QUASI-MARTINGALES AND STOCHASTIC INTEGRALS

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Chapter I: Preliminary Discussion

Let (Ω, F, P) be a probability measure space on which is defined a family of real valued random variables (r.v.'s) $\{X(t);\ t\epsilon T\}$ where T is a subset of the real line. We will always assume $E(|X(t)|) < \infty$ for every $t\epsilon T$. Let $\{F(t);\ t\epsilon T\}$ be a family of sub σ-fields of F with $F(s) \subset F(t)$ for every $s, t\epsilon T$ with $s \leq t$. The family of r.v.'s $\{X(t);\ t\epsilon T\}$ is said to be well adapted to the family of sub σ-fields $\{F(t);\ t\epsilon T\}$ if $X(t)$ is $F(t)$ measurable for every $t\epsilon T$, and we will then write $\{X(t), F(t);\ t\epsilon T\}$ to indicate this relation. The family $\{X(t), F(t);\ t\epsilon T\}$ will be referred to as a stochastic process. After specifying a particular process $\{X(t), F(t);\ t\epsilon T\}$ we will often refer to it as the "X-process," in order to simplify writing. In many cases $F(t)$ is the minimal σ-field with respect to (w.r.t.) which the family of r.v.'s $\{X(s);\ s\epsilon T,\ s \leq t\}$ is measurable. We will denote such σ-fields by $\mathcal{B}(X(s);\ s\epsilon T,\ s \leq t)$.

A process $\{X(t), F(t);\ t\epsilon T\}$ is called a martingale process if for every $s, t\epsilon T$ with $s \leq t$, $E(X(t)|F(s)) = X(s)$ with probability one (a.s., a.e.), and is called a semi-martingale (super-martingale) process if $E(X(t)|F(s)) \geq X(s)$ a.s. $(E(X(t)|F(s)) \leq X(s)$ a.s.$)$.

This can be restated as follows: Let

$$\mu_{s,t}(\Lambda) = \int_{\Lambda} X(t,\omega)\,dP(\omega) \quad \text{for } \Lambda\epsilon F(s)$$

Then by the Radon-Nikodym theorem, there exists an $F(s)$ measurable function which we denote by $E(X(t)|F(s))$ such that

ERRATA

"Quasi-Martingales and Stochastic Integrals"

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Page	Line	
3	3	replace $\cup$ by Δ = (symmetric difference).
12	1	$([\inf_n \tau_{n\nu}(\omega) \leq t] \cap \sim \Delta_\nu(t)) \cup ([\inf_n \tau_{n\nu}(\omega) \leq t] \cap \Delta_\nu(t))$.
	2	replace $\bigcup_{n=1}^{\infty}$ by $\bigcup_{n=0}^{\infty}$
	5	replace $\bigcup_{n=1}^{\infty}$ by $\bigcup_{n=0}^{\infty}$
	13	replace $\leq$ by $=$.
14	8	replace $i \leq i \leq n$ by $1 \leq i \leq N$
15	19	replace ν by v
	20	replace ν by v
16	13	$\prod_n(S,t) = (\prod_n(t) - \prod_n(S)) \cup \{\text{largest element in } \prod_n(S)\}$
18	1	$= \int_\Lambda X(t)dP - \lim_{n \to \infty} \{\int_\Lambda (\sum_{\pi_n} C_{n,j}(X)dP + \int_\Lambda (\sum_{n(S,t)} C_{n,j}(X)dP\}$.
21	15	$v(\omega) = \lim_{n \to \infty} \sum_{\pi_n} \lvert \Delta_{n,j}(X_2) \rvert$ a.s.
25	7	for each $t \in T_0$.
27	4	$X_n^*(t) = X'_{n_k}(t)$
33	9	$\sup_{t,\omega} \lvert Y_\nu(t,\omega) \rvert \leq \nu$.
39	6	$- \sum_{\pi_n} \int_{[\tau_\nu(\omega) < t_{n,j}]} \lvert C_{n,j}(X) \rvert dP$
	7	$- \sum_{\pi_n} \int_{[\tau_\nu(\omega) < t_{n,j}]} \lvert C_{n,j}(X) \rvert dP$

Page	Line	
67	2	$\operatorname{Plim}_{n\to\infty} \sum_{\Pi n} 1/2\, \Delta_{n,j}(Y_1)\, \Delta_{n,j}(X_1) = 1/8\, ([(Y_1+X_1)^2]_2(1) - [(Y_1-X_1)^2]_2(1))$
	11	$\xi_{2v} = [(Y_{1v} + X_{1v})^2]_2 \ , \ \bar{\xi}_{2v} = [(Y_{1v} - X_{1v})^2]_2$
75	9	$S(\epsilon,\eta) \leq \operatorname{Min} \lvert t_{n,j+1} - t_{n,j} \rvert$.
82	21	Theorem 3.3.2
92	4	replace possible by possibly.

$$\mu_{s,t}(\Lambda) = \int_\Lambda X(t,\omega)dP(\omega) = \int_\Lambda E(X(t)|F(s))(\omega)dP(\omega) \text{ for } \Lambda\epsilon F(s)$$

Furthermore $E(X(t)|F(s))$ is unique except on an $F(s)$ set of measure zero.

The process $\{X(t), F(t);\ t\epsilon T\}$ is then a martingale if for every $s,t\epsilon T$ with $s \leq t$,

$$\int_\Lambda X(s,\omega)dP(\omega) = \int_\Lambda X(t,\omega)dP(\omega) \text{ for } \Lambda\epsilon F(s)$$

Correspondingly we can say the process is a semi-martingale if for every $s,t\epsilon T$ with $s \leq t$,

$$\int_\Lambda X(s,\omega)dP(\omega) \leq \int_\Lambda X(t,\omega)dP(\omega) \quad \text{for every } \Lambda\epsilon F(s)$$

and the process is a super-martingale if for every $s,t\epsilon T$ with $s \leq t$,

$$\int_\Lambda X(s,\omega)dP(\omega) \geq \int_\Lambda X(t,\omega)dP(\omega) \quad \text{for every } \Lambda\epsilon F(s)$$

We will assume from now on T is a closed interval, and hence it is no further restriction to assume T is the closed unit interval $[0,1]$. This will be assumed throughout the thesis.

We can think of a process $\{X(t), F(t);\ t\epsilon T\}$ as a function X of two variables defined on the space $T\times\Omega$. For each fixed $t\epsilon T$, $X(t,\cdot)$ is a r.v. defined on $(\Omega,F,P,)$ and is measurable w.r.t. $F(t)$, and for each fixed $\omega\epsilon\Omega$, $X(\cdot,\omega)$ is a real valued function with domain T. A sample function of the process is simply a member of the family $\{X(\cdot,\omega);\ \omega\epsilon\Omega\}$ of real valued functions with domain T. We will be interested in analytic properties of the sample functions. However, in order to make probability statements about analytic properties of the sample functions, we must have separability of the process w.r.t. the class $\mathcal{A}$ of (finite or infinite) closed intervals. The process is said to be separable relative to $\mathcal{A}$ if there is a denumerable subset T_0 of T and a set $\Lambda \epsilon F$ with $P(\Lambda) = 0$ such that if $A \epsilon \mathcal{A}$, and I is an open interval, then

$$[X(t,\omega)\epsilon A;\ t\epsilon I\cap T] \cap [X(s,\omega)\epsilon A;\ t\epsilon I\cap T_0] \subset \Lambda$$

where $[X(t,\omega)\epsilon A;\ t\epsilon I\cap T] = \{\omega|\ X(t,\omega)\epsilon A;\ t\epsilon I\cap T\}$. In general we will let $[\cdots]$ denote the set of all $\omega\epsilon\Omega$ such that "$\cdots$" is true.

Separability w.r.t. $\mathcal{A}$ implies that if I is any open interval, then

$$\sup_{t\epsilon I\cap T} X(t,\omega),\quad \inf_{t\epsilon I\cap T} X(t,\omega) \text{ and } \overline{\lim_{s\to t}}\ X(s,\omega),\quad \underline{\lim_{s\to t}}\ X(s,\omega)$$

are all r.v.'s.

We remark here that any process $\{X(t),\ F(t);\ t\epsilon T\}$ which is a.s. sample continuous is separable w.r.t. $\mathcal{A}$. By a.s. sample continuity we mean there exists a set $\Lambda\epsilon F$ with $P(\Lambda) = 0$ such that if $\omega \notin \Lambda$, then

$$\lim_{s\to t} X(s,\omega) = X(t,\omega) \text{ for every } t\epsilon T.$$

Further, if T_0 is a denumerable dense subset of T, then it is a separating set.

We proceed now to the definition of a quasi-martingale process.

Definition 1.1.1

The process $\{X(t),\ F(t);\ t\epsilon T\}$ will be called a quasi-martingale process if there exists a martingale process $\{X_1(t),\ F(t);\ t\epsilon T\}$ and a process $\{X_2(t),\ F(t);\ t\epsilon T\}$ with a.e. sample function of bounded variation on T such that

$$P([X(t) = X_1(t)+X_2(t);\ t\epsilon T]) = 1$$

When we say the process $\{X_2(t),\ F(t);\ t\epsilon T\}$ has a.e. sample function of bounded variation on T we mean that except for $\omega\epsilon\Lambda$, with $P(\Lambda) = 0$, $X_2(\cdot,\omega)$ is a real valued function of bounded variation over T.

From now on we will always write $[X]_1$ or simply X_1 for the martingale, and $[X]_2$ or simply X_2 for the process of bounded variation in the decomposition of the quasi-martingale X. It is hoped that this will not be confused with the bracket notation used to indicate subsets of Ω.

We now give some simple examples of such processes.

Let $\{Z(t), F(t); t\in T\}$ be the Brownian motion process with $T = [0,1]$; i.e., the process has independent, normally distributed increments with $E(Z(t)-Z(s)) = 0$ and $E(|Z(t)-Z(s)|^2) = \sigma^2|t-s|$ where $\sigma > 0$ is fixed and $s,t\in T$. We assume $Z(0)= 0$ a.s. so that the process is a martingale process. We further assume $F(t) = \mathcal{B}(X(s); s\in T, s \leq t)$. Let $X(t) = \exp[Z(t)v]$ for every $t\in T$, where v is an arbitrary positive real number. If $u > 0$ and $t + u \leq 1$,

$$E(X(t+u)|F(t)) = E(\exp[Z(t+u)v]|F(t))$$
$$= E(\exp[(Z(t) + Z(t+u) - Z(t))v]|F(t))$$
$$= \exp[Z(t)v]\ E(\exp[(Z(t+u) - Z(t))v])$$
$$= X(t)\ \exp[\sigma^2v^2u/2]$$

If we let

$$X_2(t) = \int_0^t \frac{\sigma^2v^2}{2} X(s)dS \quad \text{for every } t\in T,$$

then the process $\{X_2(t), F(t); t\in T\}$ has a.e. sample function of bounded variation on T. That $X_2(t)$ is defined follows from the fact that the Brownian motion process is a.s. sample continuous. We show the process $\{X_1(t) = X(t) - X_2(t), F(t); t\in T\}$ is a martingale. Again assume $u > 0$ and $t + u \leq 1$, then

$$E(X_1(t+u)|F(t)) = E(X(t+u) - X_2(t+u)|F(t))$$

$$= X(t)\exp[\sigma^2 v^2 u/2] - E(X_2(t+u)|F(t))$$

Now

$$E(X_2(t+u)|F(t)) = E(\int_0^{t+u} \frac{\sigma^2 v^2}{2}\exp[Z(s)v]ds|F(t))$$

$$= X_2(t) + \int_0^u \frac{\sigma^2 v^2}{2} E(\exp[Z(t+s)v]|F(t))ds$$

$$= X_2(t) + \int_0^u \frac{\sigma^2 v^2}{2}\exp[Z(t)v]\exp[\sigma^2 v^2 s/2]ds$$

$$= X_2(t) + X(t)(\exp[\sigma^2 v^2 u/2]-1)$$

Then $E(X_1(t+u)|F(t)) = X(t)\exp[\sigma^2 v^2 u/2] - X_2(t)$

$$-X(t)(\exp[\sigma^2 v^2 u/2]-1)$$

$$= X(t) - X_2(t) = X_1(t) \quad \text{a.s.}$$

Hence the X- process is a quasi-martingale. That

$$E(\int_0^u \frac{\sigma^2 v^2}{2}\exp[Z(t+s)v]ds|F(t))$$

$$= \int_0^u \frac{\sigma^2 v^2}{2} E(\exp[Z(t+s)v]|F(t))ds$$

follows from the existence in this case of a conditional probability and we are thus only changing the order of integration which is permissible here. (One can also prove it directly by observing the Riemann-Stieltjes sums and noting these sums form a uniformly integrable sequence.)

In this example we have really just considered a continuous, convex function of a martingale process, $\exp[Z(t)v]$, and hence we have a semi-martingale process. The following will give a simple class of examples where the quasi-martingale need not be a semi-martingale.

Let $\{X(t), F(t); t\epsilon T\}$ be a process with independent increments where $F(t) = \mathcal{B}(X(s); s\epsilon T, s \leq t)$. Let $E(X(t)) = m(t)$. If $s \leq t$, then assuming $m(0) = 0$,

$$E(X(t)|F(s)) = X(s) + m(t) - m(s)$$

Define $X_2(t) = m(t)$ a.s. and $X_1(t) = X(t) - X_2(t)$ for every $t\epsilon T$. Then $\{X_1(t), F(t); t\epsilon T\}$ is a martingale process and therefore the process $\{X(t), F(t); t\epsilon T\}$ will be a quasi-martingale process when $m(t)$ is of bounded variation on T.

We will now mention some work which has been done by P. Meyer (3) on the decomposition of a continuous parameter super-martingale $\{X(t), F(t); t\epsilon[0,\infty]\}$ into the difference of a martingale process $\{X_1(t), F(t); t\epsilon[0,\infty]\}$ and a process $\{X_2(t), F(t); t\epsilon[0,\infty]$ which has a.e. sample function monotone non-decreasing. The results obtained by Meyer are the following: Let $\{X(t), F(t); t\epsilon[0,\infty]\}$ be a uniformly integrable, right continuous super-martingale. Then the process has the stated decomposition if and only if it is of class D on $[0,\infty]$, i.e., if and only if the family of r.v.'s $\{X_\tau; \tau\epsilon\mathcal{T}\}$, where $\mathcal{T}$ is the class of all stopping times for the process, is uniformly integrable.

It has been shown by Johnson and Helms (4) that there exist uniformly integrable, right continuous super-martingales $\{X(t), F(t); t\epsilon[0,\infty]\}$ which are not of class D. They have further shown that if in addition the super-martingale is a.s. sample continuous then it has the stated decomposition if and only if

$$\lim_{r\to\infty} rP([\sup_{0\le t\le\infty} |X(t,\omega)| > r]) = 0$$

The problem of decomposing a process $\{X(t), F(t); t\epsilon T\}$ into the sum of a martingale process and a process having a.e. sample function of bounded variation on T parallels the above described decomposition of a super-martingale. For if we assume $\{X(t), F(t); t\epsilon[0,\infty]\}$ is a uniformly integrable super-martingale, we may as well assume we have the super-martingale $\{X(t), F(t); t\epsilon[0,1]\}$. Then if we have the above decomposition, the process having monotone non-decreasing sample functions has a.e. sample function of bounded variation on $[0,1]$.

We will obtain necessary and sufficient conditions for a process $\{X(t), F(t); t\epsilon T\}$ to have the decomposition

$$P([X(t) = X_1(t) + X_2(t);\ t\epsilon T]) = 1$$

where $\{X_1(t), F(t); t\epsilon T\}$ is an a.s. sample continuous martingale and the process $\{X_2(t), F(t); t\epsilon T\}$ has a.e. sample function of bounded variation on T, and further if $V(\omega)$ denotes the total variation of $X_2(\cdot,\omega)$ over T, $E(V(\omega)) < \infty$

Chapter II

Section 1: Random stopping

Let $\{X(t), F(t); t \in T\}$ be a stochastic process defined on the probability space (Ω, F, P). We will be interested in obtaining a sequence of processes $\{X_\nu(t), F(t); t \in T\}$, $\nu = 1,2,\ldots$, where each process in the sequence has some specified property, such that

$$P([X_\nu(t) \neq X(t) \text{ for some } t \in T]) \longrightarrow 0$$

as $\nu \longrightarrow \infty$.

For example, we may want to define a sequence of processes $\{X_\nu(t), F(t); t \in T\}$ $\nu = 1,2,\ldots$ such that

$$\sup_{t,\omega} |X_\nu(t,\omega)| < \infty \text{ for every } \nu = 1,2,\ldots$$

and

$$P([X_\nu(t) \neq X(t) \text{ for some } t \in T]) \longrightarrow 0$$

as $\nu \longrightarrow \infty$.

Such sequences are usually obtained by a random stopping of the process $\{X(t), F(t); t \in T\}$. Therefore, we will consider briefly random stopping of a process (2, Loeve, pp. 530-535).

Let $\{X(t), F(t); t \in T\}$ be a process defined on the probability space (Ω, F, P) and let $\tau(\omega)$ be a r.v. defined on (Ω, F, P) with range T. If for each $t \in T$, $[\tau(\omega) \leq t] \in F(t)$ (and hence $[\tau(\omega) < t] \in F(t)$), the r.v. $\tau(\omega)$ is called a stopping time of the X-process. If the X-process is a.s. sample right or left continuous then we can define a new process $\{X_\tau(t), F(t); t \in T\}$ by randomly stopping the X-process according to the stopping time $\tau(\omega)$. More precisely, if the X-process is a.s. sample right continuous, define

$$X_\tau(t,\omega) = X(t,\omega) \quad t \le \tau(\omega)$$
$$= X(\tau(\omega),\omega) \quad t > \tau(\omega)$$

and then using right continuity of the process and the fact that $[\tau(\omega) \le t] \in F(t)$ it can be shown $X_\tau(t)$ is $F(t)$ measurable for every $t \in T$.

Actually, if one only requires $[\tau(\omega) < t] \in F(t)$, and defines

$$X_\tau(t,\omega) = X(t,\omega) \quad t < \tau(\omega)$$
$$= X(\tau(\omega),\omega) \quad t \ge \tau(\omega)$$

then $X_\tau(t)$ will be $F(t)$ measurable if the X-process is either a.s. sample right or left continuous.

The following is a standard theorem which we state here since it will be used extensively (2, Loeve, p. 533).

Theorem 2.1.1

If $\{X(t), F(t);\ t \in T\}$ is an a.s. sample right continuous semi-martingale, (martingale) and if τ is a stopping time of the process, then the stopped process $\{X_\tau(t), F(t);\ t \in T\}$ is also a semi-martingale (martingale).

The next two theorems will also be used extensively in later work so we will prove them in some detail.

Theorem 2.1.2

Let $\{X(t), F(t);\ t \in T\}$ be an a.s. sample continuous process. There exists a sequence of processes $\{X_\nu(t), F(t);\ t \in T\}$, $\nu = 0,1,2,\ldots$, each process in the sequence being a.s. sample equi-continuous, such that

$$P([X_\nu(t) \ne X(t) \text{ for some } t \in T]) < 2^{-\nu}$$

Proof:

By a process being a.s. sample equi-continuous we mean the following: There exists a set Λ with $P(\Lambda) = 0$, such that if $\epsilon > 0$ is given, there exists a $\delta > 0$ such that

$$|X(t,\omega) - X(s,\omega)| < \epsilon \quad \text{when } |t-s| < \delta$$

for every $\omega \notin \Lambda$. (That Λ does not depend on $t \epsilon T$ follows from the fact that $T = [0,1]$.)

We now prove the theorem.

Let $\{\epsilon_n; n \geq 0\}$ be a sequence of real numbers with $\epsilon_0 > \epsilon_1 > \ldots > \epsilon_n > \ldots > 0$ and $\lim_n \epsilon_n = 0$. For each $n \geq 0$, let $\{\delta_{n\nu}, \nu \geq 0\}$ be a sequence of real numbers with $\delta_{n0} > \delta_{n1} > \ldots > \delta_{n\nu} > \ldots > 0$ and $\lim_\nu \delta_{n\nu} = 0$. The ϵ_n's are arbitrary and the $\delta_{n\nu}$'s are to be chosen as follows: Because of the a.s. sample continuity of the X-process, for each ϵ_n, $n=0,1,\ldots$ we can find a $\delta_{n\nu} > 0$ such that

$$P([\sup_{|t-s| \leq \delta_{n\nu}} |X(t,\omega) - X(s,\omega)| \geq \epsilon_n]) < 2^{-(n+\nu)}$$

for each $\nu = 0,1,\ldots$.

Let $\tau_{n\nu}(\omega)$ be the first t such that

$$\sup_{\substack{|s-s'| \leq \delta_{n\nu} \\ s,s' \leq t}} |X(s,\omega) - X(s',\omega)| \geq \epsilon_n .$$

If no such t exists we define $\tau_{n\nu}(\omega) = 1$. Then for each $n,\nu=0,1,\ldots,$ $0 < \tau_{n\nu}(\omega) \leq 1$ a.s. $\tau_{n\nu}(\omega)$ is a stopping time of the process for every $n,\nu=0,1,\ldots$ since for any $t \epsilon (0,1]$

$$[\tau_{n\nu}(\omega) > t] = [\sup_{\substack{|s-s'| \leq \delta_{n\nu} \\ s,s' \leq t}} |X(s',\omega) - X(s,\omega)| < \epsilon_n]$$

Define,

$$\tau_\nu(\omega) = \inf_n \tau_{n\nu}(\omega), \ \nu = 0,1,\ldots.$$

Then for each $\nu = 0,1,\ldots$ $0 \leq \tau_\nu(\omega) \leq 1$.

$\tau_\nu(\omega)$ will be a stopping time for the process if $[\tau_\nu(\omega) \leq t] \epsilon F(t)$ for every $t \epsilon T$. Actually it is sufficient to require that the set $[\tau_\nu(\omega) \leq t]$ differ from an $F(t)$ set by a set of measure zero. (1, Doob, p. 365) (As was mentioned, we could require only that $[\tau_\nu(\omega) < t] \epsilon F(t)$ for every $t \epsilon T$, which is obviously the case since $[\tau_\nu(\omega) < t] = \bigcup_{n=0}^{\infty} [\tau_{n\nu}(\omega) < t]$.)

If $t \epsilon [0,1)$, then

$$P([\tau_{n\nu}(\omega) \leq t \text{ for infinitely many } n])$$

$$= P\left(\bigcap_{n=0}^{\infty} \bigcup_{m=n}^{\infty} [\tau_{m\nu}(\omega) \leq t]\right) = \lim_{n\to\infty} P\left(\bigcup_{m=n}^{\infty} [\tau_{n\nu}(\omega) \leq t]\right)$$

$$\leq \lim_{n\to\infty} \sum_{m=n}^{\infty} P([\tau_{m\nu}(\omega) \leq t]) \leq \lim_{n\to\infty} \sum_{m=n}^{\infty} P([\tau_{m\nu}(\omega) < 1])$$

$$\leq \lim_{n\to\infty} \sum_{m=n}^{\infty} P([\sup_{|t-s| \leq \delta_{m\nu}} |X(t,\omega) - X(s,\omega)| \geq \epsilon_m])$$

$$\leq \lim_{n\to\infty} 2^{-(n+\nu-1)} = 0 .$$

If $\Lambda_\nu(t) = [\tau_{n\nu}(\omega) \leq t$ for infinitely many $n]$. Then for $\omega \notin \Lambda_\nu(t)$, $\tau_\nu(\omega) \leq t$ implies $\tau_{n\nu}(\omega) \leq t$ for some n. We have $P(\Lambda_\nu(t)) = 0$ for every $t \epsilon [0,1)$ and every $\nu = 0,1,\ldots.$ Letting $\sim\Lambda_\nu(t) = \Omega - \Lambda_\nu(t)$, we have

$$[\tau_\nu(\omega) \leq t] = [\inf_n \tau_{n\nu}(\omega) \leq t]$$

$$= ([\inf_n \tau_{n\nu}(\omega) \le t] \cap \sim\Lambda_\nu(t)) \cup ([\inf_n \tau_{n\nu}(\omega) \le t] \cap \sim\Lambda_\nu(t))$$

$$= (\bigcup_{n=1}^{\infty} [\tau_{n\nu}(\omega) \le t]) \cup ([\inf_n \tau_{n\nu}(\omega) \le t] \cap \Lambda_\nu(t)),$$

so $[\tau_\nu(\omega) \le t] - (\bigcup_{n=1}^{\infty} [\tau_{n\nu}(\omega) \le t]) = [\inf_n \tau_{n\nu}(\omega) \le t] \cap \Lambda_\nu(t)$ $\subset \Lambda_\nu(t)$. Hence $[\tau_\nu(\omega) \le t]$ differs from the $F(t)$ set $\bigcup_{n=1}^{\infty} [\tau_{n\nu}(\omega) \le t]$ by a set of measure zero.

Define
$$X_\nu(t,\omega) = X(t,\omega) \text{ if } t \le \tau_\nu(\omega)$$
$$= X(\tau_\nu(\omega),\omega) \text{ if } t > \tau_\nu(\omega).$$

Then for each $\nu = 0,1,\ldots$, $X_\nu(t)$ is $F(t)$ measurable for every $t \in T$ and the process $\{X_\nu(t), F(t);\ t \in T\}$ is a.s. sample equi-continuous. For given any $\epsilon > 0$, if $\epsilon_n < \epsilon$, then for a.e. ω,

$$|X_\nu(t,\omega) - X_\nu(s,\omega)| < \epsilon \quad \text{if } |t-s| < \delta_{n\nu}.$$

Also

$$P([X_\nu(t) \ne X(t) \text{ for some } t \in T])$$
$$= P([\tau_\nu(\omega) < 1]) \le P([\inf_n \tau_{n\nu}(\omega) < 1])$$
$$= P(\bigcup_{n=0}^{\infty} [\tau_{n\nu}(\omega) < 1]) \le \sum_{n=0}^{\infty} 2^{-(n+\nu)} = 2^{-\nu}.$$

The theorem is now proved.

Theorem 2.1.3

Let $\{X(t), F(t);\ t \in T\}$ be an a.s. sample continuous process. There exists a sequence of a.s. sample continuous, uniformly bounded processes $\{X_\nu(t), F(t);\ t \in T\}$, $\nu = 0,1,\ldots$ such that

$$P([X_\nu(\nu) \neq X(t) \text{ for some } t\epsilon T]) \longrightarrow 0$$

as $\nu \longrightarrow \infty$.

If in addition the X-process is such that

$$\lim_{r \to \infty} r\, P([\sup_t |X(t,\omega)| \geq r]) = 0$$

then $E(|X_\nu(t) - X(t)|) \longrightarrow 0$ as $\nu \longrightarrow \infty$ for every $t\epsilon T$. If the X-process is uniformly integrable,

$$\sup_t E(|X_\nu(t) - X(t)|) \longrightarrow 0 \text{ as } \nu \longrightarrow \infty.$$

Proof:

Define $\tau_\nu(\omega)$ to be the first t such that $\sup_{s \leq t} |X(s,\omega)| \geq \nu$. If no such t exists, let $\tau_\nu(\omega) = 1$. Then clearly $\tau_\nu(\omega)$ defines a stopping time for the process for every $\nu = 0,1,\ldots.$ Since the process is a.s. sample continuous on the closed interval [0,1], a.e. sample function has an absolute maximum.

Define

$$X_\nu(t,\omega) = X(t,\omega) \text{ if } t \leq \tau_\nu(\omega)$$
$$= X(\tau_\nu(\omega),\omega) \text{ if } t > \tau_\nu(\omega).$$

Then for each $\nu = 0,1,\ldots$, the process $\{X_\nu(t), F(t);\ t\epsilon T\}$ is uniformly bounded by ν and is a.s. sample continuous. We have

$$P([X_\nu(t) \neq X(t) \text{ for some } t\epsilon T]) = P([\tau_\nu(\omega) < 1]) \leq P([\sup_t |X(t,\omega) \geq \nu])$$

$\longrightarrow 0$ as $\nu \to \infty$.

Assume now that $\lim_{r \to \infty} r\, P([\sup_t |X(t,\omega)| \geq r]) = 0$. Then

$$E(|X_\nu(t) - X(t)|) = \int_{[\tau_\nu(\omega) \geq t]} |X_\nu(t) - X(t)|dP + \int_{[\tau_\nu(\omega) < t]} |X_\nu(t) - X(t)|dP$$

$$= \int_{[\tau_\nu(\omega)<t]} |X_\nu(t) - X(t)|dP \leq \int_{[\tau_\nu(\omega)<1]} |X_\nu(t)|dP + \int_{[\tau_\nu(\omega)<1]} |X(t)|dP$$

$$\leq \nu P([\tau_\nu(\omega) < 1]) + \int_{[\tau_\nu(\omega)<1]} |X(t)|dP \; .$$

The conclusions are now apparent.

We note here that if we have n processes $\{X_i(t), F(t); t\epsilon T\}$ $i = 1,\ldots,n$ satisfying the conditions of Theorem 2.1.2 or 2.1.3, we can find n sequences of processes $\{X_{i\nu}(t), F(t); t\epsilon T\}$ having the specified properties and such that

$$P([X_{i\nu}(t) \neq X_i(t) \text{ for some } i \leq i \leq n \text{ or } t\epsilon T]) \longrightarrow 0 \text{ as}$$

$\nu \longrightarrow \infty$.

Section 2: Decomposition theorem.

Let $\{X(t), F(t); t\epsilon T\}$ be any real valued process with $E(|X(t)|) < \infty$ for every $t\epsilon T$.

(2.2.1) Let $\{\Pi_n, n \geq 1\}$ be a sequence of partitions of T. We denote the points of Π_n as follows:

$$0 = t_{n,0} < t_{n,1} < \dots < t_{n,N_n+1} = 1 .$$

Assume $\|\Pi_n\| = \max_{0 \leq j \leq N_n} |t_{n,j+1} - t_{n,j}| \longrightarrow 0$ as $n \longrightarrow \infty$ and let

$$\Pi_1 \subset \Pi_2 \subset \dots \subset \Pi_n \subset \dots .$$

If $m > n$, $\Pi_n \subset \Pi_m$ and we will let

$$\Pi_{nm,j} = \Pi_m \cap [t_{n,j}, t_{n,j+1}] \text{ so that}$$

$$\Pi_m = \bigcup_{j=0}^{N_n} \Pi_{nm,j} .$$

The points of $\Pi_{nm,j}$ for $j = 0,\dots,N_n$ will be denoted as follows

$$t_{n,j} = t_{nmj,0} < t_{nmj,1} < \dots < t_{nmj,k_{nmj}+1} = t_{n,j+1} .$$

We will always separate with a comma the variable subscript which denotes an arbitrary point of the partition and the subscripts indicating to which particular partition we are referrring. The variable subscripts will be written last.

If Π is a partition of a closed interval $[\alpha,\beta] \subset T$ with points $\alpha = a_0 < a_1 < \dots < a_{\nu+1} = \beta$, we will let

$$\Delta_i(X) = X(a_{i+1}) - X(a_i) \qquad 0 \leq i \leq \nu .$$

Then, for example,

$$\Delta_{n,j}(X) = X(t_{n,j+1}) - X(t_{n,j}) \qquad 0 \leq j \leq N_n$$

$$\Delta_{nmj,k}(X) = X(t_{nmj,k+1}) - X(t_{nmj,k}) \qquad 0 \le k \le k_{nmj}$$

we will also write F_i for $F(a_i)$ and we let

$$C_i(X) = E(\Delta_i(X)|F_i) \qquad 0 \le i \le v .$$

Then, for example

$$C_{n,j}(X) = E(\Delta_{n,j}(X)|F_{n,j}) \quad 0 \le j \le N_n$$

$$C_{nmj,k}(X) = E(\Delta_{nmj,k}(X)|F_{nmj,k}) \; 0 \le k \le k_{nmj}.$$

We will often write $\sum_{\pi} C_i(X)$ for $\sum_{i=0}^{v} C_i(X)$. Then, for example, we can write

$$\sum_{\pi_m} C_{m,i}(X) = \sum_{\pi_n}\sum_{\pi_{nmj}} C_{nmj,k}(X) = \sum_{i=0}^{N_n}\sum_{k=0}^{k_{nmj}} C_{nmj,k}(X).$$

We will also use the following notation throughout.

For each $t\in T$, let $\pi_n(t) = \pi_n \cap [0,t]$. For each $s,t\in T$ with $s < t$, we write

$$\pi_n(s,t) = \pi_n(t) - \pi_n(s) \cup \{\text{largest element in } \pi_n(s)\}$$

so that

$$\pi_n(t) = \pi_n(s) \cup \pi_n(s,t).$$

Then, for example, if $0 = a_0 < a_1 < \dots < a_{v+1} = 1$

$$\sum_{i=0}^{v} \sum_{\pi_n(a_i,a_{i+1})} C_{n,j}(X) = \sum_{\pi_n} C_{n,j}(X) \quad \text{for every } n = 1,2,\dots.$$

We define $\{X_n(t), F_n(t);\ t\in T\}$, $n = 1,2,\dots$ as follows:

$$X_n(t) = X(t_{n,j}) \text{ if } t_{n,j} \le t < t_{n,j+1} \quad 0 \le j \le N_n+1.$$

(2.2.2)

$$F_n(t) = F(t_{n,j}) \text{ if } t_{n,j} \le t < t_{n,j+1} \quad 0 \le j \le N_n+1.$$

We also define $\{X_{2n}(t), F_n(t);\ t\ T\}$, $n = 1,2,\ldots$ as follows:

$$X_{2n}(t) = \sum_{\Pi_n(t)} C_{n,j}(X) \qquad \text{for every } t\in T.$$

(2.2.3)

$$F_n(t) = F(t_{n,j}) \text{ if } t_{n,j} \leq t < t_{n,j+1} \quad 0 \leq j \leq N_n + 1,$$

or what is the same

$$X_{2n}(t) = \sum_{j=0}^{k-1} C_{n,j}(X) \text{ if } t_{n,k} \leq t < t_{n,k+1} \quad 0 \leq k \leq N_n + 1.$$

(2.2.4)

If we let $X_{1n}(t) = X_n(t) - X_{2n}(t)$ for every $t\in T$, the process $\{X_{1n}(t), F_n(t);\ t\in T\}$ is a martingale. Clearly, the process $\{X_{2n}(t), F_n(t);\ t\in T\}$ has a.e. sample function of bounded variation on T since a.e. sample function takes on only a finite number of distinct values. Then each process $\{X_n(t), F_n(t);\ t\in T\}$ is a quasi-martingale.

If we assume the X-process is continuous in the mean, then it is the limit in the mean of a sequence of quasi-martingales, $\{X_n(t), F_n(t);\ t\in T\}$, $n \geq 1$. Further, if for each $t\in T$, the sequence $\{X_{2n}(t);\ n \geq 1\}$ converges in the mean to a r.v. $X_2(t)$, then the process $\{X_1(t) = X(t) - X_2(t), F(t);\ t\in T\}$ is a martingale. For any $t\in T$, let $k_n(t)$ denote the last k such that $t_{n,k} \leq t$.

Let $s,t\in T$, $s \leq t$, and let $\Lambda\in F(s)$. Then

$$\int_\Lambda X_1(t)dP = \int_\Lambda X(t)dP - \int_\Lambda X_2(t)dP$$

$$= \int_\Lambda X(t)dP - \lim_{n\to\infty} \int_\Lambda \Big(\sum_{\Pi_n(s)} C_{n,j}(X)\Big)dP + \int_\Lambda \Big(\sum_{\Pi_n(s,t)} C_{n,j}(X)\Big)dP$$

$$= \int_\Lambda X(t)dP - \int_\Lambda X_2(s)dP - \lim_{n\to\infty} \int_\Lambda \Big(\sum_{\Pi_n(s,t)} C_{n,j}(X)\Big)dP.$$

Now $\int_\Lambda \Big(\sum_{\Pi_n(s,t)} C_{n,j}(X)\Big)dP = \int_\Lambda E\Big(\sum_{\Pi_n(s,t)} C_{n,j}(X) \,|\, F(s)\Big)dP$

$$= \int_\Lambda C_{n,k_n(s)}(X)dP + \int_\Lambda E\Big(\sum_{j=k_n(s)+1}^{k_n(t)-1} C_{n,j}(X) \,|\, F(s)\Big)dP$$

$$= \int_\Lambda C_{n,k_n(s)}(X)dP + \int_\Lambda \big(X(t_{n,k_n(t)}) - X(t_{n,k_n(s)+1})\big)dP$$

and hence

$$\lim_{n\to\infty} \int_\Lambda \Big(\sum_{\Pi_n(s,t)} C_{n,j}(X)\Big)dP = \int_\Lambda X(t)dP - \int_\Lambda X(s)dP$$

since $t_{n,k_n(t)} \uparrow t$, $t_{n,k_n(s)+1} \downarrow s$ and $t_{n,k_n(s)+1} - t_{n,k_n(s)} \to 0$ as $n \to \infty$.

So we have

$$\int_\Lambda X_1(t)dP = \int_\Lambda X(t)dP - \int_\Lambda X_2(s)dP - \int_\Lambda X(t)dP + \int_\Lambda X(s)dP$$

$$= \int_\Lambda X_1(s)dP \ .$$

This however still tells us nothing about the process $\{X_2(t), F(t);\ t\in T\}$ even when it exists. We now look for sufficient conditions for the X_2-process, when it exists, to have a.e. sample function of bounded variation on T.

The following lemma, though trivial, is a key starting point.

Lemma 2.2.5

If $\{\Pi_n;\ n \geq 1\}$ is defined as in 2.2.1, then for any process $\{X(t),\ F(t);\ t\in T\}$

$$E\Big(\sum_{\Pi_n} |C_{n,j}(X)|\Big)$$

is monotone non-decreasing in n.

Proof:

Let $m > n$. Then $\Pi_n \subset \Pi_m$ and we can write

$$E\Big(\sum_{\Pi_m} |C_{m,i}(X)|\Big) = E\Big(\sum_{\Pi_n}\sum_{\Pi_{nmj}} |C_{nmj,k}(X)|\Big)$$

$$= E\Big(\sum_{\Pi_n} E\Big(\sum_{\Pi_{nmj}} |C_{nmj,k}(X)|\ \big|F_{nj}\Big)\Big)$$

$$\geq E\Big(\sum_{\Pi_n} \Big|E\Big(\sum_{\Pi_{nmj}} C_{nmj,k}(X)\big)\big|F_{nj}\Big)\Big|\Big) = E\Big(\sum_{\Pi_n} |C_{n,j}(X)|\Big).$$

We make the following observation:

Let $\{X(t),\ F(t);\ t\in T\}$ be a quasi-martingale with $[X]_1 = X_1$ and $[X]_2 = X_2$. If a.e. sample function of the X_2 process is continuous, then

$$V(\omega) = \lim_{n\to\infty} \sum_{\Pi_n} |\Delta_{n,j}(X_2)|$$

is the total variation of $X_2(\cdot,\omega)$ over T for a.e. ω, and it is a random variable. Assume $E(V(\omega)) < \infty$, then

$$E\Big(\sum_{\Pi_n} |C_{n,j}(X)|\Big) = E\Big(\sum_{\Pi_n} |C_{n,j}(X_2)|\Big)$$

$$\leq E(\sum_{\pi_n}^{\neg} |\Delta_{n,j}(X_2)|) \leq E(V(\omega)) < \infty ,$$

and hence $\lim_{n\to\infty} E(\sum_{\pi_n}^{\neg} |C_{n,j}(X)|) < \infty$.

In fact, if π is any partition of T, there exists K_X, independent of π, such that

$$E(\sum_{\pi}^{\neg} |C_i(X)|) \leq K_X < \infty .$$

In view of this and Lemma 2.2.5 we restrict out attention to those processes satisfying the following condition.

(2.2.6)

There is a sequence of partitions $\{\pi_n;\ n \geq 1\}$ of T, as defined in 2.2.1 such that

$$\lim_{n} E(\sum_{\pi_n}^{\neg} |C_{n,j}(X)|) \leq K_X < \infty .$$

We could as well require the existence of a constant K_X such that for any partition π of T, $E(\sum_{\pi}^{\neg} |C_i(X)|) \leq K_X$.

Lemma 2.2.7

Let the process $\{X(t),\ F(t);\ t\epsilon T\}$ satisfy condition 2.2.6 and let $\{X_{2n}(t),\ F_n(t);\ t\epsilon T\}$, $n \geq 1$, be defined as in 2.2.3. If the process $\{X_2(t),\ F(t);\ t\epsilon T\}$ is such that

$$P([\lim_{n} X_{2n}(t) = X_2(t);\ t\epsilon T]) = 1$$

then the X_2-process has a.e. sample function of bounded variation over T. Furthermore, if the X_2-process is a.s. sample continuous, the total variation $V(\omega)$ of $X_2(\cdot,\omega)$ over T is a r.v. and

$$E(V(\omega)) \leq \lim_{n\to\infty} E(\sum_{\pi_n} |C_{n,j}(X)|) \leq K_X \cdot$$

Proof:

By 2.2.6, $K_X \geq \lim_{n\to\infty} E(\sum_{\pi_n} |C_{n,j}(X)|)$

$\geq E(\underline{\lim}_{n\to\infty} \sum_{\pi_n} |C_{n,j}(X)|)$, so that if we let $K_X(\omega) = \underline{\lim}_{n\to\infty} \sum_{\pi_n} |C_{n,j}(X)|$,

then $K_X(\omega)$ is a.s. finite and integrable.

If $0 = a_0 < a_1 < \ldots < a_{k+1} = 1$ is any partition of T, then

$$\sum_{i=0}^{k} |\Delta_i(X_2)| = \sum_{i=0}^{k} |\lim_{n\to\infty} (\sum_{\pi_n(a_i, a_{i+1})} C_{n,j}(X))|$$

$$\leq \underline{\lim}_{n\to\infty} \sum_{i=0}^{k} \sum_{\pi_n(a_i, a_{i+1})} |C_{n,j}(X)| = \underline{\lim}_{n\to\infty} \sum_{\pi_n} |C_{n,j}(X)| = K_X(\omega) < \infty$$

for a.e. ω. Since the partition was arbitrary,

$$\sup_{\pi} \sum_{\pi} |\Delta_i(X_2)| \leq K_X(\omega) \quad \text{a.s.}$$

If the X_2-process is a.s. sample continuous, $V(\omega)$ is a r.v. dominated by $K_X(\omega)$ and hence $E(V(\omega)) \leq E(K_X(\omega)) \leq K_X < \infty$. (In fact, $V(\omega)$ will be measurable if a.e. sample function has at most jump discontinuities, for even in this case

$$V(\omega) = \lim_{n\to\infty} \sum_{n} |\Delta_{n,j}(X_2)| \quad \text{a.s.)}$$

By our previous remarks and Lemma 2.2.7, if the process $\{X(t), F(t); t\in T\}$ is continuous in the mean, satisfies condition 2.2.6 and if $\{X_{2n}(t), F_n(t); t\in T\}$, $n \geq 1$, defined as in 2.2.3, is such that there exists a process $\{X_2(t), F(t);\ t\epsilon T\}$ with

$$P([\lim_{n\to\infty} X_{2n}(t) = X_2(t);\ t\epsilon T]) = 1$$

and also

$$E(|X_{2n}(t) - X_2(t)|) \longrightarrow 0 \text{ as } n \longrightarrow \infty$$

for every $t\epsilon T$, the X-process will be a quasi-martingale with $[X]_2 = X_2$. These are the basic conditions we will seek to satisfy in obtaining first sufficient conditions for a process $\{X(t), F(t);\ t\epsilon T\}$ to be a quasi-martingale.

We will need the following theorem.

Theorem 2.2.8

Let $\{X(t), F(t);\ t\epsilon T\}$ be a second order process. Let $[\alpha,\beta]$ be a closed sub interval of T and let $\alpha = a_0 < a_1 < \ldots < a_{n+1} = \beta$ be a partition of $[\alpha,\beta]$. Let $\epsilon > 0$ be given. If

$$\epsilon_{\alpha,\beta} = \sup_{\omega}{}'[\max_{0\le k\le n} |X(\beta,\omega) - X(a_k,\omega)|] < \epsilon$$

then $P([\max_{m\le n} |\sum_{k=0}^{m} c_k(X)| \ge \epsilon]) \le E(|\sum_{k=0}^{n} c_k(X)|^2)/(\epsilon-\epsilon_{\alpha,\beta})^2$

where $\sup_{\omega}{}'$ denotes the essential supremum. (i.e., the supremum over all $\omega\notin\Lambda$, where $P(\Lambda) = 0$.)

Proof:

The argument is the following: Assume $\Lambda\epsilon F$, $|A| \ge \epsilon$, and A is F measurable. Assume further $|E(B|F)| \le \delta < \epsilon$. Then

$$\int_\Lambda (A^2 + 2AB + B^2)dP = \int_\Lambda (A^2 + 2AE(B|F) + E(B^2|F))dP$$

$$\ge \int_\Lambda (A^2 + 2AE(B|F) + [E(B|F)]^2)dP$$

$$\geq \int_{\Lambda} (A^2 + \delta^2 - 2|A|\delta)dP = \int_{\Lambda} (|A| - \delta)^2 dP$$

$$\geq (\epsilon - \delta)^2 P(\Lambda) .$$

Now we let

$$\Lambda_m = [|\sum_{k=0}^{v} c_k(X)| < \epsilon \quad \text{for } v \leq m \text{ and } |\sum_{k=0}^{m} c_k(X)| \geq \epsilon] \quad \text{for } m = 0,\ldots,n.$$

Then $\Lambda_m \epsilon F_m$, since $c_k(X)$ is measurable w.r.t. F_m for $k = 0,\ldots,m$.

We also have $\Lambda_m \cap \Lambda_j = \phi$ for $m \neq j$, and

$$[\max_{m \leq n} |\sum_{k=0}^{m} c_k(X)| > \epsilon] = \bigcup_{m=0}^{n} \Lambda_m .$$

Now

$$E(|\sum_{k=0}^{n} c_k(X)|^2) = \sum_{m=0}^{n} \int_{\Lambda_m} |\sum_{k=0}^{m} c_k(X) + \sum_{k=m+1}^{n} c_k(X)|^2 dP .$$

For each $m = 0,1,\ldots,n$, we let $A_m = \sum_{k=0}^{m} c_k(X)$, $B_m = \sum_{k=m+1}^{n} c_k(X)$, and

replacing Λ and F with Λ_m and F_m and δ with $\epsilon_{\alpha,\beta}$, the above argument

gives

$$E(|\sum_{k=0}^{n} c_k(X)|^2) \geq \sum_{m=0}^{n} \int_{\Lambda_m} (\epsilon - \epsilon_{\alpha,\beta})^2 dP$$

$$= (\epsilon - \epsilon_{\alpha,\beta})^2 P([\max_{m \leq n} |\sum_{k=0}^{m} c_k(X)| > \epsilon])$$

and hence the theorem is true.

Corollary 2.2.9:

Let $\{X(t), F(t); t\epsilon T\}$ be a second order process which is a.s. sample equi-continuous. Let $\{\Pi_n; n \geq 1\}$ be a sequence of partitions of T as defined in 2.2.1. Given any $\epsilon > 0$, there exists an $n(\epsilon)$ such that if $n \geq n(\epsilon)$ then for every $m \geq n$,

$$P([\max_{0 \leq j \leq N_n} \max_{0 \leq k \leq k_{nmj}} |\sum_{i=0}^{k} C_{nmj,i}(X)| \geq \epsilon])$$

$$\leq E(\sum_{\Pi_n} |\sum_{\Pi_{nmj}} C_{nmj,k}(X)|^2) / (\epsilon - \epsilon_n)^2$$

where $\epsilon_n = \sup'_{\omega} \max_{0 \leq j \leq N_n} \sup_{t_{n,j} \leq s,t \leq t_{n,j+1}} |X(t,\omega) - X(s,\omega)| < \epsilon$

($\sup'_{\omega}$ denotes supremum over the set of equi-continuity).

Proof:

Because of the a.s. uniform sample equi-continuity of the X-process, given any $\epsilon > 0$ there exists $\delta(\epsilon)$ such that

$$\sup'_{\omega} \sup_{|t-s| \leq \delta(\epsilon)} |X(t,\omega) - X(s,\omega)| < \epsilon .$$

Let $n(\epsilon)$ be such that $\|\Pi_n\| < \delta(\epsilon)$ if $n \geq n(\epsilon)$. Then

$$\epsilon_n = \sup'_{\omega} \max_{0 \leq j \leq N_n} \sup_{t_{nj} \leq s,t \leq t_{nj+1}} |X(t) - X(s)| < \epsilon$$

and for all $m \geq n$, $\sup'_{\omega} \max_{0 \leq j \leq N_n} \max_{0 \leq k,k' \leq k_{nmj}} |X(t_{nmj,k}) - X(t_{nmj,k'})|$

$$\leq \epsilon_n < \epsilon.$$

Using Theorem 2.2.8, we then have the following:

$$P([\max_{0 \leq j \leq N_n} \max_{0 \leq k \leq k_{nmj}} |\sum_{i=0}^{k} C_{nmj,i}(X)| \geq \epsilon])$$

$$\leq \sum_{j=0}^{N_n} P([\max_{0 \leq k \leq k_{nmj}} |\sum_{i=0}^{k} C_{nmj,i}(X)| \geq \epsilon])$$

$$\leq E\Big(\sum_{\Pi_n} \Big|\sum_{\Pi_{nmj}} c_{nmj,k}(X)\Big|^2\Big)/(\epsilon-\epsilon_n)^2 .$$

We will need the following theorem.

Theorem 2.2.10

Assume $\{X_n(t);\ t\in T\}$, $n = 1,2,\ldots$ is a sequence of processes with the following properties:

i) There is a countable dense subset T_0 of T, containing the points 0 and 1, such that $P \lim_{n\to\infty} X_n(t)$ exists for each $t\in T$.

ii) Given ϵ, $\eta > 0$, there exists $n(\epsilon,\eta)$ and $\delta(\epsilon,\eta)$ such that if $n \geq n(\epsilon,\eta)$

$$P([\sup_{|t-s|\leq\delta(\epsilon,\eta)} |X_n(t) - X_n(s)| > \epsilon]) < \eta.$$

Then there exists a subsequence of processes $\{X_{n_k}(t);\ t\in T\}$ $k = 1,2,\ldots$ and a process $\{X(t);\ t\in T\}$ such that

a) $P([\lim_k X_{n_k}(t) = X(t);\ t\in T]) = 1$

and

b) The X-process is a.s. sample continuous.

Proof:

We first show that conditions i) and ii) imply

$$\overline{\lim_{m,n\to\infty}}\ P([\sup_t |X_n(t) - X_m(t)| > \epsilon]) = 0 \quad \text{for every } \epsilon > 0.$$

Let $\{T_\nu;\ \nu \geq 1\}$ be a sequence of partitions of T with $T_1 \subset T_2 \subset \ldots$ and $\bigcup_{\nu=1}^{\infty} T_\nu = T_0$. Let $\epsilon,\eta > 0$ be given. First choose $n_1(\epsilon,\eta)$, $\delta(\epsilon,\eta)$ such that $P([\sup_{|t-s|\leq\delta(\epsilon,\eta)} |X_n(t)-X_n(s)| > \frac{\epsilon}{3}]) < \frac{\eta}{3}$

for every $n \geq n_1(\epsilon,\eta)$. This can be done by condition ii). Now choose ν such that $\max_j |t_{\nu,j+1} - t_{\nu,j}| < \delta(\epsilon,\eta)$. Next, choose $n_\nu(\epsilon,\eta)$ such that

$$P([\max_j |X_n(t_{\nu,j}) - X_m(t_{\nu,j})| > \frac{\epsilon}{3}]) < \eta/3$$

for every $m > n \geq n_\nu(\epsilon,\eta)$. This is possible because there are only a finite number of points in T_ν and for each $t_{\nu,j} \epsilon T_\nu$, $P \lim_{n \to \infty} X_n(t_{\nu,j})$ exists. Now let $n(\epsilon,\eta) = \max\{n_1(\epsilon,\eta), n_\nu(\epsilon,\eta)\}$ and consider

$$P([\sup_t |X_n(t) - X_m(t)| > \epsilon])$$

$$= P([\max_j \sup_{t_{\nu,j} \leq t \leq t_{\nu,j+1}} |X_n(t) - X_m(t)| > \epsilon])$$

$$\leq P([\max_j \sup_{t_{\nu,j} \leq t \leq t_{\nu,j+1}} |X_n(t) - X_n(t_{\nu,j})| > \epsilon/3])$$

$$+ P([\max_j |X_n(t_{\nu,j}) - X_m(t_{\nu,j})| > \epsilon/3])$$

$$+ P([\max_j \sup_{t_{\nu,j} \leq t \leq t_{\nu,j+1}} |X_m(t_{\nu,j}) - X_m(t)| > \epsilon/3]) < \eta$$

Now $\lim_{n,m \to \infty} P([\sup_t |X_n(t) - X_m(t)| > \epsilon]) = 0$ for every $\epsilon > 0$ implies there exists a subsequence $\{X_{n_k}(t);\ t \epsilon T\}$, $k \geq 1$, a process $\{X(t);\ t \epsilon T\}$ and a set Λ with $P(\Lambda) = 0$ such that if $\omega \notin \Lambda$ then

$$\lim_{k \to \infty} (\sup_t |X_{n_k}(t) - X(t)|) = 0.$$

We have now established a).

We now proceed to show the process $\{X(t);\ t \epsilon T\}$ is a.s. sample continuous. Let $X'_k(t) = X_{n_k}(t)$ for every $t \epsilon T$ and $k = 1,2,\ldots.$

For every $n = 1, 2, \ldots$, by condition ii), we can find a k_n and a $\delta(n)$ such that for $k \geq k_n$

$$P([\sup_{|t-s| \leq \delta(n)} |X_k'(t) - X_k'(s)| > n^{-1}]) < 2^{-n} .$$

Let $X_n^*(t) = X_{k_n}^*(t)$ for every $t \in T$; $n = 1, 2, \ldots$. Let

$$A_n^* = [\sup_{|t-s| \leq \delta(n)} |X_n^*(t) - X_n^*(s)| > n^{-1}]$$

If $B_k^* = \bigcup_{n=k}^{\infty} A_n^*$, then $A^* = \overline{\lim}\, A_n^* = \bigcup_{k=1}^{\infty} B_k^*$ and $P(A^*) = 0$.

If $\omega \notin B_k^*$, then for every $n \geq k$,

$$\sup_{|t-s| \leq \delta(k)} |X_n^*(t) - X_n^*(s)| \leq k^{-1}.$$

Now if $\omega \notin \Lambda \cup A^*$, and if $\epsilon > 0$ is given,

$$\sup_{|t-s| \leq \delta} |X(t) - X(s)| \leq 2 \sup_t |X_k^*(t) - X(t)| + \sup_{|t-s| \leq \delta} |X_k^*(t) - X_k^*(s)| .$$

First choose $k_0(\omega)$ such that $2 \sup_t |X_k^*(t) - X(t)| < \epsilon/2$ for every $k \geq k_0(\omega)$. This can be done since $\omega \notin \Lambda$. Next choose $k_1(\omega)$ such that $\omega \notin B^*_{k_1(\omega)}$ and $k_1(\omega)^{-1} < \epsilon/2$. Then if $k(\omega) = \max\{k_0(\omega), k_1(\omega)\}$ and $\delta \leq \delta(k_1(\omega))$, we have

$$\sup_{|t-s| \leq \delta} |X(t) - X(s)| \leq \epsilon$$

Then for $\omega \notin \Lambda \cup A^*$, $P(\Lambda \cup A^*) = 0$

$$\lim_{\delta \to 0} \sup_{|t-s| \leq \delta} |X(t) - X(s)| = 0 .$$

Theorem 2.2.11

Let $\{X(t), F(t); t\in T\}$ be a uniformly bounded a.s. sample equicontinuous process satisfying condition 2.2.6. Then the process is a quasi-martingale with $[X]_2 = X_2$ where $X_2(t) = P \lim_{n\to\infty} X_{2n}(t)$ for every $t\in T$, and the processes $\{X_{2n}(t), F_n(t); t\in T\}$ $n = 1,2,\ldots$ are as defined in 2.2.3. Further, the X_2-process is a.s. sample continuous and if $V(\omega)$ denotes the variation of $X_2(\cdot,\omega)$ over T then $E(V(\omega)) < \infty$.

Proof:

Because of Theorem 2.2.10, Lemma 2.2.7, and our previous remarks we need to show the following:

i) $P \lim_{n\to\infty} X_{2n}(t)$ exists for each $t\in \bigcup_{n=1}^{\infty} \Pi_n = \Pi.$

ii) Given ϵ, $\eta > 0$, there exists $n(\epsilon,\eta)$ and $\delta(\epsilon,\eta)$ such that if $n \geq n(\epsilon,\eta)$

$$P([\sup_{|t-s| \leq \delta(\epsilon,\eta)} |X_{2n}(t) - X_{2n}(s)| > \epsilon) < \eta$$

and

iii) For each $t\in T$, the sequence $\{X_{2n}(t); n \geq 1\}$ is uniformly integrable in n.

We first show iii) is satisfied by showing $E(|X_{2n}(t)|^2) \leq K < \infty$ for every $n \geq 1$ and $t\in T$. We have

$$E(|X_{2n}(t)|^2) = E(|\sum_{\Pi_n(t)}^{-} c_{n,j}(X)|^2)$$

$$= E(\sum_{\Pi_n(t)}^{-} |c_{n,j}(X)|^2 + 2\sum_{\Pi_n(t)}^{-} c_{n,j}(X)(\sum_{k>j} c_{n,k}(X)))$$

$$= E\Big(\sum_{\Pi_n(t)} |C_{n,j}(X)|^2\Big) + 2E\Big(\sum_{\Pi_n(t)} C_{n,j}(X) E\big(\sum_{k>j} C_{n,k}(X)|F_{nj}\big)\Big)$$

$$\leq E\Big(\sum_{\Pi_n(t)} |C_{n,j}(X)||\Delta_{n,j}(X)|\Big) + 2E\Big(\sum_{\Pi_n(t)} |C_{n,j}(X)||X(t_{n,j(t)})-X(t_{n,j+1})|\Big)$$

where $j(t)$ is the last j such that $t_{n,j} \leq t$.

Then

$$E(|X_{2n}(t)|^2) \leq 6M_X E\Big(\sum_{\Pi_n(t)} |C_{n,j}(X)|\Big) \leq 6M_X K_X$$

where

$$M_X = \sup_{t,\omega}|X(t,\omega)| \text{ and } K_X \geq \lim_{n\to\infty} E\Big(\sum_{\Pi_n} |C_{n,j}(X)|\Big).$$

We now prove i) by showing

$$E(|X_{2n}(t) - X_{2m}(t)|^2) \to 0 \text{ as } n,m \to \infty \text{ for every } t\in\Pi .$$

In so doing we will pick up an inequality which will allow us to prove ii) immediately. If $t\in\Pi$, then there exists n_t such that $t\in\Pi_n$ for every $n \geq n_t$. We assume now that $m > n \geq n_t$. Then

$$X_{2n}(t) = \sum_{\Pi_n(t)} E\Big(\sum_{\Pi_{nmj}} C_{nmj,k}(X)|F_{nj}\Big)$$

and

$$X_{2m}(t) = \sum_{\Pi_n(t)} \sum_{\Pi_{nmj}} C_{nmj,k}(X) .$$

Let $\quad V_{nmj}(X) = \sum_{\Pi_{nmj}}' C_{nmj,k}(X).$

Then

$$E(|X_{2m}(t) - X_{2n}(t)|^2) = E\Big(\Big|\sum_{\Pi_n(t)} (V_{nmj} - E(V_{nmj}(X)|F_{nj}))\Big|^2\Big)$$

$$= E\Big(\sum_{\Pi_n(t)} |V_{nmj}(X) - E(V_{nmj}(X)|F_{n,j})|^2\Big)$$

$$= E\Big(\sum_{\Pi_n(t)} |V_{nmj}(X)|^2\Big) - E\Big(\sum_{\Pi_n(t)} |E(V_{nmj}(X)|F_{n,j})|^2\Big)$$

$$\leq E\Big(\sum_{\Pi_n} |V_{nmj}(X)|^2\Big)$$

$$= E\Big(\sum_{\Pi_n}\sum_{\Pi_{nmj}} |C_{nmj,k}(X)|^2\Big)$$

$$+ 2E\Big(\sum_{\Pi_n}\sum_{\Pi_{nmj}} C_{nmj,k}(X)E\Big(\sum_{i>k} C_{nmj,i}(X)|F_{nmj,k}\Big)\Big)$$

$$\leq E\Big(\max_{0\leq j\leq N_n}\ \max_{0\leq k\leq k_{nmj}} |\Delta_{nmj,k}(X)| \sum_{\Pi_n}\sum_{\Pi_{nmj}} |C_{nmj,k}(X)|\Big)$$

$$+ 2E\Big(\max_{0\leq j\leq N_n}\ \max_{0\leq k\leq k_{nmj}} |X(t_{n,j+1}) - X(t_{nmj,k+1})| \sum_{\Pi_n}\sum_{\Pi_{nmj}} |C_{nmj,k}(X)|\Big).$$

If $\epsilon_n = \sup'_{\omega}\ \max_{0\leq j\leq N_n}\ \sup_{t_{n,j}\leq s,t\leq t_{n,j+1}} |X(t,\omega) - X(s,\omega)|$, then

$$\sup'_{\omega}\ \max_{0\leq j\leq N_n}\ \max_{0\leq k,k'\leq k_{nmj}} |X(t_{nmj,k}) - X(t_{nmj,k'})| < \epsilon_n$$

for every $m \geq n$. Hence

$$E(|X_{2m}(t) - X_{2n}(t)|^2) \leq E\Big(\sum_{\Pi_n}\sum_{\Pi_{nmj}} C_{nmj,k}(X)|^2\Big) \leq 3\epsilon_n K_X \longrightarrow 0 \text{ as}$$

$m > n \longrightarrow \infty$. Hence i) is proved.

To prove ii), assume $\delta < \min_{0\leq j\leq N_n} |t_{n,j+1} - t_{n,j}|$ and then for every $m \geq n$,

$$P([\sup_{|t-s|\leq\delta}|X_{2m}(t) - X_{2m}(s)| > 3\epsilon])$$

$$\leq P([\max_{0\leq j\leq N_n}\ \max_{0\leq k\leq k_{nmj}}|\sum_{i=0}^{k} c_{nmj,i}(X)| > \epsilon])$$

$$\leq E(\sum_{\Pi_n}|\sum_{\Pi_{nmj}} c_{nmj,k}(X)|^2)/(\epsilon-\epsilon_n)^2$$

$$\leq 3\epsilon_n K_X/(\epsilon-\epsilon_n)^2 \longrightarrow 0 \text{ as } n \longrightarrow \infty .$$

Hence if ϵ,η are given, we can choose $n(\epsilon,\eta)$ such that $3\epsilon_n K_X/(\epsilon-\epsilon_n)^2 < \eta$. If we then fix $n > n(\epsilon,\eta)$ and choose $\delta(\epsilon,\eta) < \|\Pi_n\|$, then for every $m \geq n$,

$$P([\sup_{|t-s|\leq\delta(\epsilon,\eta)}|X_{2m}(t) - X_{2m}(s)| > 3\epsilon]) < \eta .$$

Hence condition ii) is satisfied and the theorem is proved.

Section 3: Main decomposition theorem:

In order to prove the main decomposition theorem we need to do more preliminary work.

We now investigate the uniqueness of the decomposition of a quasi-martingale process. We need the following lemma.

Lemma 2.3.1

Let $\{Y(t), F(t);\ t \in T\}$ be a martingale process having a.e. sample function continuous and of bounded variation on T. Then $P([Y(t) = Y(0);\ t \in T]) = 1$.

Proof:

Since the Y-process has a.e. sample function continuous and of bounded variation on T, if $V(t,\omega)$ denotes the variation of $Y(\cdot,\omega)$ over $[0,t]$ then $V(\cdot,\omega)$ is continuous and monotone non-decreasing on T for a.e. ω. Further $V(t,\cdot)$ is a random variable measurable w.r.t. $F(t)$ for every $t \in T$. As in Theorem 2.1.3, we define $\tau_\nu(t)$ to be the first t such that

$$\sup_{s \leq t} |V(s,\omega)| \geq \nu \quad \text{or} \quad \sup_{s \leq t} |Y(s,\omega)| \geq \nu .$$

If no such t exists let $\tau_\nu(\omega) = 1$. Clearly, $\tau_\nu(\omega)$ is a stopping time of both the processes $\{Y(t), F(t);\ t \in T\}$ and $\{V(t,\omega), F(t);\ t \in T\}$.

Define

$$\begin{aligned} Y_\nu(t,\omega) &= Y(t,\omega) \text{ if } t \leq \tau_\nu(\omega) \\ &= Y(\tau_\nu(\omega),\omega) \text{ if } t > \tau_\nu(\omega). \end{aligned}$$

By Theorem 2.1.1, for each $\nu = 1,2,\ldots$ the process $\{Y_\nu(t), F(t);\ t \in T\}$ is a martingale. Furthermore, for each $\nu = 1,2,\ldots$ the Y_ν-process has a.e. sample function continuous and of bounded variation on T.

As in Theorem 2.1.3, for a.e. ω, there exists $\nu(\omega)$ such that

$$Y(t,\omega) = Y_\nu(t,\omega) \text{ for every } t\in T \text{ if } \nu \geq \nu(\omega).$$

It is also clear that if $V_\nu(t,\omega)$ denotes the variation of $Y_\nu(\cdot,\omega)$ over $[0,t]$, then

$$\begin{aligned} V_\nu(t,\omega) &= V(t,\omega) \text{ if } t \leq \tau_\nu(\omega) \\ &= V(\tau_\nu(\omega),\omega) \text{ if } t > \tau_\nu(\omega). \end{aligned}$$

This follows simply from the fact that for all $s \leq t \leq \tau_\nu(\omega)$, $Y(s,\omega) = Y_\nu(s,\omega)$, and for $t \geq \tau_\nu(\omega)$, $Y_\nu(t,\omega)$ is constant. Also we have

$$\sup_{t,\omega} |V_\nu(t,\omega)| \leq \nu \;, \quad \sup_{t,\omega} |Y_\nu(t,\omega)| \leq V_\nu \;.$$

We now show that for every $\nu = 1,2,\ldots$

$$P([Y_\nu(t) = Y_\nu(0);\ t\in T]) = 1.$$

Let $\{\Pi_n;\ n \geq 1\}$ be a sequence of partitions of T as defined in 2.2.1, and let $\Pi = \bigcup_{n=1}^{\infty} \Pi_n$. Let $t\in\Pi$. Then there exists n_t such that $t\in \Pi_n$ for every $n \geq n_t$.

Assume $n \geq n_t$.

Now

$$E(|Y_\nu(t) - Y_\nu(0)|^2) = E\Big(\sum_{\Pi_n(t)} |\Delta_{n,j}(Y_\nu)|^2\Big)$$

$$\leq E\Big(\max_{0\leq j \leq N_n} |\Delta_{n,j}(Y_\nu)| \sum_{\Pi_n(t)} |\Delta_{n,j}(Y_\nu)|\Big)$$

$$\leq E\Big(\max_{0\leq j \leq N_n} |\Delta_{n,j}(Y_\nu)|\, V_\nu(1,\omega)\Big).$$

Since Y_ν and V_ν are uniformly bounded and Y_ν is a.s. sample continuous we have

$$E(|Y_\nu(t) - Y_\nu(0)|^2) \leq \overline{\lim_{n\to\infty}} E(\max_{0\leq j\leq N_n} |\Delta_{n,j}(Y_\nu)| V_\nu(1,\omega))$$

$$\leq E(\overline{\lim_{n\to\infty}} |\Delta_{n,j}(Y_\nu)| V_\nu(1,\omega)) = 0.$$

Therefore

$$P([Y_\nu(t) = Y_\nu(0);\ t\in\pi]) = 1.$$

Since π is dense in T and Y_ν is a.s. sample continuous, we have

$$P([Y_\nu(t) = Y_\nu(0);\ t\in T]) = 1,$$

and this is true for every $\nu = 1,2,\ldots$. Since for a.e. ω, when ν is sufficiently large $Y_\nu(t,\omega) = Y(t,\omega)$ for every $t\in T$, it follows that

$$P([Y(t) = Y(0);\ t\in T]) = 1 .$$

Theorem 2.3.2

If $\{X(t), F(t);\ t\in T\}$ is a quasi-martingale with the following decompositions

$$P([X(t) = X_1(t) + X_2(t);\ t\in T]) = 1$$

$$P([X(t) = X_1^*(t) + X_2^*(t);\ t\in T]) = 1$$

where the X_i and X_i^*, $i = 1,2$, are a.s. sample continuous, then

$$P([X_1(t) = X_1^*(t) + (X_1(0) - X_1^*(0));\ t\in T]) = 1.$$

In particular, if $X_1(0) = X_1^*(0)$ a.s., then

$$P([X_1(t) = X_1^*(t);\ t\in T]) = 1.$$

Proof:

Let $Y_1 = X_1 - X_1^*$, $Y_2 = X_2^* - X_2$.

Then

$$P([Y_1(t) = Y_2(t);\ t\epsilon T]) = 1$$

and hence the process $\{Y_1(t), F(t);\ t\epsilon T\}$ is a martingale process with a.e. sample function continuous and of bounded variation on T. The conclusion now follows from Lemma 2.3.1.

Suppose now $\{X(t), F(t);\ t\epsilon T\}$ is a.s. sample continuous. Let $\{\tau'_\nu(\omega);\ \nu \geq 0\}$ be the sequence of stopping times defined in Theorem 2.1.2. If we stop the X-process at $\tau'_\nu(\omega)$ then we will get a process which is a.s. sample equi-continuous. Also let $\{\tau''_\nu(\omega);\ \nu \geq 0\}$ be the sequence of stopping times defined in Theorem 2.1.3. If we stop the X-process at $\tau''_\nu(\omega)$ then we will get a process which is uniformly bounded by ν. If $\tau_\nu(\omega)$ is the minimum of $\tau'_\nu(\omega)$ and $\tau''_\nu(\omega)$, then $\tau_\nu(\omega)$ is again a stopping time of the X-process.

(2.3.3)

Define

$$\tau_\nu(\omega) = \min\ \{\tau'_\nu(\omega),\ \tau''_\nu(\omega)\}$$

and let

$$X_\nu(t,\omega) = X(t,\omega) \text{ if } t \leq \tau_\nu(\omega)$$

$$= X(\tau_\nu(\omega),\omega) \text{ if } t > \tau_\nu(\omega)\ .$$

Then for each $\nu = 0,1,\cdots$ the X_ν-process is a.s. sample equi-continuous and uniformly bounded by ν. We have

$$P([\tau_\nu(\omega) < 1]) \leq P([\tau'_\nu(\omega) < 1]) + P([\tau''_\nu(\omega) < 1]) \longrightarrow 0$$

as $\nu \longrightarrow \infty$.

Recall that

$$[\tau_\nu(\omega) < 1] = [X_\nu(t) \neq X(t) \text{ for some } t\epsilon T].$$

Then there exists a set Λ with $P(\Lambda) = 0$ such that for $\omega \notin \Lambda$, there exists $\nu(\omega)$ such that if $\nu \geq \nu(\omega)$

$$X(t,\omega) = X_\nu(t,\omega) \text{ for every } t \in T.$$

If $r\, P([\sup_t |X(t)| \geq r]) \longrightarrow 0$ as $r \longrightarrow \infty$, then

$$E(|X_\nu(t) - X(t)|) \longrightarrow 0 \text{ for every } t \in T,$$

for

$$E(|X_\nu(t) - X(t)|) = \int_{[\tau_\nu(\omega) < 1]} |X_\nu(t) - X(t)|dP + \int_{[\tau_\nu(\omega) = 1]} |X_\nu(t) - X(t)|dP$$

$$= \int_{[\tau_\nu(\omega) < 1]} |X_\nu(t) - X(t)|dP \leq \int_{[\tau_\nu(\omega) < 1]} |X_\nu(t)|dP + \int_{[\tau_\nu(\omega) < 1]} |X(t)|dP$$

$$\leq \nu P([\tau_\nu(\omega) < 1]) + \int_{[\tau_\nu(\omega) < 1]} |X(t)|dP .$$

The second term goes to zero since $P([\tau_\nu(\omega) < 1]) \longrightarrow$ as $\nu \longrightarrow \infty$ and $X(t)$ is integrable. The first term is bounded by

$$\nu P([\tau_\nu'(\omega) < 1]) + \nu P([\tau_\nu''(\omega) < 1])$$

$$\leq \nu 2^{-\nu} + \nu P([\tau_\nu''(\omega) < 1])$$

$$\leq \nu 2^{-\nu} + \nu P([\sup_t |X(t)| \geq \nu]) \longrightarrow 0 \text{ as } \nu \longrightarrow \infty .$$

We now prove another lemma which will lead us to the main theorem.

Lemma 2.3.4.

Assume $\{X(t), F(t);\ t \in T\}$ is a.s. sample continuous and is such that

$$rP([\sup_t |X(t)| \geq r]) \longrightarrow 0 \text{ as } r \longrightarrow \infty .$$

Let the sequence of processes $\{X_\nu(t), F(t);\ t \in T\}$, $\nu \geq 0$, be as defined in 2.3.3. If the X-process satisfies condition 2.2.6, then each X_ν-process also satisfies condition 2.2.6 and the bound K is independent of ν.

Proof:

We assume $\lim\limits_{n\to\infty} E(\sum_{\Pi_n} |C_{n,j}(X)|) \le K_X < \infty$ and want to show there exists $K < \infty$ such that $\lim\limits_{n\to\infty} E(\sum_{\Pi_n} |C_{n,j}(X)|) \le K < \infty$ for every $\nu \ge 0$.

Let $T_n = E(\sum_{\Pi_n} |C_{n,j}(X)|)$ and ${}_\nu T_n = E(\sum_{\Pi_n} |C_{n,j}(X_\nu)|)$. First consider

$$\sum_{\Pi_n} \int_{[\tau_\nu(\omega) \ge t_{n,j}]} (|C_{n,j}(X_\nu)| - |C_{n,j}(X)|)dP \le \sum_{\Pi_n} \int_{[\tau_\nu(\omega) \ge t_{n,j}]} |C_{n,j}(X_\nu) - C_{n,j}(X)|dP$$

$$\le \sum_{\Pi_n} \int_{[\tau_\nu(\omega) \ge t_{n,j}]} |X_\nu(t_{n,j+1}) - X(t_{n,j})|dP \le \sum_{\Pi_n} \int_{[t_{n,j} \le \tau_\nu(\omega) < t_{n,j+1}]} (|X(t_{n,j+1})| + \nu)\, dP$$

$$= \sum_{\Pi_n} \int_{[t_{n,j} \le \tau_\nu(\omega) < t_{n,j+1}]} |X(t_{n,j+1})|dP \quad + \nu P([\tau_\nu(\omega) < 1]) \; .$$

Next consider

$$\sum_{\Pi_n} \int_{[\tau_\nu(\omega) < t_{n,j}]} |C_{n,j}(X)|dP = \sum_{j=0}^{N_n} \Big(\sum_{k=0}^{j-1} \int_{[t_{n,k} \le \tau_\nu(\omega) < t_{n,k+1}]} |C_{n,j}(X)|dP \Big)$$

$$= \sum_{k=0}^{N_n - 1} \Big(\sum_{j=k+1}^{N_n} \int_{[t_{n,k} \le \tau_\nu(\omega) < t_{n,k+1}]} |C_{n,j}(X)|dP \Big)$$

$$= \sum_{k=0}^{N_n-1} \Big(\int_{[t_{n,k} \le \tau_\nu(\omega) < t_{n,k+1}]} \Big(\sum_{j=k+1}^{N_n} |C_{n,j}(X)| \Big) dP \Big)$$

$$= \sum_{k=0}^{N_n-1} \Big(\int_{[t_{n,k} \le \tau_\nu(\omega) < t_{n,k+1}]} \Big(\sum_{j=k+1}^{N_n} |E(-\operatorname{sgn} X(t_{n,k+1}) \Delta_{n,j}(X) | F_{nj})| \Big) dP \Big)$$

$$\ge \sum_{k=0}^{N_n-1} \Big(\int_{[t_{n,k} \le \tau_\nu(\omega) < t_{n,k+1}]} \Big(\sum_{j=k+1}^{N_n} E(-\operatorname{sgn} X(t_{n,k+1}) \Delta_{n,j}(X) | F_{nj}) \Big) dP \Big)$$

$$= \sum_{k=0}^{N_n-1} \Big(\int_{[t_{n,k} \le \tau_\nu(\omega) < t_{n,k+1}]} \Big(\sum_{j=k+1}^{N_n} -\operatorname{sgn} X(t_{n,k+1}) \Delta_{n,j}(X) \Big) dP \Big)$$

$$= \sum_{k=0}^{N_n-1} \Big(\int_{[t_{n,k} \le \tau_\nu(\omega) < t_{n,k+1}]} (|X(t_{n,k+1})| + [-\operatorname{sgn} X(t_{n,k+1}) X(1)]) dP \Big)$$

$$\ge \sum_{k=0}^{N_n-1} \Big(\int_{[t_{n,k} \le \tau_\nu(\omega) < t_{n,k+1}]} (|X(t_{n,k+1})| - |X(1)|) dP \Big)$$

$$= \sum_{k=0}^{N_n-1} \int_{[t_{n,k} \le \tau_\nu(\omega) < t_{n,k+1}]} |X(t_{n,k+1})| dP - \int_{[\tau_\nu(\omega) < t_{n,N_n}]} |X(1)| dP$$

$$= \sum_{k=0}^{N_n} \int_{[t_{n,k} \le \tau_\nu(\omega) < t_{n,k+1}]} |X(t_{n,k+1})| dP - \int_{[t_{n,N_n} \le \tau_\nu(\omega) < 1]} |X(1)| dP - \int_{[\tau_\nu(\omega) < t_{n,N_n}]} |X(1)| dP$$

$$= \sum_{k=0}^{N_n} \int_{[t_{n,k} \leq \tau_\nu(\omega) < t_{n,k+1}]} |X(t_{n,k+1})| dP \quad - \quad \int_{[\tau_\nu(\omega) < 1]} |X(1)| dP .$$

Now

$$_\nu T_n - T_n = E\,(\sum_{\Pi_n} |C_{n,j}(X_\nu)|) - E\,(\sum_{\Pi_n} |C_{n,j}(X)|)$$

$$= \sum_{\Pi_n} (\int_{[\tau_\nu(\omega) < t_{n,j}]} |C_{n,j}(X_\nu)| dP + \int_{[\tau_\nu(\omega) \geq t_{n,j}]} |C_{n,j}(X_\nu)| dP)$$

$$- \sum_{\Pi_n} (\int_{[\tau_\nu(\omega) < t_{n,j}]} |C_{n,j}(X)| dP + \int_{[\tau_\nu(\omega) \geq t_{n,j}]} |C_{n,j}(X)| dP)$$

$$= (\sum_{\Pi_n} \int_{[\tau_\nu(\omega) \geq t_{n,j}]} |C_{n,j}(X_\nu)| dP - \sum_{\Pi_n} \int_{[\tau_\nu(\omega) \geq t_{n,j}]} |C_{n,j}(X)| dP)$$

$$- \sum_{\Pi_n} \int_{[\tau_\nu(\omega) \geq t_{n,j}]} |C_{n,j}(X)| dP$$

$$= (\sum_{\Pi_n} \int_{[\tau_\nu(\omega) \geq t_{n,j}]} (|C_{n,j}(X_\nu)| - |C_{n,j}(X)|) dP) - (\sum_{\Pi_n} \int_{[\tau_\nu(\omega) \geq t_{n,j}]} |C_{n,j}(X)| dP)$$

$$\leq (\sum_{\Pi_n} \int_{[t_{n,j} \leq \tau_\nu(\omega) < t_{n,j+1}]} |X(t_{n,j+1})| dP \quad + \quad \nu P([\tau_\nu(\omega) < 1]))$$

$$- \Big(\sum_{\Pi_n} \int_{[t_{n,k} \le \tau_\nu(\omega) < t_{n,k+1}]} |X(t_{n,k+1})| dP \;-\; \int_{[\tau_\nu(\omega) < 1]} |X(1)| dP\Big)$$

$$= \nu P([\tau_\nu(\omega) < 1]) + \int_{[\tau_\nu(\omega) < 1]} |X(1)| dP .$$

Hence

$$_\nu T_n \le E\Big(\sum_{\Pi_n} |C_{n,j}(X)|\Big) + \nu P([\tau_\nu(\omega) < 1]) + \int_{[\tau_\nu(\omega) < 1]} |X(1)| dP .$$

or $_\nu T_n \le K_X + \nu P([\tau_\nu(\omega) < 1]) + \int_{[\tau_\nu(\omega) < 1]} |X(1) dP$ for every n. As $\nu \to \infty$, both $\nu P([\tau_\nu(\omega) < 1])$ and $\int_{[\tau_\nu(\omega)<1]} |X(1)| dP$ go to zero.

Hence if $K_\nu = \lim_{n\to\infty} E\Big(\sum_{\Pi_n} |C_{n,j}(X_\nu)|\Big)$, then $K_\nu \to K_X$ as $\nu \to \infty$.

Main Theorem 2.3.5.

In order that the a.s. sample continuous, first order process $\{X(t), F(t);\ t\epsilon T\}$ have a decomposition into the sum of two processes

$$P([X(t) = X_1(t) + X_2(t);\ t\epsilon T]) = 1,$$

where $\{X_1(t), F(t);\ t\epsilon T\}$ is an a.s. sample continuous martingale process and the process $\{X_2(t), F(t);\ t\epsilon T\}$ has a.e. sample function continuous and of bounded variation on T with $E(V(\omega)) < \infty$, where $V(\omega)$ is the total variation of $X_2(\cdot,\omega)$ over T, it is necessary and sufficient that

i) $\lim_{r \to \infty} rP([\sup_t |X(t,\omega)| \geq r]) = 0$, and

ii) For any sequence of partitions $\{\pi_n, n \geq 1\}$ of T with $\|\pi_n\| \underset{n}{\to} 0$ and $\pi_1 \subset \pi_2 \subset \ldots$.

$$\lim_{n \to \infty} E(\sum_{\pi_n} |C_{n,j}(X)|) \leq K_X < \infty$$

K_X being independent of the sequence of partitions.

Proof:

We first prove the necessity.

If $\{X(t), F(t); t \in T\}$ is a quasi-martingale with the stated decomposition, we have already indicated that ii) is true. We need then to prove i). Consider

$$rP([\sup_t |X(t,\omega)| \geq r]) = rP([\sup_t |X(t,\omega) - X(0,\omega) + X(0,\omega)| \geq r])$$

$$\leq rP([\sup_t |X(t,\omega) - X(0,\omega)| \geq r/2]) + rP([|X(0,\omega)| \geq r/2]).$$

Now $rP([|X(0,\omega)| \geq r/2]) \leq 2 \int_{[|X(0,\omega)| \geq r/2]} |X(0,\omega)| dP \to 0$ as $r \to \infty$.

Consider then

$$rP([\sup_t |X(t,\omega) - X(0,\omega)| \geq r/2])$$

$$= rP([\sup_t |(X_1(t,\omega) - X_1(0,\omega)) + (X_2(t,\omega) - X_2(0,\omega))| \geq r/2])$$

$$\leq rP([\sup_t |X_1(t,\omega) - X_1(0,\omega)| \geq r/4])$$

$$+ rP([\sup_t |X_2(t,\omega) - X_2(0,\omega)| \geq r/4]).$$

Now $\{(X_1(t) - X_1(0)), F(t); t \in T\}$ is a martingale and hence by Theorem 3.2, sec 11, Chapter VII of Doob, we have

$$P([\sup_t |X_1(t,\omega) - X_1(0,\omega)| \geq r/4]) \leq 4/r\ E(|X_1(1) - X_1(0)|)$$

$\longrightarrow 0$ as $r \longrightarrow \infty$. By the same theorem

$$rP([\sup_t |X_1(t,\omega) - X_1(0,\omega)| \geq r/4]) \leq 4 \int_{[\sup_t |X_1(t)-X_1(0)| \geq r/4]} |X_1(1) - X_1(0)|dP \longrightarrow 0$$

as $r \longrightarrow \infty$.

Consider now

$$rP([\sup_t |X_2(t) - X_2(0)| \geq r/4]) \leq rP([\sup_t |V(t,\omega)| \geq r/4])$$

$$\leq rP([V(1,\omega) \geq r/4]) < 4 \int_{[V(1,\omega) \geq r/4]} V(1,\omega)dP \longrightarrow 0 \text{ as } r \longrightarrow \infty,$$

where $V(t,\omega)$ denotes the variation of $X_2(\cdot,\omega)$ over the interval $[0,t]$. Hence, if the quasi-martingale $\{X(t), F(t);\ t \in T\}$ has the decomposition stated in the theorem, conditions i) and ii) are satisfied.

We now prove the sufficiency of i) and ii). Let the sequence of processes $\{X_\nu(t), F(t);\ t \in T\}$ $\nu = 1,2,\ldots$ be defined as in 2.3.3. Then $P([X_\nu(t) = X(t);\ t \in T]) \longrightarrow 1$ as $\nu \longrightarrow \infty$. By assumption i), $E(|X_\nu(t) - X(t)|) \longrightarrow 0$ as $\nu \longrightarrow \infty$ for every $t \in T$. By Lemma 2.3.4, each process $\{X_\nu(t), F(t);\ t \in T\}$ $\nu = 1,2,\ldots$ satisfies condition 2.2.6, the bound K being independent of ν. Then by Theorem 2.2.11, each process has the decomposition

$$P([X_\nu(t) = X_{1\nu}(t) + X_{2\nu}(t);\ t \in T]) = 1$$

where $\{X_{1\nu}(t), F(t);\ t \in T\}$ is an a.s. sample continuous martingale process, and the process $\{X_{2\nu}(t), F(t);\ t \in T\}$ has a.e. sample function of bounded variation on T. Further

$$X_{2\nu}(t) = P \lim_{n\to\infty} \sum_{\Pi_n(t)} C_{n,j}(X_\nu) \text{ for every } t\in T$$

and so if $V_\nu(t,\omega)$ denotes the variation of $X_{2\nu}(\cdot,\omega)$ over $[0,t]$ we know by Lemma 2.3.4 and Lemma 2.2.7

$$E(V_\nu(t,\omega)) \leq \lim_{n\to\infty} E\Big(\sum_{\Pi_n} |C_{n,j}(X_\nu)|\Big) \leq K < \infty$$

for every $t\in T$, $\nu = 1,2,\ldots$.

It is clear that $\tau_1(\omega) \leq \tau_2(\omega) \leq \cdots$ a.s. Let $\nu^* > \nu$ and let

$$X^*_{1\nu}(t,\omega) = X_{1\nu^*}(t,\omega) \text{ if } t \leq \tau_\nu(\omega)$$

$$= X_{1\nu^*}(\tau_\nu(\omega),\omega) \text{ if } t > \tau_\nu(\omega)$$

$$X^*_{2\nu}(t,\omega) = X_{2\nu^*}(t,\omega) \text{ if } t \leq \tau_\nu(\omega)$$

$$= X_{2\nu^*}(\tau_\nu(\omega),\omega) \text{ if } t > \tau_\nu(\omega) .$$

By Theorem 2.2.1, $X^*_{1\nu}$ is a martingale, and clearly the $X^*_{2\nu}$-process has a.e. sample function of bounded variation on T. Further $X^*_{1\nu}$ and $X^*_{2\nu}$ are a.s. sample continuous.

Since

$$X_\nu(t,\omega) = X_{\nu^*}(t,\omega) \text{ if } t \leq \tau_\nu(\omega)$$

$$= X_{\nu^*}(\tau_\nu(\omega),:) \text{ if } t > \tau_\nu(\omega)$$

we have

$$P([X_\nu(t) = X_{1\nu}(t) + X_{2\nu}(t);\ t\in T]) = 1$$

$$P([X_\nu(t) = X^*_{1\nu}(t) + X^*_{2\nu}(t);\ t\in T]) = 1 .$$

Now $X_{2\nu}(0) = P \lim_{n\to\infty} \sum_{\Pi_n(0)} C_{n,j}(X_\nu) = 0$ for every $\nu = 0,1,\ldots$ so that $X_{2\nu}(0) = X_{2\nu^*}(0) = 0$ a.s. And hence by Theorem 2.3.2, we have

$$P([X_{2\nu}(t) = X^*_{2\nu}(t);\ t\epsilon T]) = 1\ .$$

For a.e. ω, i.e., except for $\omega\epsilon\Lambda$, where $P(\Lambda) = 0$ there exists a $\nu(\omega)$ such that for all $\nu \geq \nu(\omega)$, $\tau_\nu(\omega) = 1$. Then for $\nu^* > \nu \geq \nu(\omega)$

$$X_{i\nu^*}(t,\omega) = X_{i\nu}(t,\omega) \text{ for every } t\epsilon T. \quad (i = 1,2)$$

We define, for $\omega\notin\Lambda$,

$$X_2(t,\omega) = \lim_{\nu\to\infty} X_{2\nu}(t,\omega) \text{ for every } t\epsilon T$$

$$X_1(t,\omega) = \lim_{\nu\to\infty} X_{1\nu}(t,\omega) \text{ for every } t\epsilon T.$$

And hence for every $\omega\notin\Lambda$ there exists $\nu(\omega)$ such that if $\nu \geq \nu(\omega)$

$$X_i(t,\omega) = X_{i\nu}(t,\omega) \text{ for every } t\epsilon T. \quad (i = 1,2)$$

Now, for every $\nu = 0,1,2,\ldots$ and $\omega\notin\Lambda$

$$\begin{aligned} X_{i\nu}(t,\omega) &= X_i(t,\omega) \text{ if } t \leq \tau_\nu(\omega) \\ &= X_i(\tau_\nu(\omega),\omega) \text{ if } t > \tau_\nu(\omega) \end{aligned} \qquad (i = 1,2)$$

If $V(t,\omega)$ denotes the variation of $X_2(\cdot,\omega)$ over $[0,t]$, $V(t,\omega)$ is finite since there exists $\nu(\omega)$ such that

$$X_{2\nu}(t,\omega) = X_2(t,\omega) \text{ for all } t\epsilon T$$

and hence $V(t,\omega) = V_\nu(t,\omega)$ for all $t\epsilon T$.

Clearly

$$\begin{aligned} V_\nu(t,\omega) &= V(t,\omega) \text{ if } t \leq \tau_\nu(\omega) \\ &= V(\tau_\nu(\omega),\omega) \text{ if } t > \tau_\nu(\omega) \end{aligned}$$

and

$$P([\lim_{\nu\to\infty} V_\nu(t,\omega) = V(t,\omega);\ t\epsilon T]) = 1.$$

Since $\tau_\nu(\omega)$ is a.s. non-decreasing in ν, $V_\nu(1,\omega) = V(\tau_\nu(\omega),\omega)$ is monotone non-decreasing in ν. But

$$\lim_{\nu \to \infty} E(V_\nu(1,\omega)) \leq K,$$

and hence by the monotone convergence theorem

$$\lim_{\nu \to \infty} E(V_\nu(1,\omega)) = E(V(1,\omega)).$$

Now

$$|X_{2\nu}(t,\omega) - X_2(t,\omega)| = |X_2(t,\omega) - X_2(t,\omega)| \text{ if } t \leq \tau_\nu(\omega)$$

$$= |X_2(t,\omega) - X_2(\tau_\nu(\omega),\omega)| \text{ if } t > \tau_\nu(\omega).$$

So

$$\sup_t |X_2(t,\omega) - X_{2\nu}(t,\omega)| = \sup_{t > \tau_\nu(\omega)} |X_2(t,\omega) - X_2(\tau_\nu(\omega),\omega)|$$

$$\leq \sup_{t > \tau_\nu(\omega)} (V(t,\omega) - V(\tau_\nu(\omega),\omega)) \leq V(1,\omega) - V_\nu(1,\omega).$$

And hence

$$E(\sup_t |X_2(t) - X_{2\nu}(t)|) \leq E(V(1,\omega) - V_\nu(1,\omega)) \to 0 \text{ as } \nu \to \infty.$$

Now since $E(|X_\nu(t) - X(t)|) \to 0$ and $E(|X_{2\nu}(t) - X_2(t)|) \to 0$ as $\nu \to \infty$ for every $t \in T$. We have $E(|X_{1\nu}(t) - X_1(t)|) \to 0$ as $\nu \to \infty$ and hence X_1 being the limit in the mean of a sequence of martingales is itself a martingale.

We now make a few remarks concerning the decomposition in Theorem 2.3.5.

First, the process $\{X(t), F(t); t \in T\}$ is uniformly integrable in t since the process $\{X_1(t), F(t); t \in T\}$ being a martingale process closed on the right is uniformly integrable, and $\{X_2(t), F(t); t \in T\}$ is uniformly integrable because it is dominated by $V(1,\omega)$.

Secondly, having already proved the decomposition we can easily show

$$X_2(t) = P \lim_{n\to\infty} \sum_{\Pi_n(t)}^{\neg} C_{n,j}(X) \quad \text{for every } t\in T.$$

Let $\{X_\nu(t), F(t);\ t\in T\}$ $\nu = 0,1,\ldots$ be as defined in the theorem, so that for each $\nu = 0,1,\ldots$ X_ν is, by Theorem 2.2.11, a quasi-martingale. Let $P([X_\nu(t) = X_{1\nu}(t) + X_{2\nu}(t);\ t\in T]) = 1$ where $[X_\nu]_1 = X_{1\nu}$ and $[X_\nu]_2 = X_{2\nu}$. In Theorem 2.2.11 we showed

$$P([\sup_t |\sum_{\Pi_n(t)}^{\neg} C_{n,j}(X_\nu) - X_{2\nu}(t)| > \epsilon]) \longrightarrow 0$$

as $n \longrightarrow \infty$ for every $\epsilon > 0$. In Theorem 2.3.5 we showed

$$P([\sup_t |X_{2\nu}(t) - X_2(t)| > \epsilon]) \longrightarrow 0$$

as $\nu \longrightarrow \infty$ for every $\epsilon > 0$.

We will now show

$$P([\sup_t |\sum_{\Pi_n(t)}^{\neg} (C_{n,j}(X) - C_{n,j}(X_\nu))| > \epsilon]) \longrightarrow 0$$

as $\nu \longrightarrow \infty$ <u>uniformly in n</u>.

Let $\epsilon > 0$ be given. Let $\chi_{[\ldots]}$ denote the characteristic function of the set $[\ldots]$. Then

$$P([\sup_t |\sum_{\Pi_n(t)}^{\neg} (C_{n,j}(X) - C_{n,j}(X_\nu))| > \epsilon])$$

$$= P([\chi_{[\tau_\nu(\omega)<1]} \sup_t |\sum_{\Pi_n(t)}^{\neg} (C_{n,j}(X) - C_{n,j}(X_\nu))| > \epsilon])$$

$$+ P([\chi_{[\tau_\nu(\omega)=1]} \sup_t |\sum_{\Pi_n(t)}^{\neg} (C_{n,j}(X) - C_{n,j}(X_\nu))| > \epsilon])$$

Now the first term on the right is bounded by $P([\tau_\nu(\omega) < 1])$ which we know goes to zero as $\nu \longrightarrow \infty$. The second term is bounded by

$$P([X_{[\tau_\nu(\omega)=1]} \sum_{\Pi_n} |C_{n,j}(X_2) - C_{n,j}(X_{2\nu})| > \epsilon])$$

$$\leq \frac{1}{\epsilon} \sum_{\Pi_n} \int_{[\tau_\nu(\omega)=1]} |C_{n,j}(X_2) - C_{n,j}(X_{2\nu})| dP$$

$$\leq \frac{1}{\epsilon} \sum_{\Pi_n} \int_{[\tau_\nu(\omega)\geq t_{n,j}]} E(|\Delta_{n,j}(X_2) - \Delta_{n,j}(X_{2\nu})| | F_{n,j}) dP$$

$$= \frac{1}{\epsilon} \sum_{\Pi_n} \int_{[\tau_\nu(\omega)\geq t_{n,j}]} |X_2(t_{n,j+1}) - X_{2\nu}(t_{n,j+1})| dP .$$

But $X_2(t_{n,j+1}) = X_{2\nu}(t_{n,j+1})$ unless $\tau_\nu(\omega) < t_{n,j+1}$, so that

$$\frac{1}{\epsilon} \sum_{\Pi_n} \int_{[\tau_\nu(\omega)\geq t_{n,j}]} |X_2(t_{n,j+1}) - X_{2\nu}(t_{n,j+1})| dP$$

$$= \frac{1}{\epsilon} \sum_{\Pi_n} \int_{[t_{n,j}\leq \tau_\nu(\omega)< t_{n,j+1}]} |X_2(t_{n,j+1}) - X_{2\nu}(t_{n,j+1})| dP$$

$$\leq \frac{1}{\epsilon} \sum_{\Pi_n} \int_{[t_{n,j}\leq \tau_\nu(\omega)< t_{n,j+1}]} 2V(1,\omega) dP = \frac{1}{\epsilon} \int_{[\tau_\nu(\omega)<1]} 2V(1,\omega) dP .$$

The integral on the right goes to zero as $\nu \longrightarrow \infty$ since $P([\tau_\nu(\omega) < 1]) \longrightarrow 0$ as $\nu \longrightarrow \infty$, and $V(1,\omega)$ is integrable.

Now if $\epsilon > 0$ is given,

$$P([\sup_t |\sum_{\pi_n(t)} c_{n,j}(X) - X_2(t)| > \epsilon])$$

$$\leq P([\sup_t |\sum_{\pi_n(t)} (c_{n,j}(X) - c_{n,j}(X_\nu))| > \epsilon/3])$$

$$+ P([\sup_t |\sum_{\pi_n(t)} c_{n,j}(X_\nu) - X_{2\nu}(t)| > \epsilon/3])$$

$$+ P([\sup_t |X_{2\nu}(t) - X_2(t)| > \epsilon/3]).$$

Given any $\eta > 0$, we can first choose ν such that the first and third terms on the right are less than $\eta/3$ for every $n = 1,2,\ldots$. For this fixed ν we can make the second term less than $\eta/3$ by choosing n sufficiently large.

An immediate corollary to the main theorem is the following.

Corollary 2.3.6

If $\{X(t), F(t);\ t\epsilon T\}$ is an a.s. sample continuous semi-martingale, then it has the decomposition stated in Theorem 2.3.5 if and only if

i) $\lim_{r \to \infty} rP([\sup_t |X(t,\omega)| \geq r]) = 0$

In particular, if the X-process has a.e. sample function non-negative, then i) is always satisfied.

Proof:

We need only show condition ii) is satisfied. If $\{\mathcal{T}_n;\ n \geq 1\}$ is a sequence of partitions of T as defined in 2.2.1, then

$$E\left(\sum_{\Pi_n} |C_{n,j}(X)|\right) = E\left(\sum_{\Pi_n} |E(\Delta_{n,j}(X)|F_{n,j})|\right)$$

$$= E\left(\sum_{\Pi_n} |E(X(t_{n,j+1})|F_{n,j}) - X(t_{n,j})|\right)$$

$$= E\left(\sum_{\Pi_n} (E(X(T_{n,j+1})|F_{n,j}) - X(t_{n,j}))\right)$$

$$= E(X(1)) - E(X(0)) .$$

If the X-process has a.e. sample function non-negative,

$$rP([\sup_t |X(t,\omega)| \geq r]) = rP([\sup_t X(t,\omega) \geq r])$$

$$\leq \int_{[\sup_t X(t,\omega) \geq r]} X(1)dP \quad \longrightarrow 0 \quad \text{as } r \longrightarrow \infty ,$$

and therefore i is satisfied.

If we recall the example given in Chapter I, where

$$X = \exp[Zv] \quad v > 0,$$

Z being the Brownian motion process on $[0,1]$, we see that the X-process is an a.s. sample continuous non-negative semi-martingale since $\exp[tv]$ is a continuous, convex and non-negative function. The corollary tells us the X-process is a quasi-martingale.

We note that when the X-process is a semi-martingale, our conditions coincide with those given by Johnson and Helms.

Section 4: Particular results:

In this section we concern ourselves with some special theorems and results which will be used extensively in Chapter III.

We first prove a lemma which will be quite useful and which is really just an observation.

Lemma 2.4.1

Let $\{X_i(t), F(t);\ t\epsilon T\}$, $i = 1,2,\ldots,k$ be arbitrary processes. Let $\{\pi_n,\ n \geq 1\}$ be a sequence of partitions of T as defined in 2.2.1. For each $n = 1,2,\ldots$ let $f_n(\cdot)$ denote a Baire function of $k(N_n + 2)$ real variables. Assume that for each $i = 1,2,\ldots,k$, $\{X_{i\nu}(t), F(t);\ t\epsilon T\}$, $\nu = 1,2,\ldots$ is a sequence of processes such that

$$P([X_{i\nu}(t) = X_i(t);\ t\epsilon T,\ 1 \leq i \leq k]) \longrightarrow 1$$

as $\nu \longrightarrow \infty$. If for each fixed $\nu = 1,2,\ldots,$

$$\underset{n\to\infty}{P\lim}\ f_n(X_{i\nu}(t_{n,j});\ 1 \leq i \leq k,\ 0 \leq j \leq N_n + 1)$$

$$= \underset{n\to\infty}{P\lim}\ \bar{f}_{n\nu} = \bar{f}_\nu \text{ exists,}$$

then $\underset{n\to\infty}{P\lim}\ f_n(X_i(t_{n,j});\ 1 \leq i \leq k,\ 0 \leq j \leq N_n + 1)$

$$= \underset{n\to\infty}{P\lim}\ \bar{f}_n = \bar{f} \text{ exists.}$$

Moreover, $\underset{\nu\to\infty}{P\lim}\ \bar{f}_\nu$ exists and is $\bar{f}$.

Proof:

We observe that $\bar{f}_{n\nu}$ converges to $\bar{f}_n$ in probability uniformly in n as $\nu \longrightarrow \infty$. For

$$\sup_n P([|\bar{f}_{n\nu} - \bar{f}_n| > 0]) = 1 - \inf_n P([|\bar{f}_{n\nu} - \bar{f}_n| \leq 0])$$

$$= 1 - \inf_n P([X_{i\nu}(t_{n,j}) = X_i(t_{n,j});\ 1 \le i < k,\ 0 \le j \le N_n + 1])$$

$$\le 1 - \inf_n P([X_{i\nu}(t) = X_i(t);\ t\epsilon T,\ 1 \le i \le k])$$

$$= 1 - P([X_{i\nu}(t) = X_i(t);\ t\epsilon T,\ 1 \le i \le k]) \longrightarrow 0 \text{ as } \nu \longrightarrow \infty.$$

Then $P([|\overline{f}_n - \overline{f}_m| > \epsilon]) \le P([|\overline{f}_n - \overline{f}_{n\nu}| > 0])$

$$+ P([|\overline{f}_{n\nu} - \overline{f}_{m\nu}| > \epsilon]) + P([|\overline{f}_{m\nu} - \overline{f}_m| > 0])$$

where $\epsilon > 0$ is arbitrary. Let $\delta > 0$ be given. We first choose ν such that the first and third terms on the right are less than $\delta/3$, then for that fixed ν we can choose n and m to make the second term on the right less than $\delta/3$ because $P \lim_{n\to\infty} \overline{f}_{n\nu} = \overline{f}_\nu$ exists for every $\nu = 1,2,\ldots.$ Hence

$$P \lim_{n\to\infty} \overline{f}_n = \overline{f} \text{ exists.}$$

To show $P \lim_{\nu\to\infty} \overline{f}_\nu$ exists and is $\overline{f}$, consider

$$P([|\overline{f}_\nu - \overline{f}| > \epsilon]) \le P([|\overline{f}_\nu - \overline{f}_{n\nu}| > \epsilon/3])$$

$$+ P([|\overline{f}_{n\nu} - \overline{f}_n| > \epsilon/3]) + P([|\overline{f}_n - \overline{f}| > \epsilon/3])$$

where $\epsilon > 0$ is arbitrary. Let $\delta > 0$ be given. We first choose ν such that the second term on the right is less than $\delta/3$ for every n, and for that fixed ν we can choose n such that the first and third terms on the right are less than $\delta/3$. The lemma is now complete.

With this lemma it is easy to prove a useful theorem concerning the decomposition of a particular type of semi-martingale.

Let $\{X(t),\ F(t);\ t\epsilon T\}$ be an a.s. sample continuous second order martingale process and let $\xi = X^2$. Then the process $\{\xi(t), F(t); t\epsilon T\}$ is an a.s. sample continuous, positive semi-martingale and by Corollary 2.3.6, ξ is a quasi-martingale. Let $[\xi]_1 = \xi_1$ and $[\xi]_2 = \xi_2$. Then we know ξ_2 has a.e. sample function continuous and of bounded variation over T. Further, if $V(\omega)$ denotes the total variation of $\xi_2(\cdot,\omega)$ over T, $E(V(\omega)) < \infty$, and

$$\xi_2(t) = P \lim_{n\to\infty} \sum_{\pi_n(t)} C_{n,j}(\xi) \quad \text{for every } t\epsilon T.$$

Theorem 2.4.2

If $\{\xi_2(t), F(t); t\epsilon T\}$ is the process defined above, then

$$\xi_2(t) = P \lim_{n\to\infty} \sum_{\pi_n(t)} [\Delta_{n,j}(X)]^2 \text{ for every } t\epsilon T.$$

Proof:

We know $\xi_2(t) = P \lim_{n\to\infty} \sum_{\pi_n(t)} C_{n,j}(\xi)$

$$= P \lim_{n\to\infty} \sum_{\pi_n(t)} C_{n,j}(X^2) = P \lim_{n\to\infty} \sum_{\pi_n(t)} E([\Delta_{n,j}(X)]^2|F_{n,j}) .$$

Assume first that the X-process is a.s. sample equi-continuous and uniformly bounded. Then

$$E(|\sum_{\pi_n(t)} ([\Delta_{n,j}(X)]^2 - E([\Delta_{n,j}(X)]^2|F_{n,j}))|^2)$$

$$= E(\sum_{\pi_n(t)} (|\Delta_{n,j}(X)|^4 - |E([\Delta_{n,j}(X)]^2|F_{n,j})|^2))$$

$$\le E\Big(\sum_{\pi_n(t)} |\Delta_{n,j}(X)|^4\Big) \le E\Big(\max_{0 \le j \le k_n} |\Delta_{n,j}(X)|^2 \sum_{\pi_n(t)} |\Delta_{n,j}(X)|^2\Big)$$

$$\le \epsilon_n E\Big(\sum_{\pi_n(t)} |\Delta_{n,j}(X)|^2\Big) \le \epsilon_n E(|X(1) - X(0)|^2),$$

where $\epsilon_n = \sup_{\omega} \max_{0 \le j \le k_n} |\Delta_{n,j}(X)|^2 \longrightarrow 0$ as $n \longrightarrow \infty$, because of the a.s. uniform sample equi-continuity.

Now assume the X-process is an a.s. sample continuous second order martingale process.

Let $\{X_\nu(t), \mathcal{F}(t); t \in T\}$ be as defined in 2.3.3. Then by Theorem 2.1.1, each X_ν is a uniformly bounded a.s. sample equi-continuous martingale process.

For each $\nu = 0, 1, \ldots$, if $\xi_\nu = X_\nu^2$, then ξ_ν has the decomposition

$$\xi_\nu = \xi_{1\nu} + \xi_{2\nu}$$

and by what we have just shown

$$\xi_{2\nu}(t) = P\lim_{n \to \infty} \sum_{\pi_n(t)} [\Delta_{n,j}(X_\nu)]^2 \text{ for every } t \in T.$$

We know $\xi = X^2$ has the decomposition

$$\xi = \xi_1 + \xi_2$$

and

$$P\lim_{\nu \to \infty} \xi_{2\nu}(t) = \xi_2(t) \quad \text{for every } t \in T,$$

as was seen in the proof of Theorem 2.3.[illegible]

By Lemma 2.[illegible]

$$P\lim_{n \to \infty} \sum_{\pi_n(t)} [\Delta_{n,j}(X)]^2 = \xi_2(t)$$

for every $t \in T$

Corollary 2.4.3.

Let $\{X(t), F(t);\ t\in T\}$ and $\{Y(t), F(t);\ t\in T\}$ be a.s. sample continuous second order martingales. If $\{\pi_n;\ n \geq 1\}$ is a sequence of partitions of T as defined in 2.2.1, then

$$\underset{n\to\infty}{\text{P lim}} \sum_{\pi_n(t)} \Delta_{n,j}(X)\ \Delta_{n,j}(Y) \text{ exists}$$

for every $t\in T$ and the process so defined, which we indicate by $\{Z(t), F(t);\ t\in T\}$ can be taken to have a.e. sample function of bounded variation and continuous on T. Moreover, if $V(\omega)$ denotes the total variation of $Z(\cdot,\omega)$ over T, $E(V(\omega)) < \infty$.

Proof:

We have

$$\sum_{\pi_n(t)} \Delta_{n,j}(X)\Delta_{n,j}(Y) = \sum_{\pi_n(t)} 1/4([\Delta_{n,j}(Y)+\Delta_{n,j}(X)]^2-[\Delta_{n,j}(Y)-\Delta_{n,j}(X)]^2)$$

$$= \sum_{\pi_n(t)} 1/4([\Delta_{n,j}(X+Y)]^2- [\Delta_{n,j}(X-Y)]^2) \ .$$

Let $\xi = X+Y$ and $\bar{\xi} = X-Y$. Then ξ and $\bar{\xi}$ are a.s. sample continuous second order martingales, so ξ^2 and $\bar{\xi}^2$ are a.s. sample continuous positive semi-martingales.

We write

$$\sum_{\pi_n(t)} \Delta_{n,j}(Y)\ \Delta_{n,j}(X) = \sum_{\pi_n(t)} 1/4[\Delta_{n,j}(\xi)]^2 - \sum_{\pi_n(t)} 1/4[\Delta_{n,j}(\bar{\xi})]^2$$

and hence, by Theorem 2.4.2,

$$P\lim_{n\to\infty} \sum_{\Pi_n(t)} \Delta_{n,j}(Y)\Delta_{n,j}(X) = 1/4([\xi^2]_2(t) - [\bar{\xi}^2]_2(t)) .$$

Let $Z(t) = 1/4([\xi^2]_2(t) - [\bar{\xi}^2]_2(t))$. Then the Z-process has the stated properties since both $[\xi^2]_2$ and $[\bar{\xi}^2]_2$ have these properties.

We now discuss some other results which will be used in Chapter III.

Suppose $\{X(t), F(t);\ t\epsilon T\}$ satisfies the conditions of Theorem 2.3.5. Let $[X]_1 = X_1$ and $[X]_2 = X_2$. Then X_1 and X_2 are a.s. sample continuous, $X_2(t) = P\lim_{n\to\infty} \sum_{\Pi_n(t)} C_{n,j}(X)$, and if $V(\omega)$ is the variation function of $X_2(\cdot,\omega)$ over T, $E(V(\omega)) < \infty$.

Let $\{Y(t), F(t);\ t\ T\}$ be an a.s. sample continuous process. Then for a.e. ω, the Riemann - Stieltjes integral

$$R\int_0^1 Y(t)dX_2(t) = \lim_{n\to\infty} \sum_{\Pi_n} Y(t^*_{n,j})\Delta_{n,j}(X_2)$$

exists, where $t_{n,j} \le t^*_{n,j} \le t_{n,j+1}$.

We will show

$$R\int_0^1 Y(t)dX_2(t) = P\lim_{n\to\infty} \sum_{\Pi_n} Y(t^*_{n,j})C_{n,j}(X) .$$

It will be sufficient to show

$$R\int_0^1 Y(t)dX_2(t) = P\lim_{n\to\infty} \sum_{\Pi_n} Y(t_{n,j})C_{n,j}(X)$$

since $P\lim_{n\to\infty} \sum_{\Pi_n} Y(t_{n,j})C_{n,j}(X) = P\lim_{n\to\infty} \sum_{\Pi_n} Y(t^*_{n,j})C_{n,j}(X)$

where $t_{n,j} \leq t^*_{n,j} \leq t_{n,j+1}$.

To see this, let $\sim\Lambda_{r,r'} = [\sup_{|t-s| \leq 1/r'} |Y(t) - Y(s)| \geq 1/r]$

$r, r' = 1, 2, \ldots$. Because of the a.s. sample continuity of the Y-process, $\lim_{r' \to \infty} P(\sim\Lambda_{r,r'}) = 0$ for each fixed r.

Assume n is such that $\|\Pi_n\| < 1/r'$. Then

$$P([|\sum_{\Pi_n} [Y(t^*_{n,j}) - Y(t_{n,j})] C_{n,j}(X)| > \epsilon])$$

$$= P([\chi_{\Lambda_{r,r'}} |\sum_{\Pi_n} [Y(t^*_{n,j}) - Y(t_{n,j})] C_{n,j}(X)| > \epsilon])$$

$$+ P([\chi_{\sim\Lambda_{r,r'}} |\sum_{\Pi_n} [Y(t^*_{n,j}) - Y(t_{n,j})] C_{n,j}(X)| > \epsilon])$$

$$\leq P(\sim\Lambda_{r,r'}) + 1/\epsilon \int_{\Lambda_{r,r'}} |\sum_{\Pi_n} [Y(t^*_{n,j}) - Y(t_{n,j})] C_{n,j}(X)| dP$$

$$\leq P(\sim\Lambda_{r,r'}) + 1/\epsilon \sum_{\Pi_n} \int_{\Lambda_{r,r'}} |Y(t^*_{n,j}) - Y(t_{n,j})| |C_{n,j}(X)| dP$$

$$\leq P(\sim\Lambda_{r,r'}) + 1/\epsilon \quad 1/r \quad E(\sum_{\Pi_n} |C_{n,j}(X)|) .$$

If $\eta > 0$ is given, choose r such that $1/\epsilon r \, (K_X) < \eta/2$, then choose r' such that $P(\sim\Lambda_{r,r'}) < \eta/2$. Then for all n with $\|\Pi_n\| < 1/r'$,

$$P([|\sum_{\Pi_n} [Y(t^*_{n,j}) - Y(t_{n,j})] \; C_{n,j}(X)| > \epsilon]) < \eta.$$

We first assume the X_2-process and the Y-process are uniformly bounded. Then

$$P([|\sum_{\Pi_n} Y(t_{n,j})[\Delta_{n,j}(X_2) - C_{n,j}(X)]| > \epsilon])$$

$$\leq 1/\epsilon^2 \, E(|\sum_{\Pi_n} Y(t_{n,j})[\Delta_{n,j}(X_2) - C_{n,j}(X_2)]|^2)$$

$$= 1/\epsilon^2 \, E(\sum_{\Pi_n} |Y(t_{n,j})|^2|\Delta_{n,j}(X_2) - C_{n,j}(X_2)|^2)$$

$$\leq \frac{M_Y^2}{\epsilon^2} \, E(\sum_{\Pi_n} [|\Delta_{n,j}(X_2)|^2 - |C_{n,j}(X_2)|^2])$$

$$\leq \frac{M_Y^2}{\epsilon^2} \, E(\max_{0 \leq j \leq N_n} |\Delta_{n,j}(X_2)| \sum_{\Pi_n} |\Delta_{n,j}(X_2)|)$$

$$\leq \frac{M_Y^2}{\epsilon^2} \, E(\max_{0 \leq j \leq N_n} |\Delta_{n,j}(X_2)| V(\omega)).$$

Now $\max_{0 \leq j \leq N_n} |\Delta_{n,j}(X_2)| V(\omega)$ is dominated by $2M_{X_2} V(\omega)$ where $M_{X_2} = \sup_{t,\omega} |X_2(t,\omega)|$ and hence

$$\overline{\lim_{n \to \infty}} \, P([|\sum_{\Pi_n} Y(t_{n,j})[\Delta_{n,j}(X_2) - C_{n,j}(X)]| > \epsilon])$$

$$\leq \frac{M_Y^2}{\epsilon^2} \, E(\overline{\lim_{n \to \infty}} \max_{0 \leq j \leq N_n} |\Delta_{n,j}(X_2)| V(\omega)) = 0$$

since $\max_{0 \leq j \leq N_n} |\Delta_{n,j}(X_2)| \longrightarrow 0$ a.s. as $n \longrightarrow \infty$ because of the a.s. uniform sample continuity.

If we now drop the condition of uniform boundedness on the X_2-process, we can define a sequence of uniformly bounded processes $\{X_\nu(t), F(t); t\in T\}$ $\nu = 0,1,\ldots$ as in Theorem 2.1.3 such that

$$P([X_\nu(t) = X(t);\ t\in T]) \longrightarrow 1 \quad \text{as } \nu \longrightarrow \infty.$$

For each $\nu = 0,1,\ldots,$ the X_ν-process will be a quasi-martingale with $[X_\nu]_i = X_{i\nu}$, $i = 1,2$, being the X_i-process, $i = 1,2$, stopped at $\tau_\nu(\omega)$. The X_ν-processes have all the properties stated in Theorem 2.3.5.

Still assuming the Y-process is uniformly bounded and a.s. sample continuous, we have just shown that for each $\nu = 0,1,\ldots$

$$R\int_0^1 Y(t)dX_{2\nu}(t) = P\lim_{n\to\infty} \overline{\sum_{\pi_n}} Y(t_{n,j})C_{n,j}(X_\nu).$$

For a.e. ω, there exists a $\nu(\omega)$ such that for all $\nu \geq \nu(\omega)$, $X_2(t,\omega) = X_{2\nu}(t,\omega)$ for every $t\in T$ and hence

$$\lim_{\nu\to\infty} \int_0^1 Y(t)dX_{2\nu}(t) = \int_0^1 Y(t)dX_2(t) \quad \text{a.s.}$$

If we now show that $\overline{\sum_{\pi_n}} Y(t_{n,j})C_{n,j}(X_\nu)$ converges to $\overline{\sum_{\pi_n}} Y(t_{n,j})C_{n,j}(X)$

<u>uniformly in n</u>, it will follow immediately that

$$\int_0^1 Y(t)dX_2(t) = P\lim_{n\to\infty} \sum_{\pi_n} Y(t_{n,j})C_{n,j}(X) .$$

Consider

$$P([|\sum_{\pi_n} Y(t_{n,j})[C_{n,j}(X_\nu) - C_{n,j}(X)]| > \epsilon])$$

$$\leq P([\tau_\nu(\omega) < 1]) + 1/\epsilon \int_{[\tau_\nu(\omega)=1]} \Big|\sum_{\pi_n} Y(t_{n,j})[C_{n,j}(X_\nu) - C_{n,j}(X)]\Big| dP$$

$$\leq P([\tau_\nu(\omega) < 1]) + M_Y/\epsilon \sum_{\pi_n} \int_{[\tau_\nu(\omega)=1]} |C_{n,j}(X_\nu) - C_{n,j}(X)| dP \; .$$

We know the first term goes to zero as $\nu \longrightarrow \infty$ and immediately after the proof of Theorem 2.3.5 we showed the second term goes to zero uniformly in n as $\nu \longrightarrow \infty$.

We have now shown

$$R\int_0^1 Y(t)dX_2(t) = P \lim_{n \to \infty} \sum_{\pi_n} Y(t_{n,j})C_{n,j}(X)$$

when Y is uniformly bounded and X satisfies the conditions of Theorem 2.3.5.

Suppose now Y is a.s. sample continuous but not necessarily uniformly bounded. We can define the sequence of uniformly bounded a.s. sample continuous processes $\{Y_\nu(t), F(t);\ t\in T\}$ $\nu = 0,1,\ldots,$ as in Theorem 2.1.3, such that

$$P([Y_\nu(t) = Y(t);\ t\in T]) \longrightarrow 1 \text{ as } \nu \longrightarrow \infty .$$

For each $\nu = 0,1,\ldots$ we have just shown

$$P \lim_{n \to \infty} \sum_{\pi_n} Y_\nu(t_{n,j})C_{n,j}(X) = R\int_0^1 Y_\nu(t)dX_2(t)$$

when the X-process satisfies the conditions of Theorem 2.3.5. But for a.e. ω, there exists $\nu(\omega)$ such that for $\nu \geq \nu(\omega)$, $Y(t,\omega) = Y_\nu(t,\omega)$ for all $t\in T$, and hence

$$\lim_{\nu \to \infty} \int_0^1 Y_\nu(t)dX_2(t) = \int_0^1 Y(t)dX_2(t) \quad \text{a.s.}$$

Then by Lemma 2.4.1,

$$P\lim_{n \to \infty} \sum_{\Pi_n}^{\neg} Y(t_{n,j})C_{n,j}(X) = R\int_0^1 Y(t)dX_2(t).$$

We have now proved the following theorem.

Theorem 2.4.4:

If the process $\{X(t), F(t); t\epsilon T\}$ satisfies the conditions of Theorem 2.3.5, and if $\{Y(t), F(t); t\epsilon T\}$ is any a.s. sample continuous process, then

$$R\int_0^1 Y(t)dX_2(t) = P\lim_{n \to \infty} \sum_{\Pi_n}^{\neg} Y(t^*_{n,j})C_{n,j}(X)$$

where $t_{n,j} \leq t^*_{n,j} \leq t_{n,j+1}$.

The next theorem follows almost immediately from Theorems 2.4.2, 2.4.4 and Lemma 2.4.1.

Theorem 2.4.5:

Let $\{X(t), F(t); t\epsilon T\}$ be a second order a.s. sample continuous martingale. Let $\xi = X^2$ and let $[\xi]_2 = \xi_2$. Then if $\{Y(t), F(t); t\epsilon T\}$ is any a.s. sample continuous process,

$$P\lim_{n \to \infty} \sum_{\Pi_n}^{\neg} Y(t^*_{n,j})[\Delta_{n,j}(X)]^2 = P\lim_{n \to \infty} \sum_{\Pi_n}^{\neg} Y(t^*_{n,j})C_{n,j}(X^2)$$

$$= R\int_0^1 Y(t)d\xi_2(t).$$

Proof:

We know $P \lim_{n\to\infty} \sum_{\pi_n} Y(t^*_{n,j}) C_{n,j}(X^2) = R\int_0^1 Y(t)d\xi_2(t)$

by Theorem 2.4.4. Assume first that Y is uniformly bounded and X is uniformly bounded and a.s. sample equi-continuous.

If we look at the first part of the proof of Theorem 2.4.2, we see immediately that

$$P \lim_{n\to\infty} \sum_{\pi_n} Y(t_{n,j})[\Delta_{n,j}(X)]^2 = P \lim_{n\to\infty} \sum_{\pi_n} Y(t_{n,j}) C_{n,j}(X^2)$$

$$= R\int_0^1 Y(t)d\xi_2(t).$$

Now assume X is an a.s. sample continuous second order martingale and Y is any a.s. sample continuous process. We can find two sequences of processes $\{X_\nu(t), F(t);\ t\in T\}$ and $\{Y_\nu(t), F(t);\ t\in T\}$, $\nu = 0,1,2,\ldots$ such that for every $\nu = 0,1,\ldots$ X_ν is uniformly bounded a.s. sample equi-continuous, Y_ν is uniformly bounded and

$$P([Y_\nu(t) = Y(t) \text{ and } X_\nu(t) = X(t);\ t\in T]) \longrightarrow 1$$

as $\nu \longrightarrow \infty$. Since X satisfies the condition of Theorem 2.3.5, so does X_ν for each $\nu = 0,1,\ldots$

We let $[X_\nu]_2 = X_{2\nu}$ for each $\nu = 0,1,\ldots.$ Then

$$R\int_0^1 Y_\nu(t)d\xi_{2\nu}(t) = P \lim_{n\to\infty} \sum_{\pi_n} Y_\nu(t_{n,j})[\Delta_{n,j}(X_\nu)]^2$$

for each $\nu = 0,1,\ldots,$ as we have just proved. For a.e. ω, there exists $\nu(\omega)$ such that if $\nu \geq \nu(\omega)$, $Y_\nu(t,\omega) = Y(t,\omega)$ and $\xi_{2\nu}(t,\omega) = \xi_2(t,\omega)$ for every $t\in T$. And hence

$$\lim_{\nu \to \infty} R\int_0^1 Y_\nu(t)\,d\xi_{2\nu}(t) = R\int_0^1 Y(t)\,d\xi_2(t) \quad \text{a.s.}$$

Then, by Lemma 2.4.1,

$$P\lim_{n\to\infty} \sum_{\Pi_n} Y(t_{n,j})[\Delta_{n,j}(X)]^2 = R\int_0^1 Y(t)\,d\xi_2(t) \; .$$

Chapter III: Stochastic Integrals

Section 1: General discussion:

In this chapter we will define a stochastic integral for quasi-martingales. The approach we use is that of limits in probability of Riemann - Stieltjes sums.

Ito (5) and Doob (1, Chapter IX) have defined stochastic integrals with respect to a particular type of martingale process. Doob assumes the martingale $\{X(t), F(t); t\in T\}$ has the property that there exists a monotone non-decreasing function $G(t)$ such that if $s<t$,

$$E(|X(t) - X(s)|^2) = E(|X(t) - X(s)|^2|F(s)) = G(t) - G(s)$$

with probability 1.

For every second order martingale process $\{X(t), F(t); t\in T\}$, there exists a monotone non-decreasing function $G(t)$ such that if $s<t$

$$E(|X(t) - X(s)|^2) = G(t) - G(s).$$

However, the condition

$$E(|X(t) - X(s)|^2) = E(|X(t) - X(s)|^2|F(s)) \quad \text{a.s.}$$

is a real restriction on the martingale process. As Doob points out, if $G(t) = \text{Const.}\ t$, and X is real valued and a.s. sample continuous, then the X-process is necessarily a Brownian motion process.

Doob shows that if $\{Y(t), F(t); t\in T\}$ is a measurable process, i.e., measurable w.r.t. dtdP measure, and if

$$\int_0^1 E(|Y(t)|^2)dG(t) < \infty \quad .$$

Then the stochastic integral, $D\int_0^1 Y(t)dX(t)$, can be defined as the limit in quadratic mean of a sequence of stochastic integrals of "(t,ω)" step functions, (1, Doob, p. 426).

If we let

$$Z(t) = D\int_0^t Y(s)dX(s)$$

then the process $\{Z(t), F(t);\ t \in T\}$ is always a martingale with $E(Z(t)) \equiv 0$ in t. Further, if $Z_0(t) = D\int_0^t Y_0(s)dX(s)$, then

$$E(Z(t)Z_0(t)) = \int_0^t E(Y(s)Y_0(s))dG(s) \; .$$

These properties of the integral are very nice in applications (See, for example, section 3, Chapter IV of Doob).

Unfortunately, the integral does not have some of the more common properties that one associates with the ordinary Riemann-Stieltjes integral. For example, with the Doob integral we have no integration by parts theorem. Furthermore, one of the major defects is the non-existence of a reasonable transform property. This is indicated quite easily by an example in Doob.

If the martingale process $\{X(t), F(t);\ t\ T\}$ is such that $D\int_0^1 [X(t) - X(0)]dX(t)$ exists, then it has the value

$$1/2[X(t) - X(0)]^2 - \underset{n\to\infty}{\text{l.i.m.}}\ 1/2\sum_{\pi_n}[\Delta_{n,j}(X)]^2$$

where l.i.m. indicates limit in quadratic mean.

It is readily apparent that one cannot hope to obtain a theory of stochastic integrals which parallels Riemann - Stieltjes integration if we use this definition of a stochastic integral. In the next two sections we will define a stochastic integral and give some of its properties. Although the exposition is far from complete, it is hoped it will illuminate the feasibility of obtaining a Riemann - Stieltjes type stochastic integral.

Section 2: Definition of the integral.

In what follows we will again be assuming $T = [0,1]$. Let $\{X(t), F(t);\ t\in T\}$ and $\{Y(t), F(t);\ t\in T\}$ be quasi-martingales with

$$P([X(t) = X_1(t) + X_2(t);\ t\in T]) = 1$$

$$P([Y(t) = Y_1(t) + Y_2(t);\ t\in T]) = 1$$

where as usual, $[X]_i = X_i$ and $[Y]_i = Y_i$, $i = 1,2$.

We assume X_i and Y_i, $i = 1,2$, are a.s. sample continuous.

Let $\{\pi_n,\ n \geq 1\}$ be a sequence of partitions of T as defined in 2.2.1. We will show that

(3.2.1) $$P\lim_{n\to\infty} \sum_{\pi_n} 1/2[Y(t_{n,j+1})+Y(t_{n,j})][X(t_{n,j+1})-X(t_{n,j})]$$

exists and we define $\int_0^1 Y(t)dX(t)$ to be this limit.

We will write

$$\Delta_{n,j}(Y) = [Y(t_{n,j+1}) - Y(t_{n,j})]$$

and

$$\bar{\Delta}_{n,j}(Y) = 1/2[Y(t_{n,j+1}) + Y(t_{n,j})] .$$

We will use freely the notation introduced in Section 2 of Chapter II.

We can write the sums in 3.2.1 in the form

$$\sum_{\pi_n} \bar{\Delta}_{n,j}(Y)\,\Delta_{n,j}(X) \quad .$$

First

$$\sum_{\pi_n} \bar{\Delta}_{n,j}(Y)\,\Delta_{n,j}(X) = \sum_{\pi_n} \bar{\Delta}_{n,j}(Y)\Delta_{n,j}(X_1) + \sum_{\pi_n} \bar{\Delta}_{n,j}(Y)\Delta_{n,j}(X_2) \; .$$

Since Y has a.e. sample function continuous and X_2 has a.e. sample function of bounded variation on T, the limit of the second sum exists a.s. and is the ordinary Riemann - Stieltjes integral of $Y(\cdot,\omega)$ w.r.t. $X_2(\cdot,\omega)$. We indicate this as follows

$$(3.2.2) \quad R\int_0^1 Y(t)dX_2(t) = \underset{n\to\infty}{P\lim} \sum_{\pi_n} \bar{\Delta}_{n,j}(Y)\Delta_{n,j}(X_2) \; .$$

Consider now

$$\sum_{\pi_n} \bar{\Delta}_{n,j}(Y)\Delta_{n,j}(X_1) = \sum_{\pi_n} Y(t_{n,j})\Delta_{n,j}(X_1) + \sum_{\pi_n} 1/2\,\Delta_{n,j}(Y)\Delta_{n,j}(X_1) \; .$$

The second sum on the right can be further reduced to the following

$$\sum_{\pi_n} 1/2\,\Delta_{n,j}(Y_1)\Delta_{n,j}(X_1) + \sum_{\pi_n} 1/2\,\Delta_{n,j}(Y_2)\Delta_{n,j}(X_1) \; .$$

Again, since X_1 is a.s. sample continuous and Y_2 has a.e. sample function of bounded variation on T, the second sum goes a.s. to zero.

We have now reduced the problem to showing the existence of the limits

$$\underset{n\to\infty}{P\lim} \sum_{\pi_n} 1/2\,\Delta_{n,j}(Y_1)\Delta_{n,j}(X_1), \; \underset{n\to\infty}{P\lim} \sum_{\pi_n} Y(t_{n,j})\Delta_{n,j}(X_1) \; .$$

It was proved in Corollary 2.4.3 that if Y_1 and X_1 are a.s. sample continuous second order martingales, then

$$\underset{n\to\infty}{\text{P lim}} \sum_{\pi_n}^{\neg} 1/2\, \Delta_{n,j}(Y_1)\Delta_{n,j}(X_1) = 1/8([Y_1 + X_1]_2(1) - [Y_1 - X_1]_2(1)),$$

the limit being a.s. sample continuous and having a.e. sample function of bounded variation.

If $\{Y_{1\nu}(t), F(t);\ t\in T\}$, $\{X_{1\nu}(t), F(t);\ t\in T\}$, $\nu = 0,1,\ldots$ are as in Theorem 2.1.3, then by Theorem 2.1.1 each $X_{1\nu}$ and $Y_{1\nu}$ are uniformly bounded a.s. sample continuous martingales and

$$P([Y_{1\nu}(t) \neq Y_1(t) \text{ or } X_{1\nu}(t) \neq X_1(t) \text{ for some } t\in T]) \longrightarrow 0$$

as $\nu \longrightarrow \infty$. Hence by Lemma 2.4.1

$$\underset{n\to\infty}{\text{P lim}} \sum_{\pi_n}^{\neg} 1/2\, \Delta_{n,j}(Y_1)\, \Delta_{n,j}(X_1) \text{ exists.}$$

Further, if $\xi_{2\nu} = [X_{1\nu} + X_{1\nu}]_2$, $\bar{\xi}_{2\nu} = [Y_{1\nu} - X_{1\nu}]_2$, then

(3.2.4) $$\underset{n\to\infty}{\text{P lim}} \sum_{\pi_n}^{\neg} 1/2\, \Delta_{n,j}(Y_1)\, \Delta_{n,j}(X_1) = \underset{\nu\to\infty}{\text{P lim}}\ 1/8(\xi_{2\nu}(1) - \bar{\xi}_{2\nu}(1)).$$

We define

(3.2.5) $$\int_0^1 dXdY = \underset{n\to\infty}{\text{P lim}} \sum_{\pi_n}^{\neg} 1/2\, \Delta_{n,j}(Y_1)\, \Delta_{n,j}(X_1)\ .$$

If we now show $\underset{n\to\infty}{\text{P lim}} \sum_{\pi_n}^{-} Y(t_{n,j})\Delta_{n,j}(X_1)$ exists, we will have proved the existence of the limit in (3.2.1). To prove this we first prove the following lemma.

Lemma 3.2.6.

If $\{X(t), F(t);\ t\in T\}$ is a second order martingale and

$\{Y(t), F(t);\ t\in T\}$ is a uniformly bounded, a.s. sample continuous process then

$$\lim_{n\to\infty} \sum_{\Pi_n} Y(t_{n,j})\Delta_{n,j}(X) \text{ exists in quadratic mean.}$$

Proof:

Let $m > n$. Then $\Pi_n \subset \Pi_m$ and we can write

$$Z_n = \sum_{\Pi_n} Y(t_{n,j})\Delta_{n,j}(X) = \sum_{\Pi_n}\sum_{\Pi_{nmj}} Y(t_{n,j})\Delta_{nmj,k}(X)$$

and $$Z_m = \sum_{\Pi_m} Y(t_{m,i})\Delta_{m,i}(X) = \sum_{\Pi_n}\sum_{\Pi_{nmj}} Y(t_{nmj,k})\Delta_{nmj,k}(X) .$$

Then

$$E(|Z_m - Z_n|^2) = E(|\sum_{\Pi_n}\sum_{\Pi_{nmj}} [Y(t_{nmj,k}) - Y(t_{n,j})]\Delta_{nmj,k}(X)|^2)$$

$$= E(\sum_{\Pi_n}\sum_{\Pi_{nmj}} |Y(t_{nmj,k}) - Y(t_{n,j})|^2|\Delta_{nmj,k}(X)|^2)$$

because of orthogonality.

Let $\{\epsilon_r,\ r \geq 0\}$ be a sequence of positive real numbers with $\epsilon_0 > \epsilon_1 > \ldots > 0$ and $\lim_{r\to\infty} \epsilon_r = 0$. Also let $\{\delta_{r'},\ r' \geq 0\}$ be a sequence of positive real numbers with $\delta_0 > \delta_1 > \ldots > 0$ and $\lim_{r'\to\infty} \delta_{r'} = 0$.

Define

$$\sim\Delta_{r,r'}(t) = [\sup_{\substack{|s-s'|\leq\delta_{r'}\\ s,s'\leq t}} |Y(s) - Y(s')| \leq \epsilon_r],\ r,r' = 0,1,2,\ldots.$$

Because of the a.s. uniform sample continuity

$$P(\Lambda_{r,r'}(t)) \longrightarrow 0 \quad \text{as } r' \longrightarrow \infty \quad \text{for}$$

every fixed r.

Now for fixed r and r', if $t_1 > t_0$ then

$$\sim \Lambda_{r,r'}(t_1) \subset \sim \Lambda_{r,r'}(t_0) \quad \text{and hence}$$

$$\Lambda_{r,r'}(t_1) \supset \Lambda_{r,r'}(t_0) \quad \text{when } t_1 > t_0 \ .$$

Now

$$\sum_{\pi_n} \sum_{\pi_{nmj}} E(|Y(t_{nmj,k}) - Y(t_{n,j})|^2 |\Delta_{nmj,k}(X)|^2)$$

$$= \sum_{\pi_n} \sum_{\pi_{nmj}} \int_{\Lambda_{r,r'}(t_{nmj,k})} |Y(t_{nmj,k}) - Y(t_{n,j})|^2 |\Delta_{nmj,k}(X)|^2 dP$$

$$+ \sum_{\pi_n} \sum_{\pi_{nmj}} \int_{\sim\Lambda_{r,r'}(t_{nmj,k})} |Y(t_{nmj,k}) - Y(t_{n,j})|^2 |\Delta_{nmj,k}(X)|^2 dP \ .$$

Let $M_Y = \sup_{t,\omega} |Y_{(t,\omega)}|$. Then

$$\int_{\Lambda_{r,r'}(t_{nmj,k})} |Y(t_{nmj,k}) - Y(t_{n,j})|^2 |\Delta_{nmj,k}(X)|^2 dP$$

$$\leq 4M_Y^2 \int_{\Lambda_{r,r'}(t_{nmj,k})} |\Delta_{nmj,k}(X)|^2 dP$$

$$= 4M_Y^2 \int_{\Lambda_{r,r'}(t_{nmj,k})} |X(t_{nmj,k+1})|^2 dP - 4M_Y^2 \int_{\Lambda_{r,r'}(t_{nmj,k})} |X(t_{nmj,k})|^2 dP$$

$$\leq 4M_Y^2 \int_{\Lambda_{r,r'}(t_{nmj,k+1})} |X(t_{nmj,k+1})|^2 dP - 4M_Y^2 \int_{\Lambda_{r,r'}(t_{nmj,k})} |X(t_{nmj,k})|^2 dP \;.$$

Then

$$\sum_{\Pi_n} \sum_{\Pi_{nmj}} \int_{\Lambda_{r,r'}(t_{nmj,k})} |Y(t_{nmj,k}) - Y(t_{n,j})|^2 |\Delta_{nmj,k}(X)|^2 dP$$

$$\leq 4M_Y^2 \int_{\Lambda_{r,r'}(1)} |X(1)|^2 dP \;.$$

Let $\epsilon > 0$ be given, choose r such that $\epsilon_r^2\, E(|X(1) - X(0)|^2) < \epsilon/2$. Since $P(\Lambda_{r,r'}(1)) \longrightarrow 0$ as $r' \longrightarrow \infty$ for every fixed r, we can now choose r' such that $4M_Y^2 \int_{\Lambda_{r,r'}(1)} |X(1)|^2 dP < \epsilon/2$. Now choose $n(\epsilon)$ such that $\|\Pi_{n(\epsilon)}\| < \delta_{r'}$. Then if $m > n \geq n(\epsilon)$

$$E(|Z_m - Z_n|^2) \leq \epsilon_r^2\, E(|X(1) - X(0)|^2) + 4M_Y^2 \int_{\Lambda_{r,r'}(1)} |X(1)|^2 dP$$

$$\leq \epsilon/2 + \epsilon/2 = \epsilon.$$

The lemma is now proved.

If $\{X(t), F(t);\, t\epsilon T\}$ is any a.s. sample continuous martingale process and if $\{Y(t), F(t);\, t\epsilon T\}$ is any a.s. sample continuous process then

$$P \lim_{n \to \infty} \sum_{\Pi_n} Y(t_{n,j}) \Delta_{n,j}(X) \text{ exists} \;.$$

For, if $\{X_\nu(t), F(t);\, t\epsilon T\}$ and $\{Y_\nu(t), F(t);\, t\epsilon T\}$ $\nu \geq 0$, are as defined in Theorem 2.1.3, then X_ν is a uniformly bounded a.s. sample continuous martingale and Y_ν is also uniformly bounded and a.s.

sample continuous. Then by Lemma 3.3.5

$$\lim_{n\to\infty} \sum_{\Pi_n}^{-} Y_\nu(t_{n,j})\Delta_{n,j}(X_\nu) \text{ exists in quadratic mean for}$$

every $\nu \geq 0$, and hence by Lemma 2.4.1

$$P\lim_{n\to\infty} \sum_{\Pi_n}^{-} Y(t_{n,j})\Delta_{n,j}(X) \text{ exists.}$$

We now define

$$(3.2.7) \quad D\int_0^1 Y(t)dX_1(t) = P\lim_{n\to\infty} \sum_{\Pi_n}^{-} Y(t_{n,j})\Delta_{n,j}(X_1) \; .$$

We have now established the existence of the limit in 3.2.1 and we define

$$(3.2.8) \quad \int_0^1 Y(t)dX(t) = D\int_0^1 Y(t)dX_1(t) + R\int_0^1 Y(t)dX_2(t) + \int_0^1 dY(t)dX(t)$$

$$= P\lim_{n\to\infty} \sum_{\Pi_n}^{-} \bar{\Delta}_{n,j}(Y)\,\Delta_{n,j}(X) \; .$$

Clearly, if $[\alpha,\beta]$ is any closed sub interval of T, we can define

$$(3.2.9) \quad \int_\alpha^\beta Y(t)dX(t) = P\lim_{n\to\infty} \sum_{\Pi_{\alpha\beta,n}}^{-} \bar{\Delta}_{n,j}(Y)\,\Delta_{n,j}(X),$$

for the limit will exist when $\{\Pi_{\alpha\beta,n},\ n \geq 1\}$ is a sequence of partitions of $[\alpha,\beta]$ with $\|\Pi_{\alpha\beta,n}\| \underset{n}{\to} 0$ and $\Pi_{\alpha\beta,1} \subset \Pi_{\alpha\beta,2} \subset \dots$.

If $\{\Pi_n,\ n \geq 1\}$ is a sequence of partitions of T as defined in 2.2.1 and if $\Pi_n(t) = \Pi_n \cap [0,t]$, then

$$(3.2.10) \quad \int_0^t Y(t)dX(t) = P\lim_{n\to\infty} \sum_{\Pi_n(t)}^{-} \bar{\Delta}_{n,j}(Y)\Delta_{n,j}(X) \; .$$

Section 3: Some properties of the integral.

Let $\{X(t), F(t); t\in T\}$ and $\{Y(t), F(t); t\in T\}$ be quasi-martingales with

$$P([X(t) = X_1(t) + X_2(t);\ t\in T]) = 1$$

$$P([Y(t) = Y_1(t) + Y_2(t);\ t\in T]) = 1$$

where $[X]_i = X_i$ and $[Y]_i = Y_i$, $i = 1,2$, are a.s. sample continuous.

$$\text{Let } Z(t) = \int_0^t Y(s)dX(s) = P\lim_{n\to\infty} \sum_{\pi_n(t)}^{-} \bar{\Delta}_{n,j}(Y)\,\Delta_{n,j}(X)\ .$$

Theorem 3.3.1.

The process $\{Z(t), F(t); t\in T\}$, as just defined, can be taken to be a.s. sample continuous.

Proof:

$$Z(t) = \int_0^t Y(s)dX(s) = D\int_0^t Y(s)dX_1(s) + R\int_0^t Y(s)dX_2(s)$$
$$+\int_0^t dY(s)dX(s)\ .$$

First, the integral $R\int_0^t Y(s)dX_2(s)$ as a function of its upper limit defines for a.e. ω, a real valued continuous function on T.

The integral $\int_0^t dY(s)dX(s) = P\lim_{n\to\infty} \sum_{\pi_n(t)}^{-} 1/2\,\Delta_{n,j}(Y_1)\,\Delta_{n,j}(X_1)$

$= P\lim_{\nu\to\infty} 1/8(\xi_{2\nu}(t) - \bar{\xi}_{2\nu}(t))$ where $\xi_{2\nu}$ and $\bar{\xi}_{2\nu}$ are as defined in 3.2.4 is a.s. sample continuous by Theorem 2.2.10 since both $\xi_{2\nu}$ and $\bar{\xi}_{2\nu}$ are a.s. sample continuous for every $\nu = 1,2,\dots$.

It thus remains to show the integral

$$D\int_0^t Y(s)\,dX_1(s) = P\lim_{n\to\infty} \sum_{\Pi_n(t)} Y(t_{n,j})\Delta_{n,j}(X_1)$$

as a function of its upper limit defines for a.e. ω, a real valued continuous function on T.

We first assume the X_1 and Y processes are uniformly bounded and a.s. sample equi-continuous.

By Theorem 2.2.10 it is sufficient to show that given $\epsilon,\eta > 0$ there exists $n(\epsilon,\eta)$ and $\delta(\epsilon,\eta)$ such that for all $n \geq n(\epsilon,\eta)$

$$(1)\quad P([\sup_{|t-s|\leq\delta(\epsilon,\eta)} |\sum_{\Pi_n(t)} Y(t_{n,j})\Delta_{n,j}(X_1) - \sum_{\Pi_n(s)} Y(t_{n,j})\Delta_{n,j}(X_1)| > \epsilon]) < \eta$$

First we show that if $\epsilon,\eta > 0$ are given, there exists an $n(\epsilon,\eta)$ such that if $n \geq n(\epsilon,\eta)$, then for all $m \geq n$,

$$P([\max_{0\leq j\leq N_n}\ \max_{0\leq k\leq k_{nmj}} |\sum_{i=0}^{k} Y(t_{nmj,i})\Delta_{nmj,k}(X_1)| > \epsilon]) < \eta$$

Now

$$P([\max_{0\leq j\leq N_n}\ \max_{0\leq k\leq k_{nmj}} |\sum_{i=0}^{k} Y(t_{nmj,i})\Delta_{nmj,i}(X_1)| > \epsilon])$$

$$\leq P([\max_{0\leq j\leq N_n}\ \max_{0\leq k\leq k_{nmj}} |\sum_{i=0}^{k} [Y(t_{nmj,i}) - Y(t_{n,j})]\Delta_{nmj,k}(X_1)| > \epsilon/2])$$

$$+ P([\max_{0\leq j\leq N_n}\ \max_{0\leq k\leq k_{nmj}} |Y(t_{n,j})(X_1(t_{nmj,k+1}) - X_1(t_{n,j}))| > \epsilon/2])$$

We have

$$P([\max_{0\leq j\leq N_n}\ \max_{0\leq k\leq k_{nmj}} |Y(t_{n,j})(X_1(t_{nmj,k+1}) - X_1(t_{n,j}))| > \epsilon/2])$$

$$\le P([\max_{0\le j\le N_n}\max_{0\le k\le k_{nmj}}|X_1(t_{nmj,k+1})-X_1(t_{n,j})|>\epsilon/2M_Y])$$

where $M_Y = \sup_{t,\omega}|Y(t,\omega)|$.

Because of the a.s. sample equi-continuity of the X_1-process we choose a $\delta_1 > 0$ such that $P([\sup_{|t-s|\le\delta_1}|X_1(t) - X_1(s)| > \epsilon/2M_Y]) = 0$.

If $n_1(\epsilon)$ is such that for $n \ge n_1(\epsilon)$, $\|\pi_n\| < \delta_1$, then

$$P([\max_{0\le j\le N_n}\max_{0\le k\le k_{nmj}}|X_1(t_{nmj,k+1})-X_1(t_{n,j})|>\epsilon/2M_Y]) = 0 .$$

Observe that the partial sums

$$\sum_{i=0}^{k}[Y(t_{nmj,k}) - Y(t_{n,j})]\Delta_{nmj,k}(X_1),\ k = 0, \ldots, k_{nmj}$$

form a finite martingale sequence for every $j = 0,1,\ldots,N_n$. Hence

$$P([\max_{0\le j\le N_n}\max_{0\le k\le k_{nmj}}|\sum_{i=0}^{k}[Y(t_{nmj,k})-Y(t_{n,j})]\Delta_{nmj,k}(X_1)|>\epsilon])$$

$$\le\sum_{j=0}^{N_n}P([\max_{0\le k\le k_{nmj}}|\sum_{i=0}^{k}[Y(t_{nmj,k})-Y(t_{n,j})]\Delta_{nmj,k}(X_1)|>\epsilon])$$

$$\le 1/\epsilon^2\sum_{j=0}^{N_n}\sum_{k=0}^{k_{nmj}}E(|Y(t_{nmj,k})-Y(t_{n,j})|^2|\Delta_{nmj,k}(X_1)|^2) .$$

Now choose $\delta_2(\epsilon,\eta)$ such that

$$\sup_{|t-s|\le\delta_2(\epsilon,\eta)}|Y(t)-Y(s)|^2 \le \frac{\epsilon^2}{E(|X_1(1)-X_1(0)|^2)} < \eta \quad \text{a.s.}$$

Then choose $n_2(\epsilon,\eta)$ such that $\|\pi_n\| < \delta_2(\epsilon,\eta)$ for every $n \ge n_2(\epsilon,\eta)$

Then

$$1/\epsilon^2 \sum_{j=0}^{N_n} \sum_{k=0}^{k_{nmj}} E(|Y(t_{nmj,k}) - Y(t_{n,j})|^2 |\Delta_{nmj,k}(X_1)|^2) < \eta \; .$$

Hence if $n \geq \max\{n_1(\epsilon,\eta),\ n_2(\epsilon,\eta)\}$, we have the desired result for all $m \geq n$.

We can now show (1) is true.

Assume $m > n$ and $\delta < \min_{0 \leq j \leq N_n} |t_{n,j+1} - t_{n,j}|$. Then

$$(2) \quad P([\sup_{|t-s| \leq \delta} |\sum_{\pi_m(t)} Y(t_{m,i})\Delta_{m,i}(X_1) - \sum_{\pi_m(s)} Y(t_{m,i})\Delta_{m,i}(X_1)| > 3\epsilon]$$

$$\leq P([\max_{0 \leq j \leq N_n} \max_{0 \leq k \leq k_{nmj}} |\sum_{i=0}^{k} Y(t_{nmj,i})\Delta_{nmj,i}(X_1)| > \epsilon]) \; .$$

If $n(\epsilon,\eta) = \max\{n_1(\epsilon,\eta),\ n_2(\epsilon,\eta)\}$ is as chosen above, then we can fix $n \geq n(\epsilon,\eta)$ and let $\delta(\epsilon,\eta) \leq \min |t_{n,j+1} - t_{n,j}|$. Then (2) will be less than η for all $m \geq n$. Hence

$$P \lim_{n \to \infty} \sum_{\pi_n(t)} Y(t_{n,j})\Delta_{n,j}(X_1) = D\int_0^t Y(s)dX_1(s)$$

is a.s. sample continuous when Y and X_1 are uniformly bounded and a.s. sample equi-continuous. The desired result now follows by stopping Y and X_1 according to the stopping time defined in 2.3.3 and then applying Theorem 2.2.10 and Lemma 2.4.1.

With what has already been shown, we can easily obtain conditions under which the process

$$Z(t) = \int_0^t Y(s)dX(s) \quad \text{is a quasi-martingale.}$$

Theorem 3.3.2.

Let $\{X(t), F(t); t\epsilon T\}$ and $\{Y(t), F(t); t\epsilon T\}$ be quasi-martingales. Let $[X]_i = X_i$ and $[Y]_i = Y_i$, $i = 1,2$, be a.s. sample continuous. Further assume Y is uniformly bounded and Y_1 and X_1 are second order martingales. If

$$Z(t) = \int_0^t Y(s)dX(s), \text{ for every } t\epsilon T,$$

then the process $\{Z(t), F(t); t\epsilon T\}$ is a quasi martingale with

$$[Z]_1(t) = D\int_0^t Y(s)dX_1(s) \qquad [Z]_2(t) = R\int_0^t Y(s)dX_2(s) + \int_0^t dY(s)dX(s)$$

and $[Z]_i = Z_i$, $i = 1,2$, are a.s. sample continuous.

Proof:

Since Y is uniformly bounded and X_1 is a second order martingale, by Lemma 3.2.6

$$D\int_0^t Y(s)dX_1(s) = \lim_{n\to\infty} \sum_{\pi_n(t)} Y(t_{n,j})\Delta_{n,j}(X_1)$$

where the limit is in quadratic mean. So Z_1 being the limit in quadratic mean of a sequence of martingales is again a martingale. We have further shown, in Theorem 3.3.1 that Z_1 is a.s. sample continuous. Now $R\int_0^t Y(s)dX_2(s)$ defines a continuous real valued function of bounded variation on T for a.e. ω when considered as a function of its upper limit. Also

$$\int_0^t dY(s)dX(s) = P\lim_{n\to\infty} \sum_{\pi_n(t)} 1/2\, \Delta_{n,j}(Y_1)\, \Delta_{n,j}(X_1)$$

$$= 1/8([(X_1+ Y_1)^2]_2(t) - [(X_1- Y_1)^2]_2(t))$$

is a.s. sample continuous and of bounded variation on T since X_1 and Y_1 are second order martingales. Therefore Z_2 has a.e. sample function continuous and of bounded variation on T. The theorem is now established.

We now investigate some properties of the integral which parallel the Riemann - Stieltjes integral.

Theorem 3.3.3:

Let $\{X(t), F(t);\ t\in T\}$ and $\{Y(t), F(t);\ t\in T\}$ be quasi-martingales with $[X]_i = X_i$ and $[Y]_i = Y_i$, $i = 1,2$, a.s. sample continuous. Then

$$\int_0^1 X(t)dY(t) + \int_0^1 Y(t)dX(t) = X(1)Y(1) - X(0)Y(0) \quad \text{a.s.}$$

Proof:

Observe

$$\Delta_{n,j}(XY) = \bar{\Delta}_{n,j}(X)\,\Delta_{n,j}(Y) + \bar{\Delta}_{n,j}(Y)\,\Delta_{n,j}(X), \quad \text{so that}$$

$$\sum_{\pi_n} \Delta_{n,j}(XY) = \sum_{\pi_n} \bar{\Delta}_{n,j}(X)\,\Delta_{n,j}(Y) + \sum_{\pi_n} \bar{\Delta}_{n,j}(Y)\,\Delta_{n,j}(X) .$$

Hence, taking probability limits on both sides we have the desired result.

One thing that one would expect of an integral is the following: $\{X(t), F(t);\ t\in T\}$ is a quasi martingale and if the function f is such

that $$\int_0^t f'(X(s))dX(s) \text{ exists}$$

then $$\int_0^t f'(X(s))dX(s) = f(X(t)) - f(X(0)) .$$

For the Doob integral this is not the case, as is illustrated by an example from Doob (1, p. 443). If $\{X(t), F(t);\ t\epsilon T\}$ is such that $D\int_0^1 [X(t) - X(0)]dX(t)$ exists, then

$$D\int_0^1 [X(t) - X(0)]dX(t) = 1/2[X(1) - X(0)]^2$$

$$- 1/2 \lim_{n\to\infty} \sum_{\pi_n} [\Delta_{n,j}(X)]^2$$

where the limit is in quadratic mean.

Let $\{X(t), F(t);\ t\epsilon T\}$ be a quasi martingale with $[X]_i = X_i$, $i = 1,2$, a.s. sample continuous. Let $f(t) = t^2$. Then

$$X(1)^2 - X(0)^2 = \int_0^1 2X(t)dX(t)$$

or

$$f(X(1)) - f(X(0)) = \int_0^1 f'(X(t))dX(t).$$

For, $\int_0^1 2X(t)dX(t) = 2 \text{ P} \lim_{n\to\infty} \sum_{\pi_n} 1/2\, \bar{\Delta}_{n,j}(X)\, \Delta_{n,j}(X)$

$$= \text{P} \lim_{n\to\infty} \sum_{\pi_n} \Delta_{n,j}(X^2) = X(1)^2 - X(0)^2 .$$

This property of the integral can be generalized to the following extent.

Theorem 3.3.4.

Let $\{X(t), F(t);\ t\epsilon T\}$ be a quasi-martingale with $[X]_i = X_i$, $i = 1,2$, a.s. sample continuous. If f is a real valued function of a real variable and has a continuous second derivative, then

$$f(X(1)) - f(X(0)) = \int_0^1 f'(X(s))dX(s) .$$

Proof:

We want to show

$$f(X(1)) - f(X(0)) = P\lim_{n\to\infty} \sum_{\Pi_n} \bar{\Delta}_{n,j}(f'(X))\Delta_{n,j}(X)$$

$$= P\lim_{n\to\infty} \sum_{\Pi_n} f'(X(t_{n,j}))\Delta_{n,j}(X) + P\lim_{n\to\infty} \sum_{\Pi_n} 1/2\, \Delta_{n,j}(f'(X))\Delta_{n,j}(X)$$

$$= P\lim_{n\to\infty} \sum_{\Pi_n} f'(X(t_{n,j}))\Delta_{n,j}(X) + P\lim_{n\to\infty} \sum_{\Pi_n} 1/2\, f''(X(t^*_{n,j}))[\Delta_{n,j}(X)]^2$$

where $t_{n,j} \leq t^*_{n,j} \leq t_{n,j+1}$ and where $t^*_{n,j}$ depends on ω.

We can write

$$f(X(1)) - f(X(0)) = \sum_{\Pi_n} \Delta_{n,j}(f(X))$$

$$= \sum_{\Pi_n} [f'(X(t_{n,j}))\Delta_{n,j}(X) + 1/2\, f''(X(t^{**}_{n,j}))[\Delta_{n,j}(X)]^2]$$

where again $t_{n,j} \leq t^{**}_{n,j} \leq t_{n,j+1}$, and where $t^{**}_{n,j}$ depends on ω.

Hence,

$$|f(X(1)) - f(X(0)) - \sum_{\Pi_n} \bar{\Delta}_{n,j}\ f'(X)\Delta_{n,j}(X)|$$

$$= 1/2 |\sum_{\Pi_n} [f''(X(t^*_{n,j})) - f''(X(t^{**}_{n,j}))][\Delta_{n,j}(X)]^2|$$

$$\leq 1/2 \sum_{\Pi_n} |f''(X(t^*_{n,j})) - f''(X(t^{**}_{n,j}))|\,|\Delta_{n,j}(X)|^2$$

$$\leq \frac{\epsilon_n(\omega)}{2} \sum_{\Pi_n} |\Delta_{n,j}(X)|^2$$

where $\epsilon_n(\omega) = \max_j \sup_{t_{n,j} \leq s,t \leq t_{n,j+1}} |f''(X(t)) - f''(X(s))|$.

Because of the a.s. uniform sample continuity of the X-process and the continuity of the function $f''(\cdot)$, $\epsilon_n(\omega) \longrightarrow 0$ a.s. as $n \longrightarrow \infty$.

If $\sum_{\pi_n} [\Delta_{n,j}(X)]^2$ converges in probability, then

$\sum_{\pi_n} \bar{\Delta}_{n,j}(f(X))\Delta_{n,j}(X)$ will converge in probability to $f(X(1))-f(X(0))$.

Now

$$\sum_{\pi_n} [\Delta_{n,j}(X)]^2 = \sum_{\pi_n} [\Delta_{n,j}(X_1)]^2 + \sum_{\pi_n} \Delta_{n,j}(X_1)\Delta_{n,j}(X_2)$$

$$+ \sum_{\pi_n} [\Delta_{n,j}(X_2)]^2.$$

Because of the a.s. sample continuity of the processes X_1 and X_2 and the a.s. sample bounded variation of the X_2-process, the second and third sums go to zero a.s. as $n \longrightarrow \infty$. We have previously shown the probability limit of the first sum exists. The theorem is then proved.

A natural question at this point is what additional assumptions on the function f or the quasi-martingale X will insure the process $\{f(X(t)), F(t);\ t\epsilon T\}$ is a quasi-martingale.

Assuming f and X satisfy the conditions of Theorem 3.3.4, we have

$$f(X(t)) = f(X(0)) + \int_0^t f'(X(s))dX(s) \ .$$

We write

$$\int_0^t f'(X(s))dX(s) = P \lim_{n \to \infty} \sum_{\pi_n(t)} \bar{\Delta}_{n,j}(f'(X))\Delta_{n,j}(X_1)$$

$$+ \underset{n\to\infty}{\mathrm{P\ lim}} \sum_{\Pi_n(t)} \bar{\Delta}_{n,j}(f'(X))\Delta_{n,j}(X_2).$$

Since f' is continuous and X is a.s. sample continuous,

$$\underset{n\to\infty}{\mathrm{P\ lim}} \sum_{\Pi_n(t)} \bar{\Delta}_{n,j}(f'(X))\Delta_{n,j}(X_2) = R\int_0^t f'(X(s))dX_2(s) \ .$$

Consider then

$$\sum_{\Pi_n(t)} \bar{\Delta}_{n,j}(f'(X))\Delta_{n,j}(X_1) = \sum_{\Pi_n(t)} f'(X(t_{n,j}))\Delta_{n,j}(X_1)$$

$$+ \sum_{\Pi_n(t)} 1/2\ \Delta_{n,j}(f'(X))\Delta_{n,j}(X_1) \ .$$

Assuming f' is bounded and continuous, the first sum converges in quadratic mean and therefore defines a martingale process. Last of all, consider

$$\sum_{\Pi_n(t)} 1/2\Delta_{n,j}(f'(X))\Delta_{n,j}(X_1) = \sum_{\Pi_n(t)} 1/2\ f''(X(t^*_{n,j}))[\Delta_{n,j}(X_1)]^2 \ .$$

When X_1 is an a.s. sample continuous second order martingale, we showed in Theorem 2.4.5 that

$$\underset{n\to\infty}{\mathrm{P\ lim}} \sum_{\Pi_n(t)} 1/2\ f''(X(t^*_{n,j}))[\Delta_{n,j}(X_1)]^2 = \int_0^t f''(X(s))d\xi(s)$$

where $\xi = [X_1^2]_2$, provided f" is continuous.

Then for each $t\in T$,

$$f(X(t)) = \ (X(0)) + D\int_0^t f'(X(s))dX_1(s) + R\int_0^t f'(X(s))dX_2(s)$$

$$+ \int_0^t f''(X(s))d\xi(s)$$

If we let

$$[f(X)]_1(t) = D\int_0^t f'(X(s))dX_1(s) \text{ for every } t\epsilon T, \text{ and}$$

$$[f(X)]_2(t) = f(X(0)) + R\int_0^t f'(X(s))dX_2(s) + \int_0^t f''(X(s))d\xi(s) \text{ for}$$

every $t\epsilon T$, then the process $\{f(X(t)), F(t); t\epsilon T\}$ is a quasi martingale if the above conditions are satisfied and $E(|f(X(t))|) < \infty$ for every $t\epsilon T$. We summarize these results into the statement of the next theorem.

Theorem 3.3.5.

If f is a real valued function of a real variable with f' bounded and f" continuous, and if $\{X(t), F(t); t\epsilon T\}$ is a quasi-martingale with $[X]_i = X_i$, $i = 1,2$, a.s. sample continuous and X_1 second order, then $\{f(X(t)), F(t); t\epsilon T\}$ is a quasi-martingale if $E(|f(X(t))|)$ ∞ for every $t\epsilon T$.

Further

$$[f(X)]_1(t) = D\int_0^t f(X(s))dX_1(s) \quad \text{for every } t\epsilon T$$

$$[f(X)]_2(t) = f(X(0)) + R\int_0^t f'(X(s))dX_2(s) + R\int_0^t f''(X(s))d\xi(s)$$

for every $t\epsilon T$, where $\xi = [X_1^2]_2$.

Assume now $\{X(t), F(t); t\epsilon T\}$ and $\{Y(t), F(t); t\epsilon T\}$ are quasi-martingales with $[X]_i = X_i$ and $[Y]_i = Y_i$, $i = 1,2$, a.s. sample continuous Assume further Y is uniformly bounded and X_1 and Y_1 are second order martingales.

If $Z(t) = \int_0^t Y(s)dX(s)$ for every $t\epsilon T$, then by Theorem 3.3.4, we know the process $\{Z(t), F(t); t\epsilon T\}$ is again a quasi-martingale with the following decomposition

$$[Z]_1(t) = Z_1(t) = D\int_0^t Y(s)dX_1(s)$$

$$[Z]_2(t) = Z_2(t) = R\int_0^t Y(s)dX_2(s) + 1/8(\xi(t) - \bar{\xi}(t))$$

where $\xi = [(X_1 + Y_1)^2]_2$, $\bar{\xi} = [(X_1 - Y_1)^2]_2$.

Let f be a real valued function of a real variable with f' bounded and f" bounded and continuous. Then since Z_1 is a second order martingale and Z_i, i = 1,2, are a.s. sample continuous, by Theorem 3.3.4

$$f(Z(t)) = f(Z(0)) + \int_0^t f'(Z(s))dZ(s) \text{ for every } t \in T.$$

We wish to show that

$$f(Z(1)) - f(Z(0)) = \int_0^1 f'(Z(s))dZ(s) = \int_0^1 f'(Z(s))Y(s)dX(s)$$

or symbolically, dZ = YdX.

We now write, assuming the existence of the limit,

$$\int_0^1 f'(Z(s))Y(s)dX(s) = P\lim_{n\to\infty} \sum_{\Pi_n}^{\neg} \bar{\Delta}_{n,j}(f'(Z)Y)\Delta_{n,j}(X) \quad .$$

Consider

$$\sum_{\Pi_n}^{\neg} \bar{\Delta}_{n,j}(f'(Z)Y)\Delta_{n,j}(X) = \sum_{\Pi_n}^{\neg} \bar{\Delta}_{n,j}(f'(Z)Y)\Delta_{n,j}(X_2)$$

$$+\sum_{\Pi_n}^{\neg} f'(Z(t_{n,j}))Y(t_{n,j})\Delta_{n,j}(X_1) + \sum_{\Pi_n}^{\neg} 1/2\,\Delta_{n,j}(f'(Z)Y)\Delta_{n,j}(X_1) \ .$$

The third term can be rewritten as follows:

$$\sum_{\pi_n} 1/2\, \Delta_{n,j}(f'(Z)Y)\Delta_{n,j}(X_1) = \sum_{\pi_n} 1/2\, Y(t_{n,j+1})\Delta_{n,j}(f'(Z))\Delta_{n,j}(X_1)$$

$$+ \sum_{\pi_n} 1/2\; f'(Z(t_{n,j}))\Delta_{n,j}(Y)\Delta_{n,j}(X_1)$$

$$= \sum_{\pi_n} 1/2\, \Delta_{n,j}(Y)\Delta_{n,j}(f'(Z))\Delta_{n,j}(X_1)$$

$$+ \sum_{\pi_n} 1/2\, Y(t_{n,j})\Delta_{n,j}(f'(Z))\Delta_{n,j}(X_1)$$

$$+ \sum_{\pi_n} 1/2\; f'(Z(t_{n,j}))\Delta_{n,j}(Y)\Delta_{n,j}(X_1)\;.$$

We will show that

1) $$P\lim_{n\to\infty} \sum_{\pi_n} f'(Z(t_{n,j}))Y(t_{n,j})\Delta_{n,j}(X_1)$$

$$= P\lim_{n\to\infty} \sum_{\pi_n} f'(Z(t_{n,j}))\Delta_{n,j}(Z_1)$$

2) $$P\lim_{n\to\infty} \Big[\sum_{\pi_n} \bar{\Delta}_{n,j}(f'(Z)Y)\Delta_{n,j}(X_2) + \sum_{\pi_n} 1/2\; f'(Z(t_{n,j}))\Delta_{n,j}(Y)\Delta_{n,j}(X_1)\Big]$$

$$= P\lim_{n\to\infty} \sum_{\pi_n} \bar{\Delta}_{n,j}(f'(Z))\Delta_{n,j}(Z_2) = R\int_0^1 f'(Z(t))\,dZ_2(t)$$

3) $$P\lim_{n\to\infty} \sum_{\pi_n} 1/2\, \Delta_{n,j}(f'(Z))Y(t_{n,j})\Delta_{n,j}(X_1)$$

$$= P\lim_{n\to\infty} \sum_{\pi_n} 1/2\, \Delta_{n,j}(f'(Z))\Delta_{n,j}(Z_1)$$

and finally

4) $$\operatorname*{P\,lim}_{n\to\infty} \sum_{\Pi_n} 1/2\, \Delta_{n,j}(f'(Z))\Delta_{n,j}(Y)\Delta_{n,j}(X_1) = 0 .$$

To show 1) is true, consider

$$E\Big(\Big|\sum_{\Pi_n} f'(Z(t_{n,j}))Y(t_{n,j})\Delta_{n,j}(X_1) - \sum_{\Pi_n} f'(Z(t_{n,j}))\Delta_{n,j}(Z_1)\Big|^2\Big)$$

$$= E\Big(\Big|\sum_{\Pi_n} f'(Z(t_{n,j}))[Y(t_{n,j})\Delta_{n,j}(X_1) - \Delta_{n,j}(Z_1)]\Big|^2\Big)$$

$$= E\Big(\sum_{\Pi_n} [f'(Z(t_{n,j}))]^2[Y(t_{n,j})\Delta_{n,j}(X_1) - \Delta_{n,j}(Z_1)]^2\Big)$$

$$\leq M_{f'}^2 \quad E\Big(\sum_{\Pi_n} [Y(t_{n,j})\Delta_{n,j}(X_1) - \Delta_{n,j}(Z_1)]^2\Big)$$

where $M_{f'} = \sup_{t,\omega} |f(Z(t,\omega))|$.

Now $$E\Big(\Big|Z_1(1) - \sum_{\Pi_n} Y(t_{n,j})\Delta_{n,j}(X_1)\Big|^2\Big)$$

$$= E\Big(\Big|\sum_{\Pi_n} \Delta_{n,j}(Z_1) - \sum_{\Pi_n} Y(t_{n,j})\Delta_{n,j}(X_1)\Big|^2\Big)$$

$$= E\Big(\Big|\sum_{\Pi_n} [\Delta_{n,j}(Z_1) - Y(t_{n,j})\Delta_{n,j}(X_1)]\Big|^2\Big)$$

$$= E\Big(\sum_{\Pi_n} |\Delta_{n,j}(Z_1) - Y(t_{n,j})\Delta_{n,j}(X_1)|^2\Big) \longrightarrow 0 \quad \text{as } n \longrightarrow \infty,$$

and 1) is now proved.

To prove 2) we first observe that

$$\underset{n\to\infty}{P\lim}\sum_{\pi_n} 1/2\ f'(Z(t_{n,j}))\Delta_{n,j}(Y)\Delta_{n,j}(X_1)$$

$$= \underset{n\to\infty}{P\lim}\sum_{\pi_n} 1/2\ f'(Z(t_{n,j}))\Delta_{n,j}(Y_1)\Delta_{n,j}(X_1)\ .$$

Then we can write

$$R\int_0^1 f'(Z(t))dZ_2(t) = R\int_0^1 f'(Z(t))d(R\int_0^t Y(s)dX_2(s) + 1/8(\xi(t)-\bar{\xi}(t)))$$

$$= R\int_0^1 f'(Z(t))Y(t)dX_2(t) + 1/8\ R\int_0^1 f'(Z(t))d(\xi(t)-\bar{\xi}(t))\ .$$

But $1/8\ R\int_0^1 f'(Z(t))d(\xi(t)-\bar{\xi}(t))$

$$= \underset{n\to\infty}{P\lim}\sum_{\pi_n} 1/2\ f'(Z(t_{n,j}))\Delta_{n,j}(Y_1)\Delta_{n,j}(X_1)$$

and

$$R\int_0^1 f'(Z(t))Y(t)dX_2(t) = \underset{n\to\infty}{P\lim}\sum_{\pi_n} \bar{\Delta}_{n,j}(f(Z)Y)\Delta_{n,j}(X_2)\,.$$

Hence 2) is proved.

Recall f" is bounded and Y is uniformly bounded. Now to prove 3) consider,

$$\underset{n\to\infty}{P\lim}\sum_{\pi_n} 1/2\ \Delta_{n,j}(f'(Z))Y(t_{n,j})\Delta_{n,j}(X_1)$$

$$= \underset{n\to\infty}{P\lim}\sum_{\pi_n} 1/2\ f''(Z(t^*_{n,j}))\Delta_{n,j}(Z)Y(t_{n,j})\Delta_{n,j}(X_1)$$

$$= \underset{n\to\infty}{P\lim}\sum_{\pi_n} 1/2\ f''(Z(t^*_{n,j}))\Delta_{n,j}(Z_1)Y(t_{n,j})\Delta_{n,j}(X_1)\ .$$

Now

$$E\Big(\Big|\sum_{\pi_n} 1/2\, f''(Z(t^*_{n,j}))\Delta_{n,j}(Z_1)Y(t_{n,j})\Delta_{n,j}(X_1)$$

$$-\sum_{\pi_n} 1/2\, f''(Z(t^*_{n,j}))[\,Z_1(t_{n,j})]^2\Big|^2\Big)$$

$$= E\Big(\Big|\sum_{\pi_n} 1/2\, f''(Z(t^*_{n,j}))\Delta_{n,j}(Z_1)(\Delta_{n,j}(Z_1) - Y(t_{n,j})\Delta_{n,j}(X_1))\Big|^2\Big)$$

$$\leq E\Big(\Big|\sum_{\pi_n}[1/2\, f''(Z(t^*_{n,j}))\Delta_{n,j}(Z_1)]^2\Big|^{1/2}$$

$$\Big|\sum_{\pi_n}[\Delta_{n,j}(Z_1) - Y(t_{n,j})\Delta_{n,j}(X_1)]^2\Big|^{1/2}\Big)$$

$$\leq E^{1/2}\Big(\sum_{\pi_n}[1/2\, f''(Z(t^*_{n,j}))\Delta_{n,j}(Z_1)]^2\Big)$$

$$E^{1/2}\Big(\sum_{\pi_n}[\Delta_{n,j}(Z_1) - Y(t_{n,j})\Delta_{n,j}(X_1)]^2\Big).$$

$$\leq 1/2M_{f''}\, E^{1/2}\Big(\sum_{\pi_n}[\Delta_{n,j}(Z_1)]^2\Big)\; E^{1/2}\Big(\sum_{\pi_n}[\Delta_{n,j}(Z_1) - Y(t_{n,j})\Delta_{n,j}(X_1)]^2\Big)$$

$$= 1/2M_{f''}\, E^{1/2}(|Z_1(1) - Z_1(0)|^2)\; E^{1/2}\Big(\sum_{\pi_n}[\Delta_{n,j}(Z_1) - Y(t_{n,j})\Delta_{n,j}(X_1)]^2\Big)$$

where $M_{f''} = \sup_{t,\omega} |f''(Z(t))|$.

In proving 1) we showed the term on the right goes to zero as $n \longrightarrow \infty$. Hence 3) is proved.

To show 4) we observe

$$P\lim_{n\to\infty}\sum_{\pi_n} 1/2\,\Delta_{n,j}(f'(Z))\Delta_{n,j}(Y)\Delta_{n,j}(X_1)$$

$$= P\lim_{n\to\infty}\sum_{\pi_n} 1/2\Delta_{n,j}(f'(Z))\Delta_{n,j}(Y_1)\Delta_{n,j}(X_1)$$

$$= P\lim_{n\to\infty} 1/2\sum_{\pi_n} f'(Z(t_{n,j+1}))\Delta_{n,j}(Y_1)\Delta_{n,j}(X_1)$$

$$- P\lim_{n\to\infty} 1/2\sum_{\pi_n} f'(Z(t_{n,j}))\Delta_{n,j}(Y_1)\Delta_{n,j}(X_1)$$

$$= 1/8\int_0^1 f'(Z(t))d(\xi(t)-\bar{\xi}(t)) - 1/8\int_0^1 f'(Z(t))d(\xi(t)-\bar{\xi}(t)) = 0.$$

We can now prove the following theorem.

Theorem 3.3.6.

Let $\{X(t), F(t);\ t\in T\}$ and $\{Y(t), F(t);\ t\in T\}$ be quasi-martingales with $[X]_i = X_i$ and $[Y]_i = Y_i$, $i = 1,2$, a.s. sample continuous. Let f be a real valued function of a real variable with continuous second derivative. If

$$Z(t) = \int_0^t Y(s)dX(s) \quad \text{for every } t\in T,$$

then

$$f(Z(1)) - f(Z(0)) = \int_0^1 f'(Z(t))Y(t)dX(t)\ .$$

Proof:

Let $\nu = 1,2,\ldots.$ Let $\tau_\nu(\omega)$ be the first t such that

$\sup_{s \leq t} |Z(s,\omega)| \geq \nu$ or $\sup_{s \leq t} |X(s,\omega)| \geq \nu$ or $\sup_{s \leq t} |Y(s,\omega)| \geq \nu$.

If no such t exists, let $\tau_\nu(\omega) = 1$.

Since X, Y and Z are all a.s. sample continuous

$$P([Z(t) \neq Z_\nu(t) \text{ or } Y(t) \neq Y_\nu(t) \text{ or } X(t) \neq X_\nu(t) \text{ for some } t\in T]) \longrightarrow 0$$

as $\nu \longrightarrow \infty$.

Clearly, τ_ν is a stopping time for each of the processes X, Y, and Z. Let X_ν, Y_ν and Z_ν be the processes X, Y, and Z stopped at τ_ν. Then for every $\nu = 1,2,\ldots$, X_ν, Y_ν, and Z_ν are uniformly bounded by ν and are a.s. sample continuous.

We first show, for every $t\in T$

$$Z_\nu(t) = \int_0^t Y_\nu(s)dX_\nu(s) = P \lim_{n\to\infty} \sum_{\pi_n(t)} \bar{\Delta}_{n,j}(Y_\nu)\Delta_{n,j}(X_\nu) .$$

Let $Z_\nu^*(t) = \int_0^t Y_\nu(s)dX_\nu(s)$. For each $t\in T$, we can find a subsequence of partitions $\{\pi'_k(t);\ k \geq 1\}$ such that

$$Z_\nu^*(t) = \lim_{k\to\infty} \sum_{\pi'_k(t)} \bar{\Delta}_{k,j}(Y_\nu)\Delta_{k,j}(X_\nu) \quad \text{a.s.}$$

and

$$Z(t) = \lim_{k\to\infty} \sum_{\pi'_k(t)} \bar{\Delta}_{k,j}(X)\Delta_{k,j}(X) \quad \text{a.s.}$$

If $t \leq \tau_\nu(\omega)$, then

$$Z_\nu^*(t) = \lim_{k\to\infty} \sum_{\pi'_k(t)} \bar{\Delta}_{k,j}(Y_\nu)\Delta_{k,j}(X_\nu)$$

$$= \lim_{k\to\infty} \sum_{\pi'_k(t)} \bar{\Delta}_{k,j}(Y)\Delta_{k,j}(X) = Z(t) = Z_\nu(t)$$

If $t > \tau_\nu(\omega)$, then

$$Z^*_\nu(t) = \lim_{k\to\infty} \sum_{\pi'_k(t)} \bar{\Delta}_{k,j}(Y_\nu)\Delta_{k,j}(X_\nu)$$

$$= \lim_{k\to\infty} \sum_{\pi'_k(\tau_\nu(\omega))} \bar{\Delta}_{k,j}(Y)\Delta_{k,j}(X)$$

$$+ \lim_{k\to\infty} \bar{\Delta}_{k,j(\tau_\nu(\omega))}(Y)\,\Delta_{k,j(\tau_\nu(\omega))}(X) = Z(\tau_\nu(\omega)) = Z_\nu(t),$$

since $\lim_{k\to\infty} \bar{\Delta}_{k,j(\tau_\nu(\omega))}(Y)\Delta_{k,j(\tau_\nu(\omega))}(X) = 0$. Then $Z^*_\nu(t) = Z_\nu(t)$

a.s. for every $t\in T$, so that

$$Z_\nu(t) = P\lim_{n\to\infty} \sum_{\pi_n(t)} \bar{\Delta}_{n,j}(Y_\nu)\Delta_{n,j}(X_\nu) .$$

Actually, one can take

$$P([Z^*_\nu(t) = Z_\nu(t);\ t\in T,\ \nu = 1,2,\ldots]) = 1,$$

for we get equality on an everywhere dense subset with probability one, and since Z^*_ν and Z_ν are a.s. sample continuous they must be equal for every $t\in T$ with probability one.

Since for each $\nu = 1,2,\ldots$, Z_ν is uniformly bounded and f, f' and f" are continuous, $f(Z_\nu)$, $f'(Z_\nu)$ and $f''(Z_\nu)$ are uniformly bounded. But then

$$f(Z_\nu(1)) - f(Z_\nu(0)) = \int_0^1 f'(Z_\nu(t))Y_\nu(t)dX_\nu(t).$$

For a.e. ω, there exists $\nu(\omega)$ such that $\tau_\nu(\omega) = 1$ for all $\nu \geq \nu(\omega)$, and hence

$$f(Z(1)) - f(Z(0)) = \int_0^1 f'(Z(t))Y(t)dX(t) \quad \text{a.s.}$$

The theorem is now proved.

Further properties of the integral need to be investigated extensively. Some of the theorems can be generalized somewhat, but in an obvious way.

Some of the more pertinent questions which have not been looked into to any great extent are the following:

i) Can the decomposition Theorem 2.3.5 be extended to processes $\{X(t), F(t); t \in T\}$ having a.e. sample function right (or left) continuous? Here it is felt that condition i) of Theorem 2.3.5 may have to be replaced by the condition of uniform integrability of the sequence of stopping times $\{\tau_\nu(\omega); \nu \geq 1\}$, where $\tau_\nu(\omega)$ is the first t such that $\sup_{s \leq t} |X(s,\omega)| \geq \nu$. And if no such t exists $\tau_\nu(\omega) = 1$.

ii) What functions f of a quasi-martingale $\{X(t), F(t); t \in T\}$ will again be a quasi-martingale? In terms of boundedness and differentiability conditions on f it is felt that in general f(X) need not be a quasi-martingale if f does not have a second derivative. If one investigated f(Z), where f is the integral of the Wierstrass function and Z is the Brownian motion process, one should get some indication of whether the second derivative of the function f is necessary. One could also look for other conditions on f, such as convexity

iii) What processes are integrable with respect to a quasi-martingale? Theorem 3.3.4 indicates that the integrand may not have to be a quasi-martingale.

iv) Doob (1, pp. 273-291) has given solutions of the diffusion equations on the real line. With the definition of a stochastic integral given in this thesis, can the diffusion equations be solved on a sufficiently differentiable manifold, possible a twice differentiable manifold? It seems entirely possible this is the case and should be possible with relative ease.

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