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MARKOVIAN DECISION PROCESSES WITH UNCERTAIN TRANSITION PROBABILITIES OR REWARDS

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MASSACHUSETTS INSTITUTE
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TECHNOLOGY

MARKOVIAN DECISION PROCESSES WITH UNCERTAIN TRANSITION PROBABILITIES OR REWARDS*

by

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FOREWORD

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Philip M. Morse,
Director of the Center

Ronald A. Howard,
Project Supervisor

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I would like to thank the Office of Naval Research for their financial support of my research at the M. I. T. Operations Research Center. A special word of thanks must also be given to my many colleagues at the Center, in particular, Mr. Paul Schweitzer, with whom I have had several stimulating discussions concerning the thesis.

Finally, there are the typists. Needless to say, I am indebted to Misses Constance Nicholas and Sara Lesser, in particular to the latter who did most of the typing and in such an excellent fashion.

PREFACE

There are two unusual features of the writeup of this study which should be mentioned at the outset.

First, due to the large number of symbols used, it was felt that the reader would benefit from a glossary of symbols. Therefore, such a glossary has been included at the extreme end of the report. As Appendix F has been taken from another report of the author and it contains self-explained symbols, the above mentioned glossary does not apply to that appendix.

Secondly, to make the reading of the main text easier, a considerable amount of theoretical development and numerical results have been placed in appendices. It is hoped that this and the above feature will help convey the results of the research.

MARKOVIAN DECISION PROCESSES WITH UNCERTAIN TRANSITION PROBABILITIES OR REWARDS

by

EDWARD ALLAN SILVER

Submitted to the Department of Civil Engineering on August 29, 1963 in partial fulfillment of the requirements for the degree of Doctor of Science.

ABSTRACT

In most Markov process studies to date it has been assumed that both the transition probabilities and rewards are known exactly. The primary purpose of this thesis is to study the effects of relaxing these assumptions to allow more realistic models of real world situations. The Bayesian approach used leads to statistical decision frameworks for Markov processes.

The first section is concerned with situations where the transition probabilities are not known exactly.

One approach used incorporates the concept of multi-matrix Markov processes, processes where it is assumed that one of several known transition matrices is being utilized, but we only have a probability vector on the various matrices rather than knowing exactly which one is governing the process. An explanation is given of the Bayes modification of the probability vector when some transitions are observed. Next, we determine various quantities of interest, such as mean recurrence times. Finally a discussion is presented of decision making involving multi-matrix Markov processes.

The second approach assumes more directly that the transition probabilities themselves are random variables. It is shown that the multidimensional Beta distribution is a most convenient distribution (for Bayes calculations) to place over the probabilities of a single row of the transition matrix. Several important properties of the distribution are displayed. Then a method is suggested for determining the multidimensional Beta prior distributions to use for any particular Markov process. Next we deal with the effects on various quantities of interest of having such distributions over the transition probabilities. For 2-state processes, several analytic results are derived. Despite analytic complexities, some interesting expressions are developed for N-state cases.

It is shown that for decision purposes the expected values of the steady state probabilities are important quantities. For a special 2-state situation, use of the hypergeometric function (previously utilized

in the solution of certain physics problems) permits evaluation of these expected values. Their determination for 3 or more states requires the use of simulation. Fortunately, a simple approximation technique is shown to generally give accurate estimates of the desired quantities. An entire chapter is devoted to statistical decisions in Markov processes when the transition probabilities are multidimensional Beta distributed rather than being exactly known. The main problem considered is one where we have the option of buying observations of a Markov process so as to improve our knowledge of the unknown transition probabilities before deciding whether or not to utilize the process.

In the second section of the study, we assume that the transition probabilities are exactly known, but now the rewards are random variables. First we display the Bayes modification of two convenient distributions to use for the rewards. Next, the expected rewards in various time periods are determined. Finally, an explanation is presented of how to utilize these expected rewards in making statistical decisions concerning Markov processes whose rewards are not known exactly.

Thesis Supervisor: Ronald A. Howard
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CHAPTER 1

INTRODUCTION

It is becoming increasingly evident that Markov process models are playing a more and more important role in the mathematical analysis of both military and industrial operations. This increased practical importance necessitates further research in the fundamentals of Markov process theory. This study is an effort in this direction.

Practically all Markov process applications to date have assumed that all the relevant parameters (i.e. the rewards and transition probabilities) are exactly known. In many situations this has been and will continue to be a very debatable assumption. The purpose of this dissertation is to study the effects of relaxing these assumptions to allow more realistic models of the real world situation. The approach used leads to a statistical decision framework for Markov processes.

Since Bartlett's pioneering work[1] in 1951 several theoretical studies have been devoted to the problem of estimating the transition probabilities of Markov processes (see Billingsley's article[2] for an excellent reference list). These studies have been concerned with statistical methods that result in point estimates (i.e. exact single values) of the transition probabilities. In the present research a different approach, that of a Bayesian analyst, is to be used.

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1. Bartlett, M., "The Frequency Goodness of Fit Test for Probability Chains", Cambridge Philosophical Proceedings, 1951 (47), p. 86.

2. Billingsley, P., "Statistical Methods in Markov Chains", Annals of Mathematical Statistics (U. S.), Vol. 32 (1961), No. 1, pp. 12-40.

Prior knowledge about the unknown parameters (transition probabilities or rewards) is expressed in the form of probability distributions over the unknown parameters; i.e., the parameters are considered as random variables. These prior probability distributions are modified through the use of Bayes' rule when observations (perhaps in the form of transitions or rewards) of the Markov process are made. More precisely, the a posteriori distribution is given by*

$$f_{x_1, x_2, \ldots, x_n}(y_1, y_2, \ldots, y_n | E_i) = \frac{pr(E_i | y_1, y_2, \ldots, y_n)\, \overbrace{f_{x_1, x_2, \ldots, x_n}(y_1, y_2, \ldots, y_n)}^{\text{prior}}}{pr(E_i)},$$

a direct consequence of Bayes' rule. $x_1, x_2, \ldots, x_n$ are the unknown parameters and E denotes the observations. The a posteriori distributions over the parameters, unlike point estimates, clearly reflect our uncertainty as to the exact values of the parameters and this uncertainty can be incorporated when making statistical decisions about the process. An excellent description of the philosophy of the Bayes

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*The notation is described in footnote number 12 found on page 36.

approach has been presented by Savage.[2a] Therefore, we refer the interested reader to Savage's article for a justification of this approach rather than reproducing his ideas here. Fortunately, his article also contains an extensive list of references on Bayes procedures. In fact, it is probably the best starting point to become familiar with the entire area of Bayesian statistics.

The following, in itself, is an important reason for using a Bayes approach in the study of Markov processes. Quite often prior to observational data the analyst knows considerable information about the unknown parameters, but not enough for him to state outright that the parameters can be assigned exactly known values. If the observations have significant costs associated with them, the analyst must decide exactly how many observations are worthwhile. A Bayes approach allows him to make his decision in a quantitative manner.

In Section I we shall be concerned with situations where the transition probabilities are not known exactly. More precisely, Chapter 2 assumes that we have a probability vector over several possible transition matrices.

$$a = [a_1, a_2, \ldots, a_k, \ldots, a_Q]$$

where a_k is the probability that matrix ${}^kP$ is being used $(k=1, 2, \ldots, Q)$. Then, in Chapters 3, 4 and 5 we assume more directly that the transition probabilities themselves are random variables. Chapter 6 involves a brief look at continuous time processes where the transition rates are random variables rather than being exactly known.

[2a] Savage, L. J., "Bayesian Statistics," Recent Developments in Information and Decision Processes, Macmillan, 1962.

The approach of placing probability distributions directly over the transition probabilities is probably more appealing than that used in Chapter 2 where we deal with multi-matrix Markov processes, processes where it is assumed that one of Q known transition matrices is being utilized, but we only have a probability vector on the various matrices rather than knowing exactly which one is being used. However, the multi-matrix approach does make the mathematical theory and computations considerably simpler in certain portions of the study. Also, as will be explained later, it can be thought of as a step toward the combination of Markov process theory and game theory.

In Chapter 2, with the multi-matrix framework assumed, a detailed explanation is presented of the Bayes' modification of the probability vector ($\underline{\alpha}$) when some transitions are observed. Next we determine various quantities of interest, such as mean recurrence times. Finally, decision making involving multi-matrix Markov processes is discussed.

As mentioned earlier, in Chapters 3, 4 and 5, we assume directly that the transition probabilities themselves are random variables. Unfortunately the analytic considerations become far more formidable than those encountered in Chapter 2. Still, many interesting and potentially useful results are developed.

Chapter 3 is concerned with the multidimensional Beta distribution, a most convenient distribution to place over the transition probabilities of a single row of the transition matrix. In research work concurrent with this study two other individuals, Murphy[3] and Mosimann[4], have developed this

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3 Murphy, Roy E. Jr., Adaptive Processes in Economic Systems, Technical Report No. 119, Institute for Mathematical Studies in the Social Sciences, Stanford University, 1962.

4 Mosimann, J. E., "On the Compound Multinomial Distribution, the Multivariate β-distribution, and Correlations among Proportions", Biometrika, Vol. 49 (1962), pp. 65-82.

distribution as a convenient one to place over the parameters of a multinomial distribution. However, their purposes in doing this did not include the study of Markov processes. After the important properties of this distribution are listed, a method is suggested for determining the a priori parameters. Next, the Bayes modification of the distribution is outlined. Then we concern outselves with the development of the multidimensional Beta priors for a specific N - state Markov process. Finally, two methods of simulating a multidimensional Beta distribution are suggested. This simulation is required later in the study.

Chapter 4 deals with the effects on various quantities of interest of having multidimensional Beta distributions over the transition probabilities. A particularly detailed analytical study is made of the expected values of the steady state probabilities in the 2 - state case. For 3 or more states, analytic complexities necessitate the use of simulation. Detailed results of simulations for 3 and 4 state processes are presented. First passage times, state occupancy times, transient behavior and a trapping state situation are also studied.

In Chapter 5 we attack the problem of making statistical decisions in Markov processes when the transition probabilities are not known exactly. First, the importance of the expected values of the steady state probabilities is displayed. Then we consider a specific 2 - state problem where the decision is based upon the expected reward in s periods in the steady state and we have the option of buying observations of the process so as to improve our knowledge of the unknown transition probability. The next section deals with an analogous N - state problem. After this, several other items of interest for statistical decision purposes are discussed. Finally, we look at statistical decisions under a transient situation.

In Section II we assume that the transition probabilities are exactly known but now the rewards are random variables. First, in Chapter 7, a presentation is made of two convenient distributions to use for the rewards. Bayes modification of them is also displayed. Then Chapter 8 is concerned with the determination of the expected rewards in various time periods (both steady state and transient). Chapter 9 shows how to utilize these expected rewards in making statistical decisions in Markov processes where the rewards are not known exactly.

Chapter 10, entitled "Conclusions", summarizes the more important points of the study and also suggests several related areas for further research.

SECTION I

TRANSITION PROBABILITIES NOT KNOWN EXACTLY

In this section we remove the usual Markov requirement that the transition probabilities be known exactly; rather we assume that the transition probabilities (or rates or matrices) are themselves random variables. There are three primary objectives. The first is to determine a reasonably simple method for placing convenient distributions over the transition probabilities (or rates or matrices). Secondly, we want to be able to easily modify these distributions through the use of Bayes' rule after observing some transitions. Finally, it is important to know the effects on various quantities of interest of having distributions over the transition probabilities (or rates or matrices). This latter consideration leads to a statistical decision framework for Markov processes.

CHAPTER 2

THE MULTI-MATRIX MARKOV PROCESS

2.1 Outline of the Process

Instead of placing probability distributions directly over the transition probabilities (as will be done in Chapters 3 - 5) we proceed as follows: It is assumed that the Markov process is governed by one of Q given transition matrices whose elements are exactly known. This one matrix is always used but we are not sure as to exactly which one of the Q matrices it is. In fact, we define a probability vector

$$\underline{\alpha} = [\alpha_1, \alpha_2, \ldots, \alpha_Q]$$

where α_j = probability that matrix j is governing the process. A process defined in this way is called a multi-matrix Markov process.

With this situation existing, we show in section 2.2 how to update the $\underline{\alpha}$ vector (by Bayes' rule) when several transitions of the process are observed. Next we discuss the problem of determining various quantities of interest such as the steady state probabilities. In section 2.4 consideration is given to possible cost structures for the multi-matrix Markov process. Finally, in section 2.5 we demonstrate statistical decision theory for multi-matrix Markov processes by analyzing a specific decision problem.

Although the multi-matrix Markov framework is less appealing in a physical sense than is placing probability distributions directly over the transition probabilities, it does make the mathematical theory and computations considerably simpler in certain portions of the study. Moreover, it can be thought of as a step toward the combination of Markov process theory and game theory. The multi-matrix process can be considered as follows. An opponent (or nature) selects one of Q known transition matrices with which to run the Markov process, but we only know probabilistically which one has been selected. We then have to decide upon our best course of action (as regards using the process, etc.) under these circumstances. Note, however, that it is assumed that the opponent (or nature) can not switch matrices once a choice has been made. The next step from the game theoretic point of view would be to allow switching. Such a situation will not be considered here.

2.2 Bayes Modification of the $\underline{\alpha}$ Vector

2.2.1 Ignoring the Starting State

Let B (F) be the event that a number of consecutive transitions are observed having a frequency count $F = (f_{ij})$ where f_{ij} = number of transitions from state i to state j. Two points should be noted. First, we do not require that the exact order of transitions be known. Secondly, we assume for the moment that the starting state provides no information about which matrix is being used. This assumption could be satisfied in one of two ways; either the process is forced to start in a particular state regardless of which matrix is being used, or we have absolutely no idea as to which state was the starting one.

Let the Q possible matrices be designated by ${}^1P, {}^2P, \ldots {}^kP, \ldots {}^QP$ with elements ${}^1p_{ij}, {}^2p_{ij}, \ldots, {}^kp_{ij}, \ldots {}^Qp_{ij}$ respectively. Then

$$pr\,(B(F) \mid {}^kP) = N(F) \prod_{ij=1}^{n} ({}^kp_{ij})^{f_{ij}} \qquad 0^0 \equiv 1$$

where $\prod_{ij=1}^{n} ({}^kp_{ij})^{f_{ij}}$ is the probability of a particular sequence that would produce the frequency count F given that matrix ${}^kP$ is being used, and N(F) is the number of such sequences; N(F) is independent of the P matrix used, hence will cancel out in the Bayes calculations. This is fortunate as the expression for N(F) is extremely complicated.[5]

Suppose that prior to the event B(F) we have a probability vector $\underline{a}' = (a_1', a_2', \ldots, a_k', \ldots, a_Q')$ where a_k' is the probability that matrix ${}^kP$ is being used. Then utilizing Bayes' rule the posterior probability that matrix ${}^kP$ is being used is

$$a_k'' = pr\,({}^kP \mid B(F)) = \frac{pr\,(B(F) \mid {}^kP)\; pr\,({}^kP)}{pr\,(B(F))}$$

$$= \frac{N(F) \prod_{i,j=1}^{N} ({}^kp_{ij})^{f_{ij}}\, a_k'}{\sum_{m=1}^{Q} N(F) \prod_{i,j=1}^{N} ({}^mp_{ij})^{f_{ij}}\, a_m'}$$

$$a_k'' = \frac{a_k' \prod_{i,j=1}^{N} ({}^kp_{ij})^{f_{ij}}}{\sum_{m=1}^{Q} a_m' \prod_{i,j=1}^{N} ({}^mp_{ij})^{f_{ij}}} \quad \ldots\ldots\ldots\ldots\ldots\ldots (2.1)$$

Hence, the determination of the a posteriori probabilities is a relatively simple operation.

Numerical Example

Bob confronts his friend Ray with an interesting problem. Bob has two cages with an interconnecting door. He is also the owner of two white mice that are identical in appearance. The behavior of these mice in the

---

5 Whittle, P., "Some Distribution and Moment Formulae for Markov Chains", Journal of the Royal Statistical Society, Series B, Vol. 17 (1955), pp. 235 - 242.

cages is exactly known to Ray. Mouse no. 1 has a transition matrix

$$
{}^{1}P = \begin{array}{c} \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{cc} 1 & 2 \end{array} \\ \begin{bmatrix} 0.1 & 0.9 \\ 0.7 & 0.3 \end{bmatrix} \end{array}
$$

while the corresponding matrix for mouse no. 2 is

$$
{}^{2}P = \begin{array}{c} \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{cc} 1 & 2 \end{array} \\ \begin{bmatrix} 0.5 & 0.5 \\ 0.4 & 0.6 \end{bmatrix} \end{array}
$$

Bob also informs Ray that he has selected mouse no. 1 with probability 0.8 and mouse no. 2 with probability 0.2 and once selected the mouse is never replaced, i.e. the same mouse will always be in the cages.

Now suppose Ray has observed six transitions (time periods) with frequency count

$$
F = \begin{bmatrix} 2 & 2 \\ 2 & 0 \end{bmatrix}.
$$

What should be his a posteriori probabilities that each of the two mice are in use?

This is a multi-matrix Markov process with $N = 2$ and $Q = 2$.

Using equation (2.1)

$$
a_1'' = \frac{0.8\,[\,(0.1)^2(0.9)^2(0.7)^2(0.3)^0\,]}{0.8\,[\,(0.1)^2(0.9)^2(0.7)^2(0.3)^0] + 0.2\,[\,(0.5)^2(0.5)^2(0.4)^2(0.6)^0]}
$$

$$
= .137
$$

i.e. $\underline{a}'' = [\,.137,\ .863]$

It is apparent that the six transitions have radically altered Ray's feelings about which mouse is being used. Of course, this is primarily due to the two 1 - 1 transitions which are very unlikely if mouse no. 1 is in action. However, this simple example does illustrate an important point. If the possible matrices are significantly different, considerable influence is played by the transition data in obtaining the a posteriori probabilities even if very few transitions are observed.

2.2.2 Incorporation of the Starting State

In many situations it is meaningful to incorporate the knowledge of the starting state into the Bayesian modification. Suppose that the observed transition sequence started in state s. Then, if the probability that the starting state is s is a function of the specific matrix used, this information should be utilized.

Let s_k be the probability that the starting state is s given that matrix ^kP is governing the process and let (E(F), s) be the joint event that the starting state is s and a sequence is observed having a frequency count F. Then

$$a_k'' = pr(^kP \mid E(F), s) = \frac{a_k' \, s_k \prod_{i,j=1}^{N} (^kp_{ij})^{f_{ij}}}{\sum_{m=1}^{Q} a_m' \, s_m \prod_{i,j=1}^{N} (^mp_{ij})^{f_{ij}}} \quad \ldots (2.2)$$

s_k would generally be calculated in one of two ways:

i) If we could assume that the sequence of transitions was observed when the process was in the steady state, then

$s_k = {}^k\pi_s$, the steady state probability of being in state s given that matrix ^kP is in operation.

ii) If we were told that the process was forced to be in state u n time periods before the sequence of observations, then

$s_k = {}^k\phi_{us}(n)$, the probability that the state at time n is s given that the state at time 0 is u and that matrix ${}^kP$ is being used. Howard[6] describes several methods for obtaining this quantity.

Numerical Example

For the mouse example of the previous section suppose that Ray was also informed that the observed transitions occurred when the mouse had been in the cages for a long time and the first observed transition was from cage 1.

$$ {}^1P = \begin{bmatrix} 0.1 & 0.9 \\ 0.7 & 0.3 \end{bmatrix} \qquad {}^2P = \begin{bmatrix} 0.5 & 0.5 \\ 0.4 & 0.6 \end{bmatrix} $$

$$ \underline{a}' = \begin{bmatrix} 0.8 & 0.2 \end{bmatrix} \qquad F = \begin{bmatrix} 2 & 2 \\ 2 & 0 \end{bmatrix} $$

Now we solve ${}^1\underline{\pi}\,{}^1P = {}^1\underline{\pi}$ and ${}^2\underline{\pi}\,{}^2P = {}^2\underline{\pi}$ to obtain ${}^1\underline{\pi} = [\ 7/16,\ 9/16]$ and ${}^2\underline{\pi} = [\ 4/9,\ 5/9]$ respectively. Then using equation (2.2) there results

$\underline{a}'' = [\ .135,\ .865]$

The probability of mouse no. 1 has been reduced still further by the knowledge of the starting condition.

---

6 Howard, R. A., Dynamic Probabilistic Systems, in preparation.

2.2.3 $\underline{a}'$ Unknown

When $\underline{a}'$ is completely unknown one might argue that a good approximation would be achieved by using

$$a_k' = \frac{s_k}{\sum_{m=1}^{Q} s_m}$$

However this is equivalent to assuming $a_k' = \frac{1}{Q}$ and then using the starting information as earlier, for

$$\frac{\frac{1}{Q} s_k}{\sum_{m=1}^{Q} \frac{1}{Q} s_m} = \frac{s_k}{\sum_{m=1}^{Q} s_m}$$

2.3 Determination of Various Quantities of Interest

Many deterministic quantities in a regular Markov process become random variables in the multi-matrix Markov process. Furthermore, all these random variables are of the discrete variety because of the discrete nature of the probability distribution on the matrices.

2.3.1 Mean Recurrence Times

For a known transition matrix we can easily obtain the mean recurrence times through the use of matrix or flow graph techniques[7, 8].

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7. Ibid.

8. Kemeny, J. G., and Snell, J. L., Finite Markov Chains, Van Nostrand, 1960, p. 79.

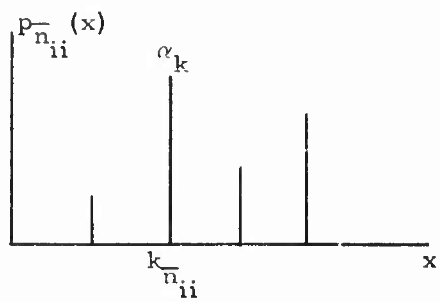
The mean recurrence time for state i is the mean time to go from that state to itself (staying in the state on the first transition results in a recurrence time of length 1).

With matrix ${}^{k}P$ we have

$$ {}^{k}\underline{\bar{n}} = ({}^{k}\bar{n}_{11}, {}^{k}\bar{n}_{22}, \ldots, {}^{k}\bar{n}_{NN}) $$

where ${}^{k}\bar{n}_{ii}$ = mean recurrence time for state i | matrix ${}^{k}P$ is being used. However, because ${}^{k}P$ is only being used with probability a_k, the mean recurrence times now become random variables with probability mass functions[9]

$$ \mathrm{pr}\,(\bar{n}_{ii} = {}^{k}\bar{n}_{ii}) = p_{\bar{n}_{ii}}({}^{k}\bar{n}_{ii}) = a_k \qquad k = 1, 2, \ldots, Q $$



This random variable will have a mean value given by

$$ E(\bar{n}_{ii}) = \sum_{k=1}^{Q} a_k \, {}^{k}\bar{n}_{ii} $$

---

9 The probability mass function is

$p_x(k)$ = pr $\{$that a discrete random variable x takes on the value k$\}$ $k = k_1, k_2, \ldots, k_n$.

Also $p_{x|y}(k|j)$ = pr. $\{$ that x takes on the value k | the variable y takes on the value j$\}$ $k = k_1, k_2, \ldots, k_n$.

Numerical Example

Again consider the mouse example. Given that a mouse is seen in cage i (i = 1, 2), what is the expected number of transitions until the moment when he first appears again in the same cage?

$$^1P = \begin{bmatrix} 0.1 & 0.9 \\ 0.7 & 0.3 \end{bmatrix} \qquad ^2P = \begin{bmatrix} 0.5 & 0.5 \\ 0.4 & 0.6 \end{bmatrix}$$

$$\underline{a} = [\ 0.8,\ 0.2]$$

It is shown in Appendix N that for

$$P = \begin{bmatrix} 1-a & a \\ b & 1-b \end{bmatrix}$$

$$\bar{n}_{11} = \frac{a+b}{b} \quad \text{and} \quad \bar{n}_{22} = \frac{a+b}{a}$$

$$\therefore \quad ^1\bar{n}_{11} = \frac{0.9 + 0.7}{0.7} = \frac{16}{7}$$, the mean time for mouse no. 1 to go from cage 1 to cage 1. Similarly

$$^1\bar{n}_{22} = \frac{16}{9}; \quad ^2\bar{n}_{11} = \frac{9}{4}; \quad ^2\bar{n}_{22} = \frac{9}{5}$$

Hence, $\bar{n}_{11}$ and $\bar{n}_{22}$ have the following probability mass functions:

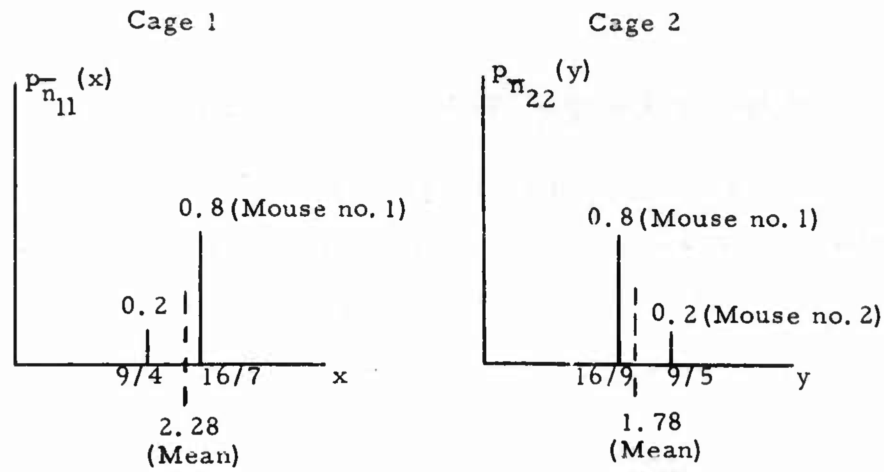


Figure 2.1 - The Probability Mass Functions of the Mean Recurrence Times in a Multi-Matrix Markov Process

Similar probability mass functions could be obtained for $\bar{\pi}_{ij}$, the mean time to go from cage i to cage j, with $i \neq j$.

2.3.2 Steady State Probabilities

When matrix ${}^{k}P$ is being used the steady state probability vector is given by

$$ {}^{k}\underline{\pi} = [{}^{k}\pi_1, {}^{k}\pi_2, \ldots, {}^{k}\pi_N] $$

As with the mean recurrence times we can think of the steady state probabilities as random variables

$$ \text{e.g.} \quad pr(\pi_i = {}^{k}\pi_i) = p_{\pi_i}({}^{k}\pi_i) = a_k \qquad k = 1, \ldots, Q $$

or more directly we can say that

$$ E(\pi_i) = \sum_{k=1}^{Q} a_k \, {}^{k}\pi_i, $$ the mean value of π_i, is the expected probability of being in state i in the steady state.

Numerical Example (same as in previous section)

$E(\pi_1) = 0.8\left(\frac{7}{16}\right) + 0.2\left(\frac{4}{9}\right) = .439$, this is the expected probability that a look at cage 1 after a long time will find a mouse in that cage during the period in which the look is taken. Also

$E(\pi_2) = 0.561$

2.3.3 Transient Behavior

As was done in section 2.2.2 let ${}^k\phi_{ij}(n) = pr.\{$ state at time n is $j \mid$ state at time 0 is i and matrix ${}^kP$ is in use $\}$. The unconditional multi-step transition probability, $\phi_{ij}(n)$, will again have a probability mass function and we can say that

$E\left[\phi_{ij}(n)\right] = \sum_{k=1}^{Q} a_k \, {}^k\phi_{ij}(n)$, the mean value of $\phi_{ij}(n)$, is

the expected probability of being in state j at time n given that the state at time 0 is i.

Numerical Example (same as in section 2.3.1).

Again using either flow graphs or matrix inversion we obtain

$${}^1\phi_{12}(n) = \frac{9}{16} - \frac{9}{16}(-0.6)^n \qquad n \geq 0.$$

This is the probability that mouse no. 1 will be in cage 2 at time n| mouse no. 1 was in cage 1 at time 0 and

$${}^2\phi_{12}(n) = \frac{5}{9} - \frac{5}{9}(0.1)^n \qquad n \geq 0.$$

This is the same as above except for mouse no. 2.

$$\therefore \quad E\left[\phi_{12}(n)\right] = 0.561 - 0.450\,(-0.6)^n - 0.111\,(0.1)^n \qquad n \geq 0$$

This is the probability that a mouse will be in cage 2 at time n | a mouse was in cage 1 at time 0.

2.3.4 Mean State Occupancy Time

Let ${}^k\bar{u}_i$ be the mean number of transitions to exit from state i | matrix ${}^kP$ is being used and the process is in state i before the first transition.

$$\therefore \quad {}^k\bar{u}_i = \frac{1}{1 - {}^kp_{ii}} \qquad \text{(mean of a geometric distribution with parameter } {}^kp_{ii}\text{)}$$

Now $\bar{u}_i$ will have a probability mass function given by

$$\mathrm{pr}\,(\bar{u}_i = {}^k\bar{u}_i) = p_{u_i}({}^k\bar{u}_i) = a_k \qquad k = 1, 2, \ldots, Q.$$

2.4 Addition of a Cost Framework

There are several possible cost frameworks for the multi-matrix Markov process. Two will be discussed in detail and a third will be mentioned.

2.4.1 Simple c_{ij} Form

Let c_{ij} = cost of assuming that matrix i is being used when in effect j is being utilized.

We have a Q x Q matrix $C = (c_{ij})$.

Let C (k) = cost of assuming that matrix ${}^kP$ is being used.

$$\text{Then } E[\,C(k)\,] = \sum_{j=1}^{Q} a_j c_{kj}$$

To minimize the expected cost we select k as follows:

$$\min_{1 \le k \le Q} \sum_{j=1}^{Q} a_j c_{kj} \quad \text{..........................} (2.3)$$

Numerical Example

For the mouse game Bob has decided to make things interesting by introducing a monetary aspect. Ray is forced to play and is confronted with the following cost matrix

$$C = (c_{ij}) = \begin{bmatrix} 0 & 5 \\ 16 & 0 \end{bmatrix} \quad ;$$

this says, for example, if he guesses that mouse no. 2 is being used when in reality no. 1 is in the cages, Ray must pay \$16 for his error. It is reasonable to have $c_{ii} = 0$ since c_{ii} is the penalty associated with making a correct decision. Using the data of section 2.2.1 we had $\underline{a}' = [0.8, 0.2]$ and after the set of transitions with frequency count

$$F = \begin{bmatrix} 2 & 2 \\ 2 & 0 \end{bmatrix} \quad \text{we had}$$

$$\underline{a}'' = [0.137, \ 0.863]$$

With $\underline{a}' = [0.8, \ 0.2]$

$$E[C(1)] = 0.8(0) + 0.2(5) = 1.0$$

$$E[C(2)] = 0.8(16) + 0.2(0) = 12.8 > 1.0$$

Therefore, prior to the observations Ray would have said that mouse no. 1 was being used, with an expected cost of 1.0. With

$$\underline{a}'' = [\,0.137 \quad 0.863\,]$$

$$E\,[C(1)] = .137\,(0) + .863\,(5) = 4.315 \quad \text{and}$$

$$E\,[C(2)] = .137\,(16) + .863\,(0) = 2.192 < 4.315$$

Therefore, after the observations, even though $c_{21} >> c_{12}$, he would say that mouse no. 2 is in the cages, with an expected cost of 2.192. It is seen that the small number of transitions has completely altered the decision and associated expected cost.

2.4.2 The Expected Net Profit per Transition

Here we assume that the important cost element is the expected net profit per transition. This quantity and its importance in statistical decision theory will be discussed at length in Chapter 5.

Let r_{ij} be the reward for each transition from state i to state j $(i, j = 1, 2, \ldots, N)$ and let c be the cost per transition for using the process. Then the expected reward per transition given that matrix ${}^kP$ is being used is ${}^kR$ where

$$ {}^kR = \sum_j \sum_j {}^k\pi_i \; {}^kp_{ij} \, r_{ij} \quad \ldots\ldots\ldots\ldots \quad (2.4)$$

with ${}^k\pi_i$ being the steady state probability of being in state i given that ${}^kP$ is in use.

With the probability distribution $\underline{a}$ over the possible matrices the expected reward per transition, $E(R)$, is given by

$$E(R) = \sum_{k=1}^{Q} a_k \, {}^kR = \sum_{k=1}^{Q} a_k \sum_{i=1}^{N} {}^k\pi_i \sum_{j=1}^{N} {}^kp_{ij} \, r_{ij} \quad \ldots\ldots (2.5)$$

and the expected net revenue per transition is

$$E(R) - c$$

Numerical Example

In the mouse problem (section 2.2.1) Ray is now confronted with a different cost structure where he may have a chance of making some money. He must pay $3.60 per time period to have a mouse in the cages but he is rewarded for mouse transitions as follows:

$$(r_{ij}) = \begin{bmatrix} 0 & 8 \\ 3 & 2 \end{bmatrix} \quad \text{(in dollars)}$$

where r_{ij} is the reward for a mouse transition from cage i to cage j. Should he now accept Bob's offer?

$${}^1P = \begin{bmatrix} 0.1 & 0.9 \\ 0.7 & 0.3 \end{bmatrix} \qquad {}^2P = \begin{bmatrix} 0.5 & 0.5 \\ 0.4 & 0.6 \end{bmatrix}$$

$$c = 3.6$$

Using the data of section 2.2.1 we had $\underline{a}' = [\,0.8 \;\; 0.2\,]$ and after the set of transitions with frequency count

$$F = \begin{bmatrix} 2 & 2 \\ 2 & 0 \end{bmatrix}$$

we had $\underline{a}'' = [\,0.137 \;\; 0.863\,]$

From equation (2.4)

$$^1R = {}^1\pi_1\,({}^1p_{11}\,r_{11} + {}^1p_{12}\,r_{12}) + {}^1\pi_2({}^1p_{21}\,r_{21} + {}^1p_{22}\,r_{22})$$

$$= \frac{7}{16}(0 + 7.2) + \frac{9}{16}(2.1 + 0.6)$$

$$= \$\,4.67$$

Similarly $^2R = 3.11$. Then using equation (2.5)

for $\underline{a}' = [\,0.8,\ \ 0.2]$ $\qquad E(R) = 4.36$

and $E(N.R.) = 4.36 - 3.6 = 0.76 > 0.$

For $\underline{a}'' = [\,0.137,\ 0.863]$ $\qquad E(R) = 3.32$

and $E(N.R.) = 3.32 - 3.6 = -0.28 < 0.$

Hence, if Ray considers it worthwhile to play the game only when E (N. R.) > 0, then it is clear that before the observations he would have been willing to play, but such is not the case afterwards.

2.4.3 A Third Possible Cost Structure

For the reader who is familiar with Howard's policy iteration problems[10] it probably has become apparent that the policy iteration model could be generalized to the situation where instead of knowing the transition probabilities exactly there is a multi-matrix framework. Hence, we have a third possible cost structure for the multi-matrix process.

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10 Howard, R. A., Dynamic Programming and Markov Processes, Technology Press and Wiley, 1960.

The analysis of the process within this cost structure appears to be quite formidable. It is hoped that successful research will be completed in this area in the near future, as inclusion of uncertainty in the transition matrices of the policy iteration problem would be a major step forward in the modeling of Markov decision processes.

2.5 A Decision Problem Involving a Multi-Matrix Markov Process

As was mentioned at the start of this chapter the primary purpose of this section is to introduce the reader to the use of statistical decision theory in multi-matrix Markov processes by considering in detail a specific decision problem.

2.5.1 Statement of the Problem

Consider a multi-matrix Markov process having Q possible matrices ${}^1P, {}^2P, \ldots, {}^QP$. Suppose that the decision maker is faced with the cost framework discussed in section 2.4.1, namely c_{ij} is the cost of assuming that matrix ${}^iP$ is being used when, in effect, ${}^jP$ is being utilized. Also let d_{rs} be the cost of observing a transition from state r to state s. Then the following general problem, as will be shown, can be formulated as a dynamic programming problem.

The process is in state i at present and we know the vector $\underline{\alpha}' = (\alpha_1', \alpha_2', \ldots, \alpha_Q')$ where α_k' is the probability that ${}^kP$ is governing the transitions. During the next n periods we have the following options:

i) State that a particular matrix is being used and pay the resulting cost - this ends the decision process.

ii) Observe the next transition paying the cost of the observation.

2.5.2 Dynamic Programming Framework of the Problem

Let $v_h(\underline{a}', i)$ = the expected cost if an optimal policy is followed and we are in state i with probability vector $\underline{a}'$ over the matrices and there are h decision periods left. Then as shown in section 2.4.1 (equation 2.3)

$$v_o(\underline{a}', i) = \min_{1 \le k \le Q} \sum_{m=1}^{Q} a_m{}' c_{km} \qquad \text{(independent of i)} \ldots (2.6)$$

Also

$$v_h(\underline{a}', i) = \min_{S,L} \begin{cases} \underline{S}\text{top} & \min\limits_{1 \le k \le Q} \sum\limits_{m=1}^{Q} a_m{}' c_{km} = v_o(\underline{a}', i) \\ \underline{L}\text{ook} & \sum\limits_{j=1}^{N} \sum\limits_{k=1}^{Q} a_k{}' \, {}^k p_{ij} \left[d_{ij} + v_{h-1}(\underline{a}'', j) \right] \end{cases} \qquad 1 \le h \le n \ldots (2.7)$$

Where $\underline{a}''$ is obtained through the use of Bayes' rule (see equation 2.1).

Theoretically the above recurrence relation and boundary conditions give us a solution to the problem. Unfortunately, the state vector is Q-dimensional for a process having Q possible matrices. Also Q - 1 of the dimensions are continuous rather than discrete variables. However, when Q = 2, even for a large number of states the computations can be carried out. An example with 2 states will now be presented.

2.5.3 A 2 - Dimensional Numerical Example

Again consider Bob and Ray's mouse game.

$$ {}^1P = \begin{bmatrix} 0.1 & 0.9 \\ 0.7 & 0.3 \end{bmatrix} \qquad {}^2P = \begin{bmatrix} 0.5 & 0.5 \\ 0.4 & 0.6 \end{bmatrix} $$

(mouse no. 1) (mouse no. 2)

Also

$$C = (c_{ij}) = \begin{bmatrix} 0 & 5 \\ 16 & 0 \end{bmatrix} \quad \text{and} \quad D = (d_{rs}) = \begin{bmatrix} 0.5 & 1 \\ 0.3 & 0 \end{bmatrix}$$

where c_{ij} = the cost of stating that mouse no. i is in use when mouse no. j is, and d_{rs} = the cost of observing a transition from cage r to cage s.

Assume that a mouse is presently in cage 1 and that Ray's decision as to which mouse is being used can be made now, after the next, the next two, or the next three transitions (i.e. n = 3). Also assume that $\underline{a} = (0.3, 0.7)$ and this includes the knowledge of the starting cage.

As there are only two possible matrices (Q = 2) we can replace $\underline{a}$ by a, the probability that mouse no. 1 is in use. Then from equation (2.6)

$$v_o(a', i) = \min_{k=1,2} [a' c_{k1} + (1-a') c_{k2}]$$

$$= \min_{1,2} \begin{cases} 1 & a'(0) + (1-a')5 = 5 - 5a' \\ 2 & a'(16) + (1-a')0 = 16a' \end{cases}$$

This says that, if no more observations are possible, Ray selects mouse no. 1

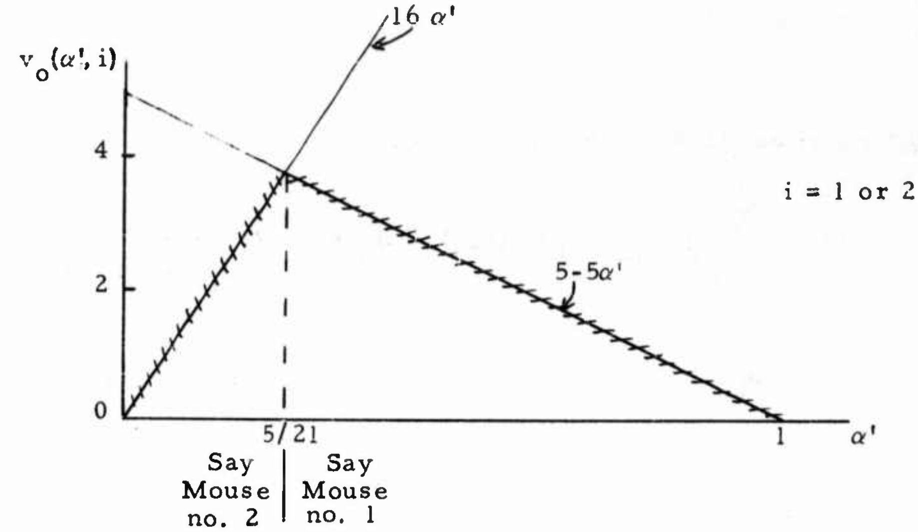
if $5 - 5a' < 16a'$

i.e if $a' > \frac{5}{21}$

and chooses mouse no. 2 if $a' < \frac{5}{21}$. When $a' = \frac{5}{21}$ the choice is immaterial.

Also

$v_o(a', i) = \min(5 - 5a', 16a')$ = Ray's expected cost if no observations remain and the probability that mouse no. 1 is being used is a'.



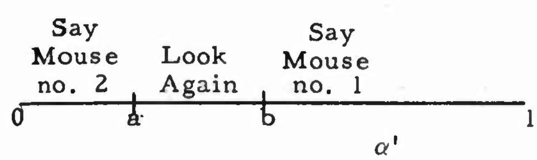
Now from equation (2. 7)

$$v_h(\alpha', i) = \min_{S,\,L} \begin{cases} S & \min\,(5-5\alpha',\ 16\alpha') \\ L & \sum_{r=1}^{2} \left[\, {}^{1}p_{ir}\,\alpha' + {}^{2}p_{ir}(1-\alpha')\right]\left[d_{ir} + v_{h-1}(\alpha'', r)\right] \end{cases} \quad \ldots (2.8)$$

Using equation (2. 1) α'' can be expressed in terms of α' as follows:

$$\alpha'' = \frac{{}^{1}p_{ir}\,\alpha'}{{}^{1}p_{ir}\,\alpha' + {}^{2}p_{ir}(1-\alpha')} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2.9)$$

As far as the variable α' is concerned it is apparent that its range (for any specific number of observations left, h, and present cage, i, will be split into 3 sections for decision purposes:



From equations (2.8) and (2.9) "a" is where

$$16a = \sum_{r=1}^{2} [\,{}^{1}p_{ir}\, a + {}^{2}p_{ir}\,(1-a)]\,[\,d_{ir} + v_{h-1}(\frac{{}^{1}p_{ir}\, a}{{}^{1}p_{ir}\, a + {}^{2}p_{ir}\,(1-a)}, r)]$$

$$= J_h(a) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2.10)$$

and "b" is where

$$5 - 5b = J_h(b) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2.11)$$

For $h = 1$, $i = 1$

Using equation (2.10) "a" is where

$$16a = [\,0.5 - 0.4a]\,[\,0.5 + v_o(\frac{0.1a}{0.5 - 0.4a}, 1)] + [\,0.5 + 0.4a]\,[\,1 + v_o(\frac{0.9a}{0.5 + 0.4a}, 2)]$$

This equation is solved for "a" using the expression obtained for $v_o(a', i)$ earlier. The solution is $a = 0.194$

Similarly equation (2.11) gives

$$5 - 5b = [\,0.5 - 0.4b]\,[\,0.5 + v_o(\frac{0.1b}{0.5 - 0.4b}, 1)] + [\,0.5 + 0.4b]\,[\,1 + v_o(\frac{0.9b}{0.5 + 0.4b}, 2)]$$

which solves for $b = 0.407$.

For $0.194 \leq a' \leq 0.407$

$\frac{0.1\, a'}{0.5 - 0.4\, a'}$ is always less than 5/21

$$\therefore\ v_o\left(\frac{0.1\alpha'}{0.5-0.4\alpha'},\ i\right) = 16\left(\frac{0.1\alpha'}{0.5-0.4\alpha'}\right) \text{ throughout the range.}$$

Also $\frac{0.9\alpha'}{0.5+0.4\alpha'}$ is always greater than 5/21

$$\therefore\ v_o\left(\frac{0.9\alpha'}{0.5+0.4\alpha'},\ i\right) = 5-5\left(\frac{0.9\alpha'}{0.5+0.4\alpha'}\right) \text{ throughout the range.}$$

Consequently for $0.194 \le \alpha' \le 0.407$, from equation (2.8)

$$v_1(\alpha', 1) = [0.5-0.4\alpha'][0.5+16\left(\frac{0.1\alpha'}{0.5-0.4\alpha'}\right)] + [0.5+0.4\alpha'][1+ 5-5\left(\frac{0.9\alpha'}{0.5+0.4\alpha'}\right)]$$

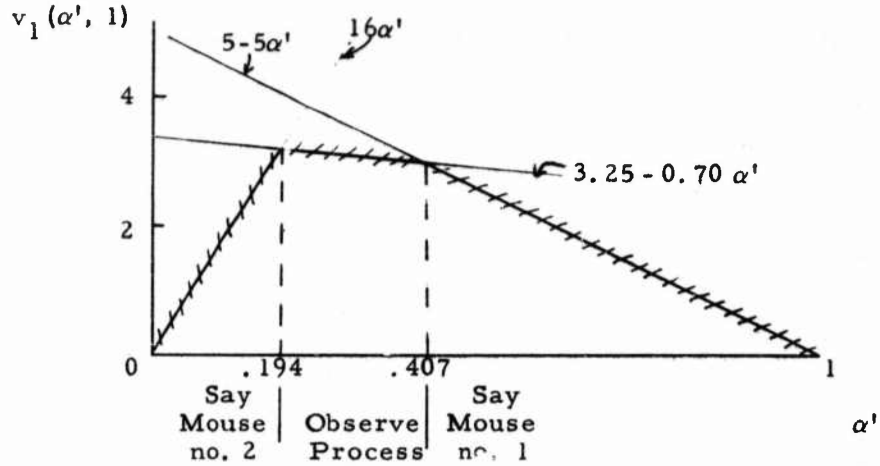
$$v_1(\alpha', 1) = 3.25 - 0.70\alpha'$$

Note the cancellation of the α' terms occurring in the denominator. Fortunately this always occurs so that it will be found that $v_h(\alpha', i)$ is a piecewise linear function of α'.

We can summarize the $h = 1$, $i = 1$ situation as follows:

α' range	Decision	$v_1(\alpha', 1)$
$0 \le \alpha' \le .194$	Stop and say mouse no. 2	$16\alpha'$
$.194 \le \alpha' \le .407$	Observe process	$3.25 - 0.70\alpha'$
$.407 \le \alpha' \le 1$	Stop and say mouse no. 1	$5 - 5\alpha'$

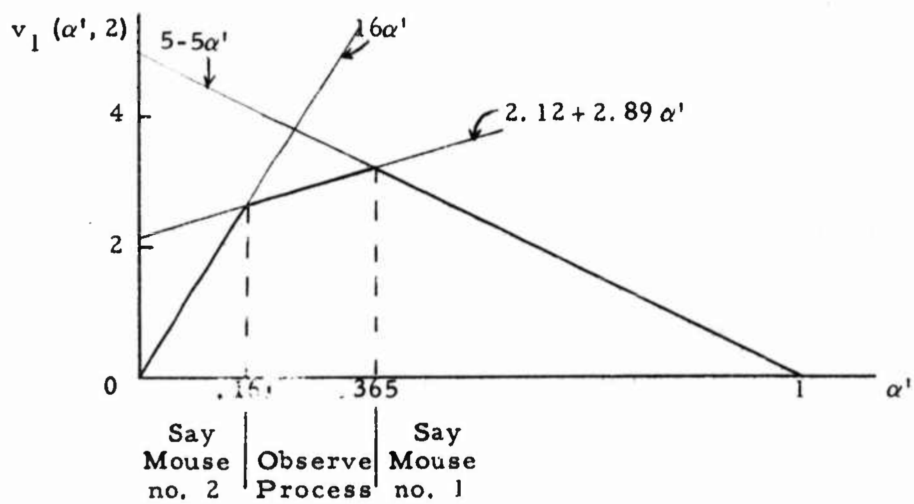
(See diagram on next page)



h = 1, i = 2

In exactly the same manner as above we find

α' range	Decision	$v_1(\alpha', 2)$
$0 \leq \alpha' \leq .161$	Stop and say mouse no. 2	$16\alpha'$
$.161 \leq \alpha' \leq .365$	Observe process	$2.12 + 2.89\alpha'$
$.365 \leq \alpha' \leq 1$	Stop and say mouse no. 1	$5 - 5\alpha'$

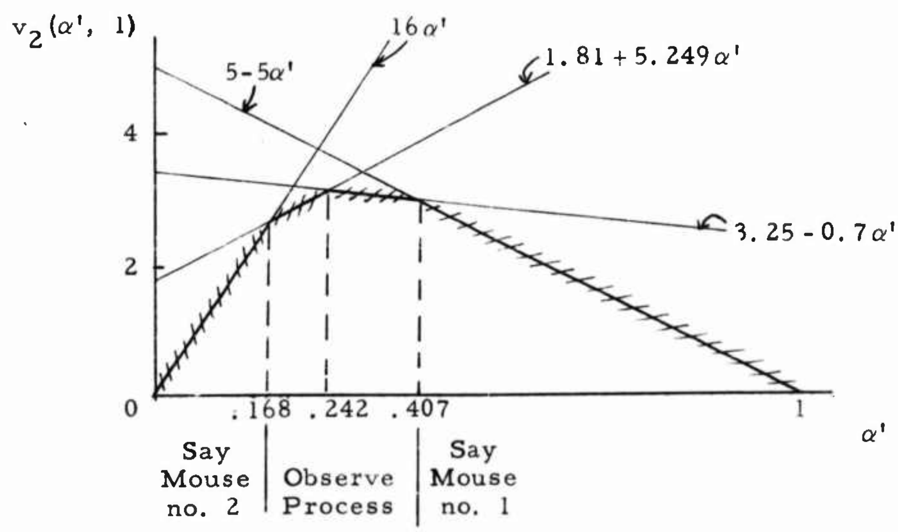


h = 2, i = 1

Equations (2.10) and (2.11) together with the above expressions for $v_1(\alpha, 1)$ and $v_1(\alpha, 2)$ allow us to obtain

α' range	Decision	$v_2(\alpha', 1)$	
$0 \le \alpha' \le .168$	Stop and say mouse no. 2	$16\alpha'$	
$.168 \le \alpha' \le .242$	Observe process	$1.81 + 5.249\alpha'$	#
$.242 \le \alpha' \le .407$	Observe process	$3.25 - 0.7\alpha'$	#
$.407 \le \alpha' \le 1$	Stop and say mouse no. 1	$5 - 5\alpha'$	

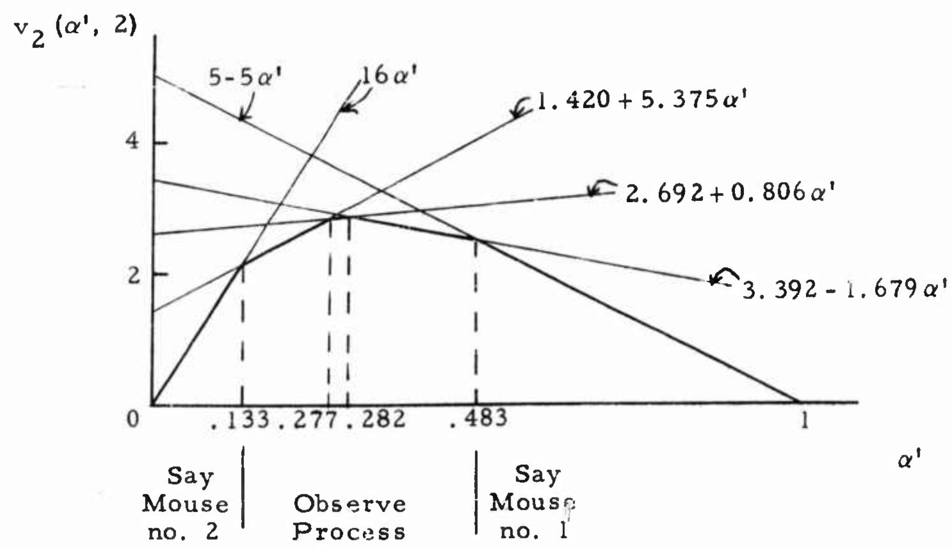
\# At $\alpha' = .242$, $\alpha'' = \dfrac{0.9\alpha'}{0.5 + 0.4\alpha'} = .365$ and therefore $v_1(\alpha'', 2)$ changes functional form (see the previous table). This in turn changes the functional form of $v_2(\alpha', 1)$.



h = 2, i = 2

In the same manner as above there results

α' range	Decision	$v_2(\alpha', 2)$
$0 \le \alpha' \le 0.133$	Stop and say mouse no. 2	$16\alpha'$
$0.133 \le \alpha' \le 0.277$	Observe process	$1.420 + 5.375\alpha'$
$0.277 \le \alpha' \le 0.282$	Observe process	$2.692 + 0.806\alpha'$
$0.282 \le \alpha' \le 0.483$	Observe process	$3.392 - 1.679\alpha'$
$0.483 \le \alpha' \le 1$	Stop and say mouse no. 1	$5 - 5\alpha'$



Now from equations (2.8) and (2.9)

$$v_3(a', 1) = \min_{S, L} \begin{cases} S & \min(5 - 5a', 16a') \\ L & [0.5 - 0.4a'][0.5 + v_2(\frac{0.1a'}{0.5-4a'}, 1)] \\ & +[0.5+0.4a'][1+v_2(\frac{0.9a'}{0.5+0.4a'}, 2)] \end{cases}$$

But Ray has been told that $a = 0.3$

$$v_3(0.3, 1) = \min_{S, L} \begin{cases} S & \min(3.5, 4.8) \\ L & 0.38[0.5 + v_2(\frac{.03}{.38}, 1)] + 0.62[1+v_2(\frac{.27}{.62}, 2)] \end{cases}$$

$$= \min_{S, L} \begin{cases} S & 3.5 \\ L & 0.38[0.5 + 16(\frac{.03}{.38})] + 0.62[1 + 3.392 - 1.679(\frac{.27}{.62})] \end{cases}$$

$$v_3(0.3, 1) = \min_{S, L} \begin{cases} S & 3.5 \\ L & 3.00 \Leftarrow \end{cases}$$

Therefore he should observe the first transition and his expected cost is 3.00. Actually the solution has given him far more than just this answer. It also shows the optimum policy to follow (with the expected cost) for either starting state, any $\underline{a}$ vector and 0, 1 or 2 looks possible.

2.5.4 Some Further Remarks

From the results of the numerical example some fairly general remarks can be made.

First, as demonstrated by induction in Appendix A, $v_h(\underline{a}', i)$ is always a piecewise linear function of the components of $\underline{a}'$. A second interesting point in the 2 - matrix example is that the range of a in which we decide to observe the next transition is a monotonic increasing function of the number of possible observations remaining. This is as expected. Furthermore, for more than two possible matrices we would expect the volume of the region of the Q - 1 dimensional space of the components of $\underline{a}$ in which we decide to observe the next transition to be a monotonic increasing function of the number of possible observations left (a monotonic increasing function is assumed to include the trivial case where the function is zero for all values of the argument).

It is clear from the dynamic programming form of the problem that the number of states is not really critical. There is one state variable for the state occupied. Increasing the number of states merely increases the range of this one state variable. On the other hand introduction of an extra possible matrix increases the number of state variables by one. Hence, the number of possible matrices is the quantity that governs the feasibility of the dynamic programming solution.

Finally, it can be stated that the numerical example has certainly demonstrated the possibility of statistical decision making for a multi-matrix Markov process.

CHAPTER 3

THE MULTIDIMENSIONAL BETA DISTRIBUTION

As was stated earlier, in this and the following two chapters we shall assume directly that the transition probabilities themselves are random variables. The present chapter is concerned with the multi-dimensional Beta distribution, a most convenient distribution to place over the transition probabilities of a single row of the transition matrix.

3.1 Conjugate Prior for a Multinomial Distribution

Consider a random variable that follows a multinomial distribution of order k, i.e., each time a draw is made it can fall into 1 of k categories.

Let p_i = probability that a particular draw will fall in the i^{th} category $(i = 1, 2, \ldots, k)$

$$\sum_{i=1}^{k} p_i = 1$$

Let E = event that in n independent draws n_i fall in the i^{th} category $(i = 1, 2, \ldots, k)$

$$\sum_{i=1}^{k} n_i = n$$

Then using the basic property of the multinomial distribution

$$\mathrm{pr}(E \mid p_1, p_2, \ldots, p_k) = \frac{n!}{\prod_{i=1}^{k} n_i!} \underbrace{p_1^{n_1} p_2^{n_2} \cdots p_k^{n_k}}_{\text{kernel}} \qquad \ldots\ldots\ldots\ldots(3.1)$$

Now following the suggestion of Raiffa and Schlaifer[11], if we want to place a prior distribution over the p_j's, it is advisable to select the prior such that it has the same kernel as the likelihood function of equation (3.1). Hence the conjugate prior should be of the form[12]

$$f_{p_1, p_2, \ldots p_k}(x_1, x_2, \ldots, x_k) = C\, x_1^{m_1-1}\, x_2^{m_2-1} \ldots x_k^{m_k-1} \quad \ldots (3.2)$$

where $\sum_{i=1}^{k} x_i = 1$ and $x_i \geq 0$

and as shown in Appendix B the proper normalizing constant to use is

$$C = \frac{1}{\beta(m_1, m_2, \ldots, m_k)} = \frac{\Gamma(m_1 + m_2 + \ldots + m_k)}{\Gamma(m_1)\Gamma(m_2)\ldots\Gamma(m_k)} \quad \ldots\ldots (3.3)$$

(3.2) and (3.3) together define the multidimensional Beta distribution with parameters $(m_1, m_2, \ldots, m_k)$.

3.2 Important Properties of the Distribution

Consider a multidimensional Beta distribution.

$$f_{p_1, p_2 \ldots, p_k}(x_1, x_2, \ldots, x_k) = \frac{1}{\beta(m_1, m_2, \ldots, m_k)}\, x_1^{m_1-1}\, x_2^{m_2-1} \ldots$$

$$\ldots x_k^{m_k-1} \quad \ldots\ldots\ldots\ldots\ldots\ldots (3.4)$$

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11 Raiffa, H., and Schlaifer, R., Applied Statistical Decision Theory, Graduate School of Business Administration, Harvard University, 1961, Chapter 3.

12 Throughout this study $f_{a,b,\ldots,d}(w, x, \ldots, z)$ will represent the joint density function of the random variables a, b, ..., d evaluated at the point (w, x, ..., z), and $f_{a|b}(w|x)$ will represent the conditional density function of the random variable "a" evaluated at the point "w" given that the random variable "b" has taken on the value "x".

$$\text{where} \sum_{i=1}^{k} x_i = 1 \text{ and } x_i \geq 0$$

In more compact notation

$$f_{p_1, p_2, \ldots, p_k}(x_1, x_2, \ldots, x_k) = f_\beta(x_1, x_2, \ldots x_k \mid m_1, m_2, \ldots, m_k).$$

Then in Appendix B the following properties are established:

i) The distribution integrates to one, i.e., we have properly chosen the normalizing constant.

ii) The marginal distribution of a specific p_j is given by

$$f_{p_j}(x) = \frac{1}{\beta(m_j, \sum_{i \neq j} m_i)} x^{m_j - 1} (1-x)^{\sum_{i \neq j} m_i - 1} \qquad 0 \leq x \leq 1$$

$$= f_\beta (x \mid m_j, \sum_{i \neq j} m_i) \qquad \ldots\ldots\ldots\ldots (3.5)$$

iii) The expected value of p_j is

$$E(p_j) = \frac{m_j}{\sum_{i=1}^{k} m_i} \qquad \ldots\ldots\ldots\ldots (3.6)$$

iv) The variance of p_j is

$$\overset{v}{p}_j = \frac{m_j (\sum_{i \neq j} m_i)}{(\Sigma m_i)^2 (1 + \Sigma m_i)} \qquad \ldots\ldots\ldots\ldots (3.7)$$

v) The covariance of p_j and p_u is

$$j \neq u \qquad \text{Cov}(p_j, p_u) = \frac{-m_j m_u}{(\Sigma m_i)^2 (1 + \Sigma m_i)} \qquad \ldots\ldots\ldots\ldots (3.8)$$

3.3 Determination of the A Priori Parameters

Ideally one would like to select the parameters of the multidimensional Beta distribution in such a way that the prior estimates of the means and ~~variances~~ of the k individual p_j's are satisfied.* However, this would require 2k-1 parameters (since $\Sigma p_j = 1$, if k-1 means are assigned, the k^{th} one is automatically determined) and we have only k parameters at our disposal. Considerable effort was made to find a suitable prior distribution that possessed 2k-1 parameters; suitable in the sense that it would allow easy Bayes modification and would be analytically tractable. Unfortunately all efforts were unsuccessful. Therefore, we restrict attention to the multidimensional Beta prior distribution (with only k degrees of freedom) which allows easy Bayes modification but is not ideal due to its deficiency in total parameters. We proceed as follows. Select the parameters such that the prior mean values of the p_j's, i.e. the $E(p_j)$'s, are satisfied exactly. This uses up k-1 of the parameters, leaving only 1 other degree of freedom. Obtain the final parameter by a least squares fit to the prior variances of the p_j's, i.e. to the $\check{p}_j$'s. With these conditions in mind in Appendix C the following expressions for the m's in terms of the $E(p_j)$'s and $\check{p}_j$'s are developed:

Define $M_{L.S.}$ = the value of $\sum_{i=1}^{k} m_i$ obtained by least squares

$$\text{Then } M_{L.S.} = \frac{\sum_{i=1}^{k} [E(p_j)]^2 [1-E(p_j)]^2}{\sum_{j=1}^{k} \check{p}_j E(p_j)[1-E(p_j)]} - 1 \quad \ldots\ldots\ldots\ldots (3.9)$$

$$\text{and } m_j = M_{L.S.} E(p_j) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (3.10)$$

*Estimating the prior means and variances is not a trivial task. Also, we do not mean to imply that this is the only method for assigning values to the k parameters.

Numerical Example - 4 category multidimensional Beta.

$E(p_1) = 0.1$ $\quad\quad$ $\check{p}_1 = 0.004$

$E(p_2) = 0.2$ $\quad\quad$ $\check{p}_2 = 0.008$

$E(p_3) = 0.3$ $\quad\quad$ $\check{p}_3 = 0.001$

$E(p_4) = 0.4$ $\quad\quad$ $\check{p}_4 = 0.004$

Using equation (3.9) we find $M_{L.S.} = 47$ (to nearest integer)

Then from equation (3.10) $m_1 \doteq 5$, $m_2 \doteq 9$, $m_3 \doteq 14$, $m_4 \doteq 19$

$$\therefore\ f_{p_1, p_2, p_3}(x_1, x_2, x_3) = \frac{1}{\beta(5, 9, 14, 19)} x_1^{4} x_2^{8} x_3^{13} (1 - x_1 - x_2 - x_3)^{18}$$

$$\sum_{i=1}^{3} x_i \leq 1 \quad \text{and } x_i \geq 0.$$

3.4 Bayes Modification of the Distribution

As in section 3.1 consider a multinomial distribution of order k with parameters $p_1, p_2, \ldots, p_k$. Suppose the p_j's are a priori jointly distributed according to the multidimensional Beta distribution.

$$f_{p_1, p_2, \ldots p_k}(x_1, x_2, \ldots, x_k) = f_\beta(x_1, x_2, \ldots, x_k | m_1, m_2, \ldots, m_k).$$

Again, as in section 3.1, let E be the event that in n independent draws from the multinomial n_i fall in the i^{th} category $(i = 1, 2, \ldots, k)$

$$\sum_{i=1}^{k} n_i = n$$

Then it is shown in Appendix D that the a posteriori distribution of the p_j's is again a multidimensional Beta, only the parameters are modified; more precisely

$$f_{p_1, p_2, \ldots p_k | E}(x_1, x_2, \ldots, x_k) = f_\beta(x_1, x_2, \ldots, x_k | m_1 + n_1, m_2 + n_2, \ldots, m_k + n_k)$$

3.5 Development of the Multidimensional Beta Priors for a Specific N - State Markov Process.

Consider state i of an N - state Markov process. If we are in state i f_i times, then the numbers of transitions from state i to state j, f_{ij} ($j = 1, 2, \ldots, N$), are multinomially distributed as follows:

$$\text{pr}[f_{i1}, f_{i2}, \ldots, f_{iN} | f_i] = \frac{f_i!}{f_{i1}! f_{i2}! \ldots f_{iN}!} p_{i1}^{f_{i1}} p_{i2}^{f_{i2}} \ldots p_{iN}^{f_{iN}}$$

if $p_{ij} = \text{pr}\{\text{state at time } n+1 \text{ is } j | \text{ state at time } n \text{ is } i\}$

and $\sum_j f_{ij} = f_i$

This multinomial behavior is the motivation for using an a priori multidimensional Beta distribution over the transition probabilities $p_{i1}, p_{i2}, \ldots, p_{iN}$. Similar reasoning would lead us to use multidimensional Beta distributions over the transition probabilities for each of the other N - 1 states. Hence we utilize N distributions of the form

$$f_{p_{i1}, p_{i2} \ldots, p_{iN}}(x_{i1}, x_{i2}, \ldots, x_{iN}) = f_\beta(x_{i1}, x_{i2}, \ldots, x_{iN} | m_{i1}, m_{i2}, \ldots, m_{iN})$$

To determine the parameters (m_{ij}'s) we consider each row separately and use our prior estimates of the individual $E(p_{ij})$'s and $\check{p}_{ij}$'s as outlined in section 3.3.

Now because of the simple Bayes modification of the multidimensional Beta prior on the probabilities of a multinomial distribution (shown in section 3.4) we know that the separate multidimensional Beta priors of a Markov process will be simply modified as a result of a series of observed transitions. More precisely, the resultant posterior distributions will be new multidimensional Beta distributions having parameters $(m_{ij} + f_{ij})$, if (m_{ij}) are the a priori parameters and (f_{ij}) are the number of observed transitions from state i to state j.

3.6 Random Sampling from the Distribution

A simple method of randomly sampling from an analytic distribution is useful for simulation purposes, both when we use the simulation to empirically determine simple functions of the distribution (precisely what is done in section 4.1.6) and also when the random variables concerned are but a small part of a complex system being simulated. Hence two methods of randomly sampling from the multidimensional Beta distribution will be described.

Method 1 - Use of Marginal and Conditional Distributions

Consider the multidimensional Beta distribution

$$f_{p_1, p_2, \ldots, p_k}(x_1, x_2, \ldots, x_k) = f_\beta(x_1, x_2, \ldots, x_k \mid m_1, m_2, \ldots, m_k)$$

Then as shown in Appendix E the following is a method of sampling from this distribution:

i) . Draw w_1 from the simple Beta $f_\beta(w_1 \mid m_1, \sum_{i=2}^{k} m_i)$

ii) . Draw w_2 from the simple Beta $f_\beta(w_2 \mid m_2, \sum_{i=3}^{k} m_i)$

.

.

.

.

k-i) w_{k-1} from the simple Beta $f_\beta(w_{k-1} \mid m_{k-1}, m_k)$

Then $x_1 = w_1$

$x_2 = w_2 (1-w_1)$

$x_3 = w_3 (1-w_2)(1-w_1)$

.

.

.

$x_{k-1} = w_{k-1}(1-w_{k-2}) \ldots (1-w_2)(1-w_1)$

and $x_k = 1 - \sum_{i=1}^{k-1} x_i$

Hence, we have reduced the problem of randomly drawing from a k-dimensional Beta distribution to one of taking a single draw from each of k - 1 different simple Beta distributions. Methods for drawing from a simple Beta distribution have been discussed in the literature[13].

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13. See, for example, Galliher, H. P., "Simulation of Random Processes", Notes on Operations Research 1959, The Technology Press, Cambridge, 1959, p. 238.

Method 2 - Use of Ratio of Gamma Distributions

Mosimann[14] has shown the following interesting result:

If t_i $(i = 1, 2, \ldots, k)$ are independent random variables having Gamma distributions with parameters m_i respectively and all with the same scale parameter q, i.e.

$$f_{t_i}(y_i) = f_\gamma(y_i \mid m_i, q) = \frac{q^{m_i}}{\Gamma(m_i)} y_i^{m_i - 1} e^{-qy_i} \qquad 0 \leq y_i < \infty ,$$

then the random variables

$$p_i \; (i = 1, \ldots, k) \; (\text{where } p_i = \frac{t_i}{\sum_{i=1}^{k} t_i})$$

have a multidimensional Beta distribution with parameters $m_1, \ldots m_k$, i.e.

$$f_{p_1, p_2, \ldots, p_k}(x_1, x_2, \ldots, x_k) = f_\beta(x_1, x_2, \ldots, x_k \mid m_1, m_2, \ldots, m_k).$$

Therefore to randomly draw from $f_\beta(x_1, x_2, \ldots, x_k \mid m_1, m_2, \ldots, m_k)$ we can take independent draws (y_i) from the k simple Gamma distributions and then

$$x_1 = \frac{y_1}{\Sigma y_i} \;, \; x_2 = \frac{y_2}{\Sigma y_i} \;, \; \text{etc.}$$

Methods for sampling from a simple Gamma distribution have been developed (in fact, we require this in method 1 as sampling from a simple Beta involves sampling from two Gamma distributions).

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14. Op. Cit.

Method 2 appears superior to method 1 as method 2 requires k draws from simple Gammas per set of k - dimensional Beta values while method 1 requires 2k - 2 draws for the same output. In any event, as mentioned earlier, being able to randomly sample from the multidimensional Beta distribution is important for two reasons. First, it allows us to obtain by simulation values of functions of multidimensional Beta distributions that are not attainable by analytic means. Secondly it permits the use of multidimensional Beta variables in complex simulations.

CHAPTER 4

EFFECTS ON QUANTITIES OF INTEREST OF HAVING MULTIDIMENSIONAL BETA DISTRIBUTIONS OVER THE TRANSITION PROBABILITIES

Having placed the transition probabilities into a framework where our uncertainty about them can be easily updated in the Bayesian sense (i.e. after observing some transitions the density functions over the transition probabilities can be simply modified according to Bayes' rule) we now turn our attention to the effects of uncertainty in the transition probabilities on various quantities of interest in the Markov model. For example, the steady state probabilities are no longer exact numbers, rather they have now become random variables since they are functions of the transition probabilities which are random variables. It is important, both for statistical decision purposes and for interest in the quantities per se, that we be able to describe their behavior when the transition probabilities are not known exactly.

4.1 Steady State Probabilities

4.1.1 General Remarks

As mentioned above the steady state probabilities are functions of the transition probabilities which are random variables, hence the steady state probabilities now become random variables. It will be assumed that the following mechanism operates. Single values for each of the transition probabilities are drawn from their distributions. With these now exactly fixed transition probabilities the system is run to the steady state producing exact steady state probabilities. The whole process is repeated many times generating various values of the steady state probabilities.

Ideally we would like to know the actual density functions of the steady state probabilities produced in this way; however, this is far more easily said than accomplished. Actually, as will be explained in Chapter 5, merely the expected values of the steady state probabilities will be adequate for most decision purposes.

For 2 - state processes exact closed-form expressions will be obtained for the expected values of the steady state probabilities. The analytic difficulties for more than 2 states are clearly illustrated by the 3 - state situation.

Consider the transition matrix

$$P = \begin{bmatrix} 1-a-b & a & b \\ c & 1-c-d & d \\ e & f & 1-e-f \end{bmatrix}$$

where $p_{ij} = pr\{$state at time $n+1 = j |$ state at time $n = i\}$

Now solving $\underline{\pi} P = \underline{\pi}$ or referring to another study done by the author (see Appendix F) tells us that the steady state probability of being in state 1 is given by

$$\pi_1 = \frac{ce + cf + de}{ad + ae + af + bc + bd + bf + ce + cf + de}$$

Now suppose the transition probabilities are random variables. Then

$$E(\pi_1) = E\left(\frac{ce + cf + de}{ad+ae+af+bc+bd+bf+ce+cf+de}\right).$$

Unfortunately, there does not appear to be any simple way of evaluating the right hand side for convenient distributions over the transition probabilities. It is clear that the problem is even more formidable when the number of states is greater than three.

4.1.2 A Special 2-State Case Where We are Able to Obtain The Density Functions of the Steady State Probabilities

Consider the transition matrix $\underset{\sim}{P} = \begin{bmatrix} 1-a & a \\ b & 1-b \end{bmatrix}$

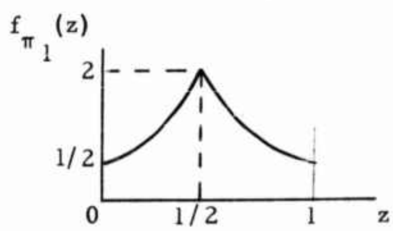
where $f_a(x) = 1 \qquad 0 \leq x \leq 1$

and $f_b(y) = 1 \qquad 0 \leq y \leq 1$

"a" and "b" are assumed independent and we know that $\pi_1 = \frac{b}{a+b}$.

Using the theory of derived distributions [15] we find that

$$f_{\pi_1}(z) = \begin{cases} \dfrac{1}{2(1-z)^2} & 0 \leq z \leq 1/2 \\ \dfrac{1}{2z^2} & 1/2 \leq z \leq 1. \end{cases}$$



This distribution has a mean of 1/2 as we would expect.

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15. See, for example, Wadsworth, G. P. and Bryan, J. G., Introduction to Probability and Random Variables, McGraw-Hill, 1959, Section 6.12.

The only reason that a closed form can be obtained for $f_{\pi_1}(\cdot)$ is that the extremely simple forms of $f_a(\cdot)$ and $f_b(\cdot)$ allow us to perform the integrations involved in deriving the new distribution. The distributions assumed for "a" and "b" are very special forms of the Beta. For the more general form of the Beta the integrations cannot be performed. Now, if we observe several transitions of the special process considered, Bayes modification will result in more general Beta forms for the distributions of "a" and "b". Hence, at that stage we would again not be able to obtain the density function of π_1. Thus it is seen that the special distributions (uniform) assumed for "a" and "b" in this section are of very limited practical value despite the fact that they allow us to obtain the density function of π_1 before any transitions occur. We really want to be able to handle the more general situation of independent Beta distributions over "a" and "b".

4.1.3. The 2-State Case Where One Transition Probability is Known Exactly While the Other is Beta Distributed

As mentioned earlier it will be demonstrated in Chapter 5 that merely the expected values of the steady state probabilities are adequate for most decision purposes. Therefore, as a start in the right direction we shall find the expected values of the steady state probabilities for the Markov process with the following special structure:

$$\mathbf{P} = \begin{bmatrix} 1 - a & a \\ & \\ b & 1 - b \end{bmatrix}$$

where "a" is assumed exactly known but "b" has the Beta distribution

$$f_b(x) = f_\beta(x|m, n)$$

Physical Application:

There is a meaningful physical application of this model. Consider customer behavior between 2 brands of a product. Let state 1 represent that the customer is buying our brand, state 2 that of our competitor. Then, from extensive past data on our own customers it is logical to assume that we know accurately the value of "a", the probability that a customer buying our brand at time n will switch to the competitor's brand on the next purchase. However, due to lack of records on the competitor's customers we only have a rough idea about the value of "b". Hence, it is reasonable to place a Beta prior distribution over "b".

Determination of $E(\pi_2)$:

For a given (a, b) pair

$$\pi_2 = \frac{a}{a+b},$$

the steady state probability of being in state 2.

$$\therefore E(\pi_2) = E\left(\frac{a}{a+b}\right) = \int_0^1 \frac{a}{a+x} f_b(x)\, dx$$

$$= \int_0^1 \frac{a}{a+x} \frac{1}{\beta(m, n)} x^{m-1} (1-x)^{n-1} dx.$$

This is not an easy integral to evaluate in its current form. However, as shown in detail in Appendix G, use of the hypergeometric function, a well known function arising in certain physics problems, enables us to obtain the following expression for $E(\pi_2)$:

$$E(\pi_2) = \frac{a}{a+1}\left[1 + \frac{n}{m+n}\left(\frac{1}{a+1}\right) + \frac{n(n+1)}{(m+n)(m+n+1)}\left(\frac{1}{a+1}\right)^2 + \dots\right.$$

$$\left. \dots + \frac{n(n+1)\dots(n+k-2)}{(m+n)(m+n+1)\dots(m+n+k-2)}\left(\frac{1}{a+1}\right)^{k-1}\right] \quad (4.1)$$

$$+ E_k$$

where

$$E_k < \frac{n(n+1)\dots(n+k-1)}{(m+n)(m+n+1)\dots(m+n+k-1)}\left(\frac{1}{a+1}\right)^k \quad (4.2)$$

It is clear that E_k can be made arbitrarily small by choosing k sufficiently large. Hence, we can come arbitrarily close to $E(\pi_2)$ by selecting a large enough k.

Finally,

$$E(\pi_1) = 1 - E(\pi_2).$$

Asymptotic Check on the $E(\pi_2)$ Formula

In section G.2 of Appendix G an asymptotic check on equation (4.1) is performed.

Monotonic Behavior of $E(\pi_2)$ as a Function of m + n for Fixed E(b)

In section G.3 of Appendix G it is shown that for fixed E(b), i.e., for a fixed mean of the prior Beta distribution on "b", $E(\pi_2)$ monotonically decreases as m + n increases; i.e., as the variance of the Beta distribution decreases ("b" becomes more and more exactly known).

Numerical Example

Consider the physical application of customer behavior between 2 brands of a product. Suppose that we know accurately that a customer buying our brand this period will switch to the competitor's brand next period with probability 0.5 (i.e., a = 0.5). Also from our limited knowledge of the competitor's customers we are willing to place the following Beta distribution on "b",

$$f_b(x) = f_\beta(x|9, 1) \qquad \text{i.e.,} \qquad m = 9, \ n = 1 .$$

To obtain $E(\pi_2)$ to 5 significant figures, equation (4.2) tells us to use $k = 7$ (a small number of terms for such a high degree of accuracy). Then equation (4.1) gives

$$E(\pi_2) \doteq 0.35882,$$

the expected value of the steady state probability that a particular customer will be buying from the competitor.

This is remarkably close to the π_2 when "b" is exactly known at 0.9. In that case

$$(\pi_2)_{exact} = \frac{0.5}{0.5 + 0.9} = 0.35714.$$

Also, for this example $m = 9$, $n = 1$ are the smallest integers for which $E(b) = 0.9$. Therefore, due to the above-stated monotonic behavior of $E(\pi_2)$ it is seen that for any combination of integers that make

$$\frac{m}{m + n} = 0.9$$

(e.g., 90 and 10) $E(\pi_2)$ must lie in the narrow range

$$0.35714 \le E(\pi_2) \le 0.35882$$

i.e., $E(\pi_2)$ is remarkably insensitive to the variance of the prior Beta distribution on "b".

Numerical Results

A program for the IBM 7090 digital computer was developed which used equations (4.1) and (4.2) to give $E(\pi_2)$ accurate to 5 decimal places for the following 9 x 9 x 14 or 1134 combinations:

$$a = 0.1,\ 0.2,\ 0.3,\ \ldots,\ 0.9$$

$$\bar{b} = 0.1,\ 0.2,\ 0.3,\ \ldots,\ 0.9$$

$$m + n = 10,\ 20,\ 30,\ 40,\ 50,\ 60,\ 80,\ 100,\ 150,\ 200,$$

$$300,\ 500,\ 1000,\ \infty.$$

Appendix H presents portions of the tabulated results (the entire tabulation would have required a prohibitive amount of typing; also graphical representation would have required 81 separate figures). Analysis of these results shows that, for a fixed $\bar{b}$, $E(\pi_2)$ is extremely insensitive to $m + n$; i.e., to the variance of the Beta distribution. Hence, we can approximate $E(\pi_2)$ by $(\pi_2)_{exact}$ where

$$(\pi_2)_{exact} = \frac{a}{a + \bar{b}}$$

(i.e., we assume "b" is exactly known at its mean value). The approximation improves as $m + n$ and/or "a" and/or $\bar{b}$ increase, but it is even quite good for low $m + n$, "a" and $\bar{b}$. The fact that $E(a/(a+b))$ deviates most from $a/(a+\bar{b})$ for low "a" and $\bar{b}$ is reasonable because under these conditions the small mean value of the denominator makes it very sensitive to variations in "b".

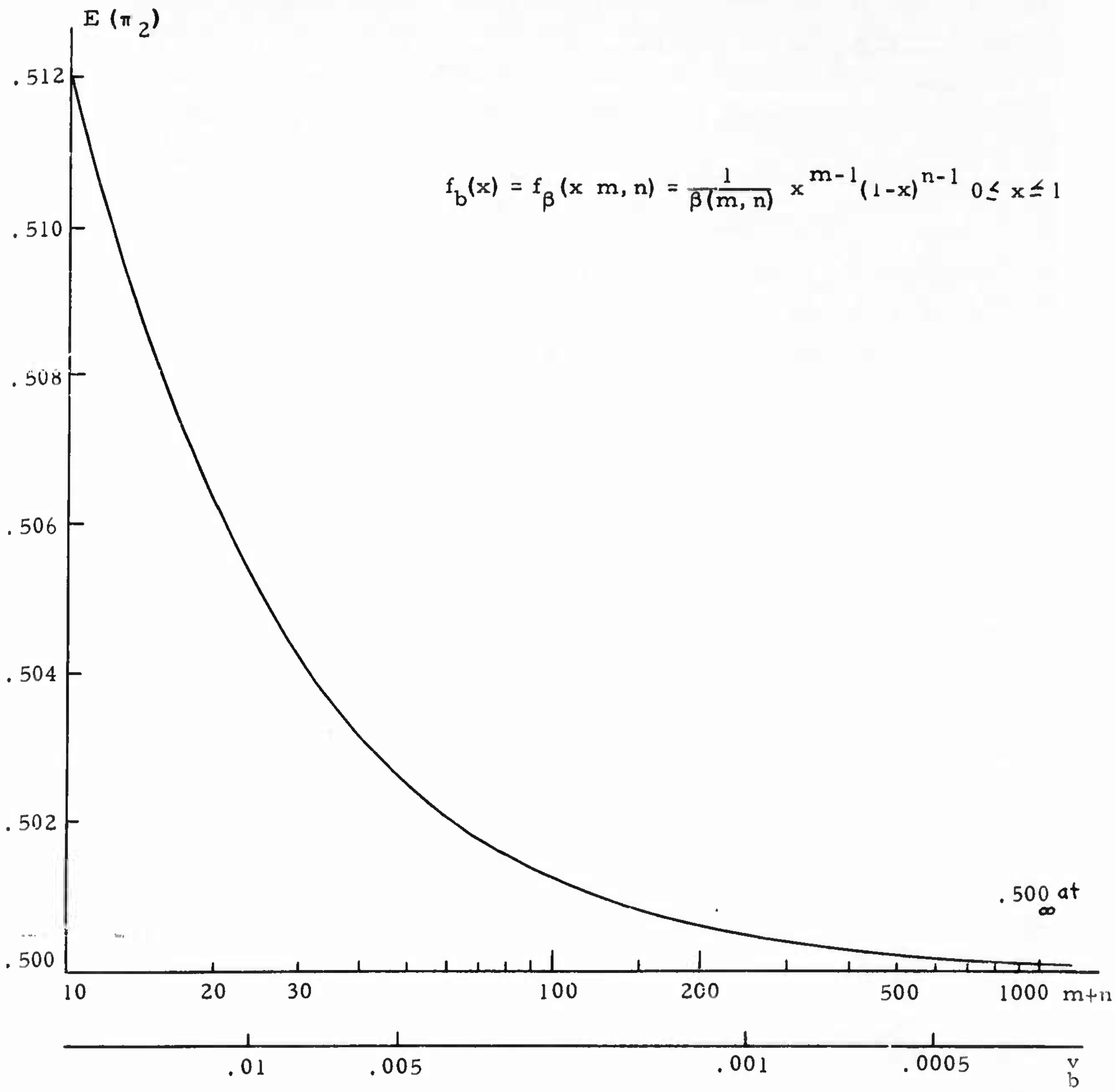

Figure 4.1: Plot of $E(\pi_2)$ vs m+n for a = 0.5, $\overline{b}$ = 0.5 (2 states).

A typical plot of $E(\pi_2)$ vs. $m + n$ for fixed "a" and $\bar{b}$ is given in Figure 4.1. The alternate horizontal scale was developed as follows:

$$\check{b} = \frac{mn}{(m+n)^2 (m+n+1)} = \frac{\bar{b}(1-\bar{b})}{m+n+1}$$

Therefore, for fixed $\bar{b}$, giving $m + n$ also prescribes b. For example, if $\bar{b} = 0.5$ and we say that $m + n = 24$, then we have prescribed the variance $\check{b}$ at the value $(0.5)(0.5)/(25) = .01$.

4.1.4. The 2-State Case Where Both Transition Probabilities are Independently Beta Distributed

Again our desire is to determine the expected values of the steady state probabilities. First, a complicated summation form for $E(\pi_2)$ will be presented. Theoretically it can be used to find $E(\pi_2)$ exactly for any integer values of the 4 Beta parameters. However, to study the general behavior of $E(\pi_2)$ it will be convenient to make use of the results of the last section.

Exact Summation Expression for $E(\pi_2)$

Here

$$P = \begin{bmatrix} 1 - a & a \\ b & 1 - b \end{bmatrix}$$

with

$$f_a(x) = f_\beta(x \mid m_1, n_1)$$

and

$$f_b(y) = f_\beta(y \mid m_2, n_2).$$

$$E(\pi_2) = \int_0^1 \int_0^1 \frac{x}{x+y} f_\beta(x \mid m_1, n_1)\, f_\beta(y \mid m_2, n_2)\, dy dx$$

As shown in Appendix I

$$\beta(m_1, n_1)\, \beta(m_2, n_2)\, E(\pi_2) = \sum_{k=0}^{m_2-2} \sum_{j=0}^{n_2-1} \sum_{r=0}^{m_2+n_2-2-k} \binom{m_2-1}{k} (-1)^{n_2+k-1-j}$$

$$\cdot \binom{n_2-1}{j} \frac{\binom{m_2+n_2-2-k}{r}}{m_2+n_2-2-k-j} \beta(m_1+k+r+1, n_1) +$$

$$\sum_{k=0}^{m_2-2} \sum_{j=0}^{n_2-1} \sum_{r=0}^{j} \frac{\binom{m_2-1}{k} (-1)^{n_2+k-j} \binom{n_2-1}{j} \binom{j}{r}}{m_2+n_2-2-k-j} \beta(m_1+m_2+n_2+r-1-j, n_1)$$

$$+ (-1)^{m_2-1} \sum_{j=0}^{n_2-2} \sum_{r=0}^{n_2-1} \frac{\binom{n_2-1}{j} (-1)^{n_2-1-j} \binom{n_2-1}{r}}{n_2-1-j} \beta(m_1+m_2+r, n_1)$$

$$+ (-1)^{m_2} \sum_{j=0}^{n_2-2} \sum_{r=0}^{j} \frac{\binom{n_2-1}{j} (-1)^{n_2-1-j} \binom{j}{r}}{n_2-1-j} \beta(m_1+m_2+n_2+r-j-1, n_1)$$

$$+ (-1)^{m_2-1} \sum_{s=0}^{n_2-1} \sum_{t=0}^{n_1-1} \binom{n_2-1}{s}\binom{n_1-1}{t} (-1)^t \mathcal{K}(m_1+m_2+s+t)$$

$$+ (-1)^{m_2-1} \sum_{s=0}^{n_2-1} \binom{n_2-1}{s} \sum_{k=0}^{n_1-1} \binom{n_1-1}{k} \frac{(-1)^k}{(m_1+m_2+s+k)^2} \tag{4.3}$$

where

$$\mathcal{K}(m_1+m_2+s+t) = \begin{cases} \dfrac{1}{m_1+m_2+s+t} \cdot \displaystyle\sum_{u=1}^{m_1+m_2+s+t} \dfrac{(-1)^{u-1}}{u} & \text{when } m_1 + m_2 + s + t \text{ is even} \\[2ex] \dfrac{2}{m_1+m_2+s+t} \ln 2 + \dfrac{1}{m_1+m_2+s+t} \cdot \displaystyle\sum_{u=1}^{m_1+m_2+s+t} \dfrac{(-1)^{u}}{u} & \text{when } m_1 + m_2 + s + t \text{ is odd.} \end{cases}$$

<u>NOTE:</u>

$\sum_{i=0}^{-1}$ means that there are no terms.

Due to the complexity of this expression it cannot be used to study the general behavior of $E(\pi_2)$ as the m's and n's are allowed to vary.

General Behavior of $E(\pi_1)$ and $E(\pi_2)$

Again consider

$$E(\pi_2) = \int_0^1 \int_0^1 \frac{1}{\beta(m_1, n_1)\,\beta(m_2, n_2)} \frac{x}{x+y} x^{m_1 - 1} (1-x)^{n_1 - 1}$$

$$\cdot\, y^{m_2 - 1} (1-y)^{n_2 - 1}\, dydx = \int_0^1 \frac{1}{\beta(m_1, n_1)} x^{m_1 - 1} (1-x)^{n_1 - 1}$$

$$\cdot \underbrace{\left(\int_0^1 \frac{1}{\beta(m_2, n_2)} \frac{x}{x+y} y^{m_2 - 1} (1-y)^{n_2 - 1}\, dy \right)}_{L(x)} dx$$

NOTATION:

Let $r\uparrow$ as $s\uparrow$ signify that r monotonically increases as s increases and $r\downarrow$ as $s\uparrow$ signify that r monotonically decreases as s increases.

Then from section 4.1.3 we know that for fixed $\bar{b} = m_2/(m_2+n_2)$

$$L(x) = E(\pi_2 | x) \downarrow \quad \text{as } m_2 + n_2 \uparrow$$

This holds for any x between 0 and 1. From above

$$E(\pi_2) = \int_0^1 f_a(x)\, L(x) dx$$

But

$f_a(x) \geq 0$

$\therefore\ E(\pi_2) \downarrow$ as $m_2 + n_2 \uparrow$,

and this is independent of m_1 and n_1. Therefore, for any Beta distribution on "a" and a fixed $\bar{b}$, $E(\pi_2) \downarrow$ as $m_2 + n_2 \uparrow$; i.e., as $\check{b}$ decreases. In fact, we can now make the following series of statements:

For	As	Then
any Beta on "a", and fixed $\bar{b}$	$m_2 + n_2 \uparrow$; i.e., $\check{b}\downarrow$	$E(\pi_1)\uparrow$ and $E(\pi_2)\downarrow$.
any Beta on "b", and fixed $\bar{a}$	$m_2 + n_2 \uparrow$; i.e., $\check{a}\downarrow$	$E(\pi_1)\downarrow$ and $E(\pi_2)\uparrow$

Suppose that both $\bar{a}$ and $\bar{b}$ are fixed. Then

$E(\pi_1)\ \uparrow$ when $\check{b}\downarrow$ and/or $\check{a}\uparrow$

$E(\pi_1)\ \downarrow$ when $\check{b}\uparrow$ and/or $\check{a}\downarrow$

$E(\pi_2)\ \uparrow$ when $\check{b}\uparrow$ and/or $\check{a}\downarrow$

and

$E(\pi_2)\ \downarrow$ when $\check{b}\downarrow$ and/or $\check{a}\uparrow$

Knowing "a" and "b" exactly corresponds to $\check{a} = \check{b} = 0$. Let the corresponding exactly known steady state probabilities be $(\pi_1)_{ex}$ and $(\pi_2)_{ex}$. Then, due to the above results, the deviations from these values (for integer m's and n's) occur as follows:

$$E(\pi_1|\infty, \infty, m_2^*, n_2^*) \le E(\pi_1|m_1, n_1, m_2, n_2) \le (\pi_1)_{ex} \le E(\pi_1|m_1^*, n_1^*, \infty, \infty)$$

or

$$E(\pi_1|\infty, \infty, m_2^*, n_2^*) \le (\pi_1)_{ex} \le E(\pi_1|m_1, n_1, m_2, n_2) \le E(\pi_1|m_1^*, n_1^*, \infty, \infty)$$

and

$$E(\pi_2|m_1^*, n_1^*, \infty, \infty) \le E(\pi_2|m_1, n_1, m_2, n_2) \le (\pi_2)_{ex} \le E(\pi_2|\infty, \infty, m_2^*, n_2^*)$$

or

$$E(\pi_2|m_1^*, n_1^*, \infty, \infty) \le (\pi_2)_{ex} \le E(\pi_2|m_1, n_1, m_2, n_2) \le E(\pi_2|\infty, \infty, m_2^*, n_2^*)$$

where m_1^* and n_1^* are the smallest integers for which

$$\bar{a} = \frac{m_1}{m_1 + n_1}$$

is satisfied.

Observe that these inequalities do <u>not</u> tell us on which side of $(\pi_1)_{ex}$ $E(\pi_1)$ falls. However, they do reveal the very important fact that the worst deviations occur in situations where one transition probability is known exactly, situations that we have already closely studied in section 4.1.3 through the use of the hypergeometric function. There we found that replacing $E(\pi_1|\infty, \infty, m_2, n_2)$ by $(\pi_1)_{ex}$ usually resulted in a very small error. Now we know that replacing $E(\pi_1|m_1, n_1, m_2, n_2)$ by $(\pi_1)_{ex}$ will produce the same or an even smaller error. Hence, we can accurately approximate $E(\pi_1 | m_1, n_1, m_2, n_2)$ by $(\pi_1)_{ex}$, the latter being easy to calculate.

4.1.5. The Special 3-State Case Where Only 1 Transition Probability is Not Known Exactly

$$P = \begin{bmatrix} 1-a-b & a & b \\ c & 1-c-d & d \\ e & f & 1-e-f \end{bmatrix}$$

Suppose "a" is the only probability not known exactly. Then it is convenient to let $t = a/(1-b)$ and to assume that

$$f_t(x) = f_\beta(x|m,n) = \frac{1}{\beta(m,n)} x^{m-1} (1-x)^{n-1} \qquad 0 \le x \le 1$$

Then Appendix J reveals that the expected values of the steady state probabilities can be expressed in terms of the hypergeometric function F (defined in equations (G.2) and (G.3) of Appendix G).

$$E(\pi_1) = \frac{A}{B+C} F\left(1, n \middle| m+n \middle| \frac{B}{B+C}\right)$$

$$E(\pi_2) = \frac{D}{B+C}\left(\frac{m}{m+n}\right) F\left(1, n \middle| m+n+1 \middle| \frac{B}{B+C}\right)$$

$$+ \frac{G}{B+C} F\left(1, n \middle| m+n \middle| \frac{B}{B+C}\right)$$

and

$$E(\pi_3) = 1 - E(\pi_1) - E(\pi_2)$$

where

$$A = \frac{ce + cf + de}{1 - b}$$

$$B = d + e + f$$

$$C = \frac{bc + bd + bf + ce + cf + de}{1 - b}$$

$$D = e + f$$

and

$$G = \frac{bf}{1 - b}$$

Appendix G also suggests a method for evaluating a hypergeometric function for given arguments; hence, we can evaluate $E(\pi_1)$, $E(\pi_2)$ and $E(\pi_3)$ for given values of m, n, b, c, d, e and f.

The Beta distribution on a/(1-b) allows simple Bayes modification of the prior. If r transitions occur from state 1 to state 2 and s from 1 to 1, then the posterior distribution on t is

$$f_t(x) = f_\beta(x \mid m+r, n+s).$$

It certainly appears that the approach used in this section could be extended to give us the expected values of the π's in larger Markov processes where only one transition probability was not known exactly.

4.1.6. Simulation of the Steady State Behavior of 3-State Markov Processes Whose Transition Probabilities are Multidimensional Beta Distributed

As discussed earlier, for

$$P = \begin{bmatrix} 1-a-b & a & b \\ c & 1-c-d & d \\ e & f & 1-e-f \end{bmatrix}$$

where a, b, c, d, e and f are random variables there is no simple analytic method of determining $E(\pi_1)$, $E(\pi_2)$ and $E(\pi_3)$. Hence, it was necessary to resort to simulation techniques.

Simulation Procedure

Consider the 3-state process having multidimensional Beta distributed transition probabilities with parameters $M = (m_{ij})$. Using one of the two techniques outlined in section 3.6, a random draw of u and v was made from $f_{a,b}(u, v) = f_\beta(u, v|m_{12}, m_{13}, m_{11})$, of w and x from $f_{c,d}(w, x) = f_\beta(w, x|m_{21}, m_{23}, m_{22})$, and of y and z from $f_{e,f}(y, z) = f_\beta(y, z|m_{31}, m_{32}, m_{33})$. Then we know (from section 4.1.1) that the corresponding steady state probabilities are

$$\pi_1 = \frac{wy + wz + xy}{ux + uy + uz + vw + vx + vz + wy + wz + xy}$$

$$\pi_2 = \frac{uy + uz + vz}{\text{denominator}}$$

and

$$\pi_3 = 1 - \pi_1 - \pi_2$$

This procedure was repeated a large number of times (the usual number was 700) using an IBM 7090 digital computer and the sample mean values of π_1, π_2 and π_3 approached $E(\pi_1)$, $E(\pi_2)$ and $E(\pi_3)$, respectively. The sample standard deviation was also calculated to obtain an idea of the spread of the distribution, hence the significance of a given deviation of the sample mean from some fixed number. A typical plot of a sample mean and sample standard deviation of the mean as a function of the sample number is given in Figure 4.2. The stabilization of the sample mean in this figure indicates that the sample size of 700 is reasonable.

Parameter Values Used and Results

For each set of parameter values the following quantities were recorded:

i) The exact steady state probabilities $(\pi_1)_{ex}$, etc., corresponding to the transition probabilities being known exactly at their mean values.

ii) The sample mean values, $\bar{\pi}_1$, $\bar{\pi}_2$, $\bar{\pi}_3$.

iii) The sample standard deviations of the means, $s_{\bar{\pi}_1}$ and $s_{\bar{\pi}_2}$.

iv) 95 per cent "confidence" regions for each of $E(\pi_1)$ and $E(\pi_2)$. (See the discussion below.)

$$\text{e.g.,}\ \bar{\pi}_1 - 1.96\, s_{\bar{\pi}_1} \le E(\pi_1) \le \bar{\pi}_1 + 1.96\, s_{\bar{\pi}_1}$$

is the 95 per cent "confidence" region for $E(\pi_1)$.

v) The total per cent deviations (T.P.D.) of the sample means from the exact steady state values

$$\text{T.P.D.} = 100 \times [\,|(\pi_1)_{ex} - \bar{\pi}_1| + |(\pi_2)_{ex} - \bar{\pi}_2| + |(\pi_3)_{ex} - \bar{\pi}_3|\,] \tag{4.4}$$

We have two measures of the difference between $(\pi_j)_{ex}$ and $E(\pi_j)$:

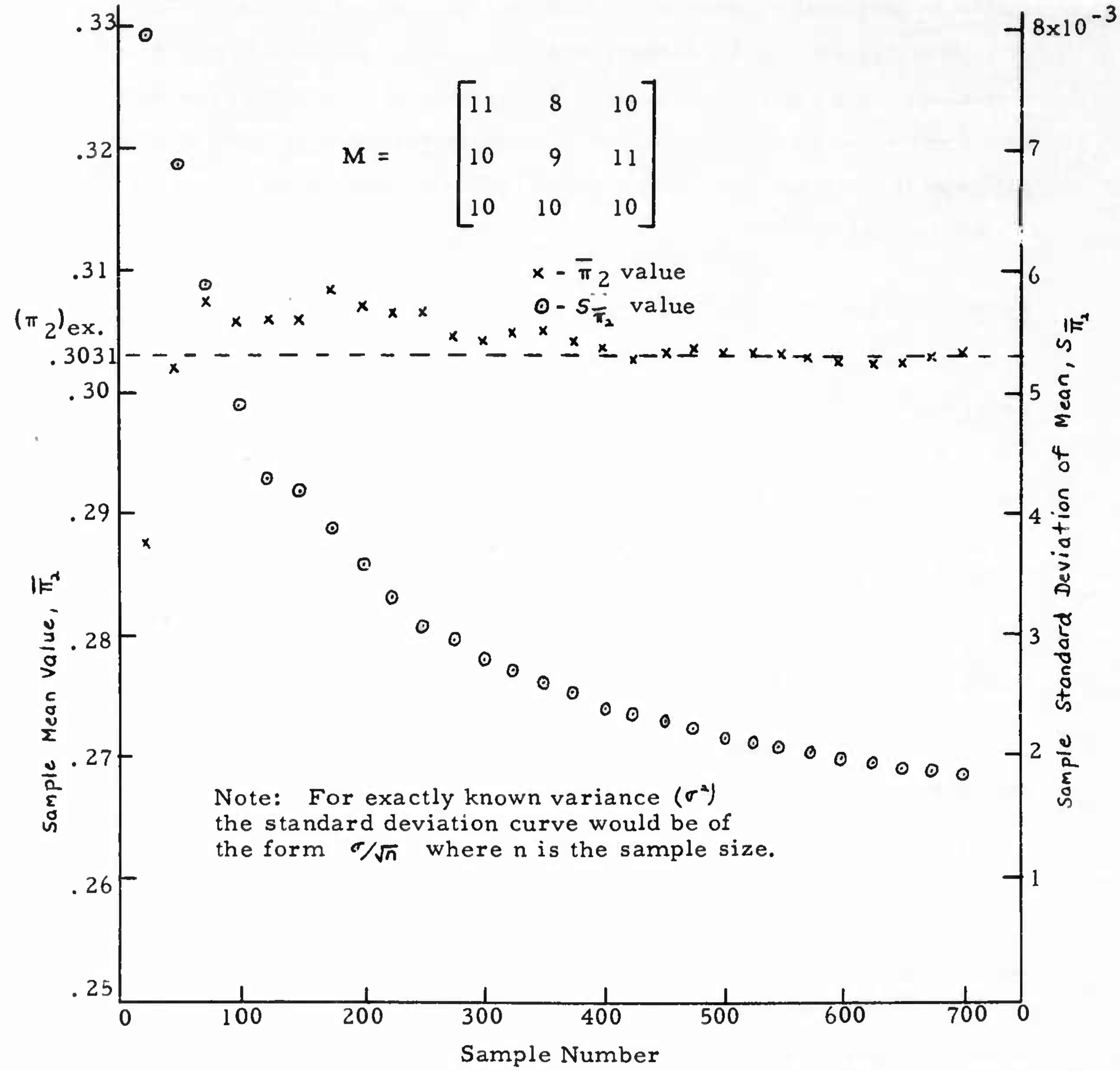

Figure 4.2: Typical Plots of $\bar{\pi}_2$ and $S_{\bar{\pi}_2}$ vs Sample Number

i) If the 95 per cent "confidence" region on $\bar{\pi}_j$ includes $(\pi_j)_{ex}$, we are more inclined to say that $E(\pi_j)$ is approximately equal to $(\pi_j)_{ex}$ than if the region does not include $(\pi_j)_{ex}$. The use of "confidence" intervals is not being advocated as this dissertation is Bayesian in nature. Fortunately for a reasonably flat prior distribution on $E(\pi_j)$ in the neighborhood of $(\pi_j)_{ex}$ (this situation can be assumed here) the a posteriori distribution on $E(\pi_j)$ is very close to the density function of $\bar{\pi}_j$ given $E(\pi_j)$; i.e., the a posteriori distribution of $E(\pi_j)$ is approximately normally distributed with mean $\bar{\pi}_j$ and standard deviation $s_{\bar{\pi}_j}$. Hence, the 95 per cent "confidence" region is the central section making up 95 per cent of the area of the a posteriori density function of $E(\pi_j)$. If it includes $(\pi_j)_{ex}$, we should be more inclined to say that $E(\pi_j) \simeq (\pi_j)_{ex}$ than if it doesn't include $(\pi_j)_{ex}$.

ii) The smaller T.P.D. is, the more likely that $(\pi_j)_{ex}$ is a good approximation to $E(\pi_j)$.

The following systematic method was used to decide on a reasonable number of (m_{ij}) sets to test. The m_{ij}'s were split into two categories, namely Low and High. So that two H's or two L's would not necessarily be the same, a random element was added as follows:

Each time an L was requested, a 1, 2, or 3 was selected, each with probability 1/3.

Each time an H was requested, a 7, 8, 9, 10, 11 or 12 was selected, each with probability 1/6.

To perform a reasonable number of tests which cover a wide variety of patterns it was decided to have each diagonal entry either high or low and each pair of off-diagonal elements in the same row both high or both low. This results in 20 essentially different patterns; permutations of states give the same pattern, e.g.,

$$\begin{bmatrix} H & L & L \\ L & L & L \\ L & L & L \end{bmatrix} \equiv \begin{bmatrix} L & L & L \\ L & H & L \\ L & L & L \end{bmatrix} \equiv \begin{bmatrix} L & L & L \\ L & L & L \\ L & L & H \end{bmatrix}.$$

Eighteen other miscellaneous experiments were performed as indicated in the legend of Appendix K. They were undertaken to provide extra points on the plot of T.P.D. (to be discussed shortly).

The detailed results are presented in Appendix K. From these Figure 4.3 was developed. It shows T.P.D. plotted as a function of two parameters, the average diagonal transition probability and the sum of the parameters of the multidimensional Beta distribution. More will be said on this plot later.

After a careful study of the results of Appendix K there is one important point that can be made. The low total per cent deviations throughout as well as the large number of times that the 95 per cent "confidence" region on $E(\pi_j)$ overlaps the corresponding $(\pi_j)_{ex}$ value clearly show the marked insensitivity of the expected values of the steady state probabilities to the variances of the multidimensional Beta distributions. By an analytic argument this fact has been shown to hold for the 2-state situation in sections 4.1.3 and 4.1.4.

At first only experiments 1-26 were performed. Close study of their results revealed two interesting points. First, for fixed values of the means of the transition probabilities the deviations of the $E(\pi_j)$'s from the $(\pi_j)_{ex}$'s decrease as the sum of the m_{ij}'s increases. This is as expected because the larger the sum of the m_{ij}'s, the closer we are to exactly known transition probabilities. Secondly, the worst deviations of the $E(\pi_j)$'s from the $(\pi_j)_{ex}$'s appear to occur when the diagonal m_{ii}'s dominate the other parameter values; i.e., when the means of the diagonal transition probabilities are high. This is directly comparable to the 2-state situation where the largest deviations occurred for small $\bar{a}$

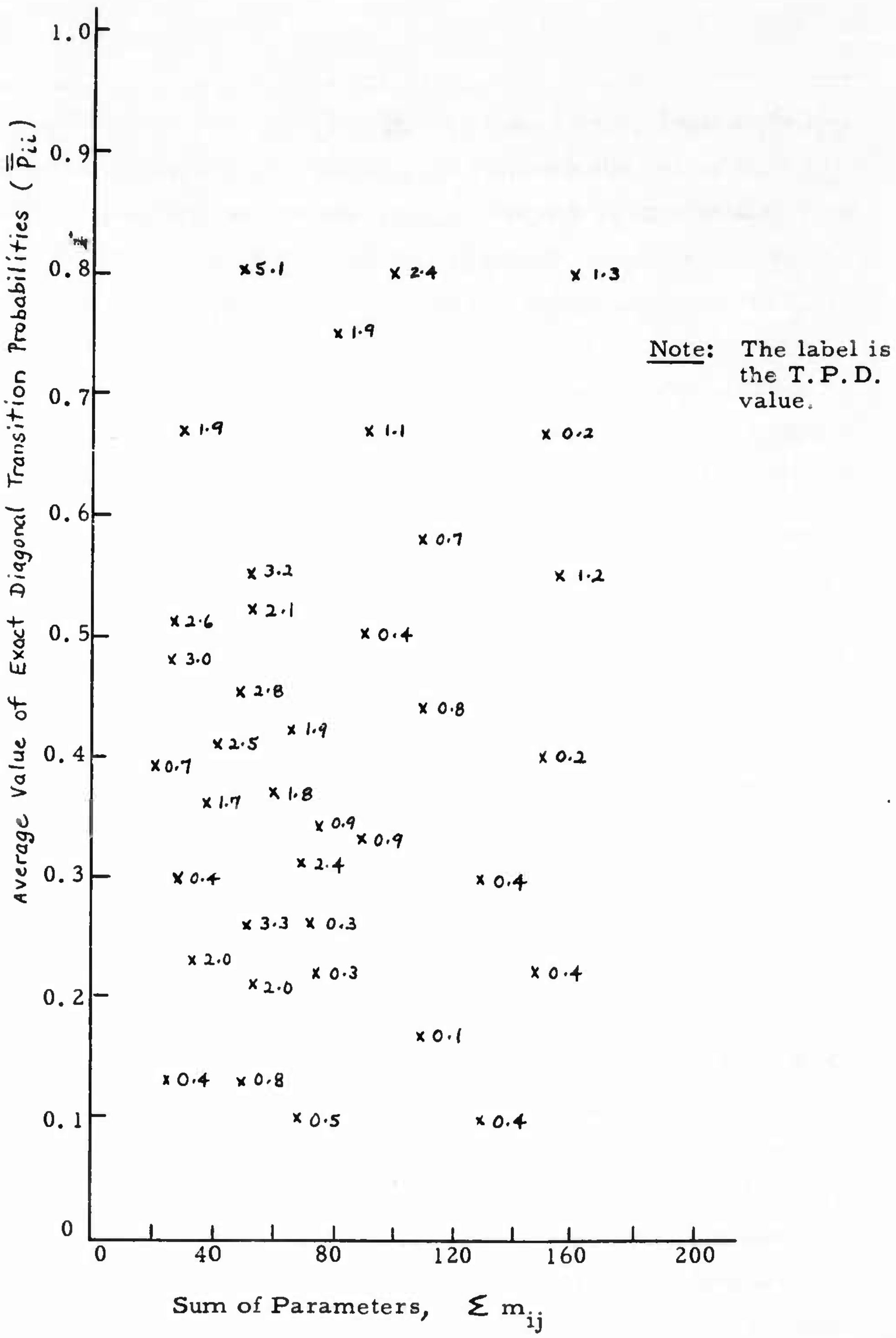

Figure 4. 3: Plot of T. P. D. for 3 State Processes

and $\bar{b}$ (the off-diagonal elements). When the means of the diagonal transition probabilities are high, all of $\bar{a}$, $\bar{b}$, $\bar{c}$, $\bar{d}$, $\bar{e}$ and $\bar{f}$ are small. Now each steady state probability is made up of a numerator over a denominator. Each of these, in turn, is the sum of a series of cross-products of a, b, c, d, e and f. Hence, if a, b, c, d, e and f are small, the numerator and denominator will be small and hence, relatively more sensitive to fluctuations in a, b, c, d, e and f than when some or all of those parameters are large. Hence, it is expected that $E(\pi_j)$ would deviate most from $(\pi_j)_{ex}$ when all of a, b, c, d, e and f were small.

In an effort to graphically portray these two important points the T.P.D. values of experiments 1-26 were plotted in Figure 4.3. It became apparent that other experiments would have to be performed to adequately cover the grid. Hence, experiments 27-38 were conducted. Their results further substantiated the three points mentioned earlier:

i) The expected values of the steady state probabilities are very insensitive to the variances of the multidimensional Beta distributions.

ii) For fixed values of the means of the transition probabilities the deviations of the $E(\pi_j)$'s from the $(\pi_j)_{ex}$'s decrease as the sum of the m_{ij}'s increases.

iii) The deviations of the $E(\pi_j)$'s from the $(\pi_j)_{ex}$'s increase as the diagonal m_{ii}'s more and more dominate the other parameter values; i.e., as the mean values of the diagonal transition probabilities become large. It was hoped that a functional relationship between T.P.D., $\sum m_{ij}$ and the average value of the exact diagonal transition probability $(\bar{\bar{p}}_{ii})$ could be developed. However, although Figure 4.3 illustrates the above points ii) and iii) qualitatively, it is clear that not even rough iso-T.P.D. curves can be drawn on the figure. If such curves could have been drawn, for a given Markov process with multidimensional

Beta distributions over the transition probabilities, we could estimate the T.P.D. value (i.e., the inaccuracy of using the $(\pi_j)_{ex}$'s for the $E(\pi_j)$'s) by the fact that we would know $\sum m_{ij}$ and $\bar{\bar{p}}_{ii}$ and would use the curves to obtain the corresponding T.P.D. value.

In Chapter 5 we shall define the expected reward per time period in the steady state as

$$E(R) = \sum_{j=1}^{N} r_j \, E(\pi_j)$$

where N is the number of states and r_j is the reward per period for being in state j. If we define

$$(R)_{ex} = \sum_{j=1}^{N} r_j \, (\pi_j)_{ex},$$

ideally we would like to know the behavior of $|E(R) - (R)_{ex}|$ as a function of the m_{ij}'s. However, this behavior would also be a function of the r_j's and therefore is quite difficult to study because of the large number of parameters involved. Thus, we have compromised by studying the behavior of

$$T.P.D. \doteq \sum_{j=1}^{N} |E(\pi_j) - (\pi_j)_{ex}| \,.$$

In summary, it appears that in most 3-state situations where the transition probabilities are multidimensional Beta distributed a good approximation to the expected values of the steady state probabilities is obtained by assuming that the transition probabilities are exactly known at their mean values and using the corresponding exact steady state probabilities (quantities that are very casy to evaluate) as the approximation. This approximation becomes questionable when the diagonal transition probabilities are large and the m_{ij}'s are relatively low in magnitude. However, as will be shown in the 4-state situation, m_{ij}'s low in magnitude are very unlikely in practice because any sort of prior knowledge about the transition probabilities will give reasonably small variances on the Betas which, in turn, will make the m_{ij}'s large.

4.1.7. Simulation of the Steady State Behavior of 4-State Markov Processes Whose Transition Probabilities are Multi-dimensional Beta Distributed

Essentially the same simulation procedure as for 3 states was used here. However, now we have 4 x 4 or 16 m_{ij}'s instead of 9. Also, the expressions for the steady state probabilities for exactly known transition probabilities are far more complex. It turns out that the denominator of each steady state probability is the sum of 64 triple cross products of the transition probabilities. More will be said on this point in Appendix L.

Part of the L-H framework of the 3-state simulation was used here but more of the m_{ij}'s were generated randomly at higher values as indicated in the legend of Appendix M.

An analysis of the results of Appendix M again reveals low total per cent deviations throughout as well as a large number of times that the 95 per cent "confidence" region on $E(\pi_j)$ overlaps the corresponding $(\pi_j)_{ex}$. As in the 3-state case this indicates that the expected values of the steady state probabilities are quite insensitive to the variances of the Beta distributions. This is most encouraging as it suggests that the same situation exists for any size Markov process. Unfortunately, simulation of processes with more than 4 states would take a prohibitively long time. Even for 4 states the computer time was becoming appreciable; e.g., experiments 52 to 56 required approximately 5 minutes on the 7090.

We again observe that for fixed values of the means of the transition probabilities $E(\pi_j)$ approaches $(\pi_j)_{ex}$ as the sum of the m_{ij}'s increases. Therefore, the approximation of using $(\pi_j)_{ex}$ for $E(\pi_j)$ will improve as the sum of the m_{ij}'s increases. Fortunately, as will now be demonstrated, in most physical situations the sum of the m_{ij}'s will be relatively large.

Consider the example presented in section 3.3 of a 4 category multidimensional Beta.

$E(p_1) = 0.1$	$\check{p}_1 = 0.004$	$\therefore \sigma_{p_1} = 0.063$
$E(p_2) = 0.2$	$\check{p}_2 = 0.008$	$\sigma_{p_2} = 0.089$
$E(p_3) = 0.3$	$\check{p}_3 = 0.001$	$\sigma_{p_3} = 0.032$
$E(p_4) = 0.4$	$\check{p}_4 = 0.004$	$\sigma_{p_4} = 0.063$

We found that $\sum m_j = 47$ with $m_1 \doteq 5$, $m_2 \doteq 9$, $m_3 \doteq 14$, $m_4 \doteq 19$. These m_j values, which are reasonably large, correspond to quite large values of the standard deviations of the p_j's. In most physical situations we would know enough a priori information about the p_j's to have smaller standard deviations than these. For the above example, keeping the σ_{p_j}'s in the same ratio we get the following results using equations (3.9) and (3.10):

	σ	0.9σ	0.75σ	0.5σ	0.1σ	0.01σ
σ_{p_1}	0.063	0.057	0.047	0.032	0.006	0.0006
σ_{p_2}	0.089	0.080	0.067	0.045	0.009	0.0009
σ_{p_3}	0.032	0.029	0.024	0.016	0.003	0.0003
σ_{p_4}	0.063	0.057	0.047	0.032	0.006	0.0006
$\sum m_j$	47	59	85	192	482	4829

The σ_{p_j} values under the 0.75σ and 0.5σ columns are of reasonable size for a physical problem and they are seen to give very high values of $\sum m_j$. Some similar calculations were done for other $E(p_j)$ and σ_{p_j} values and the results were qualitatively the same — for reasonable standard deviations of the p_j's the $\sum m_j$ value is quite large, hence the approximation of using $(\pi_j)_{ex}$ for $E(\pi_j)$ is quite good. More will be said on the approximations when they are used for statistical decision purposes in Chapter 5.

4.2. First Passage Times

Let n_{ij} be the number of transitions to get to state j for the first time given that the process is in state i before the first transition. If $i = j$, then n_{ii} is also called the recurrence time for state i.

We shall only be concerned with the mean first passage time and not with the entire probability mass function. Still, the same difficulties as those met in evaluating the expected values of the steady state probabilities are encountered for 3 or more states. This is quickly shown by recalling[16] that the mean recurrence time and steady state

[16] Howard, R. A., Dynamic Probabilistic Systems (in preparation).

probability are related by $E(n_{ii}) = 1/(\pi_i)$. Hence, we shall only study the 2 state situation where both transition probabilities are independently Beta distributed.

The mechanism assumed will again be as follows. We select a value for each of the transition probabilities from its Beta distribution. With these values we calculate the associated exactly known mean first passage times $(\bar{n}_{ij})$. We repeat this process over and over obtaining a whole series of values for the mean first passage times. We want to obtain analytic expressions for the long run average values of these series of mean first passage times; i.e., for $\overline{E(n_{ij})} \equiv \overline{\overline{n}}_{ij}$, the expected mean recurrence times.

Consider

$$P = \begin{bmatrix} 1 - a & a \\ b & 1 - b \end{bmatrix}$$

where

$$f_a(x) = f_\beta(x \mid m_1, n_1)$$

and

$$f_b(y) = f_\beta(y \mid m_2, n_2).$$

Then, as shown in Appendix N, the expected mean recurrence times are given by

$$E(\bar{n}_{11}) = 1 + \frac{m_1(m_2+n_2-1)}{(m_1+n_1)(m_2-1)} = 1 + \frac{\bar{a}\left(1 - \frac{1}{m_2+n_2}\right)}{\bar{b} - \frac{1}{m_2+n_2}}$$

$$E(\bar{n}_{22}) = 1 + \frac{m_2(m_1+n_1-1)}{(m_2+n_2)(m_1-1)} = 1 + \frac{\bar{b}\left(1 - \frac{1}{m_1+n_1}\right)}{\bar{a} - \frac{1}{m_1+n_1}}$$

$$E(\bar{n}_{12}) = \frac{m_1+n_1-1}{m_1-1} = \frac{1 - \frac{1}{m_1+n_1}}{\bar{a} - \frac{1}{m_1+n_1}} \tag{4.5}$$

and

$$E(\bar{n}_{21}) = \frac{m_2+n_2-1}{m_2-1} = \frac{1 - \frac{1}{m_2+n_2}}{\bar{b} - \frac{1}{m_2+n_2}}$$

There are several comments that can now be made. First, unlike the expected values of the steady state probabilities where we required the use of the hypergeometric function, equation (4.5) reveals that the $E(\bar{n}_{ij})$'s are very simple functions of the m_{ij}'s. Secondly, to calculate $E(\bar{n}_{11})$, although it is a function of "a", we only require $\bar{a}$. A similar comment can be made about $E(\bar{n}_{22})$ and "$\bar{b}$". Finally, the following can be said about the sensitivity of the $E(\bar{n}_{ij})$'s to the variances of the Beta distributions. The terms $1 - 1/(m_1+n_1)$ and $1 - 1/(m_2+n_2)$ rapidly approach 1; hence, the terms that determine the sensitivity are $\bar{a} - 1/(m_1+n_1)$ and $\bar{b} - 1/(m_2+n_2)$. If $\bar{a}$ is not too small, $\bar{a} - 1/(m_1+n_1)$ quickly approaches $\bar{a}$. Therefore, if $\bar{a}$ is not too small, both $E(\bar{n}_{12})$

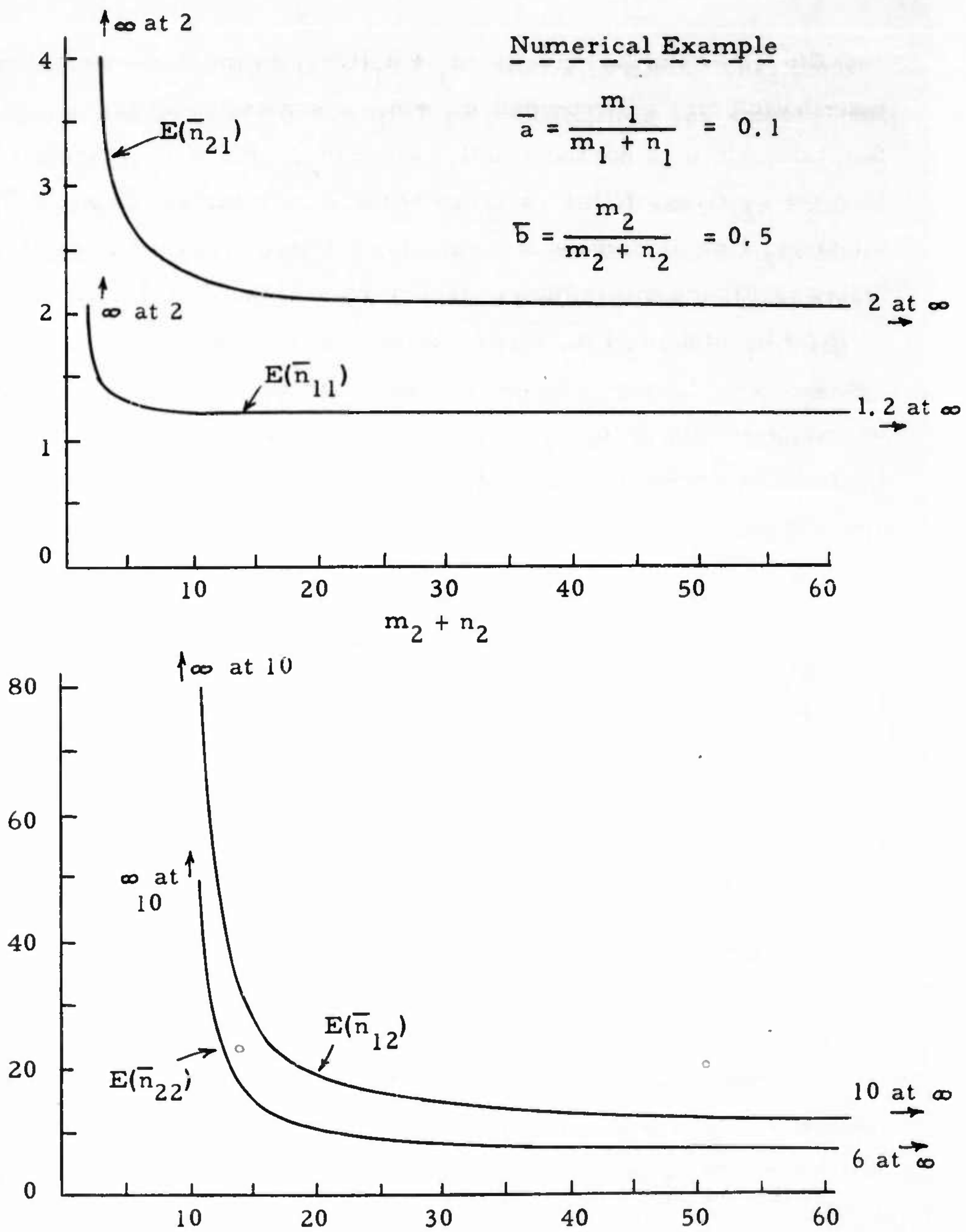

Figure 4.4: $E(\bar{n}_{ij})$'s as Functions of the Beta Parameters.

and $E(\bar{n}_{22})$ are insensitive to $m_1 + n_1$ (i.e., to the variance of the Beta distribution on "a") provided $m_1 + n_1$ is somewhat larger than $1/(\bar{a})$. Similarly, if $\bar{b}$ is not too small, both $E(\bar{n}_{21})$ and $E(\bar{n}_{11})$ are insensitive to $m_2 + n_2$ (i.e., to the variance of the Beta distribution on "b"), provided $m_2 + n_2$ is somewhat larger than $1/(\bar{b})$. This behavior is illustrated in Figure 4.4, which presents plots of the $E(\bar{n}_{ij})$'s as functions of $m_1 + n_1$ and $m_2 + n_2$ for the numerical example where $\bar{a} = 0.1$ and $\bar{b} = 0.5$. $\bar{b}$ is not too small, hence $E(\bar{n}_{21})$ and $E(\bar{n}_{11})$ are seen to be very insensitive to $m_2 + n_2$ as soon as $m_2 + n_2$ gets slightly away from the value 2. On the other hand, the low $\bar{a}$ value causes $E(\bar{n}_{12})$ and $E(\bar{n}_{22})$ to be fairly sensitive to $m_1 + n_1$ for a large range of $m_1 + n_1$.

It has been demonstrated that for a 2-state Markov process whose transition probabilities are independently Beta distributed it is a simple matter to obtain the expected values of the mean recurrence times. As the latter quantities are occasionally of importance in physical situations, this is a worthwhile discovery.

4.3. State Occupancy Times

4.3.1. The Probability Mass Function of the Occupancy Time

Again, consider an N-state Markov process with transition matrix $\mathbf{P} = (p_{ij})$ where the p_{ij}'s are multidimensional Beta distributed with parameters m_{ij}. Under these circumstances we have shown in section 3.2 that the marginal distribution on p_{ii} is given by

$$f_{p_{ii}}(x) = f_\beta\left(x \,\middle|\, m_{ii}, \sum_{j \neq i} m_{ij}\right)$$

Let u_i be the number of the transition on which the system leaves state i for the first time given that it is in state i before the first transition. Then it is well known that

$$p_{u_i|p_{ii}}(k|x) = x^{k-1}(1-x) \qquad k \geq 1 \tag{4.6}$$

Now we use the same mechanism as earlier. Select an x value from the p_{ii} distribution and determine the corresponding probability mass function on u_i. This is repeated a large number of times and we would like to know the resulting marginal probability mass function on u_i.

$$p_{u_i}(k) = \int_0^1 p_{u_i|p_{ii}}(k|x)\, f_{p_{ii}}(x)\, dx$$

Section O.1 of Appendix O reveals that

$$p_{u_i}(k) = \begin{cases} (1-\bar{p}_{ii})\left(\dfrac{\bar{p}_{ii}}{1+w_i}\right)\left(\dfrac{\bar{p}_{ii}+w_i}{1+2w_i}\right)\cdots\left(\dfrac{\bar{p}_{ii}+(k-2)w_i}{1+(k-1)w_i}\right) & k \geq 2 \\ \\ (1-\bar{p}_{ii}) & k = 1 \end{cases} \tag{4.7}$$

where

$$\bar{p}_{ii} = \frac{m_{ii}}{\sum_{j=1}^{N} m_{ij}} \quad \text{and} \quad w_i = \frac{1}{\sum_{j=1}^{N} m_{ij}} \tag{4.8}$$

<u>NOTE:</u>

For p_{ii} exactly known at $\bar{p}_{ii}$ the quantity $w_i = 0$ and equation (4.7) reduces to equation (4.6), as it should.

It is interesting that the probability of exit on the very next transition (k=1) is independent of the sum of the m_{ij}'s and only depends on $\bar{p}_{ii}$. Furthermore, it is now no longer obvious that $E(u_i)$ is finite (as in the case when p_{ii} is exactly known).

Equation (4.7) was used to develop the curves of Figures 4.5 and 4.6.

4.3.2 The Mean Occupancy Time

When p_{ii} is exactly known the mean occupancy time is given by $\frac{1}{(1-p_{ii})}$. However, when p_{ii} is Beta distributed as assumed here, the probability mass function of the occupancy time is as developed in equation (4.7). Although we cannot obtain the exact value of $E(u_i)$, section O.2 of Appendix O reveals that the mean value of the occupancy time can be bounded as follows:

$$\frac{\left(\sum_j m_{ij}-1\right)!\left(\sum_{j\neq 1} m_{ij}\right)}{(m_{ii}-1)!}\sum_{k=1}^{r}\frac{k(m_{ii}+k-2)!}{\left(\sum_j m_{ij}+k-1\right)!} < E(u_i) < \frac{\left(\sum_j m_{ij}-1\right)\left(\sum_{j\neq i} m_{ij}\right)}{\left(\sum_{j\neq i} m_{ij}-1\right)} \tag{4.9}$$

The upper bound is not very tight but it does serve the important purpose of showing that the mean value of the occupancy time is finite. This is not obvious from just looking at the probability mass function (equation (4.7)). The lower bound of equation (4.9) is obtained by merely truncating the summation

$$\sum_{k=1}^{\infty} k\, p_{u_i}(k)$$

at a finite number (r) of terms.

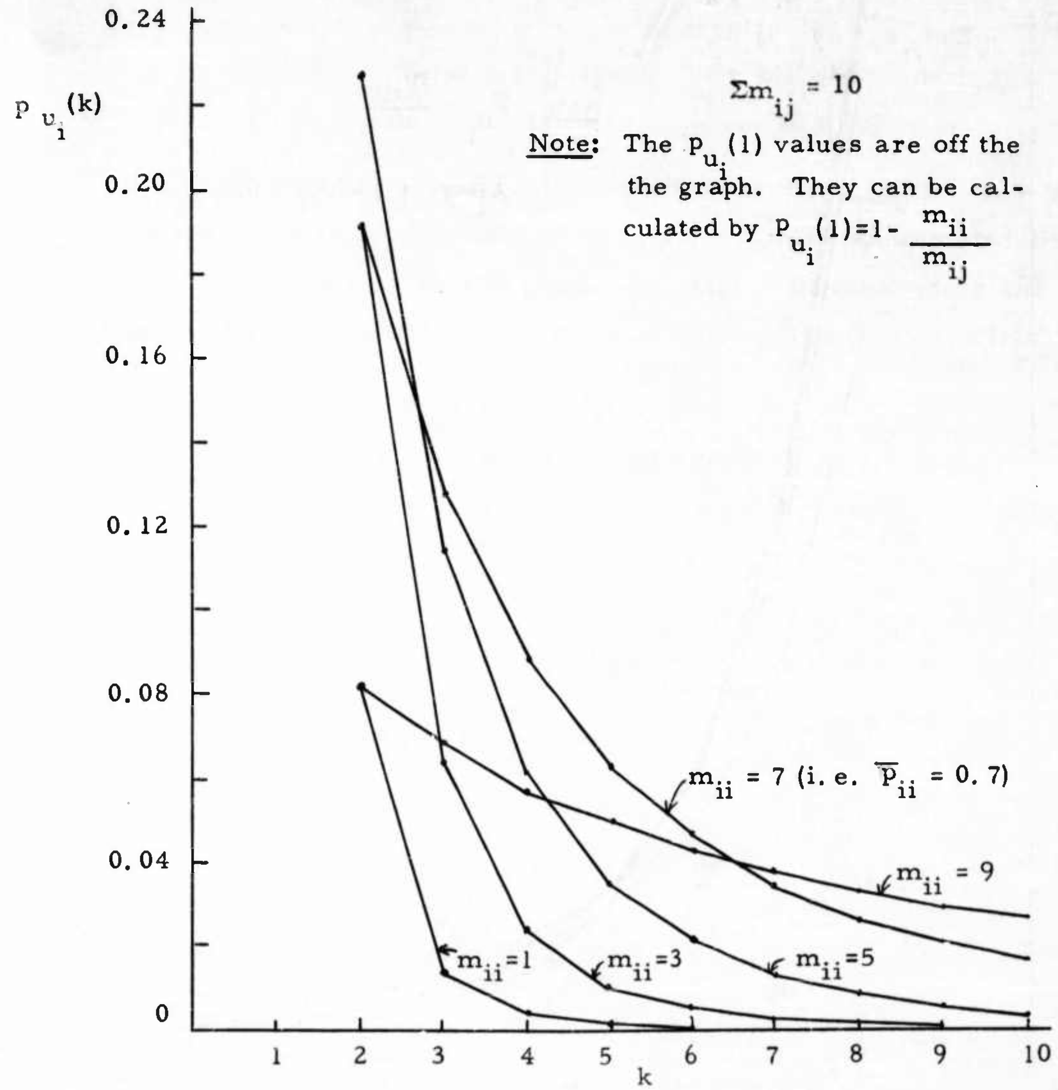

Figure 4.5: The Probability Mass Function of State Occupancy Time, $p_{u_i}(k)$, for Fixed Σm_{ij}

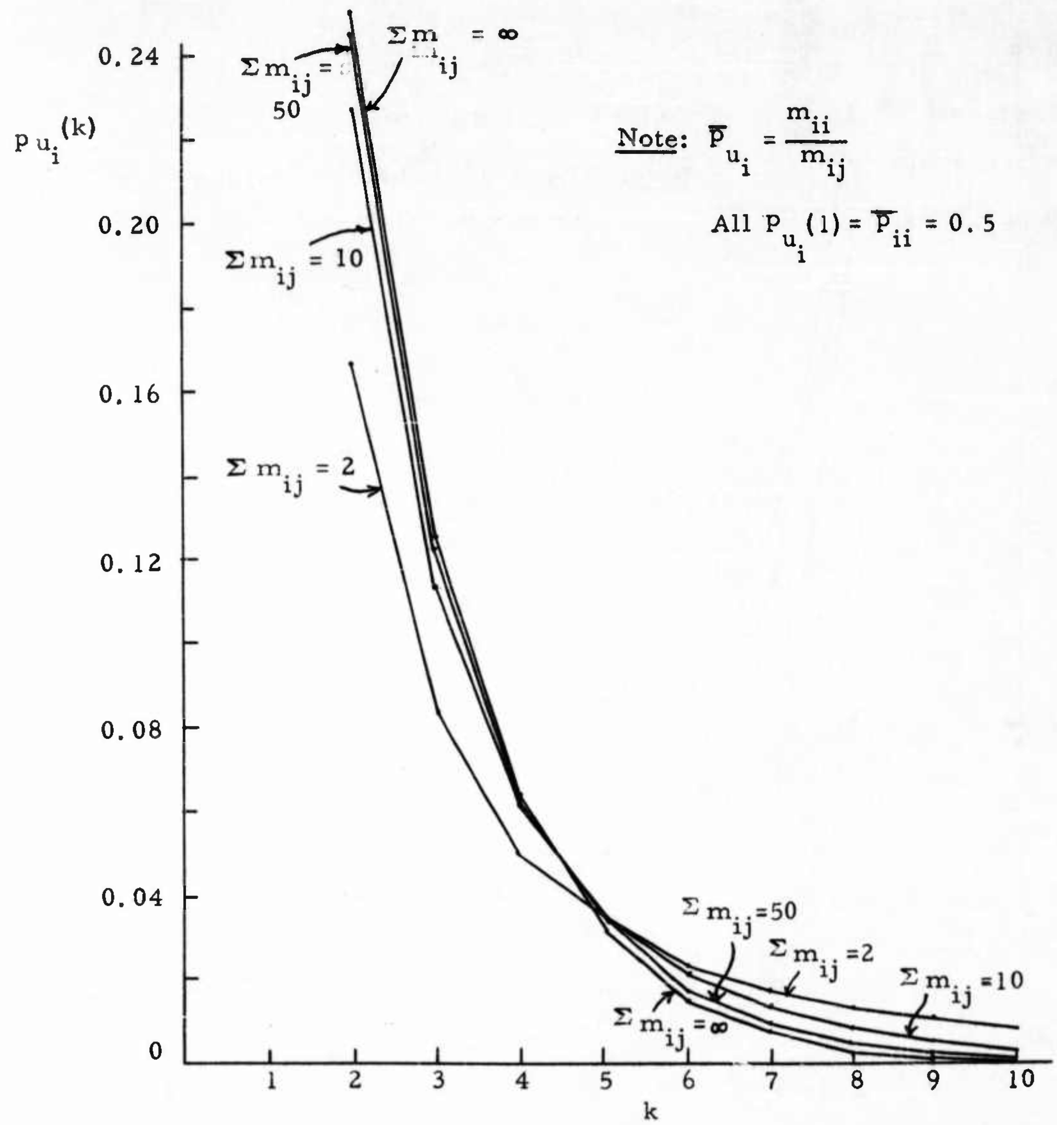

Figure 4.6: The Probability Mass Function of State Occupancy Time, $p_{u_i}(k)$, for Fixed $\overline{p}_{ii}$

Although a closed expression could not be found for the mean occupancy time, recall that we were able to obtain the probability mass function of the occupancy time when the transition probabilities are multidimensional Beta distributed. This appears to be the one important quantity that is analytically calculable in an N state Markov process when the transition probabilities are multidimensional Beta distributed.

Numerical Example

Suppose

$$f_{p_{ii}}(x) = f_\beta(x|5, 45) \qquad \text{i.e.,} \qquad m_{ii} = 5, \ \sum_j m_{ij} = 50$$

Using equation (4.9), the upper bound is given by

$$\frac{\left(\sum_j m_{ij} - 1\right)\left(\sum_{j \neq i} m_{ij}\right)}{\left(\sum_{j \neq i} m_{ij} - 1\right)} = \frac{(49)(45)}{44} = 50.1$$

The lower bound for $r = 1$ is 0.90

$r = 2$ is 1.08

$r = 3$ is 1.11

$r = 4$ is 1.12

Therefore, $1.12 < E(u_i) < 50.1$, a rather wide bound, but as stated earlier, the upper bound does serve the useful purpose of assuring us that $E(u_i)$ is, indeed, finite. For p_{ii} exactly known at $5/(50) = 0.1$, the mean occupancy time would be $1/(1-0.1)$ or 1.11.

4.4. Transient Behavior

Again, let the mechanism be as follows. Values of the transition probabilities are drawn from their distributions and the multi-step transition probabilities are evaluated conditional upon these exactly known values of the transition probabilities. If this process is repeated a large number of times, what can be said about the unconditional multi-step transition probabilities?

Define

$\phi_{ij}(n \mid \mathbf{P})$ = pr{state at time n is j | state at time 0 is i and the matrix **P**, with exactly known transition probabilities, is being used}

and

$\overline{\phi}_{ij}(n)$ = pr{state at time n is j | state at time 0 is i}.

As an example consider the 2-state process with

$$\mathbf{P} = \begin{bmatrix} 1 - a & a \\ b & 1 - b \end{bmatrix}$$

where "a" and "b" are distributed according to $f_a(x)$ and $f_b(y)$.

$$\overline{\phi}_{12}(1) = \int_0^1 x \, f_a(x) \, dx = \overline{a},$$

the mean of the distribution on "a". Similarly,

$$\overline{\phi}_{11}(1) = 1 - \overline{a}, \quad \overline{\phi}_{21} = \overline{b}, \quad \text{and} \quad \overline{\phi}_{22}(1) = 1 - \overline{b}$$

However,

$$\overline{\phi}_{12}(2) = \int_0^1 \int_0^1 [(1-x)x + x(1-y)]\, f_a(x)\, f_\beta(y)dxdy$$

$$= \int_0^1 \int_0^1 [2x - x^2 - xy]\, f_a(x)\, f_\beta(y)dxdy$$

$$= 2\overline{a} - \overline{a^2} - \overline{a}\,\overline{b} \neq 2\overline{a} - (\overline{a})^2 - \overline{a}\,\overline{b}.$$

The second moment of the "a" distribution has now become important.

More generally,

$$\overline{\phi}_{ij}(n) = \int_0^1 \int_0^1 \phi_{ij}(n|x, y)\, f_a(x)\, f_b(y)dxdy$$

i.e.,

$$\overline{\phi}_{ij}(n) = [\phi_{ij}(n|a, b)]^* \tag{4.10}$$

where * denotes that a^j has been replaced by $\overline{a^j}$ and b^k by $\overline{b^k}$. This result is useful for the 2-state case where $\phi_{ij}(n|(p_{ij}))$ has been obtained in closed form. For more than 2 states there is no closed form expression for $\phi_{ij}(n|(p_{ij}))$ in terms of power series in the transition probabilities. (The 3-state situation is illustrated in Appendix F.) However, for 3 or more states for a specific n we could find $\phi_{ij}(n)$ as a power series in the transition probabilities by obtaining the i-j element of P^n through actually raising P to the n^{th} power (where the transition probabilities would be left in symbolic form.) Power series are required to allow us to replace p_{ij}^k by $\overline{p_{ij}^k}$. For the 2-state situation, equation (4.10) was used to develop $\bar{\phi}_{ij}(n)$ for Beta distributions over "a" and "b". In the 2-state case we have[17]

$$\Phi(n|a,b) = (\phi_{ij}(n|a,b)) = \begin{bmatrix} \frac{b}{a+b} & \frac{a}{a+b} \\ \\ \frac{b}{a+b} & \frac{a}{a+b} \end{bmatrix} + (1-a-b)^n \begin{bmatrix} \frac{a}{a+b} & \frac{-a}{a+b} \\ \\ \frac{-b}{a+b} & \frac{b}{a+b} \end{bmatrix}$$

This does not appear to be expressible in terms of a^j and b^k. However, for any specific n this can be achieved by expanding the above expression. For example, suppose n = 3, then

$$\phi_{12}(3|a,b) = \frac{a}{a+b} + (1-a-b)^3 \left(-\frac{a}{a+b}\right)$$

$$= \frac{a}{a+b} [1 - 1 + 3(a+b) - 3(a+b)^2 + (a+b)^3]$$

$$= a[3 - 3(a+b) + (a+b)^2]$$

$$= a^3 + a^2(-3+2b) + a(3-3b+b^2)$$

$$\therefore \; \bar{\phi}_{12}(3) = \overline{a^3} + \overline{a^2}(-3+2\bar{b}) + \bar{a}(3-3\bar{b}+\overline{b^2}).$$

Expressions of this nature were developed for n = 1, 2, ..., 10 and then, through the use of a computer program, $\bar{\phi}_{12}(n)$ n = 1, 2, ..., 10 were evaluated for the following combinations

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[17] *Ibid*

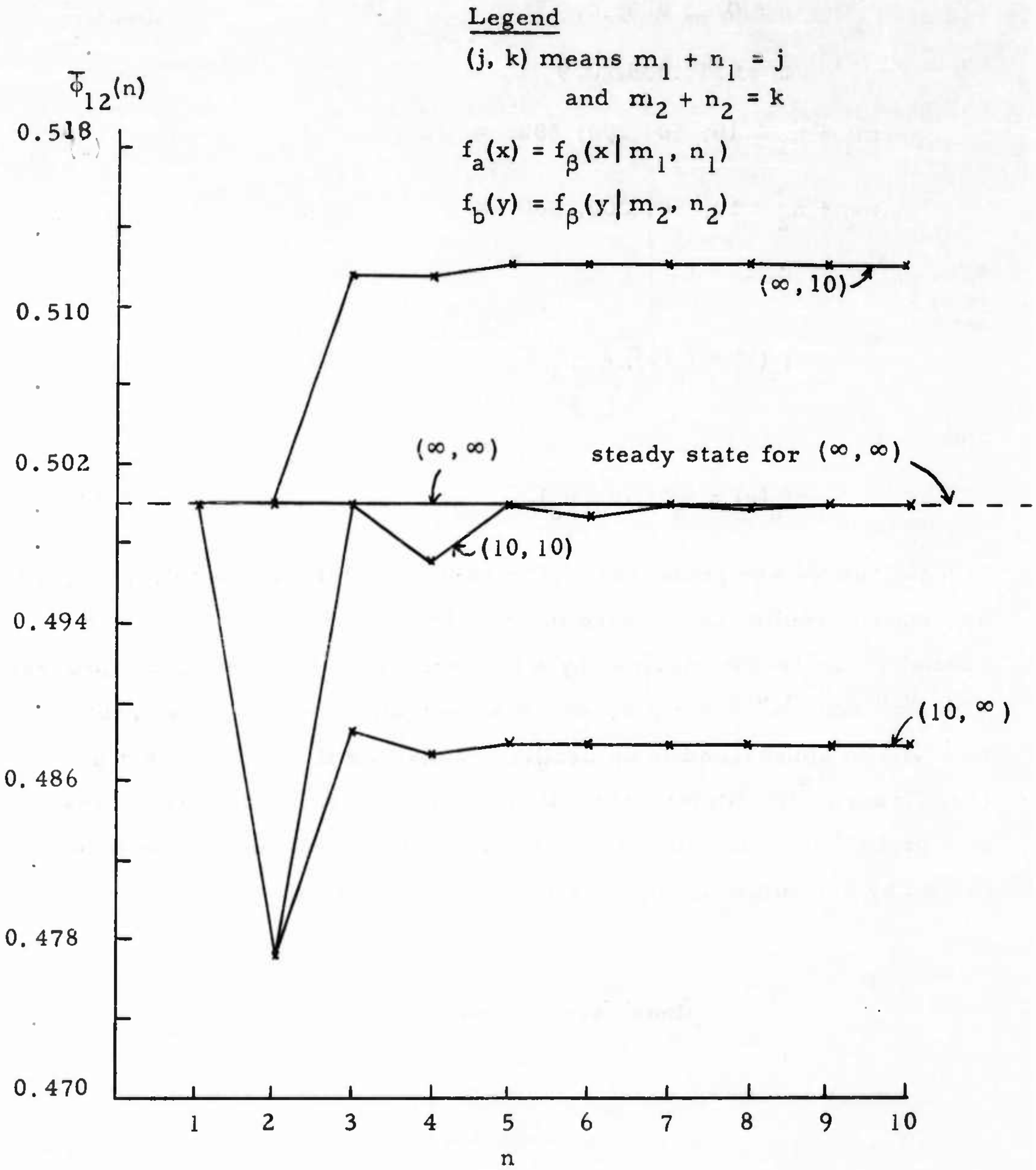

Figure 4.7: The Multi-Step Transition Probabilities for a 2-State Process When "a" and "b" are Independently Beta Distributed. $\bar{a} = 0.5$, $\bar{b} = 0.5$

$$\bar{a} = 0.1,\ 0.5,\ 0.9$$

$$\bar{b} = 0.1,\ 0.5,\ 0.9$$

$$m_1 + n_1 = 10,\ 50,\ 200,\ 500,\ \infty$$

$$m_2 + n_2 = 10,\ 50,\ 200,\ 500,\ \infty$$

where

$$f_{\tilde{a}}(x) = f_\beta(x \mid m_1, n_1)$$

and

$$f_{\tilde{b}}(y) = f_\beta(y \mid m_2, n_2).$$

Typical curves are presented in Figure 4.7 for the case $\bar{a} = 0.5$, $\bar{b} = 0.5$. The entire results (which were too detailed to be included even in an appendix) can be summarized by a few remarks. For the 2-state process with "a" and "b" exactly known we know that the steady state probabilities will be approached in an oscillatory manner if and only if $a + b > 1$ (see Howard[18]). Empirically, at least, the behavior when both transition probabilities are independently Beta distributed can best be described by the following flow chart:

(please see next page)

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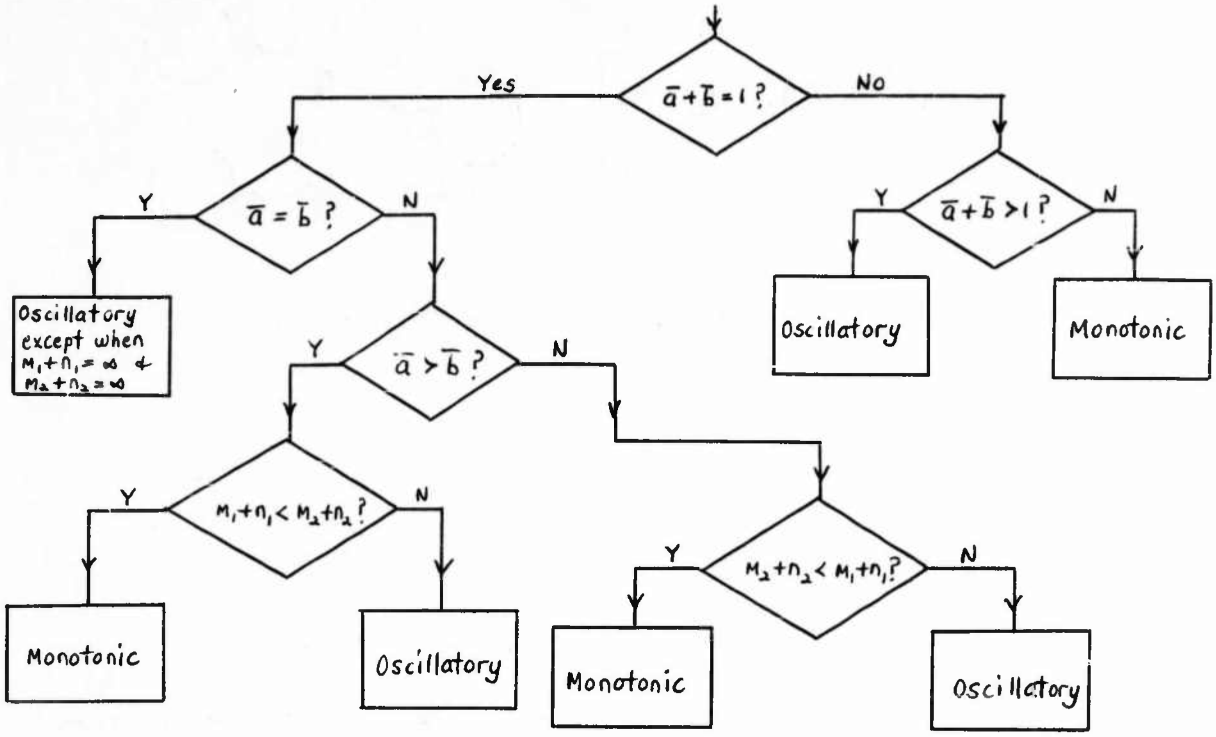
[18] <u>Ibid.</u>



$$f_a(x) = f_\beta(x|m_1, n_1); \quad f_b(y) = f_\beta(y|m_2, n_2)$$

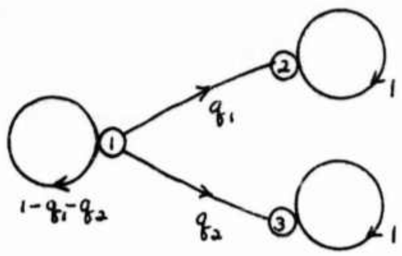
Figure 4.8. Oscillation in the transient behavior of 2-state processes when both transition probabilities are independently Beta distributed.

4.5. A Simple Trapping States Situation

4.5.1. 3-State Example

Consider the extremely simple Markov process whose transition matrix and flow graph are as follows:

$$P = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 1-q_1-q_2 & q_1 & q_2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{array}$$



where q_1 and q_2 are multidimensional Beta distributed, i.e.,

$$f_{q_1,q_2}(x,y) = f_\beta(x,y|m_2,m_3,m_1)$$

$$= \frac{1}{\beta(m_1,m_2,m_3)} (1-x-y)^{m_1-1} x^{m_2-1} y^{m_3-1}$$

$$0 \leq x,\ 0 \leq y,\ x+y \leq 1.$$

Let g_j be the probability that the process traps in state j. Then Appendix P reveals the following simple results:

$$E(g_2) = \frac{m_2}{m_2+m_3} \qquad \text{and} \qquad E(g_3) = \frac{m_3}{m_2+m_3}$$

Bayes Modification

Since the process traps in state 2 or 3 we must think of a large number of units starting in state 1 for a Bayes approach to have any meaning.

A priori distribution

$$f_{q_1,q_2}(x,y) = f_\beta(x,y|m_2,m_3,m_1)$$

Suppose we observe

n_1 transitions which return to state 1

n_2 transitions which go to state 2

n_3 " " " " " 3

Then the a posteriori distribution is

$$f_{q_1, q_2}(x, v) = f_\beta(x, y \mid m_2+n_2, m_3+n_3, m_1+n_1)$$

and the a posteriori expected values of the probabilities of trapping in states 2 and 3 are

$$E(g_2) = \frac{m_2 + n_2}{m_2 + m_3 + n_2 + n_3}$$

and

$$E(g_3) = \frac{m_3 + n_3}{m_2 + m_3 + n_2 + n_3}$$

Note that 1-1 transitions have no effect on $E(g_2)$ and $E(g_3)$. However, they would change our feelings as to the number of transitions for a trap to occur.

4.5.2. Generalization to N States

The generalization to N states is accomplished without difficulty.

$$P = \begin{bmatrix} 1 - \sum_{i=1}^{N-1} q_i & q_1 & q_2 & \cdots & q_{N-1} \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & & 0 \\ \vdots & & & & \vdots \\ 0 & & & 0 & 1 \end{bmatrix}$$

The a priori density function is

$$f_{q_1, \ldots, q_{N-1}}(x_1, \ldots, x_{N-1}) = f_\beta(x_1, \ldots, x_{N-1} | m_2, m_3, \ldots m_N, m_1)$$

then

$$E(g_j) = \frac{m_j}{\sum_{i=2}^{N} m_i} \qquad j = 2, 3, \ldots, N$$

Suppose we observe n_i transitions from state 1 to i, i = 2, ..., N. Then the a posteriori expectation of trapping in j is given by

$$\frac{m_j + n_j}{\sum_{i=2}^{N} (m_i + n_i)}$$

It is realized that the trapping state example dealt with here is practically trivial. However, it does give some insight into the problems encountered when the transition probabilities are not known exactly. Furthermore, for more complicated trapping situations we immediately become involved in the same sort of predicament as was encountered in the study of the steady state probabilities when the process contained more than 2 states.

CHAPTER 5

STATISTICAL DECISIONS IN MARKOV PROCESSES WHEN THE TRANSITION PROBABILITIES ARE NOT KNOWN EXACTLY

Before starting this chapter it is suggested that the reader scan section 5.6 to obtain an overall idea of the goals of the chapter and how they tie in with those of the previous two chapters.

5.1. The Importance of the Expected Values of the Steady State Probabilities

Consider an N-state Markov process with a given transition matrix **P**. Let the corresponding steady state probabilities be represented by $\underline{\pi} = [\pi_1, \pi_2, \ldots, \pi_N]$. Suppose that there is a reward vector $\underline{r} = [r_1, r_2, \ldots, r_N]$, where r_j = reward for being in state j for one time period. Then, in the steady state the expected reward per time period is

$$R = \sum_{j=1}^{N} \pi_j r_j$$

Now, if the p_{ij}'s are random variables, the π_j's and, therefore, also R become random variables. But

$$E(R) = \sum_{j=1}^{N} r_j E(\pi_j) \tag{5.1}$$

The quantity E(R) is central to many decision processes. Hence, it is essential that we be able to evaluate it for convenient distributions over the p_{ij}'s. However, equation (5.1) reveals that we must determine the $E(\pi_j)$'s to obtain E(R). This was the motivation for such careful study of the $E(\pi_j)$'s for multidimensional Beta distributions over the transition probabilities in section 4.1. The reasoning behind using multidimensional Beta distributions has been discussed at length in section 3.5.

5.2. A 2-State Problem

5.2.1. Description of the Problem

Consider a 2-state Markov process with transition matrix

$$P = \begin{bmatrix} 1 - a & a \\ b & 1 - b \end{bmatrix}$$

where "a" is exactly known but

$$f_b(x) = f_\beta(x|m, n) = \frac{1}{\beta(m, n)} x^{m-1} (1-x)^{n-1} \quad 0 \leq x \leq 1$$

Let

c be the fixed cost per time period for using the process.

r_j be the reward per time period for being in state j (j=1, 2)

and

d be the cost of observing a transition from state 2 regardless of the state to which it goes.

Suppose that we have the option of buying the right to observe k transitions from state 2 (the outcomes of which would modify, in a

Bayes sense, the probability distribution over "b"). For the problem to have economic meaning, let us consider that one of the following two conditions exists.

i) The process will only be run for s periods starting in the steady state.

or

ii) The process will run indefinitely but there is a known discount factor.

The second condition would be more likely to exist in the real world than the first.

5.2.2. The Situation Where the Observations Do Not Affect the Decision Procedure

Intuitively, it certainly would not be worthwhile to pay for observations if they cannot affect the decision procedure. However, in the research this fact was overlooked at first and some rather interesting, non-trivial results were developed. They are, therefore, included in this section and Appendix Q.

Let E_r be the event that in k transitions from state 2 r of them go to state 1. Then, according to Raiffa and Schlaifer[19]

$$pr(E_r | m, n, k) = p_{\beta b}(r | m, n, k)$$

$$= \frac{(m+r-1)!\,(n+k-r-1)!\,k!\,(m+n-1)!}{r!\,(m-1)!\,(k-r)!\,(n-1)!\,(m+n+k-1)!} \qquad (r=0, 1, \ldots, k) \tag{5.2}$$

This is the Beta-binomial distribution. Also, we know from section 3.4 that

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[19] Op. cit., p. 265.

$$f_b(x|E_r) = f_\beta(x|m+r, n+k-r),$$

a new member of the same Beta family.

Assume that the process will be used for s periods in the steady state. If the Beta parameters are m and n and the other transition probability is "a", then the expected net revenue is given by

$$E(N.R.|m,n,a) = s[r_1 E(\pi_1|m,n,a) + r_2 E(\pi_2|m,n,a) - c]$$

$$= s[r_1 - c + (r_2-r_1) E(\pi_2|m,n,a)].$$

From the results of section 4.1.3 and Appendix G

$$E(N.R.|m,n,a) = s\left[r_1 - c + (r_2-r_1) \frac{a}{a+1} F\left(1,n\middle|m+n\middle|\frac{1}{a+1}\right)\right] \tag{5.3}$$

Also define $E(N.R.|m,n,a;k)$ to be the expected net revenue (prior to the observations and excluding the cost of observation) if m and n are the Beta parameters, "a" is the other transition probability and k observations are to be taken. Then

$$E(N.R.|m,n,a;k) = \sum_r pr(E_r|m,n,k)\, E(N.R.|m+r, n+k-r, a)$$

i.e.,

$$E(N.R.|m,n,a;k) = \sum_{r=0}^{k} p_{\beta b}(r|m,n,k)\, E(N.R.|m+r, n+k-r, a) \tag{5.4}$$

In section Q.1 of Appendix Q it is shown by algebraic manipulation involving equations (5.2), (5.3) and (5.4) that

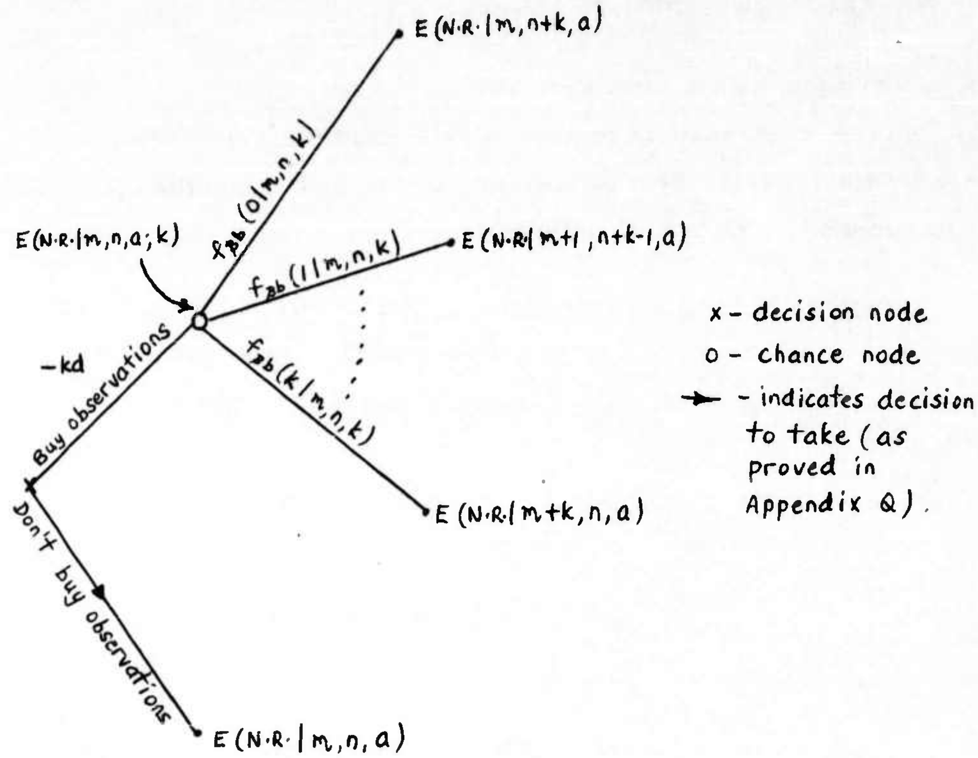


Figure 5.1. A 2-state decision problem where the observations do not affect the decision.

$$E(N.R.\,|m, n, a; k) = E(N.R.\,|m, n, a),$$

the result that was intuitively anticipated at the beginning of this section.

An alternate proof is presented in section Q.2 of Appendix Q through the use of a theorem that is of interest for decision theory in general.

5.2.3. The Situation Where the Observations Do Affect the Decision Procedure

Now let us look at the more realistic situation where we shall decide to use the process only if E(N.R.) is positive, otherwise we do not use it and the net revenue is zero. (The extension to the situation where E(N·R.) must be greater than some fixed constant is trivial.) Stated in another way, we shall act so as to maximize the expected profit. Now experimentation may be worthwhile. The case where the process can be used for s periods (in the steady state) will be considered. The rest of the required information is as outlined in section 5.2.1.

The new decision problem is shown in Figure 5.2.

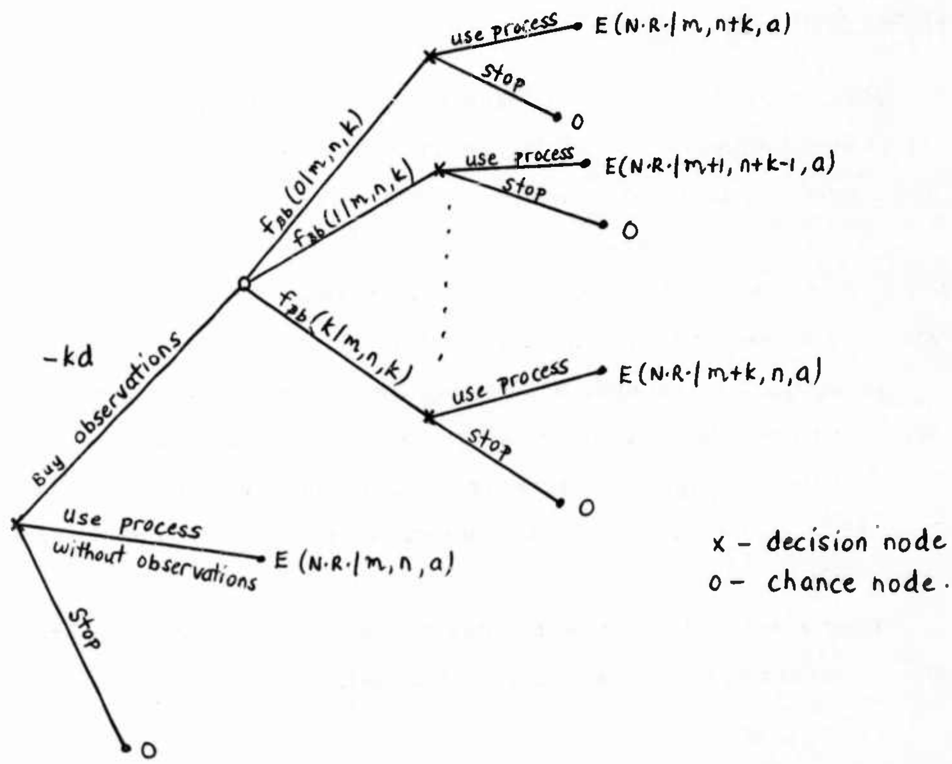


Figure 5.2. A 2-state decision problem where the observations do affect the decision.

It is clear that now

$$E(N.R.\,|m,n,a;k) = \sum_{r=0}^{k} p_{\beta b}(r\,|m,n,k)\,\max[0,\ E(N.R.|m+r,n+k-r,a)]$$

and we select

"Buy observations" if $[E(N.R.|m,n,a;k) - kd]$ is greater than both 0 and $E(N.R.\,|m,n,a)$,

"Use process without observations" if $E(N.R.\,|m,n,a)$ is greater than both 0 and $[E(N.R.\,|m,n,a;\ k) - kd]$,

"Stop" if both $[E(N.R.\,|m,n,a;k) - kd]$ and $E(N.R.\,|m,n,a)$ are less than 0.

Numerical Example

Pierre and Louie are contemplating a 5 day fishing trip, several months hence, in the wilds of northern Quebec. They, their gear and canoe will be flown in and out of the wilderness. However, realizing that there is a considerable expense involved they have decided to do some rapid calculations before making the trip. In fact, they will only make the trip if their expected net revenue is positive.

Including the transportation costs the expenses per day would be \$40. They are willing to put a dollar value on the pleasures derived from the fishing, etc. However, the value is a function of the weather. On a reasonably sunny day the value is \$60, on a cloudy or rainy day, \$20.

They are convinced that the day-to-day weather can be represented as a Markov process with transition matrix

$$P = \begin{array}{c} \\ S \\ R \end{array}\begin{array}{c} \begin{array}{cc} S & R \end{array} \\ \begin{bmatrix} 1 - a & a \\ b & 1 - b \end{bmatrix} \end{array}$$

Because the fishing area is so far in the wilderness, data on its weather conditions is extremely difficult to obtain. Pierre wrote to the Dominion Weather Bureau asking for the number of occurrences on record of the transition couplets (SS), (SR), (RS), (RR). (SS stands for sunny day followed by sunny day.) The government official sent back information on only (SS) and (SR). Furthermore, he has stated that any additional weather information of this nature will be charged at a fixed cost ($0.20) per transition couplet.

From the (SS) and (SR) information together with their prior knowledge, Pierre and Louie are satisfied with saying that "a" is exactly known at 0.9; i.e., a rainy day is very likely after a sunny one. After talking with friends, reading geography books, etc., they are willing to assume that "b" is Beta distributed with parameters $m = 9$, $n = 1$; i.e., again a mean of 0.9.

Pierre is not willing to spend any more money on weather information. Louie wants to buy one piece of information, namely the weather on a day after one specific rainy day (i.e., the outcome of a transition from the R state). Which man's decision is preferable?

Using the symbols of section 5.2.1,

$a = 0.9, \quad m = 9, \quad n = 1$

$k = 1$ (number of possible transitions observable)

$d = \$0.20$ (cost of each transition)

$c = \$40$ (cost per period for utilizing process)

$r_1 = \$60$ (reward per period for being in state 1 or S)

$r_2 = \$20$ (reward per period for being in state 2 or R)

$$\left.\begin{array}{l} p_{\beta b}(0 \mid 9, 1, 1) = 0.1 \\ p_{\beta b}(1 \mid 9, 1, 1) = 0.9 \end{array}\right\} \text{using equation (5.2)}$$

Equation (5.3) gives

$$E(N.R. \mid m, n, a) = s[r_1 - c + (r_2 - r_1) E(\pi_2 \mid m, n, a)]$$

Using this expression together with the method of evaluating $E(\pi_2 \mid m, n, a)$ by means of the hypergeometric function (outlined in section 4.1.3) we obtain

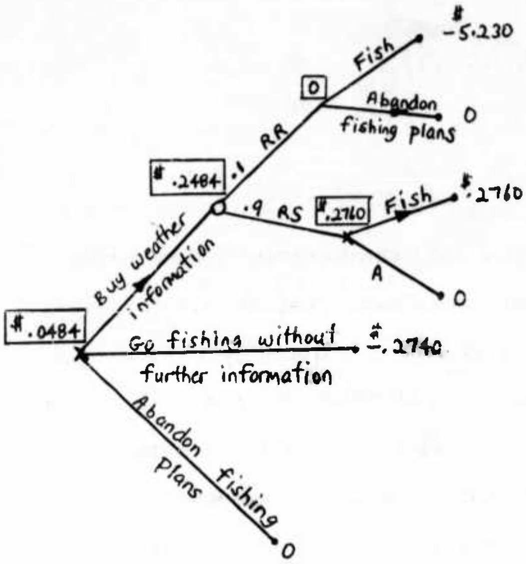
$E(N.R. \mid 9, 1, 0.9) = -\$0.2740$, the expected net revenue if they go fishing without any more information.

$E(N.R. \mid 9, 2, 0.9) = -\$5.2320$

$E(N.R. \mid 10, 1, 0.9) = \$0.2760$

The equivalent structure to Figure 5.2 is presented in the following diagram which gives the solution to the problem.

In words, they should buy the one observation with an expected net revenue of \$.0484. If the transition is from state 2 to state 2 (i.e., RR), they should not go fishing. However, if it is from state 2 to



$0.2484 is the expected net revenue after they have bought the one piece of information.

state 1 (i.e., RS), they should go fishing.

Indefinitely Long Use of the Process

Suppose that instead of being able to use the process for s periods in the steady state we can use it indefinitely long starting x periods from now in the steady state. (This is probably more meaningful in most practical situations.) The discount factor, α, is assumed known. Then, the analysis is identical to the above except we now use expected discounted present values instead of the expected net revenues. Utilizing a process with $E(R|m,n,a)$ would give a present value of rewards

$$p.v.(m, n, a) = E(R|m, n, a)[\alpha^{x} + \alpha^{x+1} + \ldots]$$

$$= \frac{\alpha^{x}}{1 - \alpha} E(R|m, n, a).$$

5.3. The Corresponding N-State Problem

It was demonstrated in section 5.1 that the determination of the expected values of the steady state probabilities is essential in obtaining the expected reward per period in the steady state, a quantity very useful for statistical decision purposes. For 2-state Markov processes we were able to show analytically (in sections 4.1.3 and 4.1.4) that for Beta distributed transition probabilities the expected values of the steady state probabilities are very insensitive to the variances of the Beta distributions (for fixed mean values). For 3 or more states with multidimensional Beta distributed transition probabilities we have arrived at the same conclusion by means of empirical results obtained through the use of simulation techniques (in sections 4.1.6 and 4.1.7). Consequently, for given multidimensional Beta distributions, we may quickly obtain a good approximation to the $E(\pi_j)$'s in the following manner: Assume that the transition probabilities are exactly known at the mean values of the multidimensional Beta distributions, then use these exact values to determine the corresponding steady state probabilities, $(\pi_j)_{ex}$'s, by standard techniques. Finally, use the $(\pi_j)_{ex}$'s as an approximation to the $E(\pi_j)$'s.

If the particular statistical problem requires extremely accurate values of the $E(\pi_j)$'s, it is always possible to obtain these by simulation techniques (as outlined in sections 4.1.6 and 4.1.7) if the $(\pi_j)_{ex}$'s are not a close enough approximation. However, being able to avoid simulation in a large scale problem would save considerable computation time.

It is worthwhile noting that, as

$$R(P) = \sum_{j=1}^{N} r_j \, \pi_j(P)$$

is a "pure" (as defined in Appendix Q) function of the p_{ij}'s, the theorem of section Q.2 of Appendix Q immediately tells us to not pay for observations if we cannot make the decision a function of the observed transitions.

5.3.1. Description of the Problem

Consider an N-state Markov process whose transition probabilities are multidimensional Beta distributed with parameters $M = (m_{ij})$. Let

c be the fixed cost per time period for using the process

r_j be the reward per time period for being in state j $(j=1, \ldots, N)$

and

d be the cost of observing each transition

Assume that the following three alternatives exist:

i) Stop entirely with zero net revenue.

ii) Do not observe the process anymore and decide to use it for s periods in the steady state.

iii) Buy observations of the next k transitions, then either stop or use the process (with no further experimentation) after the k transitions.

5.3.2. Analysis

Let B be the event that the actual transition sequence is known and the transition frequency count is $F = (f_{ij})$ where f_{ij} = number of transitions from state i to state j. For large k the number of such events possible becomes very large. The probability of event B can be determined by considering each row of F separately. Using the final result of Appendix D,

$$\mathrm{pr}(f_{i1}, \ldots, f_{iN} \text{ and known order}) = \frac{\beta(m_{i1}+f_{i1}, \ldots, m_{iN}+f_{iN})}{\beta(m_{i1}, m_{i2}, \ldots, m_{iN})}$$

i.e.,

$$\mathrm{pr}(f_{i1}, \ldots, f_{iN} \text{ and known order})$$

$$= \frac{\Gamma\left(\sum_{j=1}^{N} m_{ij}\right) \prod_{j=1}^{N} \Gamma(m_{ij}+f_{ij})}{\Gamma\left[\sum_{j=1}^{N} (m_{ij}+f_{ij})\right] \prod_{j=1}^{N} \Gamma(m_{ij})} \tag{5.5}$$

Then,

$$\mathrm{pr}[F \text{ and known order}] = \prod_{i=1}^{N} \mathrm{pr}(f_{i1}, \ldots, f_{iN} \text{ and known order}) \tag{5.6}$$

Knowing the event B, we can quickly determine the posterior density functions of the p_{ij}'s through the use of Bayes' rule. As outlined in section 3.4, the p_{ij}'s will still be multidimensional Beta distributed except that the parameters will now be $M + F = (m_{ij}+f_{ij})$. Thus we can easily calculate the expected reward for using the process after the event B has been observed.

The expected net revenue without experimentation (i.e., observations) is

$$E(N.R.\,|\,(m_{ij})) = \max_{S,U} \begin{cases} \underline{S}top & 0 \\ \\ \underline{U}se & s[E(R\,|\,(m_{ij})) - c] \end{cases} \tag{5.7}$$

where, as earlier,

$$E(R\,|\,(m_{ij})) = \sum_{t=1}^{N} r_t\, E(\pi_t\,|\,(m_{ij}))$$

is the expected reward per time period in the steady state given that the transition probabilities are multidimensional Beta distributed with parameters, $M = (m_{ij})$. (The actual calculation of $E(R\,|\,(m_{ij}))$ is described in the numerical example presented in the next section.)

The expected reward with experimentation is

$$E(N.R.\,|\,(m_{ij}); k) = \sum_{\text{all } B} pr(B)\ \max[0, s[E(R\,|\,(m_{ij}), B) - c]\,] - kd \tag{5.8}$$

If

$$E(N.R.\,|\,(m_{ij}); k) > E(N.R.\,|\,(m_{ij})),$$

we elect to observe the transitions, if not, we do not buy the observations.

To illustrate the difficulty in obtaining all possible B's and in calculating their probabilities, the following 3-state example is presented.

5.3.3. Numerical Example

A shady carnival man, Gary, always looking for an "honest" dollar, has asked his friend Mark if he would be interested in the following game: Gary has three dice — 1 red, 1 white and 1 blue. Each has two sides marked 1, two sides marked 2 and two sides marked 3. Anytime a 1 is rolled the red die is used for the next toss. Similarly, 2 and 3 force the use of the white or blue die, respectively.

Gary has made the game appear attractive by means of the following cost framework. He will charge Mark \$5.00 per roll, but will pay him \$4.00 for each 1 rolled, \$5.00 for each 2 and \$6.00 for each 3. A naive bystander quickly figures that $(4.00 + 5.00 + 6.00)/(3) = 5.00$ and hence the game is fair in the long run.

Gary realizes that the unknown bias of each die is quite important for later conquests and he doesn't want Mark to learn too much from playing. Furthermore, he will not give Mark the opportunity of starting with a particular die. Therefore, he will only let Mark play for 5 rolls and the first roll will only be after Gary himself has been rolling the dice (according to the prescribed mechanism) for quite some time (there is a third party present who will keep things honest).

To make things even more confusing, Gary has offered Mark the option of observing two rolls beginning with the red die at a cost of "d" per toss. Should Mark listen to the bystander and play the game? Should he take advantage of the option?

Unknown to Gary, Mark has seen a few rolls of the dice. Also, he is allowed to carefully scrutinize them. From these two sources of information he feels that he cannot state exact values for the transition probabilities, but he is willing to assume that they are multidimensional Beta distributed with the following parameters:

$$M = \begin{array}{c} \\ R \\ W \\ B \end{array}\begin{array}{c} \begin{array}{ccc} R & W & B \end{array} \\ \begin{bmatrix} 5 & 10 & 5 \\ 1 & 1 & 3 \\ 2 & 4 & 6 \end{bmatrix} \end{array}$$

Using the notation of section 5.3.1,

$$c = \$5.00 \qquad \underline{r} = [\$4.00 \quad \$5.00 \quad \$6.00] .$$

Without using the option Mark's expected net revenue (using equation (5.7)) is

$$E(N.R. \,|\, (m_{ij})) = \max_{S,U} \begin{cases} S & 0 \\ U & s[E(R \,|\, (m_{ij})) - c] \end{cases} \tag{5.9}$$

Now

$$\begin{aligned} s[E(R \,|\, (m_{ij})) - c] = s[&r_1 \, E(\pi_1 \,|\, (m_{ij})) + r_2 \, E(\pi_2 \,|\, (m_{ij})) \\ &+ r_3 \, E(\pi_3 \,|\, (m_{ij})) - c] \end{aligned} \tag{5.10}$$

At this stage we use the approximation (discussed earlier) that

$$E(\pi_j \,|\, (m_{ij})) \simeq (\pi_j)_{ex},$$

the exact steady state probability calculated from the transition matrix M' whose elements are the means of the multidimensional Beta distributions. That is,

$$M' = \begin{bmatrix} \frac{m_{11}}{m_{11}+m_{12}+m_{13}} & \frac{m_{12}}{m_{11}+m_{12}+m_{13}} & \frac{m_{13}}{m_{11}+m_{12}+m_{13}} \\ \frac{m_{21}}{m_{21}+m_{22}+m_{23}} & \frac{m_{22}}{m_{21}+m_{22}+m_{23}} & \frac{m_{23}}{m_{21}+m_{22}+m_{23}} \\ \frac{m_{31}}{m_{31}+m_{32}+m_{33}} & \frac{m_{32}}{m_{31}+m_{32}+m_{33}} & \frac{m_{33}}{m_{31}+m_{32}+m_{33}} \end{bmatrix}$$

Here

$$M' = \begin{bmatrix} 0.250 & 0.500 & 0.250 \\ 0.200 & 0.200 & 0.600 \\ 0.167 & 0.333 & 0.500 \end{bmatrix}$$

Using the steady state results of Appendix F

$$(\pi_1)_{ex} = 0.194, \ (\pi_2)_{ex} = 0.322, \ (\pi_3)_{ex} = 0.484$$

Therefore, from equation (5.10)

$$s[E(R|(m_{ij}) - c] \simeq 5[4(0.194) + 5(0.322) + 6(0.484) - 5.3]$$

$$= 5[-.010] = -\$.050 < 0.$$

Hence, from equation (5.9)

$$E(N.R.|(m_{ij})) = 0$$

and Mark should stop rather than playing the game without the option; i.e., Gary has duped the naive bystander (provided the latter agrees with Mark's prior parameters).

Suppose the option of observing two rolls starting with the red die is accepted. Then all the following events are possible:

B_1 111 $f_{11} = 2$, $f_{ij} = 0$ for all other i, j combinations.

B_2 112 $f_{11} = 1$, $f_{12} = 1$,

B_3 113 $f_{11} = 1$, $f_{13} = 1$,

B_4 121 $f_{12} = 1$, $f_{21} = 1$,

B_5 122 $f_{12} = 1$, $f_{22} = 1$,

B_6 123 $f_{12} = 1$, $f_{23} = 1$,

B_7 131 $f_{13} = 1$, $f_{31} = 1$,

B_8 132 $f_{13} = 1$, $f_{32} = 1$,

B_9 133 $f_{13} = 1$, $f_{33} = 1$,

Using equations (5.5) and (5.6)

$$pr(E_1) = \frac{\Gamma(20)\ \Gamma(7)\ \Gamma(10)\ \Gamma(5)}{\Gamma(22)\ \Gamma(5)\ \Gamma(10)\ \Gamma(5)} = \frac{6 \times 5}{21 \times 20} = .071$$

$$pr(B_2) = \frac{\Gamma(20)\ \Gamma(6)\ \Gamma(11)\ \Gamma(5)}{\Gamma(22)\ \Gamma(5)\ \Gamma(10)\ \Gamma(5)} = \frac{5 \times 10}{21 \times 20} = .119$$

$$pr(B_5) = \frac{\Gamma(20)\ \Gamma(5)\ \Gamma(1)\ \Gamma(5)}{\Gamma(21)\ \Gamma(5)\ \Gamma(10)\ \Gamma(5)} \times \frac{\Gamma(5)\ \Gamma(1)\ \Gamma(2)\ \Gamma(3)}{\Gamma(6)\ \Gamma(1)\ \Gamma(1)\ \Gamma(3)}$$

$$= \frac{10}{20} \times \frac{1}{5} = .100$$

similarly,

$pr(B_3) = .060$

$pr(B_4) = .100$

$pr(B_6) = .300$

$pr(B_7) = .042$

$pr(B_8) = .083$

$pr(B_9) = .125$

After each experimental outcome we have a new matrix of parameters given by M + F; e.g., after B_1,

$$M_1 = \begin{bmatrix} 5 & 10 & 5 \\ 1 & 1 & 3 \\ 2 & 4 & 6 \end{bmatrix} + \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 7 & 10 & 5 \\ 1 & 1 & 3 \\ 2 & 4 & 6 \end{bmatrix}$$

Using M_1, we proceed exactly as we did with M to obtain

$$s[E(R \mid (m_{ij}), B_1) - c] ,$$

Mark's expected return for playing the game after he has observed two R-R transitions. This is repeated for each experimental outcome. The results of the calculations are shown in Figure 5.3.

From that figure it is seen that it pays for Mark to use the option of observing two rolls starting with the red die if

$$.0325 - 2d > 0$$

i.e.,

$$d < \$.01625$$

If $d > \$.01625$, he should refuse both the option and the game. If he does buy the option, the diagram also illustrates that he should

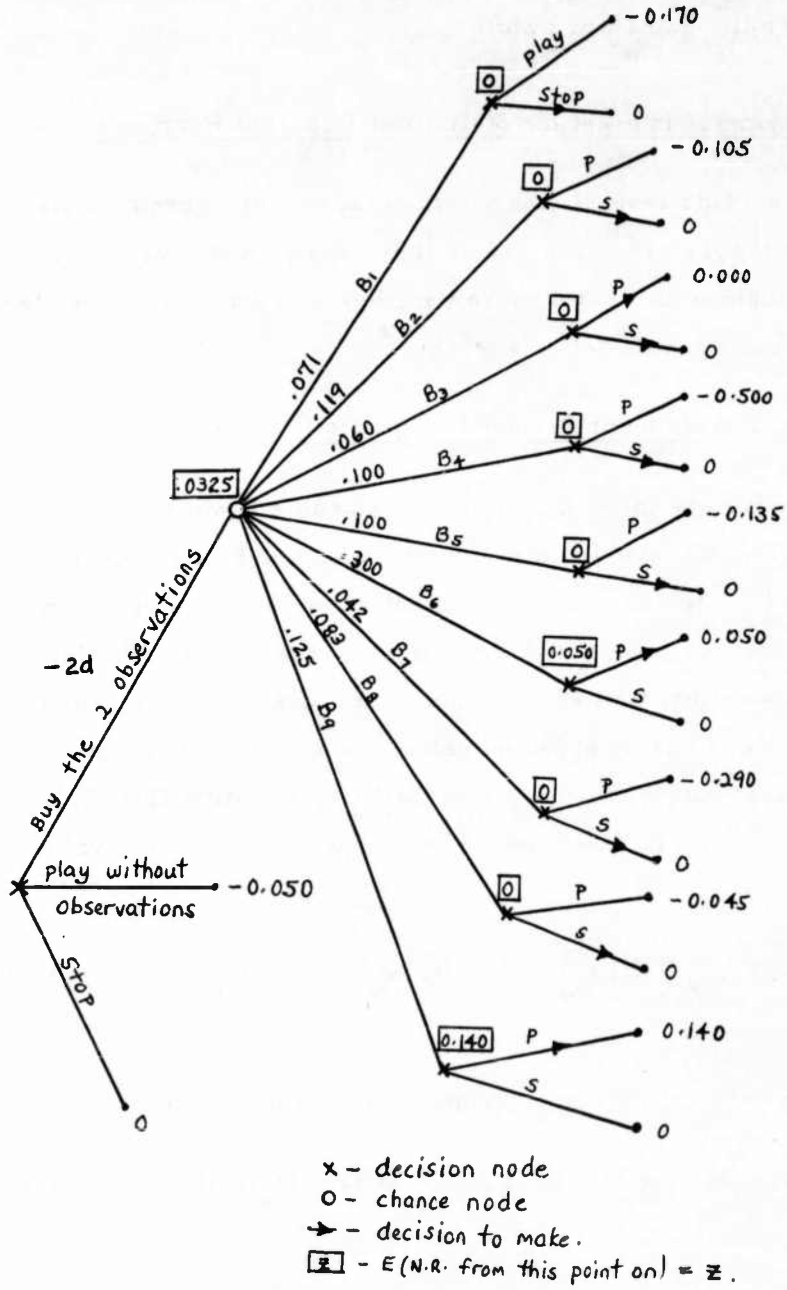


Figure 5.3. A 3-state statistical decision problem.

play Gary's game only if B_6(RWB) or B_9(RBB) occurs. These two outcomes modify the a priori parameters sufficiently in a favorable direction to make the game worthwhile.

5.4. Further Items of Interest for Statistical Decision Purposes

In this section several other items that are of interest in statistical decision theory are presented as they relate to the Markov process problems considered. For a more detailed background, the reader is referred to Raiffa and Schlaifer's work.[20]

5.4.1. The Value of Perfect Information

Suppose we are faced with a decision problem where there is a random variable, x, involved with density function $f_x(x_o)$. Assume that we are able to calculate the expected net revenue of the optimum decision procedure, $E(N.R.|f_x(x_o)$ knowing only the density function of x. Now, suppose that we were told the value of x, say x_o. Then there would be an associated expected net revenue of the optimum decision procedure knowing that $x = x_o$. Call this quantity $E(N.R.|x_o)$. Then, the a priori expected net revenue before we were told the value of x would be given by

$$E(N.R.|P.I.) = \int_{x_o} E(N.R.|x_o)\, f_x(x_o)\, dx_o \qquad (5.11)$$

and the expected value of perfect information is defined to be

$$E.V.P.I. = E(N.R.|P.I.) - E(N.R.|f_x(x_o)) \qquad (5.12)$$

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[20] Op. cit., Chapters 4 and 5.

If confronted with a proposition where we would be told the exact value of x for z dollars, we would accept if and only if $E.V.P.I. \geq z$.

This concept is simplest to illustrate for the 2-state decision problem considered in section 5.2.1. We have a 2-state Markov process with transition matrix

$$P = \begin{bmatrix} 1 - a & a \\ b & 1 - b \end{bmatrix}$$

where "a" is known exactly, but $f_b(b_o) = f_\beta(b_o | m, n)$. As before, let c be the fixed cost per period for using the process, r_j be the reward per time period for being in state j $(j=1, 2)$, and d be the cost of observing a transition from state 2.

If it is desirable, we can use the process for s periods in the steady state. Also, we have the option of buying the right to observe k transitions from state 2.

In section 5.2.3 it was shown that the expected net revenue following an optimum procedure is

$$E(N.R. | f_b(b_o)) = \max[0; E(N.R. | m, n, a); E(N.R. | m, n, a; k)] \tag{5.13}$$

To evaluate $E(N.R. | P.I.)$, Appendix R shows that we proceed as follows:

<u>Case i. $r_1 < c < r_2$</u>

$$E(N.R. | P.I.) = \int_0^{(r_2-c)a/(c-r_1)} s\left[r_1 - c + (r_2-r_1)\frac{a}{a + b_o}\right] \cdot f_\beta(b_o | m, n)\, db_o \tag{5.14}$$

This would require numerical integration or an extremely involved analytic integration. Finally, equations (5.13) and (5.14) would be used to evaluate

$$E.V.P.I. = E(N.R.\,|\,P.I.) - E(N.R.\,|\,f_b(b_o)) \qquad (5.15)$$

<u>Case ii. $r_2 < c < r_1$</u>

$$E(N.R.\,|\,P.I.) = \int_{(c-r_2)a/(r_1-c)}^{1} s\left[r_1 - c + (r_2 - r_1)\frac{a}{a+b_o}\right] \cdot f_\beta(b_o\,|\,m,n)\,db_o \qquad (5.16)$$

Then proceed as above.

<u>Case iii. $c <$ both r_1 and r_2 or $c >$ both r_1 and r_2</u>

If $c <$ both r_1 and r_2, we would always use the process, hence, the E. V. P. I. = 0. If $c >$ both r_1 and r_2, we would never use the process, hence the E. V. P. I. = 0.

<u>Numerical Example</u>

Consider, again, the Pierre-Louie fishing problem of section 5.2.3. Therefore,

$$a = 0.9,\ m = 9,\ n = 1,\ k = 1,\ d = 0.2,\ s = 5$$

$$c = \$40,\ r_1 = \$60,\ r_2 = \$20.$$

What is the expected value (to the fishermen) of perfect information about the conditional probability, "b", that day n + 1 will be sunny given that day n is sunny?

It was found that $E(N.R. | f_b(b_o)) = \$.0484$. The c and r's are seen to correspond to Case ii above. Therefore, from equation (5.16)

$$E(N.R. | P.I.) = \int_{20(0.9)/(20)}^{1} 5 \left[\$20 - \$40 \frac{0.9}{0.9 + b_o}\right] \frac{1}{\beta(9,1)} b_o^8 \, db_o$$

$$= 900 \int_{0.9}^{1} \left[b_o^8 - \frac{1.8 \, b_o^8}{0.9 + b_o}\right] db_o$$

To integrate the second term we use the substitution, $y = 0.9 + b_o$, then expand the resulting $(y-0.9)^8$ term in the numerator. We obtain

$$E(N.R. | P.I.) = \$2.7648$$

Hence, from equation (5.15)

$$E.V.P.I. = 2.7648 - .0484 = \$2.7164$$

This is the maximum amount that Pierre and Louie would be willing to pay in return for learning the exact value of "b". (The transition probability out of the rainy state.) Note that the $E(N.R. | f_b(b_o))$ value of \$.0484 is conditional upon the fact that in section 5.2.3 they were allowed to pay for observation of one transition from state R. Hence, the above E.V.P.I. value is also conditional upon that fact. If the option of observing was not available, $E(N.R. | f_b(b_o))$ would be zero as shown in section 5.2.3 and in this case

$$E.V.P.I. = \$2.7648 - 0 = \$2.7648$$

These ideas can be extended to situations where we want to know the value of perfect information concerning 2 or more random variables. Presumably we would already know

$$E(N.R. \mid f_{x_1, x_2, \ldots, x_k}(y_1, y_2, \ldots, y_k)),$$

the expected net revenue following an optimum policy when only the density function (and not the exact values) of the random variables $x_1, x_2, \ldots, x_k$ is known. Then, equations (5.11) and (5.12) become

$$E(N.R. \mid P.I.) = \int_{y_1} \ldots \int_{y_k} E(N.R. \mid y_1, y_2, \ldots, y_k)$$

$$\cdot f_{x_1, \ldots, x_k}(y_1, \ldots, y_k)\, dy_1 \ldots dy_k$$

and

$$E.V.P.I. = E(N.R. \mid P.I.) - E(N.R. \mid f_{x_1, \ldots, x_k}(y_1, y_2, \ldots, y_k))$$

However, in practice the above integration would usually be very difficult to perform.

5.4.2. The Choice of the Number of Observations to Take

Suppose that we are given a choice as to the number (k) of observations that we can buy. However, the stipulation is added that we must decide on the exact number before any observations are made. Physical constraints or an opponent could place us in this position. This, in effect, prevents the problem from becoming sequential in nature. (A sequential decision structure will be treated in the next section.)

In principal, at least, the optimum number can be determined in the following way. For any particular k we know, from earlier in this study, how to determine the expected net revenue (including the cost of observation) assuming that the observations will be made. Although it has not been proved, it seems reasonable to assume that the expected net revenue would be a unimodal function of the number of observations for the decision framework and costs considered. Hence, we determine the expected revenue for a number of values of k until the peak is detected. The corresponding k should be the optimum.

Numerical Example

For the same fishing example as that considered in sections 5.2.3 and 5.4.1, the expected net revenue as a function of the number of observations (transitions from the rainy state) is as follows:

k	Decision	E(N.R. | k)
0	Stop	0
1	Buy observation	$.0484
2	Buy observations	$.1956
3	Buy observations	$.2280
4	Buy observations	$.1858
5	Buy observations	$.0916
6	Stop	0
7	Stop	0
8	Stop	0
9	Stop	0
10	Stop	0

The behavior of the expected net revenues is depicted in Figure 5.4. In this example it is apparent that the expected net revenue is a unimodal function of the number of observations. The optimum number of

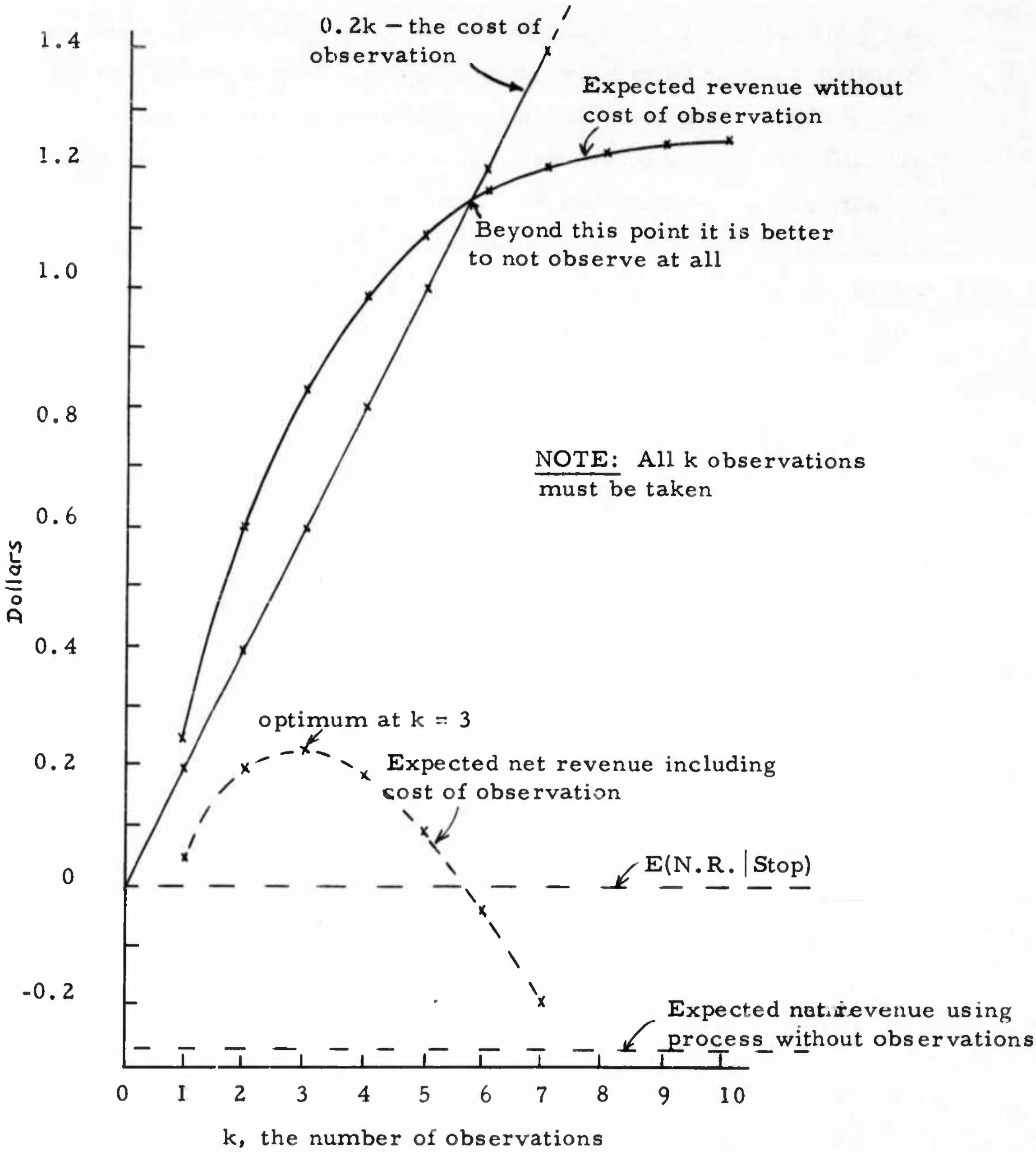

Figure 5.4. Numerical example of the choice of the optimum number of observations.

observations of transitions from rainy days is three.

5.4.3. A Sequential Decision Structure

Now we make the decision structure more flexible by dropping the assumption that all k observations must be made; i.e., we can now stop observing after any number of observations up to k. However, k is still assumed known and fixed (perhaps due to a deadline on the time of our decision to use or not to use the process). The solution of the new sequential decision problem is accomplished simply by the solution of a number of old single observation problems. Although the method will work for the N-state case, it is easiest to present using the 2-state problem where "a" is known exactly and "b" is Beta distributed. In fact, the problem considered will be identical to that of section 5.2.3 except, as stipulated above, we can now stop observing after any number of observations up to k.

Define $v_j(m, n)$ = expected net return if there are j possible observations left, the parameters of the Beta distribution are m and n, and an optimum policy is followed. Then

$$v_j(m,n) = \max_{O,\,U,\,S} \begin{cases} \underline{O}\text{bserve} & \dfrac{m}{m+n} v_{j-1}(m+1,n) + \dfrac{n}{m+n} v_{j-1}(m,n+1) - d \quad j \geq 1 \\ \underline{U}\text{se} & E(N.R. \mid m,n,a) \\ \underline{S}\text{top} & 0 \end{cases} \tag{5.17}$$

and

$$v_o(m,n) = \max_{U,S} \begin{cases} U & E(N.R. \mid m, n, a) \\ S & 0 \end{cases} \qquad (5.18)$$

The determination of $v_o(m,n)$ is seen to be identical to finding whether or not to use the process when no observation is possible. This was done in section 5.2.3. Then equation (5.17) is used recursively to solve the rest of the problem. For a particular (j, m, n) triplet, the solution of equation (5.17) is identical to solving a single observation problem except that $E(N.R. \mid m+1, n, a)$ and $E(N.R. \mid m, n+1, a)$ are replaced by $v_{j-1}(m+1, n)$ and $v_{j-1}(m, n+1)$, respectively.

It should be noted that because of the sequential nature of the decision making we now update the Beta distribution on "b" after each observation. In an N-state problem we would have to update the appropriate multidimensional Beta distribution after each observation.

Numerical Example

We again consider the same fishing example as in sections 5.2.3, 5.4.1 and 5.4.2. However, now we assume that Pierre and Louie can stop buying information about days following rainy days after 1 or 2 observations. The corresponding decision tree is shown in Figure 5.5. As would be anticipated, the expected net revenue of \$.2152 is higher than the value of \$.1956 found in section 5.4.2, where they were forced to buy both observations at once.

5.5. Statistical Decisions for a Transient Situation

In section 3.4 we outlined how to determine the multi-step transition probabilities when the p_{ij}'s are random variables instead of being exactly known. It was shown that the method was most practical

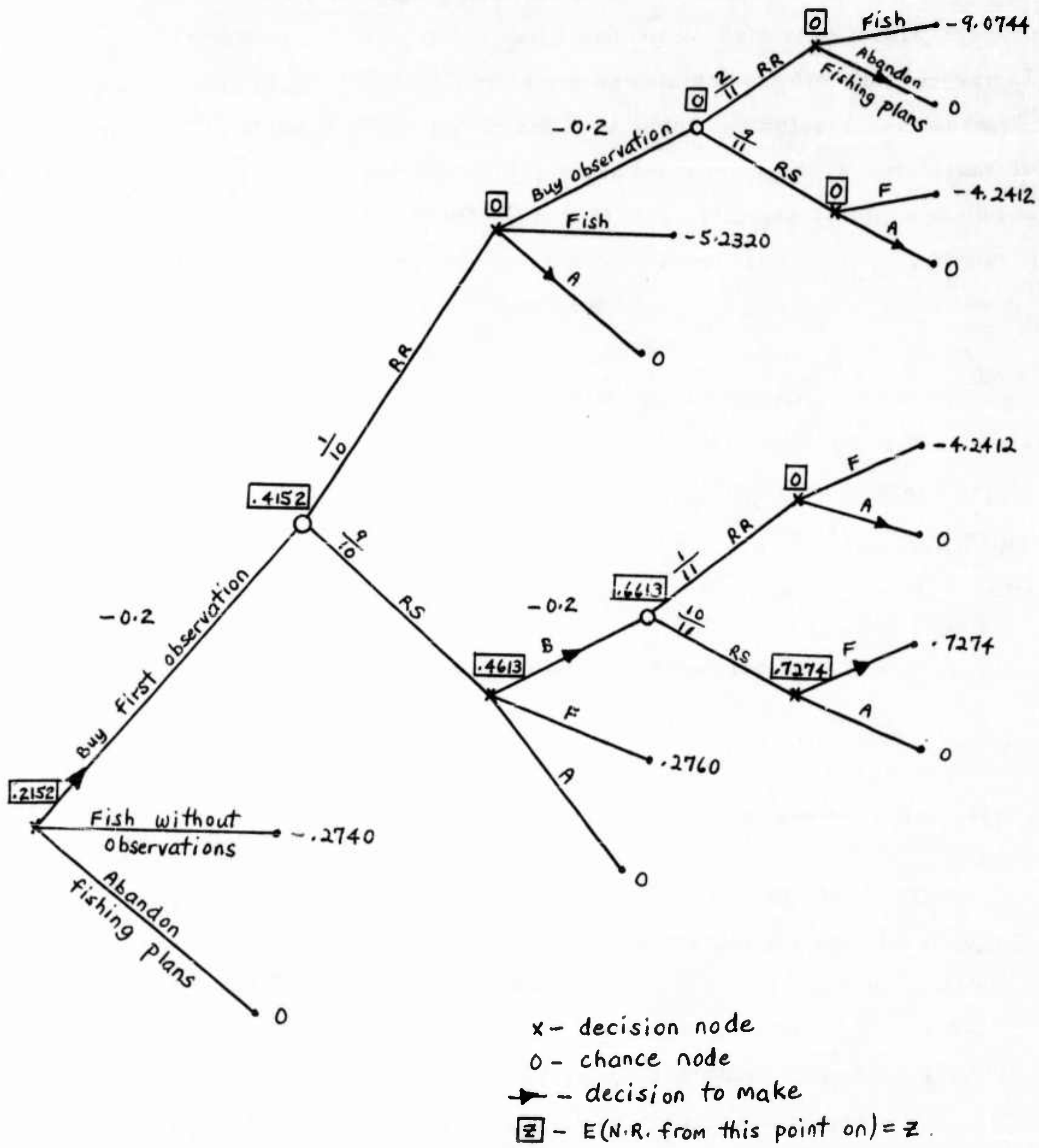

Figure 5.5. A 2-state sequential decision problem.

for 2-state processes. Hence, we shall restrict our attention here to 2-state situations.

The decision framework is the following. We have a 2-state process with both transition probabilities independently Beta distributed. The costs and rewards are all exactly known. The crucial question is: "What is the expected net revenue in the next h time periods given that we shall have a transition from state i just before the first period?" Let us denote this quantity by $E(N.R. \mid h, i, (m_{ij}))$ where (m_{ij}) are the parameters of the Beta distributions. Then

$$E(N.R. \mid h, i, (m_{ij})) = \sum_{n=1}^{h} \sum_{j=1}^{2} \overline{\Phi}_{ij}(n \mid (m_{ij}))\, r_j - hc \tag{5.19}$$

where $\overline{\Phi}_{ij}(n \mid (m_{ij}))$ = probability that the process will be in state j at time n | the state at time 0 is i and the parameters of the Beta distributions are (m_{ij}); and, as earlier

r_j = reward per period for being in state j

and

c = cost per period for using the process.

Now there is no need to elaborate in detail on the statistical decision problems for a transient situation because the methods are essentially identical to those for the cases studied earlier where the process was to be used in the steady state. We merely replace the $E(N.R. \mid (m_{ij}))$ of the steady state problem by $E(N.R. \mid h, i, (m_{ij}))$. There are other obvious minor modifications such as reducing h to h - 1 if we delay using the process for 1 period. Hence, we can handle all the following situations for a 2-state transient problem where the transition probabilities are independently Beta distributed:

i) Determination of the expected net revenue in the next h periods given that we are presently in state i.

ii) Study of whether or not to delay and pay for observations before deciding to use or not use the process (including the possibility of sequential decisions).

iii) Evaluation of the optimum number of observations to make before deciding whether or not to use the process.

iv) Determination of the expected value of perfect information about one or both of the transition probabilities.

Although the transient statistical decision problem has been so lightly covered due to the fact that its solution is very similar to that of the steady state problem, its practical importance should not be underestimated. In fact, the transient situation is probably more likely to occur in the real world than the steady state problems considered. An even more realistic extension would be the situation where the process could be used indefinitely long with discounting starting at the very next transition. Also, we have not considered the situation where once we have decided to use a process we can change our strategy (i.e., make further decisions) as we observe the process in operation during the time in which we are using it. This problem is of an adaptive control nature.

5.6. Summary

The statistical decision framework developed for a Markov process whose transition probabilities are not assumed exactly known can best be summarized with the aid of the block diagram of Figure 5.6.

In box 1 we place multidimensional Beta distributions over the transition probabilities of the Markov process considered. The procedure for doing this was discussed in section 3.5.

Next, if the process will be used in the steady state, the

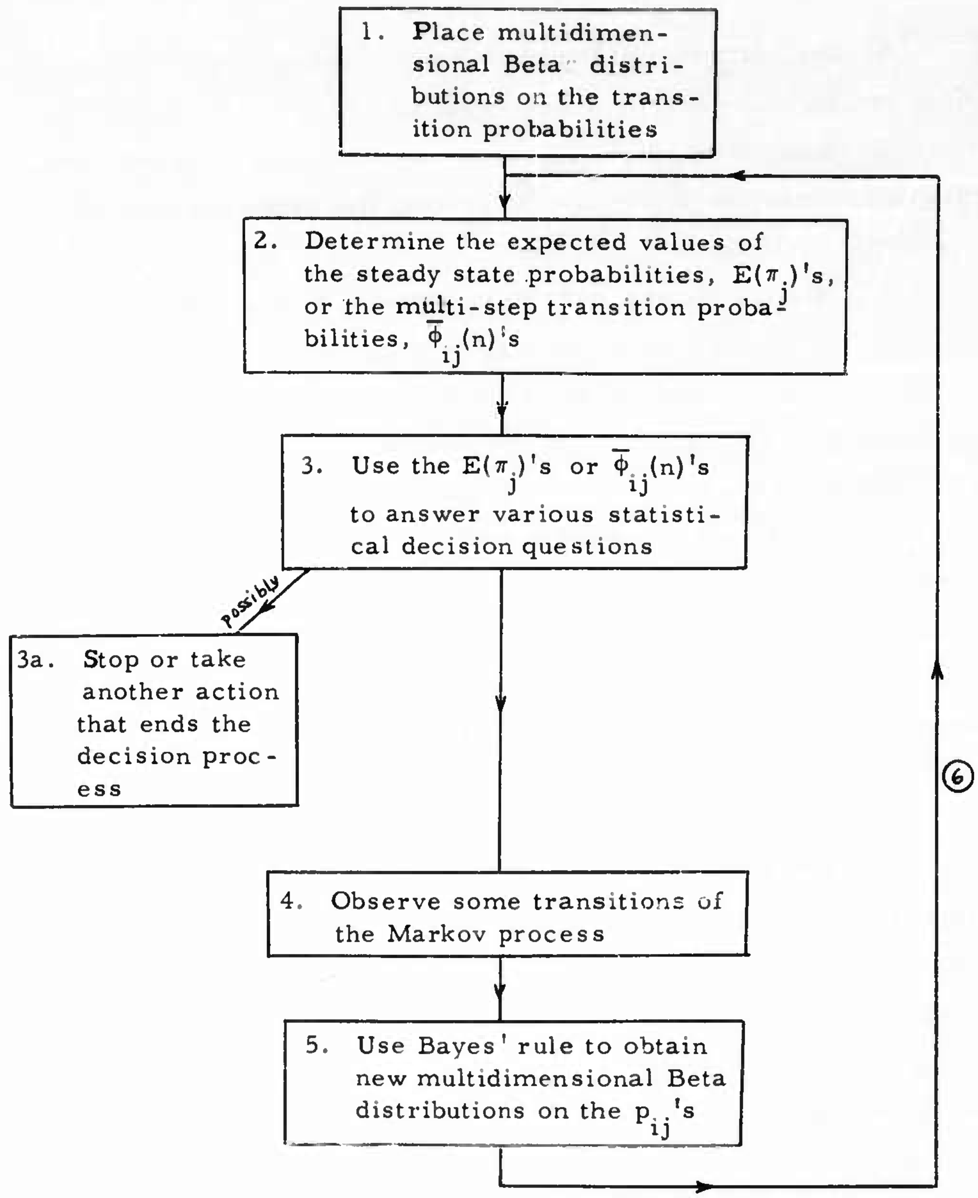

Figure 5.6. The statistical decision framework for a Markov process whose transition probabilities are not assumed exactly known.

expected values of the steady state probabilities are determined as outlined in section 4.1. For the special 2-state process where one transition probability is exactly known and the other is Beta distributed this can be done analytically (section 4.1.3). For more than 2- states we require either simulation or the approximation technique of using $(\pi_j)_{ex}$ for $E(\pi_j)$, discussed in sections 4.1.6 and 4.1.7. If the transient behavior is important, we obtain the multi-step transition probabilities, $\overline{\Phi}_{ij}(n)$, by the methods of section 4.4.

Box 3 is concerned with the use of the $E(\pi_j)$'s or $\overline{\Phi}_{ij}(n)$'s to answer various statistical decision questions (as discussed at length in the present chapter). Usually we would first evaluate the expected net revenues under various policies, then select the policy that maximizes the expected net revenue. Several other items can be obtained including the optimum number of observations to take and the expected value of perfect information about one or more of the unknown transition probabilities.

Having decided on the policy to use, we either terminate the decision process or allow the Markov process to run for one or more transitions as noted in box 4. When transitions are observed, modification of the prior distributions through the use of Bayes' rule is extremely simple because of their form (multidimensional Betas). As illustrated in section 3.4, the posterior distributions are again multidimensional Betas, only the parameters have been modified.

Now, because the transition probabilities are still multidimensional Beta distributed, we are justified in drawing the feedback (6) to box 2. The cycle is complete. We are again ready to calculate either the $E(\pi_j)$'s or the $\overline{\Phi}_{ij}(n)$'s by the exact same method, then move on to box 3 for new statistical decisions, etc.

Hence, utilization of multidimensional Beta prior distributions over the transition probabilities has enabled us to place Markov processes into a most convenient statistical decision framework.

It should be mentioned that besides their uses in the decision problems considered in this chapter, the expected rewards in various periods also allow direct comparison of the future values of several possible Markov processes if such a comparison is of interest.

CHAPTER 6

A BRIEF LOOK AT CONTINUOUS TIME PROCESSES

A continuous time Markov process is defined as follows. Given that the process is now in state i, in the next time interval dt it will make a transition to state j with probability $a_{ij}\,dt(i\neq j)$. (This implies that the state occupancy times are exponentially distributed.) For small enough dt the probability of two or more transitions is assumed to be zero. The quantity $a_{ij}(i\neq j)$ is called the transition rate from state i to state j . The process can be completely described by a matrix $A = (a_{ij})$ where we define

$$a_{ii} = -\sum_{j\neq 1} a_{ij}.$$

The primary objective of this chapter is to show that essentially the same approach as was used to incorporate uncertainty as to the values of the transition probabilities in a discrete-time process can also be utilized to take into account uncertainty as to the values of the transition rates in a continuous time process. Consequently, the treatment will be rather brief; in fact, we shall only look at the determination of the expected values of the steady state probabilities in a 2-state process and then make some general remarks about other aspects of the continuous time problem.

6.1. The Use of Gamma Prior Distributions on the Transition Rates of a 2-State Process

6.1.1. Justification for the Use of the Gamma Priors

Consider one of the states. As was mentioned earlier, the occupancy times within that state are exponentially distributed. More precisely, let t_i be the occupancy time of the state considered. Then,

$$f_{t_i|a_{ij}}(t_o|a_o) = a_o e^{-a_o t_o} \qquad 0 \le t_o$$

where a_{ij} is the transition rate from the state considered to the other state. But this is the special case of the Gamma distribution

$$f_{t|\beta}(t_o|a_o) = f_\gamma(t_o|\alpha, a_o) = \frac{a_o^\alpha t_o^{\alpha-1}}{\Gamma(\alpha)} e^{-a_o t_o} \qquad 0 \le t_o \qquad (6.1)$$

when $\alpha = 1$. Now, as shown in section S.1 of Appendix S, it is convenient for Bayes calculations to place a Gamma prior on a_o when the distribution of t is as in equation (6.1) and α is known exactly. Hence, it is advisable to use the Gamma prior on a_{ij} in the 2-state problem; i.e., we choose

$$f_{a_{12}}(x) = f_\gamma(x|v_1, w_1)$$

and (6.2)

$$f_{a_{21}}(y) = f_\gamma(y|v_2, w_2)$$

6.1.2. Bayes Modification of the Gamma Prior

Let E_1 be the event that in k_1 occupancies of state 1 the total occupancy time is T_1. Then, using the results of section S.3 of Appendix S we know that

$$f_{a_{12}}(x|E_1) = f_\gamma(x|v_1+k_1, w_1+T_1),$$

a new Gamma distribution. Similarly, if E_2 represents the event that in k_2 occupancies of state 2 the total occupancy time is T_2, then

$$f_{a_{21}}(y|E_2) = f_\gamma(y|v_2+k_2, w_2+T_2)$$

6.1.3. Selection of the A Priori Parameters

For

$$f_{t_1|a_{12}}(t_o|a_o) = a_o e^{-a_o t_o} \qquad 0 \le t_o$$

and

$$f_{a_{12}}(a_o) = f_\gamma(a_o|v_1, w_1)$$

it is shown in section S.2 of Appendix S that the marginal distribution of t_1 is

$$f_{t_1}(t_o) = \frac{v_1 w_1^{v_1}}{(t_o+w_1)^{v_1+1}} \qquad 0 \le t_o$$

and

$$E(t_1) = \frac{w_1}{v_1 - 1}$$

and

$$\check{t}_1 = \frac{v_1 w_1^2}{(v_1 - 1)^2 (v_1 - 2)} \qquad v_1 \geq 3$$

Using these last two equations we can select v_1 and w_1 so as to satisfy our prior estimates of $E(t_1)$ and $\check{t}_1$. Similar reasoning would hold for the evaluation of the two parameters of the prior Gamma distribution on the other transition rate, a_{21}.

6.2. Determination of the Expected Values of the Steady State Probabilities of a 2-State Process When the Two Transition Rates are Independently Gamma Distributed

Consider the 2-state process with transition rate matrix

$$A = \begin{bmatrix} -a_{12} & a_{12} \\ a_{21} & -a_{21} \end{bmatrix}$$

where

$$f_{a_{12}}(r) = f_\gamma(r | v_1 w_1)$$

and (6.3)

$$f_{a_{21}}(s) = f_\gamma(s | v_2, w_2)$$

For a_{12} and a_{21} exactly known it can easily be shown that the steady state probabilities are given by

$$\pi_1 = \frac{a_{21}}{a_{12} + a_{21}}$$

and (6.4)

$$\pi_2 = \frac{a_{12}}{a_{12} + a_{21}}$$

However, as a_{12} and a_{21} are random variables, π_1 and π_2 are now random variables. Again, the following mechanism will be used. We randomly select an (a_{12}, a_{21}) pair from their distributions and determine the associated exactly known steady state probabilities. If this is repeated a large number of times, we would like to know the expected values of π_1 and π_2. Interestingly enough we can evaluate the entire density functions as well as the expected values.

If π_1 and π_2 are as defined in equation (6.4) and a_{12} and a_{21} are distributed as in equation (6.3), then using the theory of derived distributions the density functions of π_1 and π_2 are

$$f_{\pi_1}(x) = \frac{w_1^{v_1} w_2^{v_2}}{\beta(v_1, v_2)} \frac{x^{v_2-1}(1-x)^{v_1-1}}{[w_1 + (w_2-w_1)x]^{v_1+v_2}} \qquad 0 \leq x \leq 1$$

and (6.5)

$$f_{\pi_2}(z) = \frac{w_1^{v_1} w_2^{v_2}}{\beta(v_1, v_2)} \frac{z^{v_1-1}(1-z)^{v_2-1}}{[w_2 + (w_1-w_2)z]^{v_1+v_2}} \qquad 0 \leq z \leq 1$$

For $w_1 = w_2$ these are seen to reduce to $f_\beta(x|v_2, v_1)$ and $f_\beta(z|v_1, v_2)$, respectively, a well known result. In that case we can directly state that

$$E(\pi_1) = \frac{v_2}{v_1 + v_2}$$

and

$$E(\pi_2) = \frac{v_1}{v_1 + v_2}$$

However, if we force the q's to be equal, we lose one degree of freedom and we then have only 3 a priori parameters with which to satisfy the prior means and variances of the occupancy times (4 quantities).

If $w_1 > w_2$, from equation (6.5)

$$E(\pi_1) = \frac{w_1^{v_1} w_2^{v_2}}{\beta(v_1, v_2)} \int_0^1 \frac{x^{v_2}(1-x)^{v_1-1}}{[w_1 + (w_2-w_1)x]^{v_1+v_2}} dx$$

$$= \frac{(w_2/w_1)^{v_2}}{\beta(v_1, v_2)} \int_0^1 x^{v_2}(1-x)^{v_1-1} \left[1 - \left(1 - \frac{w_2}{w_1}\right)x\right]^{-(v_1+v_2)} dx$$

$$= \left(\frac{w_2}{w_1}\right)^{v_2} \frac{v_2}{v_1 + v_2} \frac{\Gamma(v_1+v_2+1)}{\Gamma(v_1)\,\Gamma(v_2+1)} \int_0^1 x^{v_2+1-1} (1-x)^{v_1-1}$$

$$\cdot \left[1 - \left(1 - \frac{w_2}{w_1}\right)x\right]^{-(v_1+v_2)} dx$$

and using equation (G.2) of Appendix G, which defines the hypergeometric function F,

$$E(\pi_1) = \left(\frac{w_2}{w_1}\right)^{v_2} \frac{v_2}{v_1+v_2} F\left(v_1+v_2, v_2+1 \middle| v_1+v_2+1 \middle| \underbrace{1-\frac{w_2}{w_1}}\right)$$

$$\left(\text{convergent because } w_1 > w_2\right) \tag{6.6}$$

Then

$$E(\pi_2) = 1 - E(\pi_1).$$

If $w_2 > w_1$, we evaluate $E(\pi_2)$ first by

$$E(\pi_2) = \left(\frac{w_1}{w_2}\right)^{v_1} \frac{v_1}{v_1+v_2} F\left(v_1+v_2, v_1+1 \middle| v_1+v_2+1 \middle| 1-\frac{w_1}{w_2}\right) \tag{6.7}$$

and then use $E(\pi_1) = 1 - E(\pi_2)$.

6.3. General Remarks

We now have precisely the same framework as in the 2-state discrete case except the two transition rates are independently Gamma distributed, whereas the two transition probabilities were independently Beta distributed. Hence, we can determine many of the quantities that were obtained in the discrete case — e.g., the expected revenue per unit time in the steady state, the worthwhileness (preposterior analysis) of observing several transitions, the value of knowing a transition rate exactly, etc. Also, for more than 2 states the same analytic difficulties as those in the discrete situation would be encountered.

It should be noted that in the discrete case the hypergeometric function was required for the situation where only one transition probability was not known exactly; we did not obtain a closed form solution for the expected values of the steady state probabilities (other than a complicated summation) for the situation where the two transition probabilities were independently Beta distributed. However, in the continuous case use of the hypergeometric function has enabled us to obtain simple closed form expressions for the expected values of the steady state probabilities when <u>both</u> transition rates are independently Gamma distributed.

SECTION II

TRANSITION PROBABILITIES EXACTLY KNOWN BUT REWARDS ARE NOW RANDOM VARIABLES

In Section I it was assumed that the rewards were exactly known but that the transition probabilities (or rates or matrices) were random variables. Now, we consider the completely opposite situation; the transition probabilities are assumed exactly known but the rewards are now random variables. This new situation turns out to be easier to handle in most respects than the ones considered in Section I.

The first step is to develop convenient prior distributions to use on the rewards, convenient in the sense that they allow easy Bayes modification and simple determination of the expected values of the rewards, and their ranges satisfy the physical constraints. Also, the actual Bayes modification of a prior distribution on a reward after several sample rewards have been observed is demonstrated for two different prior distributions. Then we consider the expected rewards in steady state and transient situations, very important quantities for statistical decision purposes. Knowing how to evaluate these quantities for appropriate prior distributions on the rewards we next deal with typical statistical decision problems based on either steady state or transient situations.

CHAPTER 7

CONVENIENT PRIOR DISTRIBUTIONS TO USE ON THE REWARDS

As was the case for the $E(\pi_j)$'s in Section I, it will be demonstrated in Chapter 8 that the expected values of the rewards, $E(r_{ij})$'s, are the critical quantities to know for many Markov decision purposes. (r_{ij} = reward per transition from state i to state j.) Hence, it is important that a prior distribution on an r_{ij} allows easy determination of $E(r_{ij})$. As in Section I, the other two required properties of the prior distribution on an r_{ij} are:

i) The distribution must be convenient for Bayes calculations. Ideally, after some sample values of r_{ij} are observed we would like to obtain a posterior distribution on r_{ij} which is a member of the same family as the prior distribution.

ii) The range of the distribution must satisfy actual physical constraints. Two physical situations will be considered in detail; the first where a reward can lie anywhere between 0 and ∞, the second where the range is $-\infty$ to ∞. A third situation where the range is finite, will be briefly mentioned.

7.1. The Range of "r" is $(0, \infty)$ — Exponential-Gamma Form

7.1.1. Determination of the Form of the Prior Distribution

The probability density function of a random variable having this range can often be adequately described by an exponential distribution whose parameter is in turn a random variable. That is,

$$f_{r|\lambda}(r_o|\lambda_o) = \lambda_o e^{-\lambda_o r_o} \qquad 0 \le r_o \tag{7.1}$$

where λ, in turn, has a density function $f_{\lambda}(\lambda_o)$. This latter density function is the one that must be chosen so as to satisfy the requirements of easy determination of $E(r)$ and simple Bayes modification.

It should be recognized that a Gamma distribution on r given λ would allow greater flexibility than the exponential distribution of equation (7.1). However, as demonstrated in Appendix S, use of a Gamma distribution instead of an exponential leads to serious difficulties in assigning the prior parameters of the distribution of λ. Hence, we sacrifice some flexibility in order to simplify the assignment of the prior parameters.

The situation here is identical to that considered in Chapter 6 where we had exponential holding times whose parameters (the transition rates) were in turn random variables. There, with the use of Appendix S, we found that it was most convenient to have each transition rate Gamma distributed. Hence, in the present context the λ parameter of equation (7.1) should be Gamma distributed. That is,

$$f_{\lambda}(\lambda_o) = f_{\gamma}(\lambda_o | v, w) = \frac{w^v}{\Gamma(v)} \lambda_o^{v-1} e^{-w\lambda_o} \qquad 0 \leq \lambda_o \tag{7.2}$$

7.1.2. The Marginal Distribution of "r" and its Mean and Variance

Section S.2 of Appendix S shows that for the conditions of equations (7.1) and (7.2), the marginal density function of r is

$$\begin{aligned} f_r(r_o) &= \int_0^{\infty} f_{r|\lambda}(r_o | \lambda_o)\, f_{\lambda}(\lambda_o)\, d\lambda_o \\ &= \frac{v w^v}{(r_o + w)^{v+1}} \qquad 0 \leq r_o \end{aligned} \tag{7.3}$$

The expected value of r is

$$E(r) = \frac{w}{v - 1} \quad \text{for} \quad v \geq 2 \tag{7.4}$$

and the variance is

$$\check{r} = \frac{vw^2}{(v-1)^2 (v-2)} \quad \text{for} \quad v \geq 3 \tag{7.5}$$

7.1.3. Determination of the A Priori Parameters

Using equations (7.4) and (7.5) we can select v and w so as to satisfy our prior estimates of E(r) and $\check{r}$. These latter two marginal quantities are easier to estimate a priori than the moments of the parameter λ itself. The reason that this is mentioned is that the mean and variance of λ can be expressed in terms of v and w and an alternate method of obtaining values for v and w would be to select the values such that the estimates of the mean and variance of λ were satisfied.

7.1.4. Bayes Modification of the Gamma Prior Distribution

Consider the variable r with

$$f_{r|\lambda}(r_o|\lambda_o) = \lambda_o e^{-\lambda_o r_o} \qquad 0 \leq r_o$$

where

$$f_\lambda(\lambda_o) = f_\gamma(\lambda_o | v, w)$$

Let E represent the event that k independent values of r sum to T_k. Section S.3 of Appendix S illustrates that use of Bayes' rule gives the following a posteriori distribution on λ

$$f_\lambda(\lambda_o | E) = f_\gamma(\lambda_o | v+k, w+T_k)$$

which is another member of the same Gamma family, precisely what was wanted. Furthermore, the a posteriori marginal distribution of r is of the same form as its a priori distribution. Using the results of section 7.1.2, there follows

$$f_r(r_o | E) = \frac{(v+k)(w+T_k)^{v+k}}{(r_o+w+T_k)^{v+k+1}} \qquad 0 \leq r_o$$

and

$$E(r | E) = \frac{w + T_k}{v + k - 1}$$

Hence, the expected value of r on the next draw is still extremely easy to calculate after we have observed the event E; this was another desired consequence of the form of distribution placed on λ .

7.2. The Range of "r" is $(-\infty, \infty)$ — Normal-Normal Form

7.2.1. Determination of the Form of the Prior Distribution

For this range the logical choice for a density function for r is the Normal distribution. We could be quite general and allow both parameters of the Normal to be random variables. However, this causes the calculations to become quite involved (but they can be carried out). Instead, attention will be restricted to the case where the variance of the Normal is assumed exactly known but the mean is considered to be a random variable. In other words,

$$f_{r|\mu}(r_o|\mu_o) = f_N(r_o|\mu_o, \sigma^2) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-(r_o-\mu_o)^2/(2\sigma^2)}$$

$$-\infty < r_o < \infty \qquad (7.6)$$

and we have a density function $f_\mu(\mu_o)$. Again, this latter density function must be selected so as to satisfy the requirements of easy determination of $E(r)$ and simple Bayes modification.

Appendix T reveals that the proper choice is

$$f_\mu(\mu_o) = f_N(\mu_o|\,v', w\sigma^2) = \frac{1}{\sqrt{2\pi w}\,\sigma} e^{-(\mu_o-v')^2/(2w\sigma^2)} \qquad -\infty < \mu_o < \infty$$

$$(7.7)$$

That is, the mean should, itself, be normally distributed.

7.2.2. The Marginal Distribution of "r" and its Mean and Variance

Section T.2 of Appendix T shows that for the conditions of equations (7.6) and (7.7) the marginal density function of r is

$$f_r(r_o) = \int_{-\infty}^{\infty} f_{r|\mu}(r_o|\mu_o)\, f_\mu(\mu_o)\, d\mu_o$$

which leads to

$$f(r_o) = f_N(r_o|v', (w+1)\sigma^2) \qquad (7.8)$$

Hence, r is normally distributed with mean v and variance $(w+1)\sigma^2$

$$\therefore \; E(r) = v \tag{7.9}$$

and

$$\check{r} = (w+1)\sigma^2 \tag{7.10}$$

7.2.3. Determination of the A Priori Parameters

Using equations (7.9) and (7.10) we can select v and w so as to satisfy our prior estimates of $E(r)$ and $\check{r}$. This is possible because σ^2 is assumed exactly known.

7.2.4. Bayes Modification of the Normal Prior Distribution

Consider the random variable r with

$$f_{r|\mu}(r_o|\mu_o) = f_N(r_o|\mu_o, \sigma^2)$$

where, in turn,

$$f_{\mu}(\mu_o) = f_N(\mu_o|v, w\sigma^2)$$

Let E represent the event that k independent values of r sum to T_k. Section T.3 of Appendix T shows that use of Bayes' rule gives the following a posteriori distribution on μ.

$$f_{\mu}(\mu_o|E) = f_N\left(\mu_o \middle| \frac{v+wT_k}{kw+1}, \frac{w}{kw+1}\sigma^2\right)$$

which is another member of the same Normal family. Furthermore, the a posteriori marginal distribution of r is again a member of the Normal family. Using the results of section 7.2.2, there follows

$$f_r(r_o | E) = f_N\left(r_o \middle| \frac{v + wT_k}{kw+1}, \left(\frac{w}{kw+1}+1\right)\sigma^2\right)$$

and

$$E(r | E) = \frac{v + wT_k}{kw + 1}$$

Thus, a Normal distribution on r with its variance exactly known but its mean, in turn, normally distributed, allows us to obtain E(r) easily and also perform simple Bayes modifications when some sample values of r are observed.

7.3. The Range of "r" is Finite

In most cases a finite range on r can be adequately approximated by either an Exponential-Gamma or Normal-Normal framework with suitably chosen parameter values. Another possible method of handling this situation would be the following:

Suppose the allowable range of r is from A to B. Then, we could say that

$$f_{r|m,n}(r_o | m_o, n_o) = \frac{1}{(B-A)^{m_o+n_o-1}\,\beta(m_o, n_o)} (r_o - A)^{m_o-1} (B - r_o)^{n_o-1}$$

$$A \leq r_o \leq B$$

(The Beta distribution with arbitrary limits) where m and/or n would, in turn, be random variables having convenient distributions. Unfortunately, it appears that there are no such convenient distributions. Hence, it seems that we must resort to one of the aforementioned approximation methods when the range of r is finite.

CHAPTER 8

DETERMINATION OF THE EXPECTED REWARDS IN VARIOUS TIME PERIODS

As was explained in Chapter 5 the expected rewards in various time periods are of central importance in decision making in Markov processes. Hence, it is imperative that we be able to evaluate these expected rewards.

8.1. The Expected Reward per Period in the Steady State

8.1.1. Arbitrary Distribution on r_{ij}

Consider an N state Markov process with exactly known transition matrix, $P = (p_{ij})$, and let r_{ij} be the reward per transition from state i to state j $(i, j=1, 2, \ldots, N)$. The r_{ij}'s are random variables rather than being exactly known.

When all the r_{ij}'s are exactly known, the expected reward per transition or per period (we assume throughout that exactly one transition occurs per period) in the steady state is

$$R_t = \sum_{i=1}^{N} \pi_i \sum_{j=1}^{N} p_{ij} \, r_{ij}$$

where, again, π_i is the steady state probability of being in state i. However, the r_{ij}'s are random variables; still, we can write

$$E(R_t) = E\left(\sum_{i=1}^{N} \pi_i \sum_{j=1}^{N} p_{ij} r_{ij}\right) = \sum_{i=1}^{N} \pi_i \sum_{j=1}^{N} p_{ij}\, E(r_{ij}) \tag{8.1}$$

Two different mechanisms will produce this result:

i) As in earlier situations we select r_{ij} values from their distributions and let the process run to the steady state obtaining an R_t. This is repeated a large number of times and the long term average of R_t is as given in equation (8.1).

ii) We let the process run to the steady state just once and everytime a transition occurs from state i to state j we draw an r_{ij} value from its (unchanged) distribution. The long term expected reward per transition is again given by equation (8.1).

If, instead of rewards for transitions we used rewards for being in states, equation (8.1) would be replaced by

$$E(R) = \sum_{i=1}^{N} \pi_i\, E(r_i) \tag{8.2}$$

where E(R) is the expected reward per period in the steady state and r_i is the reward per period for being in state i (i=1, 2, . . . , N).

8.1.2. Exponential-Gamma Distribution on r_{ij}.

$$f_{r_{ij}|\lambda_{ij}}(r_o|\lambda_o) = \lambda_o e^{-\lambda_o r_o} \qquad 0 \le r_o$$

and

$$f_{\lambda_{ij}}(\lambda_o) = f_\gamma(\lambda_o | v_{ij}, w_{ij}) = \frac{w_{ij}^{v_{ij}}}{\Gamma(v_{ij})} \lambda_o^{v_{ij}-1} e^{-w_{ij}\lambda_o} \qquad 0 \leq \lambda_o$$

Then, as shown in section S.2 of Appendix S

$$f_{r_{ij}}(r_o) = \frac{v_{ij} w_{ij}^{v_{ij}}}{(r_o + w_{ij})^{v_{ij}+1}} \qquad o \leq r_o$$

and

$$E(r_{ij}) = \frac{w_{ij}}{v_{ij} - 1} \qquad v_{ij} \geq 2$$

Therefore, substituting in equation (8.1)

$$E(R_t) = \sum_{i=1}^{N} \pi_i \sum_{j=1}^{N} p_{ij} \frac{w_{ij}}{v_{ij} - 1} \tag{8.3}$$

which is the expected reward per transition in the steady state when the rewards have Exponential-Gamma prior distributions with parameters v_{ij} and w_{ij}. Unlike in section I where the p_{ij}'s were not exactly known, here we have an easily computable expression for the $E(R_t)$ of an N state process.

8.1.3. Normal-Normal Distribution on r_{ij}

$$f_{r_{ij}|\mu_{ij}}(r_o|\mu_o) = f_N(r_o|\mu_o, \sigma_{ij}^2) = \frac{1}{\sqrt{2\pi}\,\sigma_{ij}} e^{-(r_o-\mu_o)^2/(2\sigma_{ij}^2)} \qquad -\infty < r_o < \infty$$

and

$$f_{\mu_{ij}}(\mu_o) = f_N(\mu_o \mid v_{ij}, w_{ij}\sigma_{ij}^2)$$

Then, as shown in section T.2 of Appendix T,

$$f_{r_{ij}}(r_o) = f_N(r_o \mid v_{ij}, (w_{ij}+1)\sigma_{ij}^2)$$

and

$$E(r_{ij}) = v_{ij}$$

Therefore, substitution in equation (8.1) yields

$$E(R_t) = \sum_{i=1}^{N} \pi_i \sum_{j=1}^{N} p_{ij} v_{ij} \tag{8.4}$$

which is the expected reward per transition in the steady state when the rewards have Normal-Normal prior distributions with parameters v_{ij} and w_{ij}. Again, $E(R_t)$ is an easily computable quantity.

8.1.4. The Effects on $E(R_t)$ of Sample Values of the Rewards

Suppose we observe several transitions with the associated rewards.

Let f_{ij} = the number of observed transitions from state i to state j $(i, j=1, 2, \ldots, N)$, $F = (f_{ij})$, and s_{ij} = the total reward from the f_{ij} transitions; $S = (s_{ij})$.

i) For the Exponential-Gamma framework with prior parameters

v_{ij} and w_{ij} we have seen in section 7.1.4 that

$$E(r_{ij} | f_{ij}, s_{ij}) = \frac{w_{ij} + s_{ij}}{v_{ij} + f_{ij} - 1}$$

Therefore,

$$E(R_t | F, S) = \sum_{i=1}^{N} \pi_i \sum_{j=1}^{N} p_{ij} \frac{w_{ij} + s_{ij}}{v_{ij} + f_{ij} - 1} \tag{8.5}$$

ii) For the Normal-Normal framework with prior parameters v_{ij} and w_{ij}, section 7.2.4 showed that

$$E(r_{ij} | f_{ij}, s_{ij}) = \frac{v_{ij} + w_{ij} s_{ij}}{f_{ij} w_{ij} + 1}$$

Therefore,

$$E(R_t | F, S) = \sum_{i=1}^{N} \pi_i \sum_{j=1}^{N} p_{ij} \frac{v_{ij} + w_{ij} s_{ij}}{f_{ij} w_{ij} + 1} \tag{8.6}$$

Hence, both frameworks allow us to easily calculate $E(R_t)$ after several observations have been made, a very desirable property for statistical decision purposes.

8.2. The Expected Rewards in Transient Situations

We shall restrict attention to the situation where the rewards are given for being in states. For exactly known rewards let $R_i(n)$ be the expected reward in n periods given that we start in state i. Then,

$$R_i(n) = \sum_{j=1}^{N} \bar{t}_{ij}(n)\, r_j$$

where r_j is the reward per period for being in state j $(j = 1, 2, \ldots, N)$ and $\bar{t}_{ij}(n)$ is the expected number of times that the process will be in state j during the next n periods given that the present state is i. The quantity, $\bar{t}_{ij}(n)$, can be obtained by transform techniques as shown in Howard's "Dynamic Probabilistic Systems."[21]

Since the r_j's are random variables, $R_i(n)$ is now a random variable whose expected value is

$$E[R_i(n)] = E\left[\sum_{j=1}^{N} \bar{t}_{ij}(n)\, r_j\right]$$

i.e.,

$$E[R_i(n)] = \sum_{j=1}^{N} \bar{t}_{ij}(n)\, E(r_j) \tag{8.7}$$

where the quantities are defined above. This formulation assumes the following mechanism. For each j we select an r_j and run the process (starting from state i) for n periods. If this is repeated a large number of times (selecting new r_j's each time), the average reward for the n periods approaches $E[R_i(n)]$. Note that the r_j distributions are not updated as we progress through the n periods. Furthermore, it is

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[21] Op. cit.

interesting that we use here the same $E(r_{ij})$ quantities that were required under steady state conditions in section 8.1.

Numerical Example

Mike, the waiter, who is in financial trouble, is concerned about the amount of money he can expect to make on tips from the next n customers that he serves. Mike is mathematically inclined but rather moody. He figures that all customers can be split into two groups, good and bad. In fact, Mike reasons that a good customer will leave a tip that is approximately normally distributed with variance 0.2, and a mean that is, in turn, normally distributed with mean 2 and variance 0.1. The corresponding quantities for a bad customer are 0.4, 1 and 0.3. Mike's moodiness is reflected by the fact that his transitions from one customer type to another can be represented by the following probabilities.

$$P = \begin{array}{c} \\ G \\ B \end{array} \begin{array}{c} \begin{array}{cc} G & B \end{array} \\ \begin{bmatrix} 3/4 & 1/4 \\ 1/3 & 2/3 \end{bmatrix} \end{array}$$

That is, he is more likely to stay with the current type than switch. (A good tip improves his morale making his attitude better. This, in turn, improves the chance of another good tip.)

Given that Mike is now serving a good customer, what are his expected total tips in the next n customers (including the present one)?

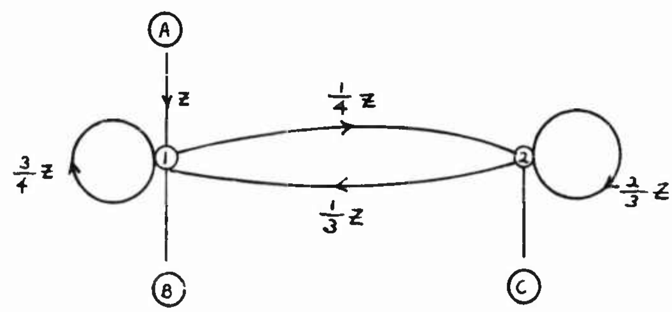
The reward structure is given by

$f_{r_1}(r_o) = f_N(r_o|\mu_1, 0.2)$ $\qquad$ $f_{r_2}(r_o) = f_N(r_o|\mu_2, 0.4)$

$f_{\mu_1}(\mu_o) = f_N(\mu_o|2, 0.1)$ $\qquad$ $f_{\mu_2}(\mu_o) = f_N(\mu_o|1, 0.3)$

$\therefore\ v_1 = 2,\ w_1 = \frac{1}{2}$ $\qquad$ $\therefore\ v_2 = 1,\ w_2 = \frac{3}{4}$

The following flow graph immediately gives us the geometric transforms of $\bar{t}_{11}(n)$ and $\bar{t}_{12}(n)$[22], the expected numbers of good and bad customers, respectively, in the next n customers.



The transmission from A to B

22 Ibid.

$$T_{AB}(z) = \overline{t}_{11}{}^{T}(z) = \sum_{n=0}^{\infty} \overline{t}_{11}(n)\, z^n$$

$$\therefore \overline{t}_{11}{}^{T}(z) = \frac{z - \frac{2}{3} z^2}{(1-z)^2 \left(1 - \frac{5}{12} z\right)}$$

Using a partial fraction expansion and inverting the transforms, we obtain

$$\overline{t}_{11}(n) = \frac{4}{7} n + \frac{36}{49} - \frac{15}{49}\left(\frac{5}{12}\right)^{n-1} \qquad n \geq 0$$

Similarly, using the transimission from Ⓐ to Ⓒ, there results

$$\overline{t}_{12}(n) = \frac{3}{7} n - \frac{36}{49} + \frac{15}{49}\left(\frac{5}{12}\right)^{n-1} \qquad n \geq 0$$

Now, from section 7.2.2,

$$E(r_1) = E(\mu_1) = v_1 = 2$$

and

$$E(r_2) = E(\mu_2) = v_2 = 1$$

Then, using equation (8.7),

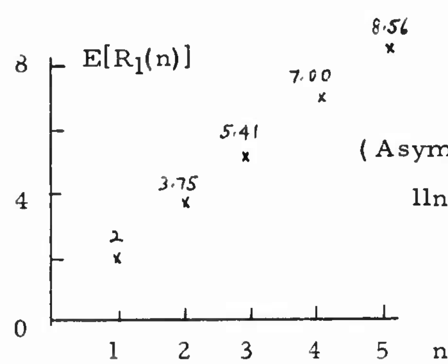
$$E[R_1(n)] = \sum_{j=1}^{2} \overline{t}_{ij}(n)\, E(r_j)$$

$$= 2\left[\frac{4}{7}n + \frac{36}{49} - \frac{15}{49}\left(\frac{5}{12}\right)^{n-1}\right]$$

$$+ 1\left[\frac{3}{7}n - \frac{36}{49} + \frac{15}{49}\left(\frac{5}{12}\right)^{n-1}\right]$$

$$= \frac{11}{7}n + \frac{36}{49} - \frac{15}{49}\left(\frac{5}{12}\right)^{n-1} \qquad n \geq 0$$

This is his expected total tip money from the next n customers.



(Asymptotic to the line
11n/7 + 36/49)

CHAPTER 9

MARKOV DECISION PROBLEMS WHEN THE TRANSITION PROBABILITIES ARE KNOWN EXACTLY BUT THE REWARDS ARE RANDOM VARIABLES

As was done in Chapter 5, we shall make the realistic assumption that, given a choice, the decision maker will use a Markov process only if the expected net revenue is positive. Hence, in the steady state situation the process will be utilized only if the expected reward per transition, $E(R_t)$, is greater than the fixed cost per period for using the process, c. In the transient situation the process will be operated only if the expected reward in the n remaining periods, $E[R_i(n)]$, is greater than the cost of operation, nc. This does not imply that all Markov decisions are made in this manner; rather it is hoped that the reader will use the analysis presented here as a guide for analyzing a situation where the decision mechanism is different.

9.1. Problems Based on Steady State Conditions

Once more it is assumed that the process will be used for only s periods in the steady state or with discounting for an indefinitely large number of periods. Only the first situation will be analyzed, the second involves a trivial extension as mentioned in section 5.2.3. Within this framework there is a possibility that observations of the process may be worthwhile.

9.1.1. Preposterior Analysis for a Single Observation

If $E(R_t)$ is the expected reward per period in the steady state, then because of the assumed decision mechanism, the expected net revenue is

$$E(N.R.) = \max[o, s[E(R_t) - c]] \qquad (9.1)$$

Suppose that the process is in state k and we decide to pay for an observation of the next transition reward. With probability p_{km} ($m = 1, 2, \ldots, N$) we shall observe a draw from the density function $f_{r_{km}}(r_o)$ of the reward r_{km}. The probability that the reward will be between x and x + dx given that the transition is to state m is clearly a function of the prior framework placed on r_{km}; it is given by $f_{r_{km}}(x)\,dx$. For the Exponential-Gamma framework with parameters v_{ij} and w_{ij}, it was shown in section 7.1.2 that

$$f_{r_{km}}(x)\,dx = \frac{v_{km} w_{km}^{v_{km}}}{(x+w_{km})^{v_{km}+1}}\,dx \qquad 0 \leq x \qquad (9.2)$$

For the Normal-Normal framework with parameters v_{km}, w_{km} and σ^2_{km}, section 7.2.2 revealed that

$$f_{r_{km}}(x)\,dx = f_N(x \mid v_{km}, (w_{km}+1)\,\sigma^2_{km})\,dx \qquad (9.3)$$

Now, when a draw falling between x and x + dx occurs from the r_{km} distribution, only the k-m element of $E(R_t)$ is changed and, again, the change, $\Delta_{km}(x)$, is a function of which framework is being used. Recall from equation (8.1) that

$$E(R_t) = \sum_{i=1}^{N} \pi_i \sum_{j=1}^{N} p_{ij}\, E(r_{ij})$$

For the Exponential-Gamma framework with parameters v_{ij} and w_{ij} prior to the observation

$$E(r_{km}) = \frac{w_{km}}{v_{km} - 1} \qquad \text{(using equation (8.3))}$$

after the observation,

$$E(r_{km} \mid 1, x) = \frac{w_{km} + x}{v_{km}} \qquad \text{(using equation (8.5))}$$

$$\therefore \Delta_{km}(x) = \pi_k \, p_{km}[E(r_{km} \mid 1, x) - E(r_{km})]$$

$$\Delta_{km}(x) = \pi_k \, p_{km} \left[\frac{w_{km} + x}{v_{km}} - \frac{w_{km}}{v_{km} - 1} \right] \qquad (9.4)$$

For the Normal-Normal framework with parameters v_{ij}, w_{ij} and σ_{ij}^2 prior to the observation

$$E(r_{km}) = v_{km} \qquad \text{(using equation (8.4))}$$

after the observation,

$$E(r_{km} \mid 1, x) = \frac{v_{km} + w_{km} x}{w_{km} + 1} \qquad \text{(using equation (8.6))}$$

$$\therefore \Delta_{km}(x) = \pi_k \, p_{km} \left[\frac{v_{km} + w_{km} x}{w_{km} + 1} - v_{km} \right] \qquad (9.5)$$

The change, $\Delta_{km}(x)$, in $E(R_t)$ may also cause a change in the expected net revenue. According to equation (9.1) prior to observation,

$$E(N.R.) = \max[0, \; E(R_t)]$$

and after it

$$E(N.R. \mid \text{observation of } x \text{ from } r_{km})$$

$$= \max[0,\ E(R_t) + \Delta_{km}(x)]$$

Finally, the expected net revenue prior to the observation, given that it will be taken, is the integral of the expected net revenue given each possible outcome, weighted by the probability of that outcome minus the cost of the observation. That is,

$$E(N.R. \mid \text{observation})$$

$$= -d_k + \sum_{m=1}^{N} p_{km} \int_x f_{r_{km}}(x) \max[0,\ E(R_t) + \Delta_{km}(x)]\ dx \qquad (9.6)$$

where d_k = the cost of observing the reward for one transition from state k.

If E(N.R. |observation) is greater than E(N.R.), we buy the observation; if not, we do not buy it.

Numerical Example

Joe, a famous local vendor, has been offered the opportunity of setting up a concession stand in Boston Gardens for 5 consecutive games late in the 1963-64 schedule of the Boston Bruins. Joe is somewhat concerned about any venture connected with the Bruins; hence, he has called in a local operations analyst for help. Joe knows for sure that his fixed costs per game will be $161. After considerable deliberation, Joe is convinced that his revenues during a particular contest will be a function of the Bruins' showing in both the previous game and the present game.

He is satisfied with separating the Bruins' behavior into "WIN", "TIE" or "LOSE." Also, from past records and forecasts of the coming season Joe and the analyst are willing to assume the transition probabilities exactly known at

$$P = \begin{array}{c} \\ W \\ T \\ L \end{array} \begin{array}{c} \begin{array}{ccc} W & T & L \end{array} \\ \begin{bmatrix} 0.1 & 0.3 & 0.6 \\ 0.2 & 0.6 & 0.2 \\ 0.3 & 0.3 & 0.4 \end{bmatrix} \end{array}$$

The corresponding steady state probability vector is

$$\begin{array}{cccc} & W & T & L \\ \underline{\pi} = [& .214 & .429 & .357] \end{array}$$

Joe isn't so confident about his revenues for various Bruins' showings. Again, after considerable thought he feels that the revenues have Exponential-Gamma frameworks. That is,

$$f_{r_{ij}|\lambda_{ij}}(r_o|\lambda_o) = \lambda_o e^{-\lambda_o r_o} \qquad 0 \leq r_o$$

and

$$f_{\lambda_{ij}}(\lambda_o) = f_\gamma(\lambda_o | v_{ij}, w_{ij})$$

His estimates of the parameters are

$$V = (v_{ij}) = \begin{bmatrix} 2 & 2 & 2 \\ 3 & 4 & 5 \\ 2 & 4 & 6 \end{bmatrix}$$

and

$$W = (w_{ij}) = \begin{bmatrix} \$300 & \$300 & \$300 \\ \$400 & \$400 & \$500 \\ \$100 & \$300 & \$300 \end{bmatrix}$$

From equation (7.4)

$$[E(r_{ij})] = \frac{w_{ij}}{v_{ij} - 1} = \begin{bmatrix} \$300 & \$300 & \$300 \\ \$200 & \$\frac{400}{3} & \$125 \\ \$100 & \$100 & \$\ 60 \end{bmatrix}$$

This says, for example, that his expected revenue in a game that the Bruins win given that they tied the previous game is \$200 (the 2-1 element).

Joe will accept the offer only if his expected net revenue is positive.

Now, Joe has a friend who has run a similar concession stand at Boston Gardens. Unfortunately, the friend only has revenue data for a single contest and without delving into his papers (for a fee) he is only willing to tell Joe that the previous game ended in a tie. For $\$d_2$ he will tell Joe his revenue and the outcome of the corresponding game. Without his friend's revenue information, from equation (8.1) we have

$$E(R_t) = .214[(0.1)\,300 + (0.3)\,300 + (0.6)\,300] + .429[(0.2)\,200 +$$

$$(0.6)\,400/3 + (0.2)\,125] + .357[(0.3)\,100 + (0.3)\,100 +$$

$$(0.4)\,60]$$

$$= \$156.40$$

$$E(R_t) - c = -\$4.60 < 0$$

Therefore, without his friend's information Joe would not accept the offer and his expected net revenue would be zero.

Now, suppose that the friend's help is accepted. As the transition is from state 2 (a tie) only the second row of $(E(r_{ij}))$ will be affected. Let us treat the 3 possible transitions separately.

<u>Transition from 2 to 1 (With Probability $p_{21} = 0.2$)</u>

Let the observed revenue be x. Then, using equation (9.4) the change in $E(R_t)$ is given by

$$(.429)(0.2)\left(\frac{400+x}{3} - 200\right)$$

or

$$.0286x - 5.72$$

From above, $E(R_t)$ must increase by at least 4.60 to make the offer worthwhile.

Therefore, $.0286x - 5.72$ must be ≥ 4.60. That is,

$$x \geq \$361.$$

Hence, when $x \geq \$361$, the Gardens' offer becomes worthwhile and

$$E(N.R.) = 5E(R_t \mid 1, x) = 5[\text{Old } E(R_t) + \Delta_{12}(x) - c]$$

$$= 5[156.40 + .0286x - 5.72 - 161.00]$$

$$= 5[-10.32 + .0286x]$$

$$\therefore E(N.R. \mid \text{transition from 2 to 1 observed}) = 5 \int_{361}^{\infty} E(R_t \mid 1, x) \cdot$$

$$f_{r_{21}}(x)\,dx = 5 \int_{361}^{\infty} [-10.32 + .0286x] \frac{3(400)^3}{(x+400)^4}\,dx \quad \text{(using equation (7.3))}$$

$$= \$7.93 \quad \text{by straight-forward integration.}$$

Similarly,

$$E(N.R. \mid \text{transition from 2 to 2 observed}) = \$12.42$$

and

$$E(N.R. \mid \text{transition from 2 to 3 observed}) = \$1.05$$

Now,

$$E(N.R. \mid \text{transition from 2 observed})$$

$$= \sum_{j=1}^{3} p_{2j}\, E(N.R. \mid \text{transition from 2 to } j) - d_2$$

$$= 0.2(7.93) + 0.6(12.42) + 0.2(1.05) - d_2$$

$$= \$(9.25 - d_2)$$

As we found that the E(N.R.) without an observation was zero, Joe would pay for his friend's information if, and only if,

$$d_2 < \$9.25$$

Furthermore, the analysis tells us that Joe would accept Boston's offer only under the following outcomes of the single revenue observed:

i) If the transition is from 2 to 1 (tie to win) and the revenue is greater than \$361.

ii) If the transition is from 2 to 2 (tie to tie) and the revenue is greater than \$205.

or

iii) If the transition is from 2 to 3 (tie to lose) and the revenue is greater than \$393.

9.1.2. Preposterior Analysis for More Than One Observation

The method in principle is the same as that used for the case of one observation discussed in the previous section. To find the expected net revenue given that the observations will be taken, we integrate the expected net revenue after each possible outcome weighted by the probability of that outcome. There is no problem in evaluating the net revenue after any particular experimental outcome. Equations (8.5) and (8.6) allow us to find the expected reward per period in the steady state after any experimental outcome for the Exponential-Gamma and Normal-Normal prior frameworks, respectively. Then, we use

$$E(N.R.) = \max[0, \; s[E(R_t) - c]]$$

to obtain the corresponding expected net revenue. The difficulty is that the probability of any particular experimental outcome is now an involved function. This makes the integration difficult. More will be said on this point in the next section.

Again, if the expected net revenue given that we shall buy the observations is higher than the expected net revenue without them, we buy the observations; if lower, we do not make the purchase.

9.1.3. Some Other Remarks of Interest

The continuous nature of the random variables causes difficulties when a sequential form of analysis is attempted. This can be illustrated by a situation where we can observe the rewards of one or two consecutive transitions with the option of not observing the second one after we have seen the first. For just one transition remaining, the problem can be analyzed as in the previous section. However, the continuous nature of the reward for the first transition forces us to consider an infinite number of possible situations when just the second transition remains. This would be tractable if the expected net revenue using an observation was a simple analytic function of the prior parameters. However, such is clearly not the case.

A possible method of avoiding the above difficulty would be to approximate each continuous reward distribution by a discrete (quantized) probability mass function; that is,

$$pr(r=x_k) = p_r(x_k) = p_k \qquad k = 1, 2, \ldots, n$$

But, in order to allow Bayes modification, we would not assume that the p_k's were known exactly; rather, we would place a multidimensional Beta distribution on them as was done with the multinomial distribution in section 3.1. Under this framework, each observation would have a finite number of outcomes and sequential analysis would thus be possible.

In the single observation case discussed in section 9.1.1 we could obtain the expected value of perfect information about one or more of the unknown parameters by following the method outlined in section 5.4.1. Therefore, there is no need to elaborate here on the value of perfect information.

9.2. Statistical Decisions in Transient Situations

In Chapter 5 it was mentioned that in the case of statistical decision problems when the transition probabilities were not exactly known, transient situations could be handled in essentially the same way as the decision problems based on steady state conditions. The exact same statement can be made for the present situation where the rewards are not exactly known.

The quantity, $s[E(R_t) - c]$ is replaced by $E[R_i(n)] - nc$, the expected net revenue in the next n periods given that the process will be used. Naturally, if we delay 1 period (perhaps to observe a reward) before using the process and the system moves to state j, the expected net revenue will then be $E[R_j(n-1)] - (n-1)c$. With these minor modifications we are able to analyze the single observation case exactly as in section 9.1.1. Also, we encounter the same predicament for a sequential decision framework as in the steady state situation. Finally, the expected value of perfect information about one or more of the unknown parameters could be evaluated by the method of section 5.4.1.

CHAPTER 10

CONCLUSIONS

10.1 Summary

The purpose of this study has been to extend the range of application of Markov process theory by removing one of the fundamental assumptions made in earlier theoretical considerations of such processes, namely that both the rewards and transition probabilities be known exactly.

In Section I we assumed that the rewards were exactly known but the transition probabilities (or matrices or rates) were random variables.

The first approach involved the concept of a multi-matrix Markov process -- this process is governed by one of several known matrices but we only know probabilistically which matrix is being used. The analysis of this situation is quite straightforward as was shown in Chapter 2. We described the Bayes modification (after some transitions are observed) of the probabilities that the various matrices are being used. The determination of various quantities, such as mean recurrence times, was illustrated. Then various cost structures were placed on multi-matrix processes. Finally, the possibility of using statistical decision theory in multi-matrix situations was revealed by consideration of a 2-state, 2-matrix example where there is an option of buying observations of transitions before stating which matrix is being used.

Chapters 3, 4, and 5 were concerned with the physically more appealing situation where the transition probabilities themselves (rather than the matrices) are considered as random variables. It was first demonstrated that it is convenient (for Bayes calculations and thus for statistical decision purposes) to have a multidimensional Beta prior distribution on the probabilities of each row of the transition matrix. This is a direct extension of the previously known result of using a Beta prior on the probability of success in a Binomial process. Chapter 3 was concerned with developing many basic properties of the multidimensional Beta distribution, several of which were to be utilized later in the study.

In Chapter 4 we attacked the problem of determining various quantities of interest when the transition probabilities are multidimensional Beta distributed. It is important, both for statistical decision purposes and for interest in the quantities per se, that we be able to describe their behavior when the transition probabilities are not known exactly. Unfortunately analytic complexities prevented a detailed theoretical treatment of the situation. The one important analytic achievement for the N state case was the derivation of the probability mass functions of the state occupancy times. Analytic results were also obtained for mean recurrence times, multi-step transition probabilities and expected steady state probabilities in 2-state processes. Finally a simple trapping states situation was analyzed.

It was illustrated that the expected values of the steady state probabilities are very important for many decision purposes. Hence, considerable time was devoted to those quantities. The analytic expression for a special 2-state situation was made possible through the use of the hypergeometric function. Due to analytic complexities simulation was required for processes with more than 2 states. However, the results were most encouraging in that they suggested that a reasonable approximation to the expected values of the steady state probabilities can be obtained by assuming that the transition probabilities are exactly known at their mean values and then using the corresponding, easily calculable, steady state probabilities. This permits us to have the multidimensional Beta priors on the transition probabilities, a situation ideal for Bayes modifications, yet we can still easily obtain a reasonable approximation to the expected steady state probabilities, the latter being essential for statistical decision purposes.

In Chapter 5 we analyzed several statistical decision situations for Markov processes whose transition probabilities are multidimensional Beta distributed. Again, it should be emphasized that no claim is made that the situations considered cover most of those that could occur in practice. Rather the intent was to provide the reader with an approach to solving certain types of decision problems arising in Markov processes in the hope that he would be able to make the suitable adjustments to fit the particular physical situation confronting him. Items such as the expected value of perfect information about a transition probability were also considered. The entire statistical decision framework established for Markov processes was summarized in Figure 5.6.

Chapter 6 involved a brief look at continuous time processes to show that essentially the same situation (including analytic difficulties) exists as in the discrete time case.

In Section II we assumed that the transition probabilities were exactly known but now the rewards were random variables. This is the exact opposite situation from that of Section I.

Chapter 7 illustrated two convenient (in the Bayes sense) prior distributions to place over the rewards. Then Chapter 8 was concerned with the determination of the expected rewards in various time periods (both steady state and transient). These quantities were much easier to obtain than were the corresponding items in Section I where the transition probabilities were not known exactly. Chapter 9 illustrated how to use these expected rewards in making statistical decisions concerning Markov processes where the rewards are not known exactly, e.g., is it worthwhile to observe some sample rewards to improve our knowledge about expected future rewards before deciding whether or not to utilize a Markov process?

10.2 Some General Remarks

There are several remarks that do not logically fit into the above "summary" section or into Section 10.3.

As in all situations where a Bayes approach is to be utilized the analyst must avoid the pitfall of developing his prior probability distributions after observing the data and then obtaining an a posteriori distribution through the use of the same data. Therefore, in using a Bayes approach to the study of Markov processes the prior probability distributions over the transition probabilities or rewards must first be developed, then the observational data is used to obtain the posterior distributions through the use of Bayes' rule.

In many sections of this study the suggested formulas or techniques involve considerable computation. Fortunately, as in other areas of applied mathematics, the use of high speed digital computers makes these methods practical whereas hand computations would be out of the question.

Again it should be stressed that the intent of this study was not to solve all practical Markov problems where either the transition probabilities or rewards are not exactly known; rather the goal was to develop mathematical expressions for some quantities of interest under this situation and to indicate by analyzing some specific examples that statistical decisions related to Markov processes can and should be made.

10.3 Suggested Related Areas for Further Research

It is apparent that a fundamental study of this nature would suggest several related research topics of varying difficulty.

In two situations in this research the lack of an ideal prior distribution restricted our progress. One restriction was of a relatively minor nature, namely in Chapter 7 where we could not obtain a convenient prior that would exactly fit the physical constraint of a reward having to lie within a finite range. However, the second restriction was of a far more serious nature. The multidimensional Beta prior used for the probabilities of a single row of the Markov transition matrix was ideal for Bayes calculations but was restrictive due to the fact that it provided only k prior parameters (for a k state process). Ideally we would like 2k - 1 parameters so that the prior estimates of the k -1 marginal means and the k marginal variances of the individual probabilities could be satisfied. The determination of a prior distribution having 2k - 1 parameters yet still allowing simple Bayes modification would be a significant extension to this study.

As mentioned in the main text an interesting related problem is Howard's policy-iteration situation but with transition probabilities and/or rewards no longer exactly known.

In many Markov process applications probabilities in different rows of the transition matrix are not independent as assumed throughout Chapters 3 to 5. An extreme situation would be a process where we know that two rows of the matrix are identical but do not know the exact values of the elements. Situations like this would require more complicated prior distributions than the multidimensional Betas used in this study.

Recently, significant research has been done in the area of semi-Markov processes, processes where the transitions are governed by a regular P matrix but the transition times have arbitrary probability distributions. A natural extension of this thesis would be to consider semi-Markov processes where the P matrix and/or the probability distributions of the transition times were not exactly known.

This study has not encompassed the situation where both the transition probabilities and the rewards are not exactly known. An analysis of this more general problem would certainly be of value.

As mentioned in Chapter 2 the multi-matrix framework is really a part of the combination of Markov process theory and game theory. Unfortunately, the complexities of game theory have made the determination of analytic results extremely difficult; hence, the use of game theory in Markov process situations would not appear imminent.

As was stated in the main text, an important decision problem not considered in this report is the situation where once we have decided to use a process we can change our strategy (i.e., make further decisions) as we observe the process in operation during the time in which we are using it. In effect, this situation can be considered as a type of adaptive control problem. We are not committed by a single decision but can adjust our actions as we become more familiar with the process.

Undoubtedly, the reader will be able to make additions to the above list of suggested related areas for additional research. In any event it is hoped that this study will stimulate further fundamental investigations in the theory of Markov processes.

APPENDIX A

PROOF THAT $v_h(\underline{a}', i)$ IS A PIECEWISE LINEAR FUNCTION OF THE COMPONENTS OF $\underline{a}'$

Equation (2.6) says

$$v_o(\underline{a}', i) = \min_{1 \le k \le Q} \sum_{m=1}^{Q} a'_m c_{km}$$

which is clearly a piecewise linear function of the a'_m's.

Using equations (2.7) and (2.1),

$$v_h(\underline{a}', i) = \min_{S, L} \begin{cases} S & v_o(\underline{a}', i) \\ & \\ L & \sum_{j=1}^{N} \sum_{k=1}^{Q} {}^k p_{ij}\, a'_k [d_{ij} + v_{h-1}(\underline{a}'', r)] \end{cases} \tag{2.7}$$

where the typical component of $\underline{a}''$ is

$$a''_k = \frac{{}^k p_{ij}\, a'_k}{\sum_{m=1}^{Q} {}^m p_{ij}\, a'_m} \tag{2.1}$$

Now, suppose that $v_{h-1}(\underline{a}'', r)$ is piecewise linear in the components of $\underline{a}''$. Then we can write,

$$v_{h-1}(\underline{a}'', r) = a + \sum_{s=1}^{Q-1} b_s \, a''_s$$

where a and the b_k's are constants. Substituting from equation (2.1),

$$v_{h-1}(\underline{a}'', r) = a + \sum_{s=1}^{Q-1} b_s \frac{{}^{s}p_{ij} \, a'_s}{\sum_{m=1}^{Q} {}^{m}p_{ij} \, a'_m}$$

Then the "L" part of equation (2.7) becomes

$$\sum_{j=1}^{N} \sum_{k=1}^{Q} {}^{k}p_{ij} \, a'_k \left[d_{ij} + a + \sum_{s=1}^{Q-1} b_s \frac{{}^{s}p_{ij} \, a'_s}{\sum_{m=1}^{Q} {}^{m}p_{ij} \, a'_m} \right]$$

or

$$\sum_{j=1}^{N} \left[\sum_{k=1}^{Q} {}^{k}p_{ij} \, a'_k (d_{ij} + a) + \sum_{s=1}^{Q-1} b_s \, {}^{s}p_{ij} \, a'_s \right]$$

which is seen to be a piecewise linear function of the components of $\underline{a}'$. Now, $v_h(\underline{a}', i)$ is either "L" or "S" but we have already shown that "S" is piecewise linear in the components of $\underline{a}'$. Therefore, by assuming that $v_{h-1}(\underline{a}', r)$ is piecewise linear in the components of $\underline{a}'$, we have shown that $v_h(\underline{a}', r)$ is piecewise linear in the components of $\underline{a}'$. Also, we have shown that $v_o(\underline{a}', i)$ is piecewise linear in the components of $\underline{a}'$. Hence, the proof that $v_h(\underline{a}', i)$ will always be a piecewise linear function of the components of $\underline{a}'$ has been completed by induction.

APPENDIX B

IMPORTANT PROPERTIES OF THE MULTIDIMENSIONAL BETA DISTRIBUTION

$$f_{p_1,p_2,\ldots,p_k}(x_1,x_2,\ldots,x_k) = f_\beta(x_1,x_2,\ldots,x_k \mid m_1,m_2,\ldots,m_k)$$

$$= \frac{1}{\beta(m_1,m_2,\ldots,m_k)} x_1^{m_1-1} x_2^{m_2-1} \cdots x_k^{m_k-1}$$

where $\sum_{i=1}^{k} x_i = 1$, $x_i \geq 0$ and $\beta(m_1,m_2,\ldots,m_k) = \dfrac{\Gamma(m_1)\Gamma(m_2)\cdots\Gamma(m_k)}{\Gamma(m_1+m_2+\cdots+m_k)}$

B.1. Check that Distribution Integrates to One

$$I = \int_{x_1}\cdots\int_{x_{k-1}} f_{p_1,p_2,\ldots,p_k}(x_1,x_2,\ldots,x_k)\,dx_1\ldots dx_{k-1}$$

$$= \int_{x_1}\cdots\int_{x_{k-1}} \frac{1}{\beta(m_1,m_2,\ldots,m_k)} x_1^{m_1-1} x_2^{m_2-1}\cdots x_{k-1}^{m_{k-1}-1}\left(1-\sum_{i=1}^{k-1}x_i\right)^{m_k-1} dx_1\ldots dx_{k-1}$$

$$= \frac{1}{\beta(m_1,m_2,\ldots,m_k)}\int_{x_1}\cdots\int_{x_{k-2}} x_1^{m_1-1}\cdots x_{k-2}^{m_{k-2}-1}\,dx_1\ldots dx_{k-2}\int_0^{1-\sum_{i=1}^{k-2}x_i} x_{k-1}^{m_{k-1}-1}$$

$$\cdot\left(1-\sum_{i=1}^{k-1}x_i\right)^{m_k-1} dx_{k-1}$$

Substitute $y = \dfrac{x_{k-1}}{1-\sum_{i=1}^{k-2}x_i}$ $\quad\therefore\quad dy = \dfrac{dx_{k-1}}{1-\sum_{i=1}^{k-2}x_i}$

and when $x_{k-1} = 0$, $y = 0$

" " $x_{k-1} = 1-\sum_{i=1}^{k-2}x_i$, $y = 1$

$$\therefore\quad I = \frac{1}{\beta(m_1,m_2,\ldots,m_k)}\int_{x_1}\cdots\int_{x_{k-2}} x_1^{m_1-1}\cdots x_{k-2}^{m_{k-2}-1}\,dx_1\ldots dx_{k-2}\int_0^1 y^{m_{k-1}-1}$$

$$\cdot\left(1-\sum_{i=1}^{k-2}x_i\right)^{m_{k-1}-1}\left[1-\sum_{i=1}^{k-2}x_i - y\left(1-\sum_{i=1}^{k-2}x_i\right)\right]^{m_k-1}\left(1-\sum_{i=1}^{k-2}x_i\right)dy$$

$$I = \frac{1}{\beta(m_1, m_2, \cdots, m_k)} \int_{x_1} \cdots \int_{x_{k-2}} x_1^{m_1-1} \cdots x_{k-2}^{m_{k-2}-1} \left(1 - \sum_{i=1}^{k-2} x_i\right)^{m_k + m_{k-1} - 1} dx_1 \cdots$$

$$\cdots dx_{k-2} \int_0^1 y^{m_{k-1}-1} (1-y)^{m_k - 1} dy$$

$$= \frac{\beta(m_{k-1}, m_k)}{\beta(m_1, m_2, \cdots, m_k)} \int_{x_1} \cdots \int_{x_{k-2}} x_1^{m_1-1} \cdots x_{k-2}^{m_{k-2}-1} \left(1 - \sum_{i=1}^{k-2} x_i\right)^{m_k + m_{k-1} - 1} dx_1 \cdots dx_{k-2}$$

Letting $z = \dfrac{x_{k-2}}{1 - \sum_{i=1}^{k-3} x_i}$,

the next integration gives

$$I = \frac{\beta(m_{k-1}, m_k)\, \beta(m_{k-2}, m_k + m_{k-1})}{\beta(m_1, m_2, \cdots, m_k)} \int_{x_1} \cdots \int_{x_{k-3}} x_1^{m_1-1} \cdots x_{k-3}^{m_{k-3}-1} \left(1 - \sum_{i=1}^{k-3} x_i\right)^{m_k + m_{k-1} + m_{k-2} - 1} dx_1 \cdots dx_{k-3}$$

Iterating in this manner we end up with

$$I = \frac{\beta(m_{k-1}, m_k)\, \beta(m_{k-2}, m_k + m_{k-1}) \cdot \cdots \cdot \beta(m_1, m_k + m_{k-1} + \cdots + m_2)}{\beta(m_1, m_2, \cdots, m_k)}$$

$$= \frac{\dfrac{\Gamma(m_{k-1})\Gamma(m_k)}{\Gamma(m_{k-1} + m_k)} \cdot \dfrac{\Gamma(m_{k-2})\Gamma(m_k + m_{k-1})}{\Gamma(m_{k-2} + m_{k-1} + m_k)} \cdots\cdots \dfrac{\Gamma(m_1)\Gamma(m_k + \cdots + m_2)}{\Gamma(m_1 + m_2 + \cdots + m_k)}}{\dfrac{\Gamma(m_1)\Gamma(m_2) \cdots\cdots \Gamma(m_k)}{\Gamma(m_1 + m_2 + \cdots + m_k)}}$$

Complete cancellation occurs and $I = 1$.

B.2. The Marginal Distribution of p_j and Its Moments

By the same approach as that used in section B.1 (except that all variables are integrated before p_j instead of merely proceeding by subscript order) we find that

$$f_{p_j}(x) = \frac{1}{\beta(m_j, \sum_{i \neq j} m_i)} x^{m_j - 1} (1-x)^{\sum_{i \neq j} m_i - 1} \qquad 0 \leq x \leq 1$$

$$= f_\beta(x \mid m_j, \sum_{i \neq j} m_i)$$

$$E(p_j) = \int_0^1 x\, f_{p_j}(x)\,dx = \int_0^1 \frac{1}{\beta(m_j, \sum_{i\neq j} m_i)}\, x^{m_j}(1-x)^{\sum_{i\neq j} m_i - 1}\, dx$$

$$= \frac{\beta(m_j+1, \sum_{i\neq j} m_i)}{\beta(m_j, \sum_{i\neq j} m_i)}$$

$$= \frac{\Gamma(m_j+1)\, \Gamma(\sum_{i\neq j} m_i)\, \Gamma(\sum_{i=1}^{k} m_i)}{\Gamma(1+\sum_{i=1}^{k} m_i)\, \Gamma(m_j)\, \Gamma(\sum_{i\neq j} m_i)}$$

$$E(p_j) = \frac{m_j}{\sum_{i=1}^{k} m_i}$$

Similarly we can obtain $E(p_j^2)$ and use it to determine

$$\check{p}_j = E(p_j^2) - [E(p_j)]^2 = \frac{m_j(\sum_{i\neq j} m_i)}{(\sum m_i)^2(1+\sum m_i)}$$

B.3. Joint Distributions and Covariances

Proceeding as above we obtain

$$f_{p_j, p_u}(x_j, x_u) = \frac{1}{\beta(m_j, m_u, \sum_{\substack{i\neq j\\ i\neq u}} m_i)}\, x_j^{m_j-1}\, x_u^{m_u-1}\, (1-x_j-x_u)^{\sum_{\substack{i\neq j\\ i\neq u}} m_i - 1} \qquad j\neq u$$

with $x_j + x_u \leq 1$, $x_j \geq 0$, $x_u \geq 0$.

i.e. $f_{p_j, p_u}(x_j, x_u) = f_\beta(x_j, x_u \mid m_j, m_u, \sum_{\substack{i\neq j\\ i\neq u}} m_i)$

and $$\operatorname{cov}(p_j, p_u) = \frac{-m_j m_u}{(\sum m_i)^2(1+\sum m_i)} \qquad j\neq u$$

APPENDIX C

DETERMINATION OF THE PRIOR PARAMETERS OF A MULTIDIMENSIONAL BETA DISTRIBUTION BY LEAST SQUARES

$$f_{p_1, p_2, \ldots, p_k}(x_1, x_2, \ldots, x_k) = \frac{1}{\beta(m_1, m_2, \ldots, m_k)} x_1^{m_1 - 1} \cdot x_2^{m_2 - 1} \cdots x_k^{m_k - 1}$$

where

$$\sum_{i=1}^{k} x_i = 1$$

As stated in section 3.3, we wish to select the parameters (m_i's) such that the prior mean values of the p_j's; i.e., the $E(p_j)$'s are satisfied exactly. This uses up k-1 of the parameters, leaving only 1 other degree of freedom. The final parameter is to be obtained by a least squares fit to the prior variances of the p_j's; i.e., to the $\check{p}_j$'s.

Define

$$M = \sum_{i=1}^{k} m_i$$

From equations (3.6) and (3.7),

$$E(p_j) = \frac{m_j}{M} \tag{C.1}$$

and

$$\check{p}_j = \frac{m_j(M-m_j)}{M^2(M+1)} \tag{C.2}$$

Equation (C.1) gives

$$m_j = ME(p_j)$$

Substituting in equation (C.2),

$$\check{p}_j = \frac{ME(p_j)[M - ME(p_j)]}{M^2(M+1)}$$

$$= \frac{E(p_j)[1 - E(p_j)]}{(M+1)}$$

Let $\check{p}_j$ be the prior estimated value of the variance of p_j and let $\tilde{\check{p}}_j$ be the value of the variance of p_j obtained when M is used.

Then, the problem is to select M so as to minimize

$$D = \sum_{j=1}^{k} (\check{p}_j - \tilde{\check{p}}_j)^2$$

$$D = \sum_{j=1}^{k} \left[\check{p}_j - \frac{E(p_j)[1 - E(p_j)]}{M + 1} \right]^2$$

We set $dD/dM = 0$ (the necessary condition for a minimum) and solve the resulting equation for the least-squares value of M, denoted by $M_{L.S.}$. This quantity is given by

$$M_{L.S.} = \frac{\sum_{j=1}^{k} [E(p_j)]^2 [1 - E(p_j)]^2}{\sum_{j=1}^{k} \check{p}_j E(p_j)[1 - E(p_j)]} - 1 \tag{C. 3}$$

and from equation (C. 1)

$$m_j = M_{L.S.} E(p_j) \tag{C. 4}$$

APPENDIX D

BAYES MODIFICATION OF THE MULTIDIMENSIONAL BETA DISTRIBUTION

As in section 3.1, consider a multinomial distribution of order k with parameters $p_1, p_2, \ldots, p_k$. Suppose the p_j's are a priori jointly distributed according to the multidimensional Beta distribution

$$f_{p_1, p_2, \ldots, p_k}(x_1, x_2, \ldots, x_k) = f_\beta(x_1, x_2, \ldots, x_k | m_1, m_2, \ldots, m_k).$$

Let E be the event that in n independent draws from the multinomial n_i fall in the i^{th} category (i=1, 2, ..., k)

$$\sum_{i=1}^{k} n_i = n$$

$$pr(E | x_1, x_2, \ldots, x_k) = \frac{n!}{n_1! \cdot n_2! \cdot \ldots \cdot n_k!} x_1^{n_1} \cdot x_2^{n_2} \cdot \ldots \cdot x_k^{n_k}$$

Using Bayes' rule

$$f_{p_1, p_2, \ldots, p_k | E}(x_1, x_2, \ldots, x_k)$$

$$= \frac{pr(E | x_1, x_2, \ldots, x_k) \, f_{p_1, p_2, \ldots, p_k}(x_1, x_2, \ldots, x_k)}{pr(E)}$$

$$f_{p_1, p_2, \ldots, p_k | E}(x_1, x_2, \ldots, x_k)$$

$$= \frac{x_1^{n_1} x_2^{n_2} \cdots x_k^{n_k} \cdot x_1^{m_1-1} x_2^{m_2-1} \cdots x_k^{m_k-1}}{\left(\int_{x_1} \cdots \int_{x_{k-1}} x_1^{n_1} x_2^{n_2} \cdot \ldots \cdot x_{k-1}^{n_{k-1}} \left(1 - \sum_{i=1}^{k-1} x_i\right)^{n_k} x_1^{m_1-1} \cdot \ldots \cdot x_{k-1}^{m_{k-1}-1} \cdot \left(1 - \sum_{i=1}^{k-1} x_i\right)^{m_k-1} dx_1 \cdot \ldots \cdot dx_{k-1} \right)} \qquad \text{(D.1)}$$

where some terms independent of the x_i's have been cancelled from the numerator and denominator.

Proceeding exactly as in Appendix B this reduces to

$$f_{p_1, p_2, \ldots, p_k | E}(x_1, x_2, \ldots, x_k)$$

$$= \frac{1}{\beta(m_1+n_1, m_2+n_2, \ldots, m_k+n_k)} x_1^{m_1+n_1-1}, x_2^{m_2+n_2-1} \cdot \ldots \cdot x_k^{m_k+n_k-1}$$

$$= f_\beta(x_1, x_2, \ldots, x_k | m_1+n_1, m_2+n_2, \ldots, m_k+n_k)$$

This is seen to be a multidimensional Beta distribution with modified parameters.

NOTE:

The denominator of equation (D.1) was (before the above-mentioned cancellation of terms)

$$\text{pr}(E) = \frac{n!}{n_1!\, n_2! \cdots n_k!} \frac{\beta(m_1+n_1, m_2+n_2, \ldots, m_k+n_k)}{\beta(m_1, m_2, \ldots, m_k)}$$

Let E_* be the event E with the added stipulation that we know the order of the occurrences. Then, clearly

$$\text{pr}(E_*) = \frac{\beta(m_1+n_1, m_2+n_2, \ldots, m_k+n_k)}{\beta(m_1, m_2, \ldots, m_k)}$$

APPENDIX E

A METHOD FOR SAMPLING FROM THE MULTIDIMENSIONAL BETA DISTRIBUTION

Consider the variables $p_1, p_2, \ldots, p_k$ having the multidimensional Beta distribution

$$f_{p_1, p_2, \ldots, p_k}(x_1, x_2, \ldots, x_k) = f_\beta(x_1, x_2, \ldots, x_k | m_1, m_2, \ldots, m_k) \tag{E.1}$$

As shown in equation (3.5) the marginal distribution of p_1 is given by

$$f_{p_1}(x_1) = f_\beta\left(x_1 \middle| m_1, \sum_{i \neq 1} m_i\right)$$

The conditional distribution of $p_2, p_3, \ldots, p_k$ given p_1 is defined by

$$f_{p_2, p_3, \ldots, p_k | p_1}(x_2, x_3, \ldots, x_k | x_1)$$

$$= \frac{f_{p_1, p_2, \ldots, p_k}(x_1, x_2, \ldots, x_k)}{f_{p_1}(x_1)}$$

$$\therefore f_{p_2, p_3, \ldots, p_k | p_1}(x_2, x_3, \ldots, x_k | x_1)$$

$$= \frac{\dfrac{1}{\beta(m_1, m_2, \ldots, m_k)} x_1^{m_1 - 1} x_2^{m_2 - 1} \cdots x_k^{m_k - 1}}{\dfrac{1}{\beta\left(m_1, \sum_{i \neq 1} m_i\right)} x_1^{m_1 - 1} (1 - x_1)^{\sum_{i \neq 1} m_i - 1}}$$

This simplifies to

$$f_{p_2, p_3, \ldots, p_k | p_1}(x_2, x_3, \ldots, x_k | x_1)$$

$$= \frac{1}{\beta(m_2, m_3, \ldots, m_k)} \left(\frac{1}{1-x_1}\right)^{k-1} \left(\frac{x_2}{1-x_1}\right)^{m_2-1} \left(\frac{x_3}{1-x_1}\right)^{m_3-1} \cdots \left(\frac{x_k}{1-x_1}\right)^{m_k-1}$$

where

$$\sum_{i=2}^{k} x_i = 1 - x_1$$

and

$$x_i \geq 0$$

Making the substitutions

$$r_j = \frac{p_j}{1 - p_1} \qquad \left(\text{also } y_j = \frac{x_j}{1-x_1}\right) \qquad j = 1, 2, \ldots, k$$

and noting that the Jacobian

$$J = \left| \frac{\partial(p_2, p_3, \ldots, p_k)}{\partial(r_2, r_3, \ldots, r_k)} \right| = (1-p_1)^{k-1}$$

we obtain

$$f_{r_2, r_3, \ldots, r_k | p_1}(y_2, y_3, \ldots, y_k | x_1)$$

$$= \frac{1}{\beta(m_2, m_3, \ldots, m_k)} y_2^{m_2 - 1}, y_3^{m_3 - 1}, \ldots, y_k^{m_k - 1}$$

$$= f_\beta(y_2, y_3, \ldots, y_k | m_2, m_3, \ldots, m_k)$$

which is a multidimensional Beta distribution of one lower dimension.

Now we know that the marginal distribution of r_2 will be the simple Beta

$$f_{r_2 | p_1}(y_2 | x_1) = f_\beta\left(y_2 \middle| m_2, \sum_{i=3}^{k} m_i\right)$$

and letting

$$s_j = \frac{r_j}{1 - r_2} \qquad j = 3, 4, \ldots, k$$

we obtain the conditional distribution

$$f_{s_3, s_4, \ldots, s_k | p_1, r_2}(z_3, z_4, \ldots, z_k | x_1, y_2)$$

$$= f_\beta(z_3, z_4, \ldots, z_k | m_3, m_4, \ldots, m_k).$$

Continuing in this way the conditional distribution will, eventually, be reduced to a simple Beta. This suggests the following method of sampling from the k-dimensional Beta distribution, $f_\beta(x_1, x_2, \ldots, x_k | m_1, m_2, \ldots, m_k)$:

i) Draw w_1 from the simple Beta $f_\beta\left(w_1 \middle| m_1, \sum_{i=2}^{k} m_i\right)$

ii) Draw w_2 from the simple Beta $f_\beta\left(w_2 \middle| m_2, \sum_{i=3}^{k} m_i\right)$

iii)

.

.

.

k-1) Draw w_{k-1} from the simple Beta $f_\beta(w_{k-1} | m_{k-1}, m_k)$

Then

$$
\begin{aligned}
x_1 &= w_1 \\
x_2 &= w_2(1-w_1) \\
x_3 &= w_3(1-w_2)(1-w_1) \\
&\vdots \\
x_{k-1} &= w_{k-1}(1-w_{k-2}) \ldots (1-w_2)(1-w_1)
\end{aligned}
$$

and

$$x_k = 1 - \sum_{i=1}^{k-1} x_i$$

As shown in equation (E.1), x_1, x_2, ..., x_k are sample values of p_1, p_2, ..., p_k, respectively.

APPENDIX F

THE TRANSIENT SOLUTIONS FOR 3-STATE, DISCRETE TIME, MARKOV PROCESSES

This appendix presents a portion of the results of another study performed by the author.[23]

Consider a 3-state, discrete time, Markov process having the transition matrix

$$P = \begin{bmatrix} 1 - a - b & a & b \\ c & 1 - c - d & d \\ e & f & 1 - e - f \end{bmatrix}$$

Define $\phi_{ij}(n)$ to be equal to the probability that the system will be in state j at time n given that it is in state i at time zero. Also, let

$$P_1 = \begin{bmatrix} 1 - 2a & a & a \\ a & 1 - 2a & a \\ a & a & 1 - 2a \end{bmatrix}$$

and

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[23] Silver, E. A., The Transient Solutions for 3-State, Discrete Time, Markov Processes. Technical Note 1. M. I. T. Operations Research Center. 1963.

$$P_2 = \begin{bmatrix} 1 - 2a & a & a \\ b & 1 - 2b & b \\ c & c & 1 - 2c \end{bmatrix}$$

Then the following flow chart and equations give $\phi_{ij}(n)$ for any combination of the parameters a, b, c, d, e and f.

The Relevant Equations

$$\phi_{ij}(n) = \frac{1}{3} - \frac{1}{3}(1-3a)^n \qquad i \neq j$$

$$n \geq 0 \qquad \text{(F.1)}$$

$$\phi_{ii}(n) = \frac{1}{3} + \frac{2}{3}(1-3a)^n$$

....

Convention:

We shall denote a transition probability by $p_{\alpha\beta}$ = pr(state is β at time $n + 1 |$ state was α at time n). Then, if we are looking for $\phi_{ij}(n)$, $i \neq j$, we denote the third state by k; if we are looking for $\phi_{ii}(n)$, we arbitrarily denote the other two states by j and k.

For the matrix, P_2, we can make the notation even simpler by noting that

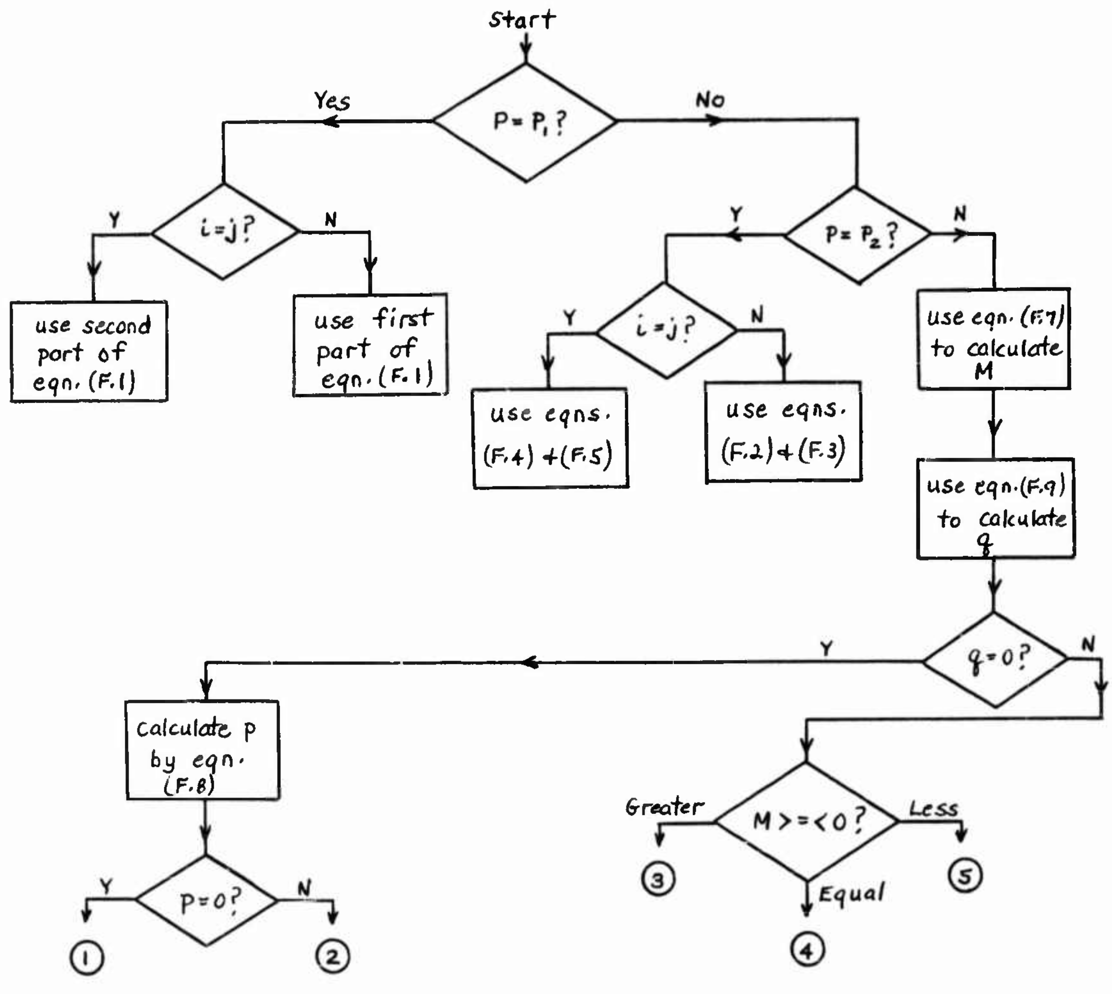
$$p_{ij} = p_{ik} = p_{i*}$$

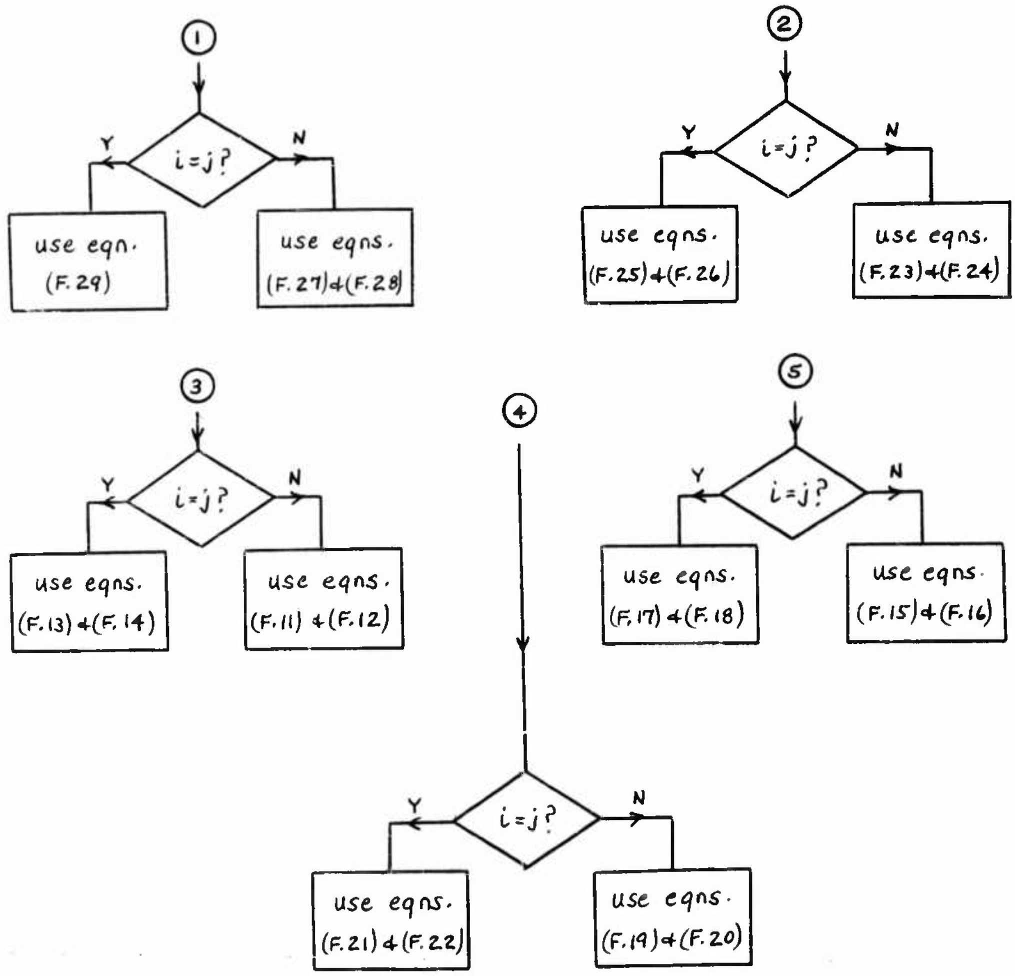
$$p_{ji} = p_{jk} = p_{j*}$$

and

$$p_{ki} = p_{kj} = p_{k*}$$

FLOW CHART OF METHOD FOR OBTAINING A PARTICULAR $\phi_{ij}(n)$, GIVEN P.





1
i=j?
Y
N
use eqn. (F.29)
use eqns. (F.27)+(F.28)
2
i=j?
Y
N
use eqns. (F.25)+(F.26)
use eqns. (F.23)+(F.24)
3
i=j?
Y
N
use eqns. (F.13)+(F.14)
use eqns. (F.11)+(F.12)
4
i=j?
Y
N
use eqns. (F.21)+(F.22)
use eqns. (F.19)+(F.20)
5
i=j?
Y
N
use eqns. (F.17)+(F.18)
use eqns. (F.15)+(F.16)

$$\phi_{ij}(n) = \frac{p_{i*}\, p_{k*}}{L} - \frac{1}{6L}\left[\left(\frac{-p-v}{2}\right)^n \left(\alpha+\frac{\beta}{v}\right) + \left(\frac{-p+v}{2}\right)^n \left(\alpha-\frac{\beta}{v}\right)\right] \qquad \begin{matrix} n \geq 0 \\ i \neq j \end{matrix}$$

(F. 2)

where

$$L = p_{i*}\, p_{j*} + p_{i*}\, p_{k*} + p_{j*}\, p_{k*}$$

$$p = 2(p_{i*}+p_{j*}+p_{k*}-1)$$

$$v = 2\sqrt{p_{i*}^2 + p_{j*}^2 + p_{k*}^2 - L}$$

$$\alpha = 3p_{i*}\, p_{k*}$$

(F. 3)

and

$$\beta = 6p_{i*}(p_{i*}p_{j*}-p_{k*}^2)$$

$$\phi_{ii}(n) = \frac{p_{j*}\, p_{k*}}{L} - \frac{1}{6L}\left[\left(\frac{-p-v}{2}\right)^n \left(w+\frac{x}{v}\right) + \left(\frac{-p+v}{2}\right)^n \left(w-\frac{x}{v}\right)\right] \qquad n \geq 0$$

(F. 4)

where L, p, and v are as in equation (F. 3)

$$w = -3p_{i*}(p_{j*}+p_{k*})$$

and (F. 5)

$$x = 6p_{i*}(p_{j*}^2+p_{k*}^2-p_{i*}p_{j*}-p_{i*}p_{k*})$$

$$\alpha = p_{ij}\, p_{ki} + p_{ij}\, p_{kj} + p_{ik} p_{kj}$$

$$\begin{aligned} \gamma = {} & p_{ij}\, p_{jk} + p_{ij}\, p_{ki} + p_{ij}\, p_{kj} + p_{ik}\, p_{ji} \\ & + p_{ik}\, p_{jk} + p_{ik}\, p_{kj} + p_{ji}\, p_{ki} \\ & + p_{ji}\, p_{kj} + p_{jk}\, p_{ki} \end{aligned} \tag{F.6}$$

$$M = p^2 - 4q \tag{F.7}$$

where

$$p = \rho - 2 \tag{F.8}$$

$$q = 1 - \rho + \gamma \tag{F.9}$$

$$\rho = a + b + c + d + e + f \tag{F.10}$$

and γ is as defined in equation (F.6)

$$\phi_{ij}(n) = \frac{\alpha}{\gamma} - \frac{1}{2\gamma}\left[\left(\frac{-p-v}{2}\right)^n\left(\alpha+\frac{\beta}{v}\right) + \left(\frac{-p+v}{2}\right)^n\left(\alpha-\frac{\beta}{v}\right)\right] \qquad \begin{matrix} n \geq 0 \\ i \neq j \end{matrix} \tag{F.11}$$

where

$$v = \sqrt{M}$$

M is defined in equation (F.7)

$$\begin{aligned} \beta = {} & 2p_{ij}^2\, p_{jk} + p_{ij}^2\, p_{ki} + p_{ij}^2\, p_{kj} + 2p_{ij}\, p_{ik}\, p_{ji} \\ & + 2p_{ij}\, p_{ik}\, p_{jk} + p_{ij}\, p_{ji}\, p_{ki} + p_{ij}\, p_{ji}\, p_{kj} \\ & + p_{ij}\, p_{jk}\, p_{ki} \end{aligned}$$

$$- p_{ij} p_{ik} p_{ki} - p_{ij} p_{jk} p_{kj} - 2p_{ij} p_{ki} p_{kj}$$

$$- p_{ij} p_{ki}^2 - p_{ij} p_{kj}^2 - p_{ik} p_{ji} p_{kj} - p_{ik} p_{jk} p_{kj}$$

$$- p_{ik} p_{ki} p_{kj} - p_{ik}^2 p_{kj} - p_{ik} p_{kj}^2 \qquad \text{(F.12)}$$

γ and α are as defined in equation (F.6) and p is as defined in equation (F.8).

$$\phi_{ii}(n) = \frac{\delta}{\gamma} - \frac{1}{2\gamma}\left[\left(\frac{-p-v}{2}\right)^n \left(w+\frac{x}{v}\right) + \left(\frac{-p+v}{2}\right)^n \left(w-\frac{x}{v}\right)\right] \qquad n \geq 0 \qquad \text{(F.13)}$$

where

$$\delta = p_{ji} p_{ki} + p_{jk} p_{ki} + p_{ji} p_{kj}$$

$$w = \delta - \gamma$$

$$x = p_{ik}^2 p_{kj} - p_{ik}^2 p_{ji} - p_{ik}^2 p_{jk} - p_{ij}^2 p_{ki} - p_{ij}^2 p_{kj}$$

$$- p_{ij}^2 p_{jk} + p_{ij} p_{ki}^2 + p_{ik} p_{kj}^2 + p_{ij} p_{kj}^2$$

$$+ p_{ik} p_{ji}^2 + p_{ik} p_{jk}^2 + p_{ij} p_{jk}^2 - p_{ik} p_{ij} p_{ki}$$

$$- 2p_{ik} p_{ij} p_{kj} - 2p_{ik} p_{ij} p_{jk} \qquad \text{(F.14)}$$

$$- p_{ik} p_{ki} p_{ji} - p_{ik} p_{ki} p_{jk} - p_{ik} p_{ij} p_{ji}$$

$$- p_{ij} p_{ki} p_{ji} - p_{ij} p_{kj} p_{ji} + p_{ik} p_{ki} p_{kj}$$

$$+ 2p_{ij} p_{ki} p_{kj} + 2p_{ik} p_{kj} p_{jk} + 2p_{ij} p_{kj} p_{jk}$$

$$+ 2p_{ik} p_{ji} p_{jk} + p_{ij} p_{ji} p_{jk}$$

and γ, p and v are as defined in equations (F.6), (F.8) and (F.12), respectively.

$$\phi_{ij}(n) = \frac{\alpha}{\gamma} - \frac{q^{n/2}}{\gamma}\,[\alpha \cos n\theta + \beta/y \sin n\theta] \qquad \begin{matrix} n \geq 0 \\ i \neq j \end{matrix} \qquad (F.15)$$

where

$$y = \sqrt{-M}$$

M is defined in equation (F.7) (F.16)

$$\theta = \tan^{-1}(-y/(-p))$$

and α, β, γ, q and p are defined in equations (F.6), (F.12), (F.6), (F.9) and (F.8), respectively.

$$\phi_{ii}(n) = \frac{\delta}{\gamma} - \frac{q^{n/2}}{\gamma}\,[-w \cos n\theta - x/(y) \sin n\theta] \qquad n \geq 0 \qquad (F.17)$$

where δ, w and x are defined in equation (F.14); y and θ are defined in equation (F.16); and γ and p are defined in equations (F.6) and (F.8), respectively. (F.18)

$$\phi_{ij}(n) = \frac{\alpha}{\gamma} + \frac{2\mu n \tau^{n}}{(\rho-2)\rho} - \frac{\alpha}{\gamma}\tau^{n} \qquad \begin{matrix} n \geq 0 \\ i \neq j \end{matrix} \qquad (F.19)$$

where

$$\mu = p_{ij}(-p_{ij}-p_{ik}-p_{ji}-p_{jk}+p_{ki}+p_{kj}) + 2p_{ik}\,p_{kj}$$

$$\tau = \frac{2-\rho}{2} \qquad (F.20)$$

and α, γ and ρ are defined in equations (F.6), (F.6), and (F.10), respectively.

$$\phi_{ii}(n) = \frac{\delta}{\gamma} + \frac{\eta n \tau^{n}}{(\rho - 2)\rho} + \frac{\gamma - \delta}{\gamma} \tau^{n} \qquad n \geq 0 \tag{F.21}$$

where

$$\eta = p_{ij}^2 + p_{ik}^2 - p_{ji}^2 - p_{jk}^2 - p_{ki}^2 - p_{kj}^2 + 2(p_{ij}p_{ik} + p_{ji}p_{ki}$$

$$+ p_{ji}p_{kj} + p_{jk}p_{ki} - p_{ji}p_{jk} - p_{jk}p_{kj} - p_{ki}p_{kj}) \tag{F.22}$$

and δ, γ, ρ and τ are defined in equations (F.14), (F.6), (F.10), and (F.20), respectively.

$$\phi_{ij}(n) = \frac{(p_{ij} - \alpha)}{\rho - 2} \delta(n) + \frac{\alpha}{\rho - 1} + \frac{(\rho - 1)\, p_{ij} - \alpha}{\rho - 1} (2 - \rho)^{n-1} \qquad n \geq 0,\ i \neq j \tag{F.23}$$

where α and ρ are defined in equations (F.6) and (F.10)

$$\text{and } \delta(k) = \begin{cases} 1 & \text{for } k = 0 \\ 0 & \text{otherwise} \end{cases} \tag{F.24}$$

$$\phi_{ii}(n) = \frac{(\rho - 2 + p_{ii} - \delta)}{\rho - 2} \delta(n) + \frac{\delta}{\rho - 1} + \frac{(\rho - 1)\, p_{ii} - \delta}{\rho - 1} (2 - \rho)^{n-1} \qquad n \geq 0 \tag{F.25}$$

where δ, ρ, and $\delta(k)$ are defined in equations (F.14), (F.10) and (F.24), respectively. (F.26)

$$\phi_{ij}(n) = (-\alpha)\, \delta(n) + (p_{ij} - \alpha)\, \delta(n-1) + \alpha \qquad n \geq 0,\ i \neq j \tag{F.27}$$

where α and $\delta(k)$ are defined in equations (F.6) and (F.24). (F.28)

$$\phi_{ii}(n) = (1 + \delta)\, \delta(n) + (1 - p_{ij} - p_{jk} - \delta)\, \delta(n-1) + \delta \qquad n \geq 0 \tag{F.29}$$

where δ and $\delta(k)$ are defined in equations (F.14) and (F.24).

APPENDIX G

THE USE OF THE HYPERGEOMETRIC FUNCTION IN THE DETERMINATION OF THE EXPECTED VALUES OF THE STEADY STATE PROBABILITIES IN A SPECIAL 2-STATE MARKOV PROCESS.

$$P = \begin{bmatrix} 1 - a & a \\ b & 1 - b \end{bmatrix}$$

where "a" is assumed exactly known, but "b" has the Beta distribution

$$f_b(x) = f_\beta(x \mid m, n)$$

G.1. Determination of the Expected Values of the Steady State Probabilities

For a given (a, b) pair

$$\pi_2 = \frac{a}{a + b},$$

the steady state probability of being in state 2.

$$\therefore \quad E(\pi_2) = E\left(\frac{a}{a+b}\right) = \int_0^1 \frac{a}{a + x} f_b(x)\, dx$$

$$= \int_0^1 \frac{a}{a + x} \frac{1}{\beta(m, n)} x^{m-1} (1 - x)^{n-1}\, dx$$

This is not an easy integral to evaluate in its current form. However, the substitution $w = 1 - x$ leads to

$$E(\pi_2) = \frac{a}{a+1} \frac{1}{\beta(m,n)} \int_0^1 w^{n-1} (1-w)^{m-1} \left[1 - w\left(\frac{1}{a+1}\right)\right]^{-1} dw \quad (G.1)$$

It is fortunate that this integral has appeared elsewhere, namely in the solution of differential equations arising in certain physics problems. More precisely, we have the hypergeometric function, F, defined by[24]

$$F(a, c-b|c|z) = \frac{\Gamma(c)}{\Gamma(b)\,\Gamma(c-b)} \int_0^1 w^{c-b-1} (1-w)^{b-1} (1-wz)^{-a} dw \quad (G.2)$$

F is tabulated only for values of the 4 parameters relevant to the physics problems for which the function was first conceived. Unfortunately, those values are not appropriate for the Markov process analysis. Therefore, we must find a convenient way of calculating F.

F is expressible as a convergent series[25]

$$F(p,q|r|z) = 1 + \frac{pqz}{r1!} + \frac{p(p+1)\,q(q+1)}{r(r+1)\,2!} z^2 + \ldots . \quad (G.3)$$

(convergent provided $|z| < 1$).

Using equations (G.1) and (G.2), there results

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[24]Morse, P. M. and Feshbach, H., Methods of Theoretical Physics, Part I, McGraw-Hill, 1961, p. 591.

[25]Ibid, p. 388.

$$E(\pi_2) = \frac{a}{a+1} \; F\left(1, n \middle| m+n \middle| \frac{1}{a+1}\right)$$

Then, utilizing equation (G.3) (valid $\because 1/(a+1) < 1$), we obtain

$$E(\pi_2) = \frac{a}{a+1}\left[1 + \frac{n}{m+n}\left(\frac{1}{a+1}\right) + \frac{n(n+1)}{(m+n)(m+n+1)}\left(\frac{1}{a+1}\right)^2 + \cdots \right.$$

$$\left. \cdots + \underbrace{\frac{n(n+1)\ldots(n+k-2)}{(m+n)(m+n+1)\ldots(m+n+k-2)}}_{k^{th} \text{ term counting the "1" as the first term}}\left(\frac{1}{a+1}\right)^{k-1} + R_k\right] \qquad \text{(G. 4)}$$

where

$$R_k = \frac{n(n+1)\ldots(n+k-1)}{(m+n)(m+n+1)\ldots(m+n+k-1)}\left(\frac{1}{a+1}\right)^k \left[1 + \frac{n+k}{m+n+k}\left(\frac{1}{a+1}\right)\right.$$

$$\left. + \frac{(n+k)(n+k+1)}{(m+n+k)(m+n+k+1)}\left(\frac{1}{a+1}\right)^2 + \ldots\right]$$

But

$$\frac{n+j}{m+n+j} < 1$$

$$\therefore R_k < \frac{n(n+1)\ldots(n+k-1)}{(m+n)(m+n+1)\ldots(m+n+k-1)}\left(\frac{1}{a+1}\right)^k \left[1 + \left(\frac{1}{a+1}\right) + \left(\frac{1}{a+1}\right)^2 + \ldots\right]$$

$$= \frac{n(n+1)\ldots(n+k-1)}{(m+n)(m+n+1)\ldots(m+n+k-1)}\left(\frac{1}{a+1}\right)^k\left(\frac{a+1}{a}\right)$$

Therefore, substituting in equation (G. 4) there follows

$$E(\pi_2) = \frac{a}{a+1}\left[1 + \frac{n}{m+n}\left(\frac{1}{a+1}\right) + \frac{n(n+1)}{(m+n)(m+n+1)}\left(\frac{1}{a+1}\right)^2 + \ldots \right.$$

$$\left. \ldots + \frac{n(n+1)\ldots(n+k-2)}{(m+n)(m+n+1)\ldots(m+n+k-2)}\left(\frac{1}{a+1}\right)^{k-1}\right] + E_k \qquad (G.5)$$

where

$$E_k = \frac{a}{a+1} R_k < \frac{n(n+1)\ldots(n+k-1)}{(m+n)(m+n+1)\ldots(m+n+k-1)}\left(\frac{1}{a+1}\right)^k \qquad (G.6)$$

It is clear that E_k can be made arbitrarily small by choosing k sufficiently large. Hence, we can come arbitrarily close to $E(\pi_2)$ by selecting a large enough k.

Finally,

$$E(\pi_1) = 1 - E(\pi_2)$$

G.2. Asymptotic Check on the $E(\pi_2)$ Formula

In equation (G.5) suppose we fix $m/(m+n) = \bar{b}$ and let m and n both tend to infinity (this is equivalent to saying that "b" is exactly known). Then,

$$\lim_{m,n\to\infty} E(\pi_2) = \lim_{m,n\to\infty} \frac{a}{a+1}\left[1 + \frac{n}{m+n}\,\frac{1}{a+1} + \frac{n}{m+n} \cdot \frac{\frac{n}{m+n} + \frac{1}{m+n}}{1 + \frac{1}{m+n}}\left(\frac{1}{a+1}\right)^2 + \ldots\ldots\right]$$

$$= \frac{a}{a+1}\left[1 + (1-b)\,\frac{1}{a+1} + (1-b)\,\frac{1-b+0}{1+0}\left(\frac{1}{a+1}\right)^2 + \ldots\right]$$

$$= \frac{a}{a+1}\left[1 + \frac{1-b}{a+1} + \left(\frac{1-b}{a+1}\right)^2 + \ldots\ .\right]$$

$$= \frac{a}{a+1} \cdot \frac{1}{1 - \frac{1-b}{a+1}} = \frac{a}{a+1} \cdot \frac{a+1}{a+1-1+b}$$

$$= \frac{a}{a+b}$$

which is the exact steady state probability when "a" and "b" are known exactly.

G.3. Monotonic Behavior of $E(\pi_2)$ as a Function of $m + n$ for Fixed $E(b)$

$$\frac{n+j}{m+n+j} = \frac{\frac{n}{m+n} + \frac{j}{m+n}}{1 + \frac{j}{m+n}}$$

$$= \frac{1 - \bar{b} + \frac{j}{m+n}}{1 + \frac{j}{m+n}} \qquad (m/(m+n) = E(b) = \bar{b})$$

$$\frac{d\left(\frac{n+j}{m+n+j}\right)}{d(m+n)} = \frac{\left(1+\frac{j}{m+n}\right)\left(-\frac{j}{(m+n)^2}\right) - \left(1-\bar{b}+\frac{j}{m+n}\right)\left(-\frac{j}{(m+n)^2}\right)}{\left(1+\frac{j}{m+n}\right)^2}$$

$$= \frac{-\bar{b}j}{(m+n)^2\left(1+\frac{j}{m+n}\right)^2} < 0 \qquad \text{for } j > 0$$

Therefore, $(n + j)/(m+n+j)$ decreases monotonically as $m + n$ increases.

But, we observe from equation (G.5) that each term in $E(\pi_2)$ is a multiple of factors of the form $(n+j)/(m+n+j)$ and terms that don't depend on $m+n$. Hence, for fixed $E(b)$, $E(\pi_2)$ monotonically decreases as $m+n$ increases.

APPENDIX H

PORTIONS OF THE $E(\pi_2)$ VALUES OBTAINED THROUGH THE USE OF THE HYPERGEOMETRIC FUNCTION

Consider the 2-state Markov process with

$$P = \begin{bmatrix} 1 - a & a \\ b & 1 - b \end{bmatrix}$$

where "a" is exactly known but $f_b(x) = f_\beta(x | m, n)$. The following tables give values of $E(\pi_2)$ accurate to 5 significant figures for the various combinations of a, $\bar{b} = m/(m + n)$, and m + n. These are only portions of the results obtained using a computer program and equations (4.1) and (4.2). (Presentation of the entire results would have required a prohibitive amount of typing).

a = 0.2

m + n \ $\bar{b}$	0.2	0.4	0.6	0.8
10	.54432	.35584	.25988	.20341
30	.51606	.34080	.25319	.20109
50	.50979	.33780	.25190	.20065
100	.50495	.33556	.25094	.20032
200	.50249	.33445	.25047	.20016
500	.50100	.33378	.25019	.20006
1000	.50050	.33355	.25009	.20003
∞	.50000	.33333	.25000	.20000

a = 0.4

m+n \ $\overline{b}$	0.2	0.4	0.6	0.8
10	.69196	.51785	.40962	.33714
30	.67601	.50616	.40321	.33458
50	.67240	.50372	.40192	.33408
100	.66958	.50187	.40096	.33370
200	.66813	.50093	.40048	.33352
500	.66726	.50037	.40019	.33341
1000	.66696	.50019	.40009	.33337
∞	.66667	.50000	.40000	.33333

a = 0.6

m+n \ $\overline{b}$	0.2	0.4	0.6	0.8
10	.76598	.61338	.50813	.43208
30	.75590	.60469	.50276	.42974
50	.75362	.60284	.50166	.42927
100	.75184	.60143	.50083	.42892
200	.75093	.60072	.50041	.42875
500	.75037	.60029	.50016	.42864
1000	.75018	.60014	.50008	.42860
∞	.75000	.60000	.50000	.42857

a = 0.8

m+n \ $\bar{b}$	0.2	0.4	0.6	0.8
10	.81095	.67687	.57815	.50308
30	.80403	.67027	.57373	.50104
50	.80247	.66885	.57282	.50062
100	.80126	.66777	.57212	.50031
200	.80063	.66722	.57178	.50015
500	.80025	.66689	.57157	.50006
1000	.80013	.66678	.57150	.50003
∞	.80000	.66667	.57143	.50000

APPENDIX I

SUMMATION EXPRESSION FOR $E(\pi_2)$ FOR A 2-STATE PROCESS WHERE BOTH TRANSITION PROBABILITIES ARE INDEPENDENTLY BETA DISTRIBUTED

$$P = \begin{bmatrix} 1-a & a \\ b & 1-b \end{bmatrix} \qquad \begin{matrix} f_a(x) = f_\beta(x \mid m_1, n_1) \\ f_b(y) = f_\beta(y \mid m_2, n_2) \end{matrix}$$

where m_1, n_1, m_2 and n_2 are positive integers

$$E(\pi_2) = \int_0^1 \int_0^1 \frac{x}{x+y} f_\beta(x \mid m_1, n_1)\, f_\beta(y \mid m_2, n_2)\, dy\, dx$$

$$\beta(m_1,n_1)\,\beta(m_2,n_2)\,E(\pi_2) = \int_0^1 x^{m_1}(1-x)^{n_1-1} \underbrace{\int_0^1 \frac{1}{x+y}\, y^{m_2-1}(1-y)^{n_2-1}\, dy}_{I}\, dx \quad \ldots(I.1)$$

Substitute $z = x+y$

$$\therefore\ I = \int_x^{1+x} \frac{1}{z}(z-x)^{m_2-1}(1+x-z)^{n_2-1}\, dz$$

$$= \int_x^{1+x} \frac{1}{z}\left[\sum_{k=0}^{m_2-1}\binom{m_2-1}{k}(-x)^k z^{m_2-1-k}\right]\left[\sum_{j=0}^{n_2-1}\binom{n_2-1}{j}(1+x)^j(-z)^{n_2-1-j}\right] dz$$

$$= \sum_{k=0}^{m_2-1}\binom{m_2-1}{k}(-x)^k \sum_{j=0}^{n_2-1}\binom{n_2-1}{j}(1+x)^j(-1)^{n_2-1-j}\int_x^{1+x} z^{m_2+n_2-k-j-3}\, dz$$

The term where $k = m_2-1$ and $j = n_2-1$ will produce a logarithm. Therefore, we must separate it.

$$I = \sum_{k=0}^{m_2-2}\binom{m_2-1}{k}(-x)^k \sum_{j=0}^{n_2-1}\binom{n_2-1}{j}(1+x)^j(-1)^{n_2-1-j}\left[\frac{(1+x)^{m_2+n_2-2-k-j}}{m_2+n_2-2-k-j} - \frac{x^{m_2+n_2-2-k-j}}{m_2+n_2-2-k-j}\right] + (-x)^{m_2-1}\sum_{j=0}^{n_2-2}\binom{n_2-1}{j}(1+x)^j(-1)^{n_2-1-j}$$

$$\cdot\left[\frac{(1+x)^{n_2-1-j}}{n_2-1-j} - \frac{x^{n_2-1-j}}{n_2-1-j}\right] + (-x)^{m_2-1}(1+x)^{n_2-1}\ln\frac{1+x}{x}$$

Substituting this expression into (I.1) leads to

$$\beta(m_1,n_1)\,\beta(m_2,n_2)\,E(\pi_2) = \int_0^1 x^{m_1-1}(1-x)^{n_1-1}\Big\{\sum_{k=0}^{m_2-2}\binom{m_2-1}{k}(-1)^k x^k$$

$$\cdot\Big[(1+x)^{m_2+n_2-2-k}\sum_{j=0}^{n_2-1}\frac{\binom{n_2-1}{j}(-1)^{n_2-1-j}}{m_2+n_2-2-k-j} - \sum_{j=0}^{n_2-1}\binom{n_2-1}{j}(1+x)^j(-1)^{n_2-1-j}$$

$$\cdot\frac{x^{m_2+n_2-2-k-j}}{m_2+n_2-2-k-j}\Big] + (-1)^{m_2-1}x^{m_2-1}\Big[(1+x)^{n_2-1}\sum_{j=0}^{n_2-2}\frac{\binom{n_2-1}{j}(-1)^{n_2-1-j}}{n_2-1-j}$$

$$-\sum_{j=0}^{n_2-2}\binom{n_2-1}{j}(1+x)^j(-1)^{n_2-1-j}\frac{x^{n_2-1-j}}{n_2-1-j}\Big] + x^{m_2-1}(-1)^{m_2-1}(1+x)^{n_2-1}\ln\frac{1+x}{x}\Big\}dx$$

Expanding the $(1+x)^u$ terms, we obtain

$$\beta(m_1,n_1)\beta(m_2,n_2)\,E(\pi_2) = \sum_{k=0}^{m_2-2}\sum_{j=0}^{n_2-1}\sum_{r=0}^{m_2+n_2-2-k}\binom{m_2-1}{k}(-1)^{n_2+k-1-j}\frac{\binom{n_2-1}{j}\binom{m_2+n_2-2-k}{r}}{m_2+n_2-2-k-j}$$

$$\cdot\int_0^1 x^{m_1+k+r}(1-x)^{n_1-1}dx$$

$$+\sum_{k=0}^{m_2-2}\sum_{j=0}^{n_2-1}\sum_{r=0}^{j}\frac{\binom{m_2-1}{k}(-1)^{n_2+k-j}\binom{n_2-1}{j}\binom{j}{r}}{m_2+n_2-2-k-j}\int_0^1 x^{m_1+m_2+n_2+r-2-j}(1-x)^{n_1-1}dx$$

$$+(-1)^{m_2-1}\sum_{j=0}^{n_2-2}\sum_{r=0}^{n_2-1}\frac{\binom{n_2-1}{j}(-1)^{n_2-1-j}\binom{n_2-1}{r}}{n_2-1-j}\int_0^1 x^{m_1+m_2+r-1}(1-x)^{n_1-1}dx$$

$$+(-1)^{m_2}\sum_{j=0}^{n_2-2}\sum_{r=0}^{j}\frac{\binom{n_2-1}{j}(-1)^{n_2-1-j}\binom{j}{r}}{n_2-1-j}\int_0^1 x^{m_1+m_2+n+r-j-2}(1-x)^{n_1-1}dx$$

$$+(-1)^{m_2-1}\sum_{s=0}^{n_2-1}\sum_{t=0}^{n_1-1}\binom{n_2-1}{s}\binom{n_1-1}{t}(-1)^t\int_0^1 x^{m_1+n_2-1+s+t}\ln(1+x)\,dx$$

The first four terms involve Beta integrals; the last two are evaluated through the use of the following formulas taken from integral tables[26]

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[26] DeHaan, D. B., Nouvelles Tables D'Intégrales Définies, Hafner, 1957, Tables 106-107.

$$\int_0^1 \ln(1+y)\, y^{2a}\, dy = \frac{2}{2a+1} \ln 2 + \frac{1}{2a+1} \sum_{u=1}^{2a+1} \frac{(-1)^u}{u}$$

$$\int_0^1 \ln(1+y)\, y^{2a-1}\, dy = \frac{1}{2a} \sum_{u=1}^{2a} \frac{(-1)^{u-1}}{u}$$

and $\int_0^1 (\ln y)^b (1-y^q)^a y^{p-1}\, dy = (-1)^b\, b! \sum_{u=0}^{a} \binom{a}{u} \frac{(-1)^u}{(p+uq)^{b+1}}$

Then

$$\beta(m_1,n_1)\,\beta(m_2,n_2)\,E(\pi_2) = \sum_{k=0}^{m_2-2} \sum_{j=0}^{n_2-1} \sum_{r=0}^{m_2+n_2-2-k} \frac{\binom{m_2-1}{k}(-1)^{n_2+k-1-j}\binom{n_2-1}{j}\binom{m_2+n_2-2-k}{r}}{m_2+n_2-2-k-j}$$

$$\cdot\, \beta(m_1+k+r+1,\, n_1)$$

$$+ \sum_{k=0}^{m_2-2} \sum_{j=0}^{n_2-1} \sum_{r=0}^{j} \frac{\binom{m_2-1}{k}(-1)^{n_2+k-j}\binom{n_2-1}{j}\binom{j}{r}}{m_2+n_2-2-k-j}\, \beta(m_1+m_2+n_2+r-1-j,\, n_1)$$

$$+ (-1)^{m_2-1} \sum_{j=0}^{n_2-2} \sum_{r=0}^{n_2-1} \frac{\binom{n_2-1}{j}(-1)^{n_2-1-j}\binom{n_2-1}{r}}{n_2-1-j}\, \beta(m_1+m_2+r,\, n_1)$$

$$+ (-1)^{m_2} \sum_{j=0}^{n_2-2} \sum_{r=0}^{j} \frac{\binom{n_2-1}{j}(-1)^{n_2-1-j}\binom{j}{r}}{n_2-1-j}\, \beta(m_1+m_2+n_2+r-j-1,\, n_1)$$

$$+ (-1)^{m_2-1} \sum_{s=0}^{n_2-1} \sum_{t=0}^{n_1-1} \binom{n_2-1}{s}\binom{n_1-1}{t}(-1)^t\, \mathcal{K}(m_1+m_2+s+t)$$

$$+ (-1)^{m_2-1} \sum_{s=0}^{n_2-1} \binom{n_2-1}{s} \sum_{k=0}^{n_1-1} \frac{\binom{n_1-1}{k}(-1)^k}{(m_1+m_2+s+k)^2}$$

where $\mathcal{K}(m_1+m_2+s+t)$

$$= \begin{cases} \dfrac{1}{m_1+m_2+s+t} \displaystyle\sum_{u=1}^{m_1+m_2+s+t} \frac{(-1)^{u-1}}{u} & \text{when } m_1+m_2+s+t \text{ is even} \\[2ex] \dfrac{2}{m_1+m_2+s+t} \ln 2 + \dfrac{1}{m_1+m_2+s+t} \displaystyle\sum_{u=1}^{m_1+m_2+s+t} \frac{(-1)^u}{u} & \text{when } m_1+m_2+s+t \text{ is odd.} \end{cases}$$

APPENDIX J

DETERMINATION OF THE EXPECTED VALUES OF THE STEADY STATE PROBABILITIES FOR A SPECIAL 3 - STATE PROCESS

$$\underline{P} = \begin{bmatrix} 1-a-b & a & b \\ c & 1-c-d & d \\ e & f & 1-e-f \end{bmatrix}$$

Where b, c, d, e and f are all exactly known, but $t = \frac{a}{1-b}$ is Beta distributed, i.e.

$$f_t(x) = f_\beta(x|m,n) = \frac{1}{\beta(m,n)} x^{m-1}(1-x)^{n-1} \qquad 0 \leq x \leq 1$$

From section 4.1.1, we know that

$$\pi_1 = \frac{ce+cf+de}{ad+ae+af+bc+bd+bf+ce+cf+de} = \frac{\frac{ce+cf+de}{1-b}}{(d+e+f)\frac{a}{1-b} + \frac{bc+bd+bf+ce+cf+de}{1-b}}$$

$$\text{and } \pi_2 = \frac{ae+af+bf}{\text{same denominator}} = \frac{(e+f)\frac{a}{1-b} + \frac{bf}{1-b}}{\text{same denominator}}$$

$$\therefore \pi_1 = \frac{A}{Bt+C} \quad \text{and} \quad \pi_2 = \frac{Dt+G}{Bt+C}$$

Where A, B, C, D and G are constants defined by

$$A = \frac{ce+cf+de}{1-b}$$

$$\dots (J.1)$$

$$B = d+e+f$$

$$C = \frac{bc + bd + bf + ce + cf + de}{1-b}$$

$$D = e + f \qquad \ldots (J.1)$$

and $G = \frac{bf}{1-b}$

Fortunately the situations here are closely related to those studied in Appendix G.

$$E(\pi_1) = E\left(\frac{A}{Bt + C}\right) = \frac{A}{B} \int_0^1 \frac{1}{t + \frac{C}{B}} \frac{t^{m-1}(1-t)^{n-1}}{\beta(m, n)} \, dt$$

Setting $w = 1 - t$ and simplifying gives

$$E(\pi_1) = \frac{A}{B + C} \frac{1}{\beta(m, n)} \int_0^1 w^{n-1}(1-w)^{m-1} \left[1 - w \frac{B}{B + C}\right]^{-1} dw$$

$$\therefore E(\pi_1) = \frac{A}{B + C} \quad F\left(1, n \middle| m+n \middle| \frac{B}{B + C}\right) \qquad \text{(convergent because } \frac{B}{B + C} < 1\text{)}$$

Where A, B and C are defined in equation (J. 1).

$$\text{Now } E(\pi_2) = E\left(\frac{Dt}{Bt + C}\right) + E\left(\frac{G}{Bt + C}\right)$$

Clearly, the second term presents no difficulty as it is similar in form to that appearing in $E(\pi_1)$. The other term is treated as follows.

$$E\left(\frac{Dt}{Bt + C}\right) = \frac{D}{B} \frac{1}{\beta(m, n)} \int_0^1 \frac{t}{t + \frac{C}{B}} t^{m-1}(1-t)^{n-1} \, dt$$

Substituting $w = 1-t$ and performing some algebraic manipulations leads to

$$E\left(\frac{Dt}{Bt + C}\right) = \frac{D}{B + C}\left(\frac{m}{m + n}\right) \frac{\Gamma(m + n + 1)}{\Gamma(m+1)\Gamma(n)} \int_0^1 w^{n-1}(1-w)^m \left[1 - \frac{B}{B+C} w\right]^{-1} dw$$

Using the definition of F (equation (G. 2) of Appendix G)

$$E\left(\frac{Dt}{Bt + C}\right) = \frac{D}{B + C}\left(\frac{m}{m + n}\right) F\left(1, n \middle| m + n + 1 \middle| \frac{B}{B + C}\right)$$

Hence, $E(\pi_2) = \frac{D}{B+C}\left(\frac{m}{m+n}\right) F\left(1, n \middle| m+n+1 \middle| \frac{B}{B+C}\right) + \frac{G}{B+C} F\left(1, n \middle| m+n \middle| \frac{B}{B+C}\right)$

Finally, $E(\pi_3) = 1 - E(\pi_1) - E(\pi_2)$

APPENDIX K

TABLE OF SIMULATION RESULTS FOR 3-STATE STEADY STATE BEHAVIOR WHEN THE TRANSITION PROBABILITIES ARE MULTIDIMENSIONAL BETA DISTRIBUTED

Expt. No.	Sample Size	$(\pi_1)_{ex}$	$\bar{\pi}_1$	$S_{\bar{\pi}_1}$ x 10^{-3}	Within 95% Conf. Region	$(\pi_2)_{ex}$	$\bar{\pi}_2$	$S_{\bar{\pi}_2}$ x 10^{-3}	Within 95% Conf. Region	T.P.D.
1	700	.3368	.3370	4.32	Yes	.2211	.2176	3.81	Y	0.7
2	700	.2528	.2470	2.88	Y	.3199	.3159	3.16	Y	2.0
3	700	.2988	.2929	2.20	No	.3382	.3342	2.24	Y	2.0
4	750	.4094	.4101	1.38	Y	.2894	.2873	1.06	N	0.4
5	700	.3807	.3685	5.49	N	.2749	.2720	3.65	Y	3.0
6	700	.4361	.4278	3.67	N	.2349	.2387	3.34	Y	1.7
7	700	.5617	.5652	5.63	Y	.2043	.1900	2.84	N	2.5
8	700	.3099	.3005	2.90	N	.2949	.2878	2.60	N	3.3
9	700	.6450	.6572	3.85	N	.1512	.1471	1.99	Y	2.4
10	700	.4094	.4082	2.04	Y	.2894	.2885	1.65	Y	0.3
11	700	.4775	.4908	5.75	Y	.3708	.3691	5.63	Y	2.6
12	700	.1470	.1385	3.00	N	.5245	.5223	5.42	Y	2.1
13	700	.4266	.4406	4.91	N	.3961	.3907	4.55	Y	2.8
14	700	.3327	.3307	2.43	Y	.3434	.3410	2.36	Y	0.9
15	700	.2523	.2512	2.73	Y	.4997	.5090	3.87	N	1.8
16	700	.3467	.3455	2.35	Y	.3552	.3569	1.93	Y	0.3
17	1200	.2727	.2980	5.27	N	.5455	.5306	5.90	N	5.1
18	700	.2527	.2378	3.13	N	.2780	.2772	4.11	Y	3.2
19	700	.2316	.2288	2.61	Y	.2692	.2624	2.59	N	1.9
20	700	.3494	.3537	2.00	Y	.3031	.3033	1.84	Y	0.9
21	600	.1404	.1377	2.24	Y	.1930	.1913	2.57	Y	0.8

Expt. No.	Sample Size	$(\pi_1)_{ex}$	$\bar{\pi}_1$	$S_{\bar{\pi}_1}$ x 10^{-3}	Within 95% Conf. Region	$(\pi_2)_{ex}$	$\bar{\pi}_2$	$S_{\bar{\pi}_2}$ x 10^{-3}	Within 95% Conf. Region	T.P.D.
22	800	.3882	.3901	2.69	Y	.2824	.2820	2.70	Y	0.4
23	800	.3882	.3922	2.09	Y	.2824	.2783	2.09	Y	0.8
24	1050	.2727	.2848	4.00	N	.5455	.5362	4.55	Y	2.4
25	700	.2527	.2467	1.85	N	.2780	.2808	2.45	Y	1.2
26	750	.4094	.4101	1.38	Y	.2894	.2873	1.06	Y	0.4
27	1000	.2000	.2012	3.77	Y	.4000	.4085	5.01	Y	1.9
28	1000	.2000	.1955	2.26	N	.4000	.3992	3.02	Y	1.1
29	1000	.2000	.1996	1.76	Y	.4000	.4011	2.31	Y	0.2
30	1000	.2857	.2864	1.87	Y	.2987	.3016	2.21	Y	0.7
31	1000	.3929	.3916	1.97	Y	.3750	.3740	2.05	Y	0.4
32	1000	.2368	.2377	1.56	Y	.4211	.4209	1.34	Y	0.2
33	1000	.2148	.2142	1.21	Y	.4003	.3989	1.16	Y	0.4
34	1000	.3593	.3592	0.95	Y	.3353	.3351	1.01	Y	0.1
35	1000	.2527	.2506	0.98	N	.3571	.3585	0.87	Y	0.4
36	1000	.2710	.2691	2.84	Y	.2897	.2907	2.82	Y	0.4
37	1000	.2963	.2970	3.35	Y	.3333	.3421	4.16	N	1.9
38	1000	.2353	.2416	3.03	N	.5882	.5858	3.63	Y	1.3

Note: $(\pi_3)_{ex}$ and $\bar{\pi}_3$ can be obtained from $\pi_1 + \pi_2 + \pi_3 = 1$.
T. P. D. is defined in equation (4. 4).

(See Legend for Experiment Numbers on next page.)

Legend for Experiment Numbers of the Above Table

Note: To save space a, b, c/d, e, f/g, h, i $\equiv \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$

Expt. No.	M	Category
1	3, 2, 3/ 2, 2, 3/ 2, 1, 3	L, L, L/L, L, L/L, L, L
2	2, 12, 7/ 2, 1, 3/ 2, 2, 3	L, H, H/L, L, L/L, L, L
3	3, 12, 8/11, 3, 8/ 2, 3, 3	L, H, H/H, L, H/L, L, L
4	3, 8, 9/11, 1, 11/11, 11, 3	L, H, H/H, L, H/H, H, L
5	9, 3, 2/ 2, 2, 3/ 1, 2, 3	H, L, L/L, L, L/L, L, L
6	11, 8, 8/ 2, 1, 1/ 3, 1, 3	H, H, H/L, L, L/L, L, L
7	9, 2, 1/11, 2, 11/ 1, 2, 2	H, L, L/H, L, H/L, L, L
8	7, 9, 10/10, 2, 8/ 1, 2, 2	H, H, H/H, L, H/L, L, L
9	12, 1, 3/12, 3, 12/12, 12, 2	H, L, L/H, L, H/H, H, L
10	12, 11, 9/ 9, 3, 11/ 9, 7, 3	H, H, H/H, L, H/H, H, L
11	7, 1, 2/ 3, 8, 1/ 2, 3, 1	H, L, L/L, H, L/L, L, L
12	12, 12, 10/ 1, 9, 3/ 1, 2, 3	H, H, H/L, H, L/L, L, L
13	8, 2, 2/ 2, 8, 3/11, 11, 2	H, L, L/L, H, L/H, H, L
14	12, 12, 10/10, 11, 11/3, 3, 3	H, H, H/H, H, H/L, L, L
15	9, 8, 10/ 2, 8, 3/ 7, 9, 3	H, H, H/L, H, L/H, H, L
16	11, 8, 8/ 7, 7, 12/ 7, 10, 2	H, H, H/H, H, H/H, H, L
17	8, 1, 1/ 1, 18, 1/ 3, 3, 14	H, L, L/L, H, L/L, L, H
18	11, 8, 9/ 3, 8, 3/ 2, 1, 7	H, H, H/L, H, L/L, L, H
19	8, 7, 8/ 8, 9, 12/ 2, 3, 8	H, H, H/H, H, H/L, L, H
20	11, 8, 10/10, 9, 11/10, 10, 10	H, H, H/H, H, H/H, H, H
21	10, 10, 20/10, 15, 25/ 2, 3, 15	Test Sequence
22	1, 2, 2/ 4, 1, 5/ 6, 3, 1	Low Diagonal, reasonably low elsewhere
23	2, 4, 4/ 8, 2, 10/12, 6, 2	2 x No. 22
24	16, 2, 2/ 2, 36, 2/ 6, 6, 28	2 x No. 17
25	33, 24, 27/ 9, 24, 9/ 6, 3, 21	3 x No. 18

Expt. No.	M	Category
26	24, 22, 18/18, 6, 22/18, 14, 6	2 x No. 10
27	6, 2, 2/ 1, 7, 2/ 1, 2, 7	Miscellaneous
28	18, 6, 6/ 3, 21, 6/ 3, 6, 21	"
29	30, 10, 10/ 5, 35, 10/ 5, 10, 35	"
30	15, 6, 9/ 8, 24, 8/ 8, 6, 26	"
31	15, 9, 6/12, 15, 3/ 6, 9, 15	"
32	25, 10, 15/10, 20, 20/ 5, 30, 15	"
33	10, 22, 8/10, 15, 25/ 8, 16, 14	"
34	3, 15, 12/24, 4, 12/16, 16, 8	"
35	4, 20, 16/ 8, 4, 28/20, 25, 5	"
36	3, 1, 6/ 2, 3, 5/ 3, 4, 3	"
37	14, 2, 4/ 2, 16, 2/ 6, 4, 30	"
38	32, 4, 4/ 3, 54, 3/ 6, 12, 42	"

APPENDIX L

THE FORM OF THE STEADY STATE PROBABILITIES IN AN N-STATE MARKOV PROCESS

For the simulations of the steady state behavior of 3 and 4 state processes (see sections 4.1.6 and 4.1.7), it was necessary to write out expressions for the steady state probabilities for the cases of exactly known transition probabilities. The purpose of this appendix is to point out some general properties of the form of the steady state probabilities which became evident from a study of the 3 and 4 state cases.

In general, to obtain the steady state probability vector $\underline{\pi} = (\pi_1, \pi_2, \ldots, \pi_N)$ for a given transition matrix P we solve the set of equations $\underline{\pi} P = \underline{\pi}$ and $\sum_{j=1}^{N} \pi_j = 1$.

For 3 states with $$P = \begin{bmatrix} 1-a_1-a_2 & a_1 & a_2 \\ b_1 & 1-b_1-b_2 & b_2 \\ c_1 & c_2 & 1-c_1-c_2 \end{bmatrix}$$

we obtain $$\pi_1 = \frac{b_1c_1 + b_1c_2 + b_2c_1}{a_1b_2 + a_1c_1 + a_1c_2 + a_2b_1 + a_2b_2 + a_2c_2 + b_1c_1 + b_1c_2 + b_2c_1}$$

$$\pi_2 = \frac{a_1c_1 + a_1c_2 + a_2c_2}{\text{denom.}}$$

$$\pi_3 = \frac{a_1b_2 + a_2b_1 + a_2b_2}{\text{denom.}}$$

For 4 states with

$$P = \begin{bmatrix} 1-a_1-a_2-a_3 & a_1 & a_2 & a_3 \\ b_1 & 1-b_1-b_2-b_3 & b_2 & b_3 \\ c_1 & c_2 & 1-c_1-c_2-c_3 & c_3 \\ d_1 & d_2 & d_3 & 1-d_1-d_2-d_3 \end{bmatrix}$$

$$\pi_1 = \frac{\begin{matrix}(b_1c_1d_1 + b_1c_1d_2 + b_1c_1d_3 + b_1c_2d_1 + b_1c_2d_2 + b_1c_2d_3 + b_1c_3d_1 + b_1c_3d_2 + \\ b_2c_1d_1 + b_2c_1d_2 + b_2c_1d_3 + b_2c_3d_1 + b_3c_1d_1 + b_3c_1d_3 + b_3c_2d_1 + b_3c_3d_1\end{matrix}}{D}$$

Where D = sum of 4 expressions (64 terms in all) similar to the numerator of π_1.

(The other 3 expressions are the numerators of π_2, π_3 and π_4).

Also, for 2 states with

$$P = \begin{bmatrix} 1-a & a \\ b & 1-b \end{bmatrix}$$

$$\pi_1 = \frac{b}{a+b}$$

and

$$\pi_2 = \frac{a}{a+b}$$

From the above results for the 2, 3 and 4 state systems, we can now suggest the following general statements about the form of the steady state probabilities of an N-state Markov process.

i) For a given process, every steady state probability will have the same denominator D, i.e. $\pi_j = \frac{N_j}{D}$ where $D = \sum_{j=1}^{N} N_j$

ii) For an N-state process, D is the sum of N^{N-1} terms - these are all the possible $(N-1)^{st}$ order cross-products obtainable by taking one off-diagonal transition probability from each of N-1 rows of the transition matrix in such a way that no cross-product involving $p_{ij}\, p_{ji}$ is obtained for any i and j.

iii) The numerator N_j is made up of all those terms of D which do not include a factor from the j^{th} row of the matrix. N_j will have N^{N-2} terms.

iv) Any one off-diagonal transition probability occurs in N^{N-2} terms of D, i.e., in $\frac{1}{N}^{th}$ of the terms.

APPENDIX M

TABLE OF SIMULATION RESULTS FOR 4-STATE STEADY STATE BEHAVIOR WHEN THE TRANSITION PROBABILITIES ARE MULTI-DIMENSIONAL BETA DISTRIBUTED.

Expt. No.	Sample Size	$(\pi_1)_{ex.}$	$\bar{\pi}_1$	$(\pi_2)_{ex.}$	$\bar{\pi}_2$	$(\pi_3)_{ex.}$	$\bar{\pi}_3$	No. (out of 3) within 95% Conf. Region.	T. P. D.
39	700	.2601	.2487	.1938	.1876	.2442	.2475	1	3.5
40	700	.2032	.1873	.2579	.2443	.2787	.2844	0	5.9
41	700	.2926	.2926	.1688	.1655	.2653	.2728	3	1.5
42	700	.2654	.2567	.2450	.2358	.2676	.2653	1	4.0
43	1000	.2654	.2560	.2450	.2350	.2676	.2661	1	4.2
44	700	.2757	.2722	.2496	.2455	.2183	.2177	1	1.6
45	700	.2438	.2322	.2463	.2336	.2481	.2475	1	5.0
46	700	.2889	.2792	.2029	.1823	.2375	.2366	1	6.2
47	700	.2889	.2822	.2029	.1838	.2375	.2344	1	5.8
48	700	.2598	.2533	.2322	.2235	.2632	.2663	1	3.0
49	700	.2619	.2631	.2523	.2513	.2630	.2631	3	0.2
50	700	.2334	.2334	.2632	.2625	.2757	.2766	3	0.2
51	700	.2363	.2349	.2455	.2447	.2508	.2525	3	0.4
52	1000	.2514	.2517	.3086	.3093	.2640	.2617	3	0.5
53	1000	.2514	.2510	.3086	.3091	.2640	.2625	3	0.4
54	1000	.2514	.2504	.3086	.3084	.2640	.2642	3	0.2
55	1000	.2514	.2511	.3086	.3094	.2640	.2631	3	0.2
56	1000	.2514	.2510	.3086	.3088	.2640	.2644	3	0.1
57	700	.2593	.2597	.2827	.2843	.2352	.2343	2	0.4
58	700	.2672	.2669	.2251	.2254	.2892	.2907	3	0.3
59	700	.2661	.2656	.2459	.2454	.2231	.2235	3	0.2
60	700	.2511	.2512	.2812	.2810	.2390	.2380	3	0.2
61	700	.2638	.2639	.2425	.2418	.2332	.2339	3	0.2
62	700	.2407	.2412	.2586	.2579	.2555	.2558	3	0.2

$(\pi_4)_{ex.}$ and $\bar{\pi}_4$ can be obtained from $\pi_1 + \pi_2 + \pi_3 + \pi_4 = 1$

T. P. D. is defined in equation (4. 4).

Legend for Experiment Numbers of the Above Table

Expt. No.	M	Comments
39	1, 2, 2, 3/ 3, 1, 1, 3/ 2, 2, 2, 3/ 2, 1, 2, 1	L throughout
40	1, 12, 10, 11/ 8, 3, 11, 7/ 7, 7, 3, 12/ 7, 11, 11, 2	L diagonal, H off-diagonal
41	7, 2, 1, 3/ 3, 8, 3, 1/ 3, 1, 10, 1/ 3, 1, 2, 11	H diagonal, L off-diagonal
42	9, 7, 12, 9/12, 7, 8, 8/ 9, 12, 7, 7/ 8, 9, 12, 8	H throughout
43	18, 14, 24, 18/24, 14, 16, 16/18, 24, 14, 14/16, 18, 24, 16	2 x No. 42
44	20, 11, 8, 9/11, 21, 10, 11/12, 13, 22, 13/13, 8, 9, 21	H off-diagonal, higher diagonal
45	8, 20, 19, 23/22, 8, 22, 22/21, 22, 10, 19/18, 19, 19, 11	H diagonal, higher off-diagonal
46	4, 8, 3, 2/ 9, 1, 7, 6/ 6, 1, 9, 7/ 5, 3, 2, 7	Random 1- 9
47	8, 16, 6, 4/18, 2, 14, 12/12, 2, 18, 14/10, 6, 4, 14	2 x No. 46
48	17, 15, 18, 15/15, 15, 20, 22/23, 20, 22, 19/21, 18, 17, 16	Random 15-24
49	24, 16, 21, 15/23, 20, 20, 18/16, 19, 19, 18/15, 20, 18, 15	" "
50	15, 23, 23, 18/23, 24, 24, 16/19, 19, 19, 15/15, 16, 20, 22	" "
51	19, 22, 15, 22/15, 17, 24, 19/22, 23, 21, 23/22, 19, 22, 24	" "
52	1, 4, 3, 2/ 3, 2, 3, 1/ 4, 4, 2, 4/ 3, 4, 3, 1	Random 1 - 4
53	2, 8, 6, 4/ 6, 4, 6, 2/ 8, 8, 4, 8/ 6, 8, 6, 2	2 x No. 52
54	5, 20, 15, 10/15, 10, 15, 5/20, 20, 10, 20/15, 20, 15, 5	5 x No. 52
55	10, 40, 30, 20/30, 20, 30, 10/40, 40, 20, 40/30, 40, 30, 10	10 x No. 52
56	15, 60, 45, 30/45, 30, 45, 15/60, 60, 30, 60/45, 60, 45, 15	15 x No. 52
57	22, 36, 24, 29/37, 30, 28, 20/36, 36, 32, 39/29, 33, 29, 21	Random 20 - 40
58	36, 25, 34, 26/36, 36, 39, 34/34, 36, 40, 28/33, 21, 38, 27	" "
59	24, 32, 36, 29/27, 33, 21, 35/37, 25, 22, 24/34, 23, 23, 33	" "
60	36, 36, 33, 22/28, 39, 25, 29/27, 35, 33, 36/39, 33, 33, 31	" "
61	49, 39, 42, 33/42, 34, 30, 35/42, 43, 43, 42/30, 34, 30, 49	Random 30 - 50
62	49, 46, 47, 39/31, 38, 38, 39/30, 39, 45, 49/48, 45, 36, 31	" "

APPENDIX N

THE EXPECTED MEAN RECURRENCE TIMES OF A 2-STATE PROCESS WHEN THE TRANSITION PROBABILITIES ARE INDEPENDENTLY BETA DISTRIBUTED

Consider the 2-state Markov process with

$$\mathbf{P} = \begin{bmatrix} 1-a & a \\ b & 1-b \end{bmatrix}$$

where

$$f_a(x) = f_\beta(x \mid m_1, n_1)$$

and

$$f_b(y) = f_\beta(y \mid m_2, n_2).$$

For "a" and "b" exactly known, the exact mean recurrence times ($\bar{n}_{ij}$'s) are found as follows:

$$\bar{n}_{11} = \overbrace{(1-a)\cdot 1}^{\substack{\text{come right}\\ \text{back to}\\ \text{state 1}}} + \overbrace{a(\bar{n}_{21}+1)}^{\substack{\text{go to state}\\ 2}}$$

and

$$\bar{n}_{21} = (b)\cdot 1 + (1-b)(\bar{n}_{21}+1)$$

These two equations yield

$$\bar{n}_{21} = \frac{1}{b}$$

and

$$\bar{n}_{11} = \frac{a+b}{b} = \frac{1}{\pi_1}$$

Similarly,

$$\bar{n}_{12} = \frac{1}{a}$$

and

$$\bar{n}_{22} = \frac{a+b}{a} = \frac{1}{\pi_2} \ .$$

Now, let "a" and "b" be independent Beta variables as indicated above. Then,

$$E(\bar{n}_{11}) = E\left(\frac{a+b}{b}\right) = 1 + E\left(\frac{a}{b}\right) = 1 + E(a)\, E\left(\frac{1}{b}\right) \quad \because \text{ a and b are independent}$$

$$= 1 + \frac{m_1}{m_1 + n_1} \; E\left(\frac{1}{b}\right).$$

$$E\left(\frac{1}{b}\right) = \int_0^1 \frac{1}{y} \, \frac{1}{\beta(m_2, n_2)} \, y^{m_2-1} (1-y)^{n_2-1} \, dy$$

$$= \frac{\beta(m_2-1, n_2)}{\beta(m_2, n_2)} = \frac{(m_2-2)!\, \cancel{(n_2-1)!}}{(m_2+n_2-2)!} \cdot \frac{(m_2+n_2-1)!}{(m_2-1)!\, \cancel{(n_2-1)!}}$$

$$= \frac{m_2 + n_2 - 1}{m_2 - 1}$$

$$\therefore \; E(\bar{n}_{11}) = 1 + \frac{m_1(m_2+n_2-1)}{(m_1+n_1)(m_2-1)} = 1 + \frac{\bar{a}\left(1 - \frac{1}{m_2+n_2}\right)}{\bar{b} - \frac{1}{m_2+n_2}}$$

Similarly,

$$E(\bar{n}_{22}) = 1 + \frac{m_2(m_1+n_1-1)}{(m_2+n_2)(m_1-1)} = 1 + \frac{\bar{b}\left(1 - \frac{1}{m_1+n_1}\right)}{\bar{a} - \frac{1}{m_1+n_1}} \qquad \text{(N.1)}$$

$$E(\bar{n}_{12}) = \frac{m_1 + n_1 - 1}{m_1 - 1} = \frac{1 - \frac{1}{m_1 + n_1}}{\bar{a} - \frac{1}{m_1 + n_1}}$$

and

$$E(\bar{n}_{21}) = \frac{m_2 + n_2 - 1}{m_2 - 1} = \frac{1 - \frac{1}{m_2 + n_2}}{\bar{b} - \frac{1}{m_2 + n_2}}$$

APPENDIX O

STATE OCCUPANCY TIMES WHEN THE TRANSITION PROBABILITIES ARE MULTIDIMENSIONAL BETA DISTRIBUTED

O.1. Determination of the Probability Mass Function of the Occupancy Time

Consider an N-state Markov process with transition matrix, $\mathbf{P} = (p_{ij})$ where the p_{ij}'s are multidimensional Beta distributed with parameters m_{ij}.

Let u_i be the number of the transition on which the system leaves state i for the first time given that it is in state i before the first transition.

From equation (4.6),

$$p_{u_i | p_{ii}}(k|x) = x^{k-1}(1-x) \qquad k \geq 1 \tag{O.1}$$

We want to know the marginal probability mass function on u_i,

$$p_{u_i}(k) = \int_0^1 p_{u_i|p_{ii}}(k|x)\, f_{p_{ii}}(x)\, dx \tag{O.2}$$

But

$$f_{p_{ii}}(x) = f_\beta\left(x \middle| m_{ii}, \sum_{j \neq i} m_{ij}\right) \tag{O.3}$$

(from section 3.2). Therefore, substituting equation (O.1) and equation (O.3) into equation (O.2), we obtain

$$p_{u_i}(k) = \int_0^1 x^{k-1} (1-x) \frac{1}{\beta\left(m_{ii}, \sum_{j\neq i} m_{ij}\right)} x^{m_{ii}-1} (1-x)^{\sum_{j\neq i} m_{ij} - 1} dx$$

$$p_{u_i}(k) = \frac{\left(\sum_j m_{ij}-1\right)! \left(\sum_{j\neq i} m_{ij}\right)}{(m_{ii}-1)!} \frac{(m_{ii}+k-2)!}{\left(\sum_j m_{ij}+k-1\right)!} \qquad k \geq 1 \qquad (O.4)$$

This can also be written as

$$p_{u_i}(k) = \begin{cases} \left(\sum_{j\neq i} m_{ij}\right) \dfrac{(m_{ii})(m_{ii}+1) \ldots (m_{ii}+k-2)}{\left(\sum_j m_{ij}\right)\left(\sum_j m_{ij}+1\right) \ldots \left(\sum_j m_{ij}+k-1\right)} & k \geq 2 \\ \dfrac{\sum_{j\neq i} m_{ij}}{\sum_j m_{ij}} & k = 1 \end{cases} \qquad (O.5)$$

This last form can be normalized by letting

$$\bar{p}_{ii} = \frac{m_{ii}}{\sum_{j=1}^{N} m_{ij}}$$

and (O.6)

$$w_i = \frac{1}{\sum_{j=1}^{N} m_{ij}}$$

for then

$$p_{u_i}(k) = \begin{cases} (1-\bar{p}_{ii})\left(\frac{\bar{p}_{ii}}{1+w_i}\right)\left(\frac{\bar{p}_{ii}+w_i}{1+2w_i}\right)\cdots\left(\frac{\bar{p}_{ii}+(k-2)w_i}{1+(k-1)w_i}\right) & k \geq 2 \\ (1-\bar{p}_{ii}) & k = 1 \end{cases} \tag{O. 7}$$

O. 2. Bounds on the Mean Occupancy Time

Using equation (O. 4), the mean occupancy time is

$$E(u_i) = \sum_{k=1}^{\infty} k\, p_{u_i}(k) = \sum_{k=1}^{\infty} k \frac{\left(\sum_j m_{ij}-1\right)!\left(\sum_{j\neq i} m_{ij}\right)(m_{ii}+k-2)!}{(m_{ii}-1)!\left(\sum_j m_{ij}+k-1\right)!} \tag{O. 8}$$

There is no apparent way to obtain this summation in closed form. However, a reasonable bounding technique has been developed. We are primarily concerned with an upper bound since a lower bound can easily be obtained by truncating the summation at a finite number of terms.

The process must, eventually, leave state i. Therefore,

$$\sum_{k=1}^{\infty} p_{u_i}(k) = 1$$

Hence, from equation (O. 5)

$$c \sum_{k=1}^{\infty} \frac{(m_{ii}+k-2)!}{\left(\sum_j m_{ij}+k-1\right)!} = 1$$

where

$$c = \frac{\left(\sum_j m_{ij}-1\right)!\left(\sum_{j\neq i} m_{ij}\right)}{(m_{ii}-1)!} \tag{O.9}$$

is independent of k. Therefore,

$$\sum_{k=1}^{\infty} \frac{(m_{ii}+k-2)!}{\left(\sum_j m_{ij}+k-1\right)!} = \frac{1}{c} = \frac{(m_{ii}-1)!}{\left(\sum_j m_{ij}-1\right)!\left(\sum_{j\neq i} m_{ij}\right)}$$

or, more generally,

$$\sum_{k=1}^{\infty} \frac{(a+k-2)!}{(b+k-1)!} = \frac{(a-1)!}{(b-1)!\,(b-a)} \qquad (b>a\geq 1) \tag{O.10}$$

Now, equation (O.8) gives

$$E(u_i) = c \sum_{k=1}^{\infty} \frac{k(m_{ii}+k-2)!}{\left(\sum_j m_{ij}+k-1\right)!}$$

$$< c \sum_{k=1}^{\infty} \frac{(m_{ii}+k-2)!}{\left(\sum_j m_{ij}+k-2\right)!}$$

$$= c \frac{(m_{ii}-1)!}{\left(\sum_j m_{ij}-2\right)!\left(\sum_{j\neq i} m_{ij}-1\right)} \qquad \text{(using equation (O.10))}$$

That is,

$$E(u_i) < \frac{\left(\sum_j m_{ij} - 1\right)! \left(\sum_{j \neq i} m_{ij}\right) \cancel{(m_{ii}-1)!}}{\cancel{(m_{ii}-1)!} \left(\sum_j m_{ij} - 2\right)! \left(\sum_{j \neq i} m_{ij} - 1\right)}$$

or

$$E(u_i) < \frac{\left(\sum_j m_{ij} - 1\right)\left(\sum_{j \neq i} m_{ij}\right)}{\left(\sum_{j \neq i} m_{ij} - 1\right)}$$

Hence, $E(u_i)$ can be bounded as follows:

$$\frac{\left(\sum_j m_{ij} - 1\right)! \left(\sum_{j \neq i} m_{ij}\right)}{(m_{ii}-1)!} \sum_{k=1}^{r} \frac{k(m_{ii}+k-2)!}{\left(\sum_j m_{ij} + k - 1\right)!} < E(u_i) < \frac{\left(\sum_j m_{ij} - 1\right)\left(\sum_{j \neq i} m_{ij}\right)}{\left(\sum_{j \neq i} m_{ij} - 1\right)} \qquad (O.11)$$

APPENDIX P

DETERMINATION OF THE EXPECTED VALUES OF THE TRAPPING PROBABILITIES IN A SPECIAL TRAPPING STATES PROBLEM

Consider the Markov process with the transition matrix

$$P = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 1 - q_1 - q_2 & q_1 & q_2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{array}$$

where q_1 and q_2 are multidimensional Beta distributed. That is,

$$f_{q_1, q_2}(x, y) = f_\beta(x, y \mid m_2, m_3, m_1)$$

$$= \frac{1}{\beta(m_1, m_2, m_3)} (1-x-y)^{m_1-1} x^{m_2-1} y^{m_3-1}$$

$$0 \le x; \; 0 \le y; \; \text{and } x + y \le 1.$$

For known q_1 and q_2, let g_j = pr(process traps in state j). Then,

$$g_2 = q_1 + (1-q_1-q_2) q_1 + (1-q_1-q_2)^2 q_1 + \ldots \qquad (P.1)$$

$$= q_1[1 + (1-q_1-q_2) + (1-q_1-q_2)^2 + \ldots]$$

$$= \frac{q_1}{q_1 + q_2}$$

When q_1 and q_2 become random variables, from equation (P.1)

$$E(g_2) = E(q_1) + E[q_1(1-q_1-q_2)] + E[q_1(1-q_1-q_2)^2] + \ldots$$

But

$$E[q_1(1-q_1-q_2)^k] = \frac{1}{\beta(m_1, m_2, m_3)} \int_0^1 \int_0^{1-x} (1-x-y)^{m_1+k-1} x^{m_2} y^{m_3-1} \, dy dx$$

Substituting $z = y/(1-x)$ and performing the double integration gives

$$E[q_1(1-q_1-q_2)^k] = \frac{\beta(m_3, m_1+k)}{\beta(m_1, m_2, m_3)} \beta(m_2+1, m_1+m_3+k)$$

$$\therefore E(g_2) = \sum_{k=0}^{\infty} \frac{\beta(m_3, m_1+k)\, \beta(m_2+1, m_1+m_3+k)}{\beta(m_1, m_2, m_3)}$$

This simplifies to

$$E(g_2) = \frac{m_2 \, \Gamma(m_1+m_2+m_3)}{\Gamma(m_1)} \sum_{k=0}^{\infty} \frac{(m_1+k-1)!}{(m_1+m_2+m_3+k)!} \qquad \text{(P. 2)}$$

Similarly,

$$E(g_3) = \frac{m_3 \, \Gamma(m_1+m_2+m_3)}{\Gamma(m_1)} \sum_{k=0}^{\infty} \frac{(m_1+k-1)!}{(m_1+m_2+m_3+k)!}$$

But we know that $E(g_2) + E(g_3) = 1$. Therefore,

$$\frac{\Gamma(m_1+m_2+m_3)(m_2+m_3)}{\Gamma(m_1)} \sum_{k=0}^{\infty} \frac{(m_1+k-1)!}{(m_1+m_2+m_3+k)!} = 1$$

or

$$\sum_{k=0}^{\infty} \frac{(m_1+k-1)!}{(m_1+m_2+m_3+k)!} = \frac{\Gamma(m_1)}{(m_2+m_3)\ \Gamma(m_1+m_2+m_3)} \qquad \text{(P. 3)}$$

NOTE:

This checks with the identity (equation (O.10)) developed in Appendix O if in the identity we substitute $a = m_1 + 1$ and $b = m_1 + m_2 + m_3 + 1$.

Substituting equation (P.3) into equation (P.2), there follows

$$E(g_2) = \frac{m_2}{m_2 + m_3}$$

and

$$E(g_3) = \frac{m_3}{m_2 + m_3}$$

APPENDIX Q

PROOF THAT $E(N.R.\,|m,n,a;k) = E(N.R.\,|m,n,a)$

Q.1. ALGEBRAIC PROOF

Equations (5.4), (5.2) and (5.3) give

$$E(N.R.\,|m,n,a;k) = \sum_{r=0}^{k} p_{\beta b}(r|\,m,n,k) \cdot E(N.R.\,|m+r,n+k-r,a) \qquad (5.4)$$

$$p_{\beta b}(r|\,m,n,k) = \frac{(m+r-1)!\,(n+k-r-1)!\,k!\,(m+n-1)!}{r!\,(m-1)!\,(k-r)!\,(n-1)!\,(m+n+k-1)!} \quad (r=0,1,\ldots,k) \qquad (5.2)$$

and

$$E(N.R.\,|m,n,a) = s\left[r_1 - c + (r_2-r_1)\frac{a}{a+1}F\left(1,n\,|\,m+n\,|\,\frac{1}{a+1}\right)\right] \qquad (5.3)$$

Substituting equation (5.2) and equation (5.3) into equation (5.4) gives

$$E(N.R.\,|m,n,a;k) = s(r_1-c) + s(r_2-r_1)\frac{a}{a+1}\sum_{r=0}^{k}\frac{(m+r-1)!\,(n+k-r-1)!\,k!\,(m+n-1)!}{r!\,(m-1)!\,(k-r)!\,(n-1)!} \cdot \frac{F(1,n+k-r\,|\,m+n+k\,|\,1/(a+1))}{(m+n+k-1)!} \qquad (Q.1)$$

Now, from equation (G. 3) of Appendix G we know that

$$F(1, p|q|1/(a+1)) = 1 + \frac{p}{q}\left(\frac{1}{a+1}\right) + \frac{p(p+1)}{q(q+1)}\left(\frac{1}{a+1}\right)^2 + \ldots \qquad (Q.2)$$

That is, a power series in (1/(a+1)). Therefore, both E(N.R. | m, n, a; k) and E(N.R. | m, n, a) can be thought of as power series in (1/(a+1)).

Let A_j = coefficient of $(1/(a+1))^j$ in E(N.R. | m, n, a; k). Then, from equations (Q. 1) and (Q. 2)

$$A_j = sa(r_2 - r_1) \sum_{r=0}^{k} \frac{(m+r-1)!\,(n+k-r-1)!\,k!\,(m+n-1)!}{r!\,(m-1)!\,(k-r)!\,(n-1)!\,(m+n+k-1)!}$$

$$\cdot \frac{(n+k-r)(n+k-r+1)\ldots(n+k-r+j-2)}{(m+n+k)(m+n+k+1)\ldots(m+n+k+j-2)}$$

$$= sa(r_2 - r_1) \sum_{r=0}^{k} \frac{(m+r-1)!\,(n+k+j-r-2)!\,k!\,(m+n-1)!}{r!\,(m-1)!\,(k-r)!\,(n-1)!\,(m+n+k+j-2)!}$$

Multiplying and dividing by

$$\frac{(n+j-2)!}{(m+n+j-2)!}$$

there results

$$A_j = \frac{sa(r_2 - r_1)(n+j-2)!\,(m+n-1)!}{(m+n+j-2)!\,(n-1)!} \cdot$$

$$\underbrace{\sum_{r=0}^{k} \frac{(m+r-1)!\,(n+k+j-r-1)!\,k!\,(m+n+j-2)!}{r!\,(m-1)!\,(k-r)!\,(m+n+k+j-1)!\,(n+j-2)!}}_{p_{\beta b}(r\,|\,m,\,n+j-1,\,k)\ \therefore\ \sum = 1}$$

Therefore,

$$A_j = \frac{sa(r_2 - r_1)n\,(n+1)\,\ldots\,(n+j-2)}{(m+n)(m+n+1)\,\ldots\,(m+n+j-2)}$$

Now, using equations (5.3) and (Q.2), this is seen to be the coefficient of $(1/(a+1))^j$ in $E(N.R.\,|\,m, n, a)$. Hence,

$$E(N.R.\,|\,m, n, a; k) = E(N.R.\,|\,m, n, a),$$

the result that was intuitively anticipated at the beginning of section 5.2.2. An interesting side issue is that the following identity has been proved

$$F(1, n\,|\,m+n\,|\,a) = \sum_{r=0}^{k} p_{\beta b}(r\,|\,m, n, k)\ F(1, n+k-r\,|\,m+n+k\,|\,a)$$

Q.2. Alternate Proof of Why $E(N.R.\,|\,m, n, a; k)$ is Independent of k, the Number of Transitions Observed

The following theorem (that is of interest for decision theory in general) will enable us to justify the independence of $E(N.R.\,|\,m, n, a; k)$ from k.

Theorem: Consider a random variable x. Suppose R(x) is a "pure" function of x; that is, it is independent of the distribution of x (this rules out such situations as $R(x) = c(x-\bar{x})^2$ because the mean, $\bar{x}$, is a function of the distribution). Let there be an experiment G whose outcome y is a function of x; more precisely, the likelihood function is $f_{y|x}(y_o|x_o)$. Then the expected posterior mean of R before performing the experiment, given that it will be performed, is equal to the mean of R without experimentation.

Proof:

$$E(R|y_o) = \int_{x_o} R(x_o)\, f_{x|y}(x_o|y_o)\, dx_o$$

$$= \int_{x_o} R(x_o) \frac{f_{y|x}(y_o|x_o)\, f(x_o)}{f_y(y_o)}\, dx_o \qquad \text{(Bayes)}$$

$$\text{expected posterior mean} = \int_{y_o} E(R|y_o)\, f_y(y_o)\, dy_o$$

$$= \int_{y_o}\int_{x_o} R(x_o) \frac{f_{y|x}(y_o|x_o)\, f_x(x_o)}{\cancel{f_y(y_o)}}\, dx\, \cancel{f_y(y_o)}\, dy_o$$

$$= \int_{x_o} R(x_o)\, f_x(x_o) \Big(\underbrace{\int_{y_o} f_{y|x}(y_o|x_o) dy_o}_{=1} \Big) dx_o^{\,*}$$

$$= \int_{x_o} R(x_o)\, f_x(x_o)\, dx_o$$

$$= E(R) \qquad \text{Q.E.D.}$$

*NOTE:

If R(x) was a function of the distribution of x, $R(x_o)$ would have been a function of the experimental outcome y_o and we could not have performed the simple integration on y_o. Hence, the proof would not be valid.

Now in the 2-state problem considered, because we cannot control the decision mechanism after the observations, E(N.R.) becomes a "pure" function of the random variable "b". Hence, $E(N.R. | m, n, a; k) = E(N.R. | m, n, a)$.

APPENDIX R

DETERMINATION OF THE EXPECTED VALUE OF PERFECT INFORMATION IN A SPECIAL 2-STATE MARKOV PROCESS

Consider a 2-state Markov process with transition matrix

$$P = \begin{bmatrix} 1-a & a \\ b & 1-b \end{bmatrix}$$ where "a" is known exactly but $f_b(b_o) = f_\beta(b_o | m, n)$

Let c be the fixed cost per period for using the process.

r_j, the reward per time period for being in state j. (j = 1, 2).

and d, the cost of observing a transition from state 2.

If it is desirable (i. e. the expected revenue is positive), we can use the process for s periods in the steady state. Also, we have the option of buying the right to observe k transitions from state 2.

In section 5. 2. 3 it was shown that the expected net revenue following an optimum procedure is

$$E[N.R. | f_b(b_o)] = \max_{O,U,S} \begin{cases} \underline{O}\text{bserve} & E(N.R. | m, n, a; k) \\ \underline{U}\text{se} & E(N.R. | m, n, a) \\ \underline{S}\text{top} & 0 \end{cases} \qquad \ldots . (R.1)$$

To evaluate E (N. R. | P. I.), we proceed as follows.

For a known value of "b", say b_o, $(\pi_2)_{ex} = \frac{a}{a + b_o}$, the exact probability of being in state 2 in the steady state and the expected reward per period in the steady state is

$$R = r_1(\pi_1)_{ex} + r_2(\pi_2)_{ex} - c$$

$$= r_1 - c + (r_2 - r_1)(\pi_2)_{ex}$$

$$= r_1 - c + (r_2 - r_1)\frac{a}{a + b_o}$$

We would use the process if $R > 0$

i.e. $r_1 - c + (r_2 - r_1) \frac{a}{a + b_o} > 0$

$$(r_1 - c)(a + b_o) + (r_2 - r_1) \; a > 0$$

or use only if $b_o (r_1 - c) + (r_2 - c) a > 0$ .... (R. 2)

If $c <$ both r_1 and r_2, we will always use the process.

If $c >$ both r_1 and r_2, we will never use the process.

Therefore, we need only consider c between r_1 and r_2.

<u>Case i</u> $r_1 < c < r_2$

Inequality (R. 2) becomes — use process when $b_o < \frac{(r_2 - c)a}{c - r_1}$

Therefore, if $b_o < \frac{(r_2 - c)a}{c - r_1}$, the $E(N.R. | b_o) = s\,[r_1 - c + (r_2 - r_1) \frac{a}{a+b_o}]$

if $b_o > \frac{(r_2 - c)a}{c - r_1}$, the $E(N.R. | b_o) = 0$

Now, from equation (5. 11)

$$E(N.R. | P.I.) = \int_{b_o} E(N.R. | b_o) \, f_b(b_o) \, db_o$$

$$= \int_0^{\frac{(r_2-c)a}{c-r_1}} s\,[r_1 - c + (r_2 - r_1)\frac{a}{a+b_o}] \, f_\beta(b_o | m, n) \, db_o$$

$$+ \int_{\frac{(r_2-c)a}{c-r_1}}^{1} (0) \, f_\beta(b_o | m, n) \, db_o$$

i.e. $E(N.R.\,|P.I.) = \int_{0}^{\frac{(r_2-c)a}{c-r_1}} s[r_1-c+(r_2-r_1)\frac{a}{a+b_o}]\,f_\beta(b_o|m,n)\,db_o$ ...(R. 3)

Finally, (R. 1) and (R. 3) would be used to give

$E.V.P.I. = E(N.R.\,|P.I.) - E[N.R.\,|f_b(b_o)]$ ... (R. 4)

<u>Case ii</u> $r_2 < c < r_1$

Inequality (R. 2) becomes — use process when $b_o > \frac{(c-r_2)a}{r_1-c}$

Consequently, $E(N.R.\,|P.I.) = \int_{\frac{(c-r_2)a}{r_1-c}}^{1} s[r_1-c+(r_2-r_1)\frac{a}{a+b_o}]\,f_\beta(b_o|m,n)\,db_o$ ...(R. 5)

Then proceed as above.

APPENDIX S

THE USE OF A GAMMA PRIOR ON THE SECOND PARAMETER OF A GAMMA DISTRIBUTION[27]

S.1 Justification for the Gamma Prior.

Consider a Gamma variable t with parameters α and β where α is assumed exactly known but β will be a random variable.

$$f_{t|\beta}(x|y) = f_{\alpha}(x|\alpha, y) = \frac{y^{\alpha}\, x^{\alpha - 1}\, e^{-yx}}{\Gamma(\alpha)} \qquad 0 \leq x \qquad \ldots (S.1)$$

Let E be the event that k independent draws from this distribution take on the values $x_1, x_2, \ldots, x_k$

$$\text{Clearly pr}(E|y) = \prod_{i=1}^{k} \frac{y^{\alpha}\, x_i^{\alpha - 1}\, e^{-yx_i}}{\Gamma(\alpha)}$$

$$= \left(\frac{y^{\alpha}}{\Gamma(\alpha)}\right)^k (x_1 x_2 \ldots x_x)^{\alpha - 1}\, e^{-y \sum_{i=1}^{k} x_i}$$

$$\text{pr}(E|y) = \frac{(x_1 x_2 \ldots x_x)^{\alpha - 1}}{[\Gamma(\alpha)]^k} \underbrace{y^{\alpha k}\, e^{-y \sum_{i=1}^{k} x_i}}_{\text{Kernel}} \qquad \ldots (S.2)$$

This suggests using a prior on β of the form

$$C\, \beta^{v-1}\, e^{-w\beta}$$

but this is seen to be a Gamma distribution with parameters v and w, i.e., the conjugate prior is

$$f_{\beta}(y) = f_{\gamma}(y|v, w) = \frac{w^v\, y^{v-1}\, e^{-wy}}{\Gamma(v)} \qquad 0 \leq y \qquad \ldots (S.3)$$

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[27] Part of the results of this appendix have been stated (but without proof) in an article by Scarf.

Scarf, H. E., "Some Remarks on Bayes Solutions to the Inventory Problem", Naval Research Logistics Quarterly, Vol. 7 (1960), pp. 591-6.

S. 2 The Marginal Distribution of "t" and its Moments

$$f_t(x) = \int_0^\infty f_{t|\beta}(x|y)\, f_\beta(y)\, dy$$

$$= \int_0^\infty \frac{y^\alpha x^{\alpha-1} e^{-yx}}{\Gamma(\alpha)} \quad \frac{w^v y^{v-1} e^{-wy}}{\Gamma(v)} \quad dy$$

$$= \frac{w^v x^{\alpha-1}}{\Gamma(\alpha)\Gamma(v)} \int_0^\infty y^{\alpha+v-1} e^{-(x+w)y} \quad dy$$

$$= \frac{w^v x^{\alpha-1}}{\Gamma(\alpha)\ \Gamma(v)} \cdot \frac{\Gamma(\alpha+v)}{(x+w)^{\alpha+v}}$$

$$f_t(x) = \frac{w^v}{\beta(\alpha, v)} \cdot \frac{x^{\alpha-1}}{(x+w)^{\alpha+v}} \qquad 0 \leq x \qquad \ldots (S.4)$$

$$E(t) = \int_0^\infty x\, f_t(x)\, dx = \int_0^\infty \frac{w^v x^\alpha}{\beta(\alpha, v)(x+w)^{\alpha+v}} \quad dx$$

This can be integrated to obtain

$$E(t) = \frac{w}{\beta(\alpha, v)} \cdot \sum_{k=0}^{\alpha} \binom{\alpha}{k} \frac{(-1)^{\alpha-k}}{\alpha+v-k-1} \qquad \text{provided } v \geq 2.$$

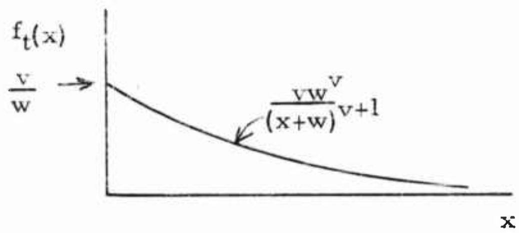
Unfortunately, this is a rather complicated function of the parameters v and w. The situation with the variance will be even worse. As was done in the discrete time problem, we would like to choose v and w to satisfy our prior estimates of E (t) and $\check{t}$. This would be very difficult with such complicated expressions for E (t) and $\check{t}$. Hence, we restrict attention to the special case of $\alpha = 1$, i.e. the density function of t given β is

$$f_{t|\beta}(x|y) = f_e(x|y) = ye^{-yx} \qquad 0 \leq x \qquad \ldots \text{(S. 5)},$$

the exponential distribution with mean $\frac{1}{y}$.

Setting $\alpha = 1$ in (S. 4), we obtain

$$f_t(x) = \frac{vw^v}{(x+w)^{v+1}} \qquad 0 \leq x \qquad (\alpha = 1) \qquad \ldots \text{(S. 6)}$$



Then $E(t) = \int_0^{\infty} \frac{vw^v x}{(x+w)^{v+1}} dx$ which integrates to

$$E(t) = \frac{w}{v-1} \qquad \text{for } v \geq 2 \text{ and } \alpha = 1 \qquad \ldots \text{(S. 7)}$$

In the same way we find that

$$E(t^2) = \frac{2w^2}{(v-2)(v-1)}$$

$$\therefore \; V_t = E(t^2) - [E(t)]^2 \quad \text{which simplifies to}$$

$$V_t = \frac{vw^2}{(v-1)^2(v-2)} \qquad \text{for } v \geq 3 \text{ and } \alpha = 1 \qquad \ldots \text{(S. 8)}.$$

S. 3 Bayes Modification of the Gamma Prior

Let the event E be as defined in Section S. 1

$$\text{Now } f_\beta (y | E) = \frac{pr (E | y) \; f_\beta (y)}{pr (E)}$$

$$f_\beta (y | E) = \frac{\cancel{\dfrac{(x_1 x_2 \ldots x_k)^{\alpha-1}}{[\Gamma(\alpha)]^k}} \; y^{\alpha k} \; e^{-y \sum_{i=1}^{k} x_i} \; \cancel{w^v} \; y^{v-1} \; e^{-wy}}{\int_0^\infty \cancel{\dfrac{(x_1 x_2 \ldots x_k)^{\alpha-1}}{[\Gamma(\alpha)]^k}} \; y^{\alpha k} \; e^{-y \sum_{i=1}^{k} x_i} \; \cancel{w^v} \; y^{v-1} \; e^{-wy} \, dy}$$

$$= \frac{(w + \sum_{i=1}^{k} x_i)^{v + \alpha k}}{\Gamma (v + \alpha k)} \; y^{v+\alpha k - 1} \; e^{-y(w + \sum_{i=1}^{k} x_i)}$$

i.e. $f_\beta (y | E) = f_\gamma (y | v + \alpha k, \; w + \sum_{i=1}^{k} x_i)$, a Gamma distribution with modified parameters. Note that the individual x_i values are not important; all we need know is their sum.

APPENDIX T

THE USE OF A NORMAL PRIOR ON THE MEAN OF A NORMAL DISTRIBUTION[28]

T.1 Justification for the Normal Prior.

Consider a Normal variable whose variance σ^2 is assumed exactly known, but whose mean μ will be a random variable.

$$f_{r|\mu}(x \mid y) = f_N(x \mid y, \sigma^2) = \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left(\frac{-(x-y)^2}{2\sigma^2}\right) \quad -\infty < x < \infty \qquad ..(T.1)$$

Let E be the event that k independent draws from this distribution take on the values $x_1, x_2, \ldots, x_k$.

$$\text{Clearly, } pr(E \mid y) = \prod_{i=1}^{k} \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left(-\frac{(x_i - y)^2}{2\sigma^2}\right)$$

$$= \left(\frac{1}{\sqrt{2\pi}\,\sigma}\right)^k \prod_{i=1}^{k} \exp\left(-\frac{(x_i^2 - 2x_i y + y^2)}{2\sigma^2}\right)$$

which can be expressed as

$$pr(E \mid y) = C \exp\left(\frac{-(y-a)^2}{2b\,\sigma^2}\right) \quad \text{where a, b and c are constants}$$

Kernel

This suggests using a prior on μ of the form

$$C \exp\left(\frac{-(\mu-v)^2}{2w\sigma^2}\right)$$

---

[28] A portion of the results of this appendix have been given by Raiffa and Schlaifer.[29] However, for completeness, they are rederived here with slight notational changes.

[29] Op. Cit, pp. 294-6

This is merely a Normal distribution with mean v and variance $w\sigma^2$, i.e., the conjugate prior is

$$f_{\mu}(y) = f_N(y \mid v,\ w\sigma^2) = \frac{1}{\sqrt{2\pi w}\,\sigma} \exp\left(\frac{-(y-v)^2}{2w\sigma^2}\right) \qquad -\infty < y < \infty \qquad \ldots \text{(T. 2)}$$

T. 2 The Marginal Distribution of "r" and its Moments.

$$f_r(x) = \int_{-\infty}^{\infty} f_{r\mid\mu}(x \mid y)\ f_{\mu}(y)\ dy$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left(\frac{-(x-y)^2}{2\sigma^2}\right) \frac{1}{\sqrt{2\pi w}\,\sigma} \exp\left(\frac{-(y-v)^2}{2w\sigma^2}\right) dy$$

After performing the integration on y and considerable simplification there results,

$$f_r(x) = \frac{1}{\sqrt{2\pi(w+1)}\,\sigma} \exp\left(\frac{-(x-v)^2}{2(w+1)\sigma^2}\right) = f_n(x \mid v,\ (w+1)\sigma^2) \qquad \ldots \text{(T. 3)}$$

Hence, r is normally distributed with mean v and variance $(w+1)\sigma^2$.

Therefore, $E(r) = v$ and $V_r = (w+1)^2$.

T. 3 Bayes Modification of the Normal Prior.

Let the event E be as defined in section T. 1.

Now $f_{\mu}(y \mid E) = \dfrac{pr(E \mid y)\ f_{\mu}(y)}{pr(E)}$

$$= \frac{\left(\frac{1}{\sqrt{2\pi}\,\sigma}\right)^k \prod_{i=1}^{k} \exp\left(\frac{-(x_i - 2x_i y + y^2)}{2\sigma^2}\right) \frac{1}{\sqrt{2\pi w}\,\sigma} \exp\left(\frac{-(y-v)^2}{2w\sigma^2}\right)}{\left(\frac{1}{\sqrt{2\pi}\,\sigma}\right)^k \prod_{i=1}^{k} \exp\left(\frac{-(x_i - 2x_i y + y^2)}{2\sigma^2}\right) \frac{1}{\sqrt{2\pi w}\,\sigma} \exp\left(\frac{-(y-v)^2}{2w\sigma^2}\right) dy}$$

After considerable straight-forward manipulations (such as completing the square in "y") there follows

$$f_\mu(y\,|\,E) = \frac{1}{\sqrt{2\pi \frac{w}{kw+1}}\,\sigma} \exp\left(\frac{-\left(y - \frac{v+w\sum_{i=1}^{k} x_i}{kw+1}\right)^2}{\frac{2w}{kw+1}\sigma^2}\right)$$

i.e. $f_\mu(y\,|\,E) = f_N\left(y \,\middle|\, \frac{v+w\sum_{i=1}^{k} x_i}{kw+1}, \frac{w}{kw+1}\sigma^2\right)$, a Normal distribution with modified parameters. Again note that the individual x_i values are not important; all we need know is their sum.

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GLOSSARY OF SYMBOLS

Symbol	Meaning	Page
a_{ij}	Transition rate from state i to state j in a continuous time process ($i \neq j$)	127
A	$A = (a_{ij})$, matrix	127
B	Event	137
B(F)	Event	9
(B(F), s)	Joint event that B(F) occurs and the starting state is s	12
c	Cost per period for using a Markov process	93
c_{ij}	Cost of assuming that matrix ${}^iP$ is in use when ${}^jP$ is really governing the process	19
C(k)	Cost of assuming matrix ${}^kP$	19
cov(p, q)	Covariance of p and q	37
d	Cost of observing a transition	93
d_{rs}	Cost of observing a transition from state r to state s	24
E, E_r	Events	35, 94
E_k	Error in $E(\pi_2)$ when only k terms are used in the hypergeometric expansion	50
$E(x) = \overline{x}$	Expected value of the random variable x	37

Symbol	Meaning	Page
$E(N.R. \mid m, n, a)$	Expected net revenue in a 2-state process given that one transition probability is known exactly at "a" and the other is Beta distributed with parameters "m" and "n"	95
$E(N.R. \mid m, n, a; k)$	Same as above except k observations of transitions related to the unknown probability will be taken (does not include the cost of observations)	95
$E(N.R. \mid (m_{ij}))$	Expected net revenue following an optimum policy without experimentation when the transition probabilities are multidimensional Beta distributed with parameters (m_{ij})	105
$E(N.R. \mid (m_{ij}), k)$	Same as above except k transitions will be observed	105
$E(N.R. \mid f_x(x_o))$	Expected net revenue following an optimum policy given that the random variable x has the density function shown	112
$E(N.R. \mid h, i, (m_{ij}))$	Expected net revenue in the next h periods given that a transition will occur from state i just before the first period and the transition probabilities are multidimensional Beta distributed with parameters (m_{ij})	122
$E(N.R. \mid P.I.)$	Expected net revenue following an optimum policy given perfect information about the unknown parameter(s)	112
E.V.P.I.	Expected value of perfect information	112

Symbol	Meaning	Page
$E(\pi_2 \mid m, n, a)$	Expected value of the steady state probability of being in state 2 of a 2-state process where "a" is known exactly but "b" is Beta distributed with parameters m and n.	95
$E(\pi_j \mid m_1, n_1, m_2, n_2)$	Expected value of the steady state probability of being in state j of a 2-state Markov process whose transition probabilities are multidimensional Beta distributed with parameters $\begin{bmatrix} n_1 & m_1 \\ m_2 & n_2 \end{bmatrix}$	59
$f_{a,b,\ldots,d}(w, x, \ldots, z)$	Joint density function of the random variables a, b, . . . , d evaluated at the point (w, x, . . . , z)	36
$f_{a \mid b}(w \mid x)$	Conditional density function of the random variable "a" evaluated at the point "w" given that the random variable "b" has taken on the value "x"	36
f_i	Number of times that the process has been in state i	40
f_{ij}	Number of transitions from state i to state j	40
$f_N(\bar{r}_o \mid \mu_o, \sigma^2)$	Normal distribution	140
$f_\beta(x_1, x_2, \ldots, x_k \mid m_1, m_2, \ldots, m_k)$	Multidimensional Beta distribution	37

Symbol	Meaning	Page
$f_\gamma(x\|v, w)$	Gamma distribution	128
$F(a, c-b\|c\| z)$	Hypergeometric function (integral form)	197
$F(p, q\|r\|z)$	Hypergeometric function (series form)	197
g_j	Probability of trapping in state j	88
$M = (m_{ij})$	Parameters of the multidimensional Beta prior distributions on the probabilities of a Markov transition matrix	103
$M_{L.S.}$	Value assigned to the sum of the parameters of a single multidimensional Beta distribution by the least squares technique of Appendix C	38
$m_1^* + n_1^*$	Smallest integers for which $\bar{a} = m_1/(m_1+n_1)$ takes on a specified value	59
n_i	Number of occurrences of the i^{th} category of a multinomial distribution	35
n_{ij}	Number of transitions to get to state j for the first time given that the process is in state i before the first transition	72
${}^k\bar{n}_{ii}$	Mean recurrence time for state i given that matrix ${}^kP$ is being used	15
N	Number of states in a Markov process	---

Symbol	Meaning	Page
$N(F)$	Number of possible sequences producing the transition frequency count $F = (f_{ij})$	10
$\check{p}$	Variance of p	37
$\tilde{\check{p}}$	Value assigned to the variance of p by least squares technique	178
p_i	Probability that a random draw falls in the i^{th} category of a multinomial	35
p_{ij}	Probability that the state at time $n+1$ is j given that the state at time n is i	40
$\bar{p}_{ii}$	Expected value of diagonal transition probability of i^{th} row of matrix; $\bar{p}_{ii} = m_{ii} / \sum_j m_{ij}$	68
$p_x(k)$	Probability that the discrete random variable "x" takes on the value "k"	15
$p_{x\|y}(k\|j)$	Conditional probability that the discrete random variable "x" takes on the value "k" given that the variable "y" assumes the value "j"	15
$p_{\beta b}(r\|m, n, k)$	Beta-binomial probability mass function	94
${}^k\mathbf{P} = ({}^k p_{ij})$	k^{th} possible transition matrix in a multi-matrix Markov process	9
Q	Number of possible transition matrices in a multi-matrix Markov process	8

Symbol	Meaning	Page
r_j	Reward per period for being in state j	69
r_{ij}	Reward per transition from state i to state j	21
R	Expected reward per period in the steady state when the transition probabilities and rewards are known exactly	92
$R_i(n)$	Expected reward in the next n periods given that the present state is i and the transition probabilities and rewards are known exactly	148
R_t	Expected reward per transition in the steady state when the transition probabilities and rewards are known exactly	144
s	Number of periods in the steady state that the process will be used	94
s_k	Probability that the process starts in state s given that matrix ${}^kP$ is being used	12
$s_{\overline{\pi}_j}$	Sample standard deviation of the mean of the steady state probability of being in state j	63
s_{ij}	Total observed reward in f_{ij} transitions from state i to state j	147
$\bar{t}_{ij}(n)$	Expected number of times in state j in the next n periods given that the present state is i	149

Symbol	Meaning	Page
T_k	$T_k = \sum_{i=1}^{k} r_i$	139
T.P.D.	Total per cent deviations of the sample means from the exact steady state values $T.P.D. = 100 \sum_{j=1}^{N} \lvert (\pi_j)_{ex} - \bar{\pi}_j \rvert$	63
u_i	Number of the transition on which the system leaves state i for the first time given that it is in state i before the first transition	76
${}^k\bar{u}_i$	Mean number of transitions to first leave state i given that the process is in state i before the first transition and ${}^k\mathbf{P}$ is the transition matrix	19
$v_h(\underline{a}', i)$	Expected cost if an optimal policy is followed and we are in state i with probability vector $\underline{a}'$ over the matrices of a multi-matrix process and there are h decision periods left	25
v_{ij}, w_{ij}	Parameters of the prior distributions in the Exponential-Gamma and Normal-Normal frameworks	146, 147
$v_i(m, n)$	Value function of a special 2-state Markov decision problem	119
w_i	$w_i = \frac{1}{\sum_j m_{ij}}$	77
a	i) Discount factor ii) Probability that ${}^1\mathbf{P}$ is being	101 26

Symbol	Meaning	Page
	used when there are only 2 possible matrices in a multi-matrix process	
α_k	Probability that ${}^kP$ is being used in a multi-matrix process	8
$\beta(m_1, m_2, \ldots, m_k)$	Generalized Beta function	36
$\Delta_{km}(x)$	Change in $E(R_t)$ caused by the observation of a reward of value x to x + dx for a transition from state k to state m	155
${}^k\pi_s$	Steady state probability of being in state s given that matrix ${}^kP$ is being used.	12
$\bar{\pi}_j$	Sample mean value of π_j	63
$(\pi_j)_{ex}$	Steady state probability of being in state j when the transition probabilities are assumed exactly known at their mean values	63
σ_x	Standard deviation of x	---
$\phi_{ij}(n\|P)$	Probability that the state at time n is j given that the state at time 0 is i and the transition matrix is P	82
$\bar{\phi}_{ij}(n)$	Unconditional (marginal) probability that the state at time n is j given that the state at time 0 is i	82
${}^k\phi_{ij}(n)$	Probability that the state at	13

Symbol	Meaning	Page
	time n is j given that the state at time 0 is i and ^kP is being used	
$[\phi_{ij}(n \mid a,b)]^*$	* denotes that a^j is replaced by $\bar{a}^j$ and b^k by $\bar{b}^k$	83
$x\downarrow$ as $y\uparrow$	x monotonically decreases as y increases	57

BIOGRAPHICAL SKETCH

Edward A. Silver was born in Montreal, Canada on June 13, 1937. He attended grammar school and West Hill High School in Montreal.

Mr. Silver's undergraduate education was taken at McGill University where in May, 1959 he received a Bachelor of Civil Engineering (Applied Mechanics Option). While at McGill, Mr. Silver was the recipient of several scholarships and at the 1959 convocation he was named a university scholar and was awarded the C. Michael Morssen Gold Medal for Great Distinction and Engineering Promise, a British Association Medal for Great Distinction and the Robert Forsyth Prize for Theory of Structures.

Since entering graduate school in September, 1959, Mr. Silver has been a research assistant in the M. I. T. Operations Research Center. His work in this capacity has led to the presentation of a paper entitled "The Use of the Hypergeometric Function as Part of Bayesian Estimation in a Two-State Markov Process" at the May 1963 National meeting of the Operations Research Society of America in Cleveland, Ohio. He is also the author of "The Transient Solutions for 3-State, Discrete Time, Markov Processes", a publication of the M. I. T. Operations Research Center. During his first year at M. I. T., Mr. Silver held a Johnson Foundation Fellowship.

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