qquad \bar{W} = \psi_2^2 + \psi_4^2 - m^2(1 + \alpha\psi_5)^2$$

$$(22') \qquad u(t) = \begin{cases} K\varphi/|\varphi| & \bar{W} > 0, \\ 0 & \bar{W} < 0. \end{cases}$$

It follows that, for the equations of motion (18), the K-optimal transfer functions (we have not yet, of course, shown that such exist) are characterized in exactly the same way as for the linearized equations (1). That is

<u>For a K-optimal transfer function (for (18)) one is always either firing at maximal thrust</u> ($|u| = K$ <u>for</u> $\bar{W} > 0$) <u>or "coasting"</u> ($u = 0$ <u>for</u> $\bar{W} < 0$). <u>Further, if</u> τ_K <u>denotes the amount of firing time during the trip then</u> $K\tau_K$ <u>is monotonically decreasing and</u>
$$\tau_K = \mathcal{O}(1/K).$$

Unfortunately, it is no longer possible in the non-linear case
to find the general solution of the auxiliary system (19) independent
of $\bar{x}$ and u nor can we write down, as with (16), a simple expression
for the manifold of possible $\bar{W}$. Thus, the method used above to demon-
strate a bound (independent of K) on the number of firing intervals
cannot be employed here. We have no proof that for a given K the
determining function $\bar{W}$ will have only a finite number of roots, much
less an explicit bound independent of K.

The existence theorem given above for the case of the linearized
equations (1) no longer applies as the manifold M cannot be charac-
terized as in (3') - the proof is valid modulo a lemma to the effect
that M is weakly closed. It is possible, however, to prove existence
by the use of a general existence theorem due to Roxin [4] which for our
purposes may be stated as follows:

Theorem: Let $\dot{x} = R(\bar{x}, u)$ where $\bar{x}, R \in E_m$ and $u \in U \subset E_n$ for each t.
Let $S_{\bar{x}} = \{R(\bar{x}, u) : u \in U\}$ be convex for each $\bar{x} \in E_m$. Assume that R is
defined on $E_m \times U$ so as to be continuous in u and Lipschitz continuous
in $\bar{x}$ (uniformly for $u \in U$). Specify initial conditions for $x(t_0)$ and
terminal conditions $\bar{x}(t_1) \in E$ where E is compact. Then there exists
an optimal control, i.e., a measurable function $u(t)$ such that the condi-
tions on $\bar{x}$ are satisfied and $x_0(t_1)$ is a minimum.

The right hand side $R(\bar{x}, u)$ in (18) is not uniformly Lipschitz
continuous, as it has a singularity at $x_1 = x_3 = 0$ and at $x_5 = 0$. This
is easy to fix up, however, - if, as we assume, the capsule is already

in an approximate orbit it is clear that the trajectory determined by
any optimal control will be bounded away from the singularity of the field
(the center of the earth - actually, of course, it must be bounded away
from the earth's surface) and x_5 is, of course, always greater than the
payload so that $R(\bar{x},u)$ can be re-defined within that bound arbitrarily
so as to make R Lipschitz continuous in $\bar{x}$ without affecting the
physically significant problem.

That $S_{\bar{x}}$ is not convex for each $\bar{x}$ (it turns out to be part of the
surface of a cone) is a more difficult matter which can, fortunately, be
gotten around by a trick. Consider the new set of equations (the boundary
conditions are the same as in (18) except that we place some a priori
bound on $x_0(t_1)$ to make the terminal set E compact)

(18')
$$
\begin{aligned}
\dot{x}_0 &= R_0' = & u_0 \\
\dot{x}_1 &= R_1' = x_2 \\
\dot{x}_2 &= R_2' = - x_1(x_1^2 + x_3^2)^{-3/2} + x_5^{-1} u_1 \\
\dot{x}_3 &= R_3' = x_4 \\
\dot{x}_4 &= R_4' = - x_3(x_1^2 + x_3^2)^{-3/2} + x_5^{-1} u_2 \\
\dot{x}_5 &= R_5' = & - \alpha \quad u_0 \\
\dot{x}_6 &= R_6' = 1
\end{aligned}
$$

where the new control $u' = (u_0,u_1,u_2)$ lies in the set $U' \subset E_3$ defined
by

(25) $0 \le (u_1^2 + u_2^2)^{\frac{1}{2}} \le u_0 \le K.$

Modify R', of course, as described above so as to make it Lipschitz
continuous even for small x_1,x_3,x_5. Then this problem satisfies all the

conditions of Roxin's existence theorem and an optimal control $u'(t)$ must exist.

If we could show that necessarily $u_0 = (u_1^2 + u_2^2)^{\frac{1}{2}}$ for this optimal control then the solution to this problem would also give a K-optimal control for the problem posed by (18). We make use of the Pontryagin Maximum Principle again and note that (letting ψ be the solution to the auxiliary system corresponding to (18') - which happens to be the system (19) again) the function $H = H(\psi, \bar{x}, u')$ is now given by

(21') $H = [-u_0 + x_5^{-1}(\psi_2 u_1 + \psi_4 u_2) - \alpha \psi_5 u_0]$

$$+ (\psi_1 x_2 + \psi_3 x_4) - (x_1^2 + x_3^2)^{-3/2}(\psi_2 x_1 + \psi_4 x_3) + \psi_6$$

and this is clearly maximized, for $u' \varepsilon U$, by letting

$$(u_1, u_2) = u_0 \varphi/|\varphi|, \quad u_0 = (u_1^2 + u_2^2)^{\frac{1}{2}} = \begin{cases} K & \bar{w} > 0 \\ 0 & \bar{w} < 0 \end{cases}$$

so that u_0 can be replaced in (18') by $|u|$ and (18') reduces to (18). Thus $u = (u_1, u_2)$ provides a K-optimal control for (18) and existence is demonstrated.

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