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# MONTE CARLO METHODS AND THE PERT PROBLEM

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### I. Introduction and Summary

Project schedules of many kinds can be represented as a directed acyclic network in which activities of the project correspond to the arcs in the network with lengths equal to the duration of the activities. The structure of the network represents the order in which these activities may be performed, and the duration of the total project is simply the length of the longest path through the network. If the relevant activity durations were known with certainty, finding the longest path (often called critical path in this context) is a trivial matter even for very large networks. Unfortunately in many projects, especially research and development projects, the time durations for various activities are known only with a high degree of uncertainty. It was to cope with this aspect of network planning that the PERT system was created. The approach was then to consider the arc lengths (activity durations) as random variables with known distributions. This approach gives rise to two important problems. The problem of determining the distributions for the arcs and the problem of simply solving the model, i.e., finding something that corresponds, in some sense, to the project duration and critical path in the deterministic case. Both these problems were avoided to some extent by approximating the stochastic problem by a problem of the deterministic form. The purpose of this paper is to try to characterize the qualitative aspects of individual networks which indicate whether the reduction of a stochastic longest path problem to a deterministic one is adequate and to

*Parts of the material in Sections 2-7 and the Appendix were written while the author was a summer consultant to the RAND Corporation and appeared in slightly different form in [12]. The author is grateful to T. E. Harris, B. L. Fry, F. S. Pardee, E. Scheuer, K. R. MacKrimmon, and C. A. Ryavec, all of RAND Corporation for their helpful discussion and encouragement in the preparation of this paper.

some extent, to estimate the magnitude of the errors involved.

To accomplish this, we turned to Monte-Carlo simulation. A bonus from this approach was the realization that Monte-Carlo solution techniques could be used to solve PERT networks more accurately and to gain estimates for quantities not obtainable from the standard PERT approach. In particular, the "criticality" of an activity, i.e., the probability of an activity being on the longest path, can be calculated. This appears to be an exceedingly useful measure of the degree of attention an activity should receive by management, and is not as misleading as the critical path concept now used.

In Section 2, we characterize in more detail the PERT model and define relevant terms. In Section 3 the solution technique now used is analyzed and the important assumptions emphasized. Section 4 describes the use of Monte-Carlo simulation and Section 5 describes the statistical analysis used. Section 6 relates shortcuts for the computation, one of which has the novel feature of reducing cost by increasing the variance of the estimate. Section 7 evaluates the utility of simulation for solving networks, and the Appendix illustrates many of the points with examples.

## 2. The PERT Model*

We are given a finite acyclic directed network of arcs, which we call _activities_, and their nodes, which we call _events_. Associated with each activity is a non-negative random variable, which we call its _duration_. There are two special events, the _initial_ event and the _terminal_ event, such that no activity leads into the initial event or out of the terminal event and such that each activity is contained in a directed path leading from the initial event to the terminal event. We usually assume that the durations have independent distributions, each with a finite range. A _realization_ of

*For further details see References 1, 4, 5, and 7.

the network is the network with a fixed value for each of its durations. For a particular realization of the network, we call a longest path from the initial to the terminal event a critical path and its length the project duration. We often speak also of the random variable corresponding to the project duration and of its distribution.

## 3. The PERT System†

The basic data required for the PERT system is the distribution of the activity durations. It is here that the approximations are particularly gross, if to some extent unavoidable. The data for these distributions are obtained from technical people who have had some experience with the type of activities involved, although in research and development projects, for example, the activity in question may never before have been attempted. The most one can expect are estimates for a few parameters of the distribution, commonly the range and the mode. Since, as will be seen later, only the mean and variance of the distributions are used in current calculation methods, the character of the distribution is treated somewhat cavalierly. The distribution is assumed to be a beta distribution with standard deviation equal to 1/6 the range. These are of course highly arbitrary assumptions and should not be taken too seriously.

The procedure in current use for solving PERT problems is as follows:

(a) Reformulate the problem as a deterministic problem, using the expected value of an activity duration as its deterministic length.*

(b) Find the corresponding critical path $P_E$.

(c) Use the associated project duration $\ell_E$ as the mean of the project

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*Actually, an approximation for the mean is used which can lead to an error as large as 4% of the range. See [1, p. C-10, 11].

†For a more complete discussion of the problems raised in this section see Reference 5.

duration distribution; and use the sum of the activity variances along the path as the variance of the true project duration, which is then assumed to be normal.

(d) Calculate the probability that schedule dates will be met.

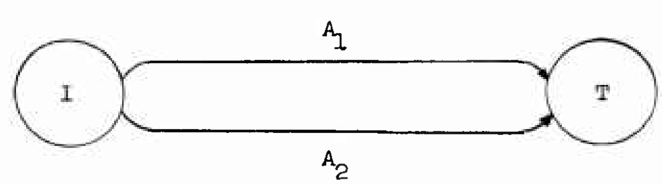
The assumptions used to justify this approach are mainly the following:

(A) A critical path $P_E$ of the derived problem is assumed to be "enough longer"* than any other path so that the probability of a realization having a different critical path is negligible.

(B) The critical path $P_E$ has enough activities so that a central limit theorem applies.

We note, however, that assumption (B) is not needed for calculating the expected length of the critical path or to calculate its variance, but is used only for making inferences about probabilities of meeting schedule times.

To illustrate what can happen when these assumptions are not satisfied, we consider the following simple network:




Here the arc durations $A_1$ and $A_2$ are normal,† with means $\mu_1$ , $\mu_2$ and variances $\sigma_1^2$ , $\sigma_2^2$ , respectively. Clark[2] shows that for the random variable

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*A very loose measure of this for individual alternative paths can be obtained using Tchebysheff's inequality.

†We should realize, of course, that strictly speaking we should not use normal distributions because they do not have finite ranges.

which is the maximum of $A_1$ and $A_2$, the mean is given by

$$\mu_M = \mu_1 \Phi(\alpha) + \mu_2 \Phi(-\alpha) + a\phi(\alpha) \quad ,$$

and the variance by

$$\sigma_M^2 = (\mu_1^2 + \sigma_1^2) \Phi(\alpha) + (\mu_2^2 + \sigma_2^2) \Phi(-\alpha) + (\mu_1 + \mu_2) a\phi(\alpha) - \mu_M^2 \quad ,$$

where

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2} dt \quad , \quad \phi(t) = \frac{1}{\sqrt{2\pi}} e^{-t^2/2}$$

$$a = \sqrt{\sigma_1^2 + \sigma_2^2} \quad , \quad \alpha = \frac{\mu_1 - \mu_2}{a} \quad ,$$

and $\mu_M$ and $\sigma_M$ are the mean and variance of the distribution of the maximum. Assume for simplicity that $\mu_1 = \mu_2$; then $\alpha = 0$ and $\mu_M = \mu_1 + (1/\sqrt{2\pi}) \sqrt{\sigma_1^2 + \sigma_2^2}$. The usual procedure (p. 3) gives $\mu_M$ "=" $\mu_1$. So as $\sigma_1^2$ or $\sigma_2^2$ increases, we can make the difference, $\Delta = (1/\sqrt{2\pi})\sqrt{\sigma_1^2 + \sigma_2^2}$, between the correct mean and the mean calculated using the PERT approximation arbitrarily large. Similarly, we have

$$\sigma_M^2 = \frac{1}{2} (\sigma_1^2 + \sigma_2^2) \quad ,$$

where the above procedure would consider $\sigma_M^2$ "=" $\sigma_1^2$ or $\sigma_2^2$, so that the estimate could be 100% too high or arbitrarily too small. Similar exercises could be indulged in for other distributions than the normal. The point is simply that it is trivial to construct examples violating (A) for which the above technique will yield almost arbitrary errors.*

More interesting questions are: What does "enough longer" mean? and, Given a network, how can we discover if (A) is satisfied? Two factors enter here; one is the "closeness" of two paths, and the other is the number of paths that may become critical. Even if each alternative path has a small probability of being critical, if there are enough of these alternative paths the error can accumulate significantly. In comparing two paths for criticality, the following considerations are important:

(a) The means of the path lengths.

(b) The variances of the path lengths.

(c) The correlation of paths, i.e., the activities the paths have in common. Examples to illustrate these factors are given in the Appendix.

Probably a large fraction of networks encountered in practice will satisfy assumption (A) at least to the extent needed for the desired accuracy, but general analytic methods of determining this are hard to come by because of the strong dependence on the particular network involved. The problems connected with assumption (B) are classical and will not be gone into here except to note that the probabilities assigned to meeting schedule dates depend on all the assumptions and are therefore especially unreliable. Nevertheless, the current procedure is extremely simple and rapid, and any other solution method yielding significantly improved results must be expected to

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*This is not entirely true; for example, the estimate of expected project-duration time will always be optimistic under the above procedure. This can be shown in several ways. One informative way is to write the problem as a linear program with a random constant vector, the components of which are simply the activity durations. Then the general results of Madansky [4] for stochastic linear programming problems can be applied. In particular, for any linear programming problem with a random right-hand side the answer obtained by using expected values is always too optimistic. This result is also established in [3]. Another connection between linear programming and PERT is the relation between sensitivity analysis and the short cut of Sec. 6.

take considerably more computation time.

The purpose of the PERT system is to help management spot bottlenecks and overruns before they occur, so that corrective action can be taken before it is too late. Emphasis is on providing a tool for "management by exception," that is, to point up the few activities that are particularly critical and require close managerial supervision. Currently emphasis is put on the activities on the "critical path" in the network with the activity distributions replaced by their means.

Current PERT systems essentially give as output "early" and "late" times for each activity, and the probability of meeting given bench-marks or schedule dates. Since these are expected values, a variance for these numbers is also given to indicate the reliability of the estimates. Generally in this paper we will restrict our attention to the expectation of the project duration, its variance, and the distribution itself. All the other commonly given output is obtained using essentially the same methods.

One of the more misleading aspects of current PERT solution methods is the implication that there is a unique longest path. In general any of a number of paths could be longest, depending on the particular realization of the random activity durations that actually occurs. Thus it makes sense to talk about a "criticality index," which is simply the probability that an arc will be on the longest path. Current solution procedures by their very nature cannot give any information on this. We see in the next section that Monte Carlo techniques can give answers to this and other questions.

## 4. Solving the PERT Model Using Monte Carlo Techniques

When we say that we "Monte Carlo" a PERT network, we simply mean that we apply the longest path algorithm to a long series of realizations each one obtained by assigning a sample value to every activity drawn from its proper distribution. Each realization obtained in this way is considered as a sample value for the simulation. Given this information, we use standard statistical methods to estimate the distribution and parameters of interest. This technique can be used in two basically different ways. The first involves a relatively small sample size and is used to test, for example, the classical type of hypothesis as to whether the true mean equals $\ell_E$ , to give some indication as to whether assumption (A) is violated. These statistical tests could be sequential and involve the application of standard statistical techniques. This approach has the disadvantage that it gives only incidental information about the distribution of the project duration.

The second application of the technique offers more fascinating opportunities, and that is what will be considered here. That is to take a large number of samples and Monte Carlo the network to actually obtain the answers rather than to run a check on traditional methods.

In the Monte Carlo framework, the model can be generalized to some extent. Including resource allocation would be complicated but could be accomplished, especially for certain cases. For example, if two activities use the same scarce resource, things could be set up in such a way that the first activity to start would have to finish before the second could begin, or, for three activities, the last to start might have to wait for the first two to finish. Alternatively, the three could be constrained to go one at a time, etc. Correlation between activities could very easily be handled with Monte Carlo techniques.

The data-collection problems are just as grievous when treated by Monte Carlo techniques as when treated by the usual methods, if not more so.

In current solution methods, the output does not depend on the structure of the activity duration distributions but only on their means and variances. The Monte Carlo approach, in order to gain extra accuracy, does depend on the shape of the distribution. On the other hand, the Monte Carlo approach has greater flexibility in that any distribution can be used for activity durations — beta, normal, triangular, uniform, or discrete in any sort of mix. This flexibility allows one, in particular, to try different distributions and observe the effect of neglecting or making highly arbitrary assumptions on the shape of these distributions. An example illustrating this point is given in the Appendix.

It is in the area of computational accuracy, however, that the Monte Carlo techniques yield the greatest benefits. For example, the estimate for the mean using procedures of Section 3 is always low, whereas the Monte Carlo procedure gives an unbiased estimate. Again, while the earlier procedure is the least accurate in making estimates of the probability of meeting schedule dates, we shall see in the next section that the Monte Carlo technique yields very good answers indeed and in particular makes no recourse to central limit theorems.

Finally, with respect to output, this approach yields all the output of previous methods plus information on the "criticality indexes," or probabilities of activities being on the longest path.

## 5. Accuracy of Monte Carlo Techniques

The only type of statistical analysis we shall investigate here is predicated on a fixed number of trials determined before sampling. We then ask what is the probability that our estimators for the various parameters and distributions will be "close" to the correct ones.

The first parameter we shall consider is the mean of the project duration. Each complete realization is considered as one sample observation, and the estimator we consider is the sample mean. More formally, let $X$ be the random variable (r.v.) corresponding to the project duration. Let $X_i$ $i = 1,\ldots,N$, be $N$ independent r.v.'s each with the same distribution as $X$. Then $(X_1,\ldots,X_N)$ is our sample, The r.v.

$$\hat{\mu} = \frac{\Sigma X_i}{N}$$

is the estimator of

$$\text{Exp}\,\{X\} \equiv \mu_x \quad .$$

Let

$$\sigma_x^2 = \text{Var}\,\{X\} \quad ;$$

then we have

$$\text{Var}\,\{\hat{\mu}\} = \sigma_x^2/N \quad .$$

Since we shall be talking in terms of thousands of samples, the central limit theorem talls us that $\hat{\mu}$ is to a very good approximation normal with mean $\mu_x$ and variance $\sigma_x^2/N$, denoted Nor $(\mu_x, \sigma_x^2/N)$. Clearly as $N$ increases the uncertainty in our estimate of Exp $\{X\}$ decreases. Since the standard deviation, i.e., the square root of the variance, is a measure of spread with the same units as $X$, we see that roughly speaking the accuracy increases inversely with the square root of $N$.

EXAMPLE: Suppose with probability 0.95 we want our estimate of the mean to be within $\sigma_x/50$ of the true mean. That is, we want to choose $N$ so that

$$\Pr\{\mu_x - \frac{\sigma_x}{50} < \hat{\mu} < \mu_x + \frac{\sigma_x}{50}\} = 0.95 \quad .$$

Now we have

$$\Pr\{\mu_x - \frac{\sigma_x}{50} < \hat{\mu} < \mu_x + \frac{\sigma_x}{50}\} = \Pr\left\{\frac{-\sqrt{N}}{50} < \frac{\hat{\mu} - \mu_x}{\sigma_x/\sqrt{N}} \leq \frac{N}{50}\right\}$$

and

$$\frac{\hat{\mu} - \mu_x}{\sigma_x/\sqrt{N}}$$

is distributed Nor (0,1) , which implies that one should choose $t$ such that

$$\Phi(t) - \Phi(-t) = 0.95 \quad ,$$

where $\sqrt{N}/50 = 1.96$ from tables of the normal distribution. This gives $N \doteq 10{,}000$ .

In general, however, we do not know the variance of the distribution, so this also must be estimated. To do this, we use as our estimator

$$S^2 = \frac{1}{N} \Sigma (X_i - \hat{\mu})^2 \quad .$$

Assuming the distribution of $X$ is approximately normal, $NS^2/\sigma^2$ is $\chi^2_{N-1}$ , so we can get rough estimates of the error involved. [13]

EXAMPLE: Suppose we wish to know with probability 0.95 that our estimate of $\sigma^2$ is correct within 5%; that is, we wish to chose $N$ such that

$$\Pr\{0.95\sigma^2 < S^2 \leq 1.05\sigma^2\} = 0.95 \quad .$$

But

$$\Pr\{0.95\sigma^2 < S^2 \leq 1.05\sigma^2\} = \Pr\{0.95N < \frac{NS^2}{\sigma^2} \leq 1.05N\} \quad .$$

For large $N$, $\chi^2_{N-1}$ is approximately $\text{Nor}\,(N - 1\ ,\ 2(N - 1))$, whence we have, upon some more manipulations,

$$\Pr\{0.95N < \frac{NS^2}{\sigma^2} \leq 1.05N\} = \Pr\left\{\frac{1 - 0.05N}{\sqrt{2N - 2}} < \frac{(NS^2/\sigma^2) - N + 1}{\sqrt{2N - 2}} \leq \frac{1 + 0.05N}{\sqrt{2N - 2}}\right\} = 0.95 \quad ,$$

where

$$\frac{(NS^2/\sigma^2) - N+1}{\sqrt{2N - 1}}$$

is approximately $\text{Nor}\,(0,1)$. Proceeding as before, we find

$$\frac{1 + 0.05N}{\sqrt{2N - 2}} = 1.96 \quad ,$$

which yields $N \doteq 3000$.

For the "criticality index," one is simply sampling from a binomial distribution. The estimator is the ratio of the number of sample realizations for which the arc is critical to the total sample size. The mean is $NP$ and the variance $NP(1 - P)$, where $P$ is the probability of being critical. Since the number of samples is fixed, the estimator is simply a sum of independent random variables and therefore asymptotically normal.

EXAMPLE: Suppose with probability 0.95 we want our estimator of P to be correct within 0.01. Now $(\hat{P} - P)/\sqrt{P(1 - P)/N}$ is asymptotically Nor (0,1), where P is our estimator. We want

$$Pr\{\,|P - P| > 0.01\} = 0.95 \quad ,$$

or

$$Pr\left\{\frac{-0.01}{\sqrt{P(1 - P)/N}} < \frac{\hat{P} - P}{\sqrt{P(1 - P)/N}} \leq \frac{+0.01}{\sqrt{P(1 - P)/N}}\right\} = 0.95 \quad .$$

Since $P(1 - P) \leq 1/4$, if

$$Pr\left\{-0.02\sqrt{N} < \frac{\hat{P} - P}{\sqrt{P(1 - P)/N}} \leq 0.02\sqrt{N}\right\} = 0.95$$

the result will be even better. This implies

$$.02\sqrt{N} = 1.96 \quad , \quad \text{or} \quad N \doteq 10{,}000 \quad .$$

Finally, we look at estimates of meeting specific schedule dates. For any given fixed schedule date, the problem is again sampling from a binomial distribution, and the previous analysis goes through. On the other hand, we may not want to limit our attention to any fixed date, and may want to know the accuracy for any date we choose; i.e., we may want to know, in some sense, how well our sample project duration distribution "fits" the real one. If we are willing to assume that the project duration cumulative distribution function is continuous, we can make probability statements about the greatest absolute differences between the sample cumulative distribution function (c.d.f.) and the true one independent of the distribution itself.[13]

Kolmogorov in 1933 gave the asymptotic results that are tabulated, for example, in [8].

EXAMPLE: Let $D_n = \sup |F_n - F|$ , where $F$ is the c.d.f. of the parent population and $F_n$ is the sample cumulative function. We find tabulated in tables $Z$ , $L(Z)$ , where

$$L(Z) = \lim_{n \to \infty} PR\{\sqrt{n}\, D_n \leq Z\} .$$

Suppose with probability 0.95 we want the maximum deviation to be less than 0.01 . We find for $L(Z) = 0.9505$ that $Z = 1.36$. Solving for $N$ in $\sqrt{N}\,(.01) = 1.36$ , we find $N \doteq 18{,}500$ .

The statistical analysis for other outputs of interest is carried through using the same methods found in this section.

## 6. Computational Short Cuts

As in most Monte Carlo applications, there are techniques available for reducing computational expense in solving PERT problems. In solving PERT problems using Monte Carlo methods, most of the time is consumed in generating random numbers. Our approach will be to avoid sampling activity duration distributions that are rarely if ever on the critical path. These activities do not significantly affect the solution. The trick of course is to find these activities. Two methods will be outlined here. The first is an analytic approach, based on a suggestion of Kenneth Mac Crimmon, and the second is a statistical approach.

Because the distributions of the activity durations have finite ranges, it may turn out that many paths can never be critical. One way to discover this is the following: Consider the realization obtained by setting all the activity durations at the lowest point of their ranges, and find the corresponding critical path P and path duration $\ell$. Now set all the activity durations not on the critical path at their highest value. Then only the activities used in paths with length strictly greater than $\ell$ can ever be critical. The remaining activities may be eliminated. References [9,11] give algorithms for finding these paths. This procedure does not, in general, remove all the arcs that can never be critical; moreover, if the number of paths longer than $\ell$ is very large, then the algorithm for finding them can become quite unwieldy.

The statistical approach is quite a bit easier and can be applied to activity duration with infinite ranges. The approach here is to sample a relatively few times using all the activities and then stop and eliminate all activities that were never critical. For a given network the size of the initial sample determines the probability of making a mistake, that is of eliminating an arc that "should have" remained. For instance, if the initial sample were 1000, then the probability that a given activity with probability 0.01 of being critical would not be critical is $(0.99)^{1000} \doteq 7.3 \times 10^{-5}$. Another way to prevent the elimination of the wrong activities is not to remove the activities at all but to use each random sample for the activities with low criticality for K realizations before drawing a new one. In other words, draw new activity durations for these arcs only every $K^{th}$ realization. The same estimators as before are respectively unbiased estimators of the project duration distribution, its mean, and the criticality probabilities for the arcs. The estimate for the variance of the project duration is no longer unbiased but is consistent. The variance is not decreased but as a matter of fact is very slightly increased. The savings are obtained by having to sample the "unimportant" activities only $(1/k)^{th}$ the number of times that the more important activities are sampled. Under this approach it is not necessary to separate only those activities that were never critical in the initial sample, but more generally all those activities that were on the critical path less than some given number of realizations could be distinguished.

Finally, consider a situation in which the exact distribution of the project-duration distribution is not desired, but only its mean and variance. In this case an activity which is always critical can be replaced by a deterministic activity with duration equal to the mean of the original distribution. The same procedure is gone through as before, except that the original variances of all the changed activities must be added to the estimated project-duration variance.

## 7. Feasibility of Method and Proposals for Future Investigation

An experimental computer code has been coded by R.J. Clasen for the IBM 7090 computer to test the feasibility of Monte Carlo evaluation of PERT networks. Runs of 10,000 samples and up to 45 activities were run. The code can handle 1000 activities and can generate uniform, triangular, and beta-duration-distribution functions. The output is the project-duration density function, and its mean and variance, and the criticality index for each of the activities.

Experience with the program has indicated that most of the time is consumed generating random numbers, so that the running time is essentially a linear function of the number of random numbers generated — or, for fixed sample size, a linear function of the number of activities in the network. This is especially true for larger sized networks. For a 200-activity network and 10,000 samples, the running time would be about 20 minutes for triangular distributions and about 5 minutes for uniform distributions. This is to be compared with analytic methods, the difficulty of which usually increases exponentially with the number of arcs.

## APPENDIX

On all the simulations illustrated here 10,000 realizations were used. The first figure illustrates a distribution on which the arc distributions for most of the examples were based. Figures A-2 and A-3 illustrate the effect on the expected project duration of having two paths with approximately the same length. $\mu_p$ and $\sigma_p^2$ represent the project duration mean and variance. In Figure A-3 the cumulative project duration distribution is plotted for $\mu_1 = 5$, 9.5 and 10. The shaded bar on the graph indicates a 95% confidence band (see Sec. 5). Similarly Figures A-4 and A-5 illustrate the effect of a multiplicity of paths of comparable length. The confidence regions are 95% confidence intervals. Figures A-6 and A-7 illustrate the effect of correlation between alternate paths. Since each arc in these first examples is intended to represent paths made up of many activities, it would be perhaps more realistic to assume normal distributions for the arc lengths. In this case the table of Figure A-2 can be redone using the results of Clark.[2] This is done in Table A-1.

TABLE A-1

| $\mu_1$ | $\mu_p$ | $\sigma_p^2$ |
|---|---|---|
| 5 | 10 | 1 |
| 8 | 10.05 | .896 |
| 9.5 | 10.35 | .703 |
| 10 | 10.56 | .681 |

Similarly from the literature of order statistics (e.g., [10] Chapter 10) we can redo the table in A-4. The results are tabulated in Table A-2.

TABLE A-2

| n | $\mu_p$ | $\sigma_p^2$ |
|---|---|---|
| 1 | 10 | 1 |
| 2 | 10.56 | .681 |
| 3 | 10.84 | .559 |
| 5 | 11.16 | .447 |
| 10 | 11.54 | .344 |
| "∞" | ∞ | 0 |

Figure A-8 demonstrates how the sum of independent uniformly distributed random variables approaches normality. This was done using the uniform distribution so the computations could be accomplished analytically. In each sum the distribution is normalized to have variance 1.

In Figure A-9 we see where an arc, 1-2, not on the "critical path," 1-3-4, in the PERT sense is, in the stochastic sense, more often critical than any other arc.

Figure A-10 shows an example given in [1] p. V-2. The three numbers assigned to each activity are respectively the optimistic, most likely and pessimistic estimates for each activity.

The expected project duration time using PERT is 66.0* with a variance of 60.27. The Monte Carlo method using 10,000 realizations yields a mean of 67.0 ± .13 and a variance of 42.39 ± 2.†

The distributions used for the activity durations were the beta distributions with end points and modes given by the three parameters indicated in Fig. A-10 The standard deviation was taken to be 1/6 the range.

*Actually when adjustment is made for an approximation used for calculating the mean of a beta distribution the PERT estimate is 65.5

†Expressions of the form $X = N \pm \Delta$ denote the statement, "the probability that $N$ is within $\Delta$ of $X$ is greater than 0.95.

In Fig. A-11 a normal distribution with mean 66.0 and variance 60.27 is compared with the distribution obtained by Monte Carlo. Figure A-12 displays the network with the criticality index for each activity. The heavy line is "<u>the</u> critical path" calculated using expected values for the activity durations.

Carrying through the analysis of Sec. 5 we see that with $N = 10{,}000$ we can expect with probability 0.95 that our estimated mean is within $\sigma_x/50$ of the true mean. Using as an approximation of $\sigma_x$ the square root of the estimated variance, we obtain

$$\sigma_x/50 \doteq \sqrt{42}/50 = .13 \quad .$$

Since the Monte Carlo estimate differs by 1.0 from the PERT estimate the difference is certainly statistically significant at the 95% level. Finding the shortest and longest possible project durations by using for activity durations respectively the optimistic and pessimistic times we find that the discrepancy, 1 , is on the order of $1\frac{1}{2}$ per cent of the range. On the other hand, for the variance of the distribution the variance estimated by PERT methods is 1.42 times as large as the Monte Carlo estimate, but as we saw in Sec. 5 the probability is better than 0.95 that the Monte Carlo estimate is within 5% of the correct value. Finally, in Fig. A-13 the probability that the project will be completed by X is charted as estimated by the PERT technique and as estimated by the Monte Carlo method. With $N = 10{,}000$ in $N(D_n) = 1.36$, we see that with probability 0.95 the sample cumulative distribution is within 0.0136 of the true distribution at every point, (see p. 13).

We also note that activities 6-8 and 8-9 were never critical. In fact, they cannot be critical, but the deterministic algorithm given in Sec. 6 will not reveal this. So in this case the statistical approach is clearly preferable.

Finally in Figures A-14 to A-17 we see the effect on the project distribution of using various activity distributions. The distributions used are uniform and triangular applied to the network of Figure A-10. Recalculating the project mean and variance as predicted by PERT using the correct activity means and variance we obtain, in the triangular case $\mu_p = 69$ and $\sigma_p^2 = 92.77$. The actual values as determined by the simulation are $\mu_p = 71.70 \pm .15$ and $\sigma_p^2 = 58.94 \pm 3$. When we use uniform distiribution the PERT approach yields $\mu_p = 72.0$ and $\sigma_p^2 = 172$. Using simulation we obtain $\mu_p = 77.61 \pm .2$ and $\sigma_p^2 = 92.31 \pm 5$. We notice however, that for planning purposes the results in all three cases are essentially the same, in each network the same activities have high criticality indices.

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TYPICAL ARC DISTRIBUTION

MEAN = 10 VARIANCE = 1 MODE = 9.56

RANGE 14.06 8.06 = 6

$\sigma = 1/6$ RANGE

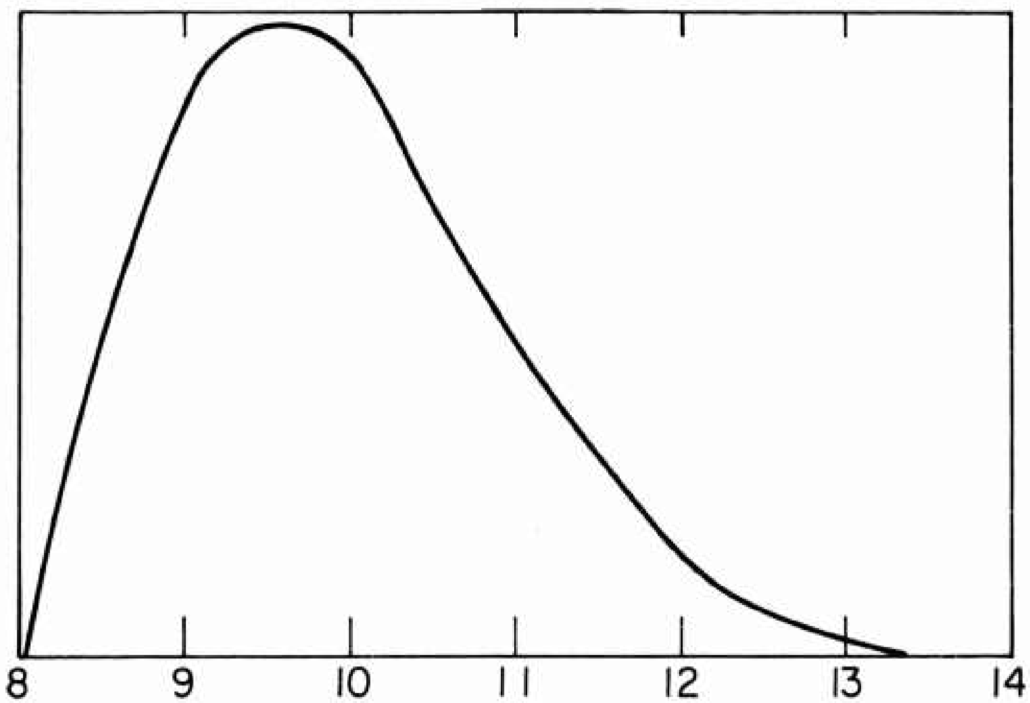
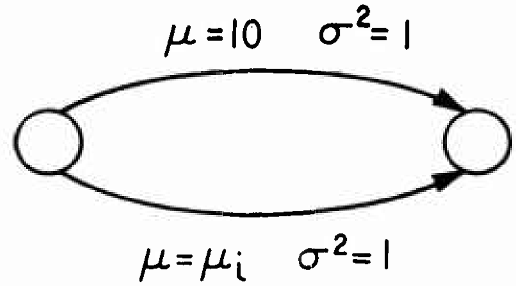



FIGURE A-1

PATHS OF NEARLY EQUAL LENGTH




| $\mu_i$ | $\mu_p \pm .02$ | $\sigma_p^2 \pm .05$ |
|---|---|---|
| 5 | 10.00 | 1.0 |
| 8 | 10.04 | 0.96 |
| 9.5 | 10.35 | 0.84 |
| 10 | 10.56 | 0.84 |

FIGURE A-2

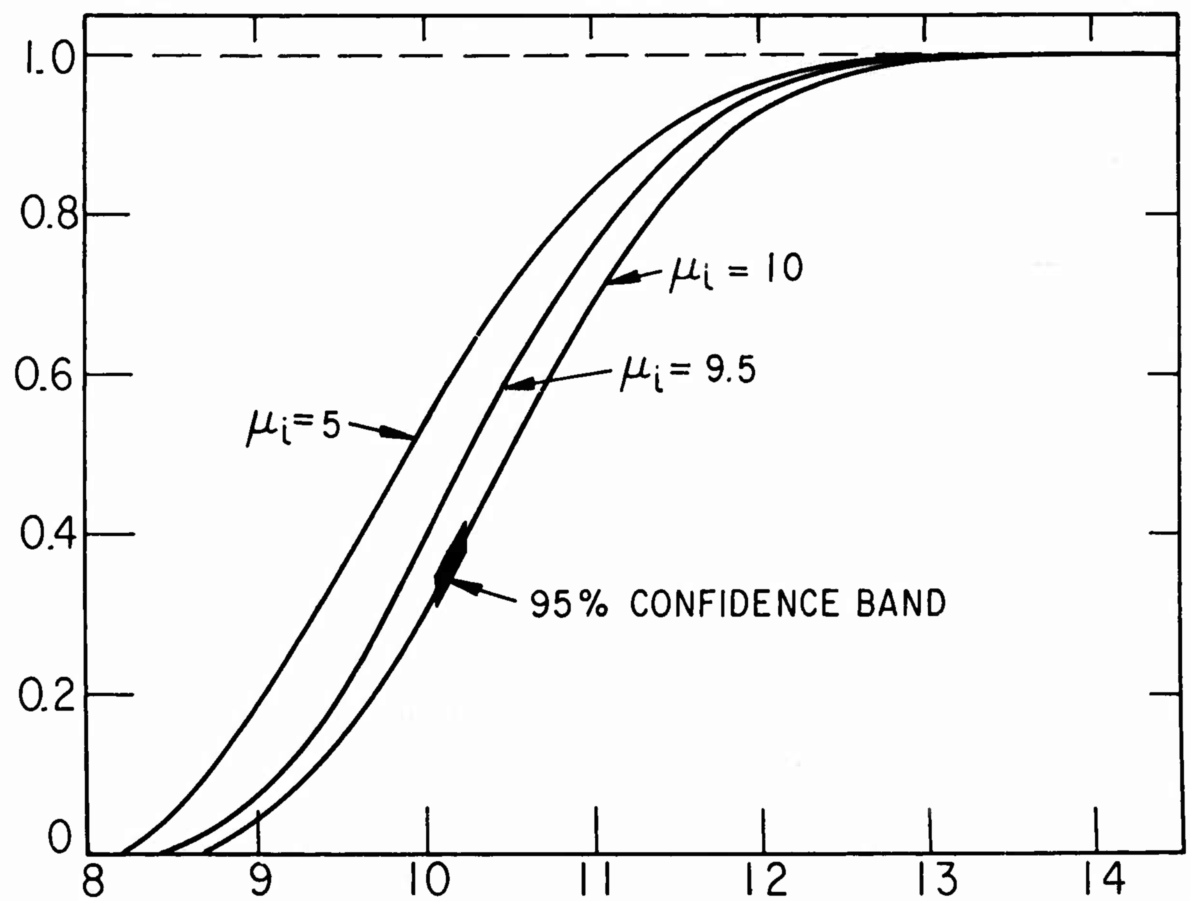
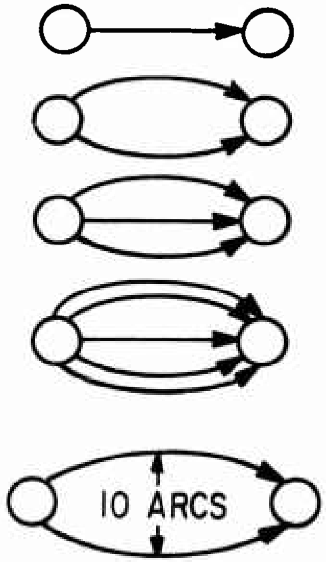



FIGURE A-3

## EFFECT OF NUMBER OF PARALLEL PATHS

FOR ALL ARCS $\mu = 10$ $\sigma^2 = 1$




| $n$ | $\mu_p \pm .02$ | $\sigma_p^2 \pm .05$ |
|---|---|---|
| 1 | 10.00 | 1.00 |
| 2 | 10.55 | 0.82 |
| 3 | 10.86 | 0.71 |
| 5 | 11.24 | 0.57 |
| 10 | 11.65 | 0.41 |
| $\infty$ | 14.06 | 0 |

FIGURE A-4

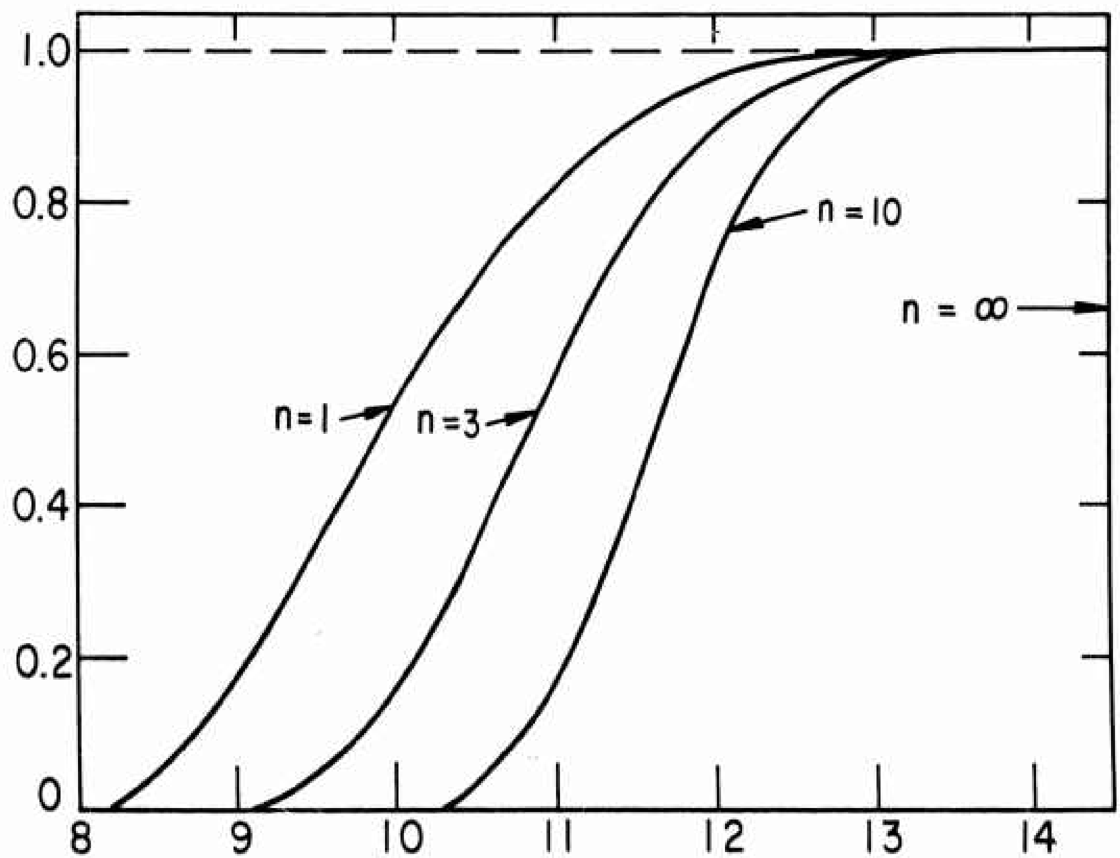



FIGURE A-5

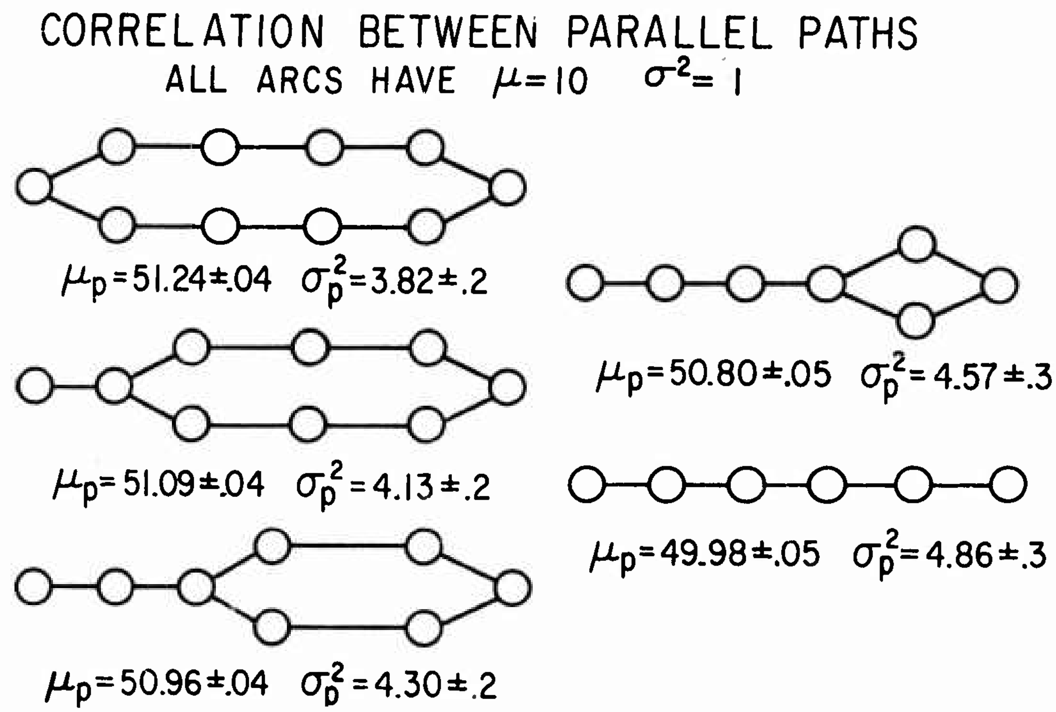



FIGURE A-6

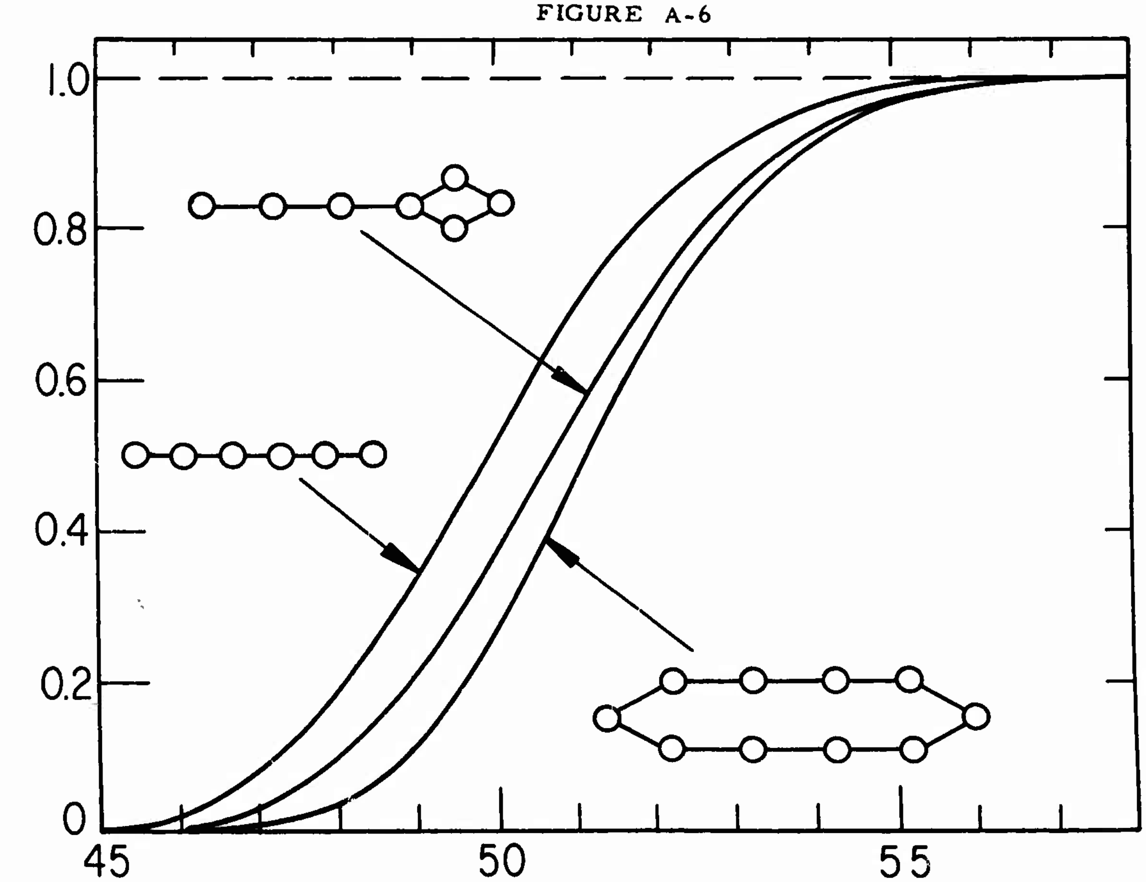
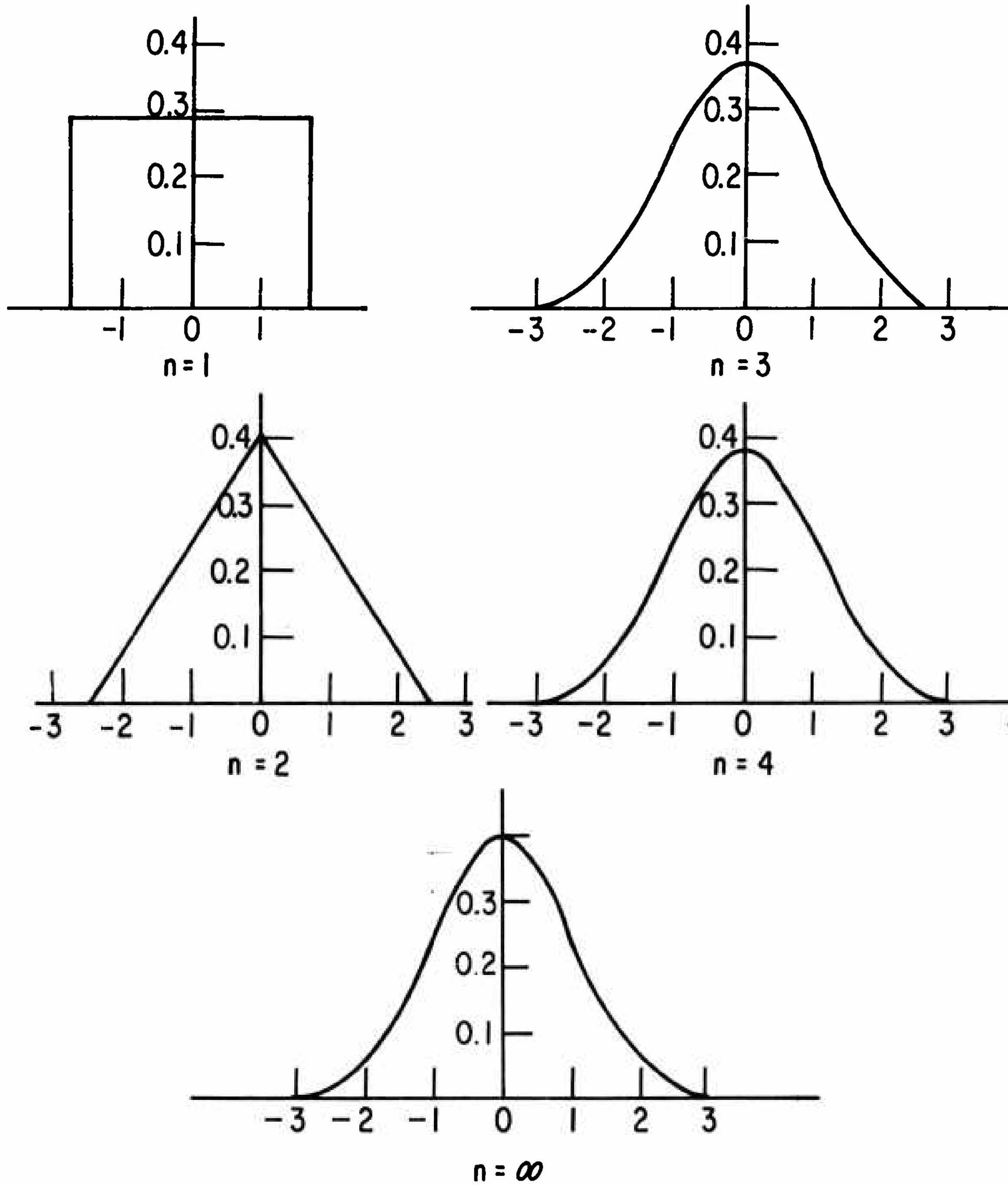



FIGURE A-7




ASSUMPTION B

SUM OF INDEPENDENT RANDOM VARIABLES

FIGURE A-8

CRITICALITY INDEX
C

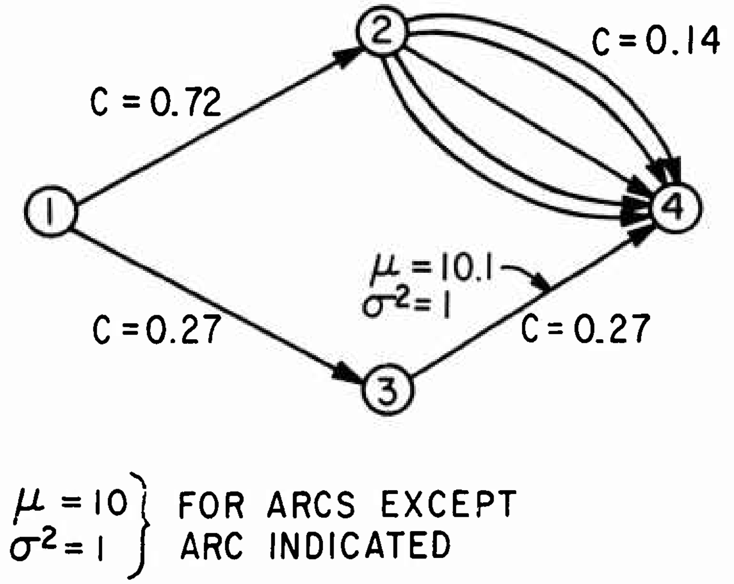



FIGURE A-9

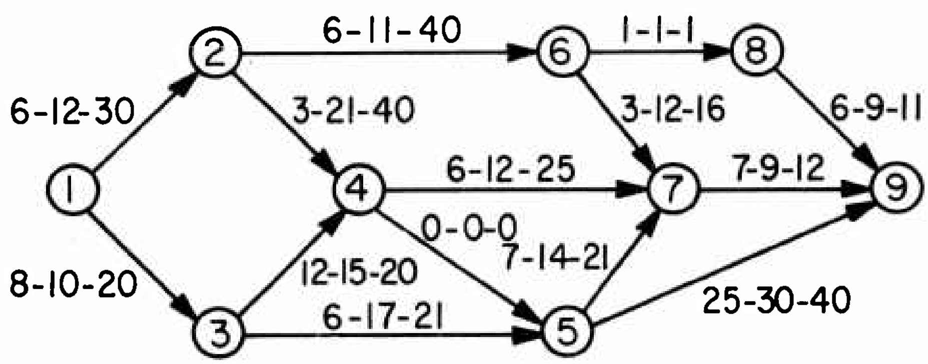
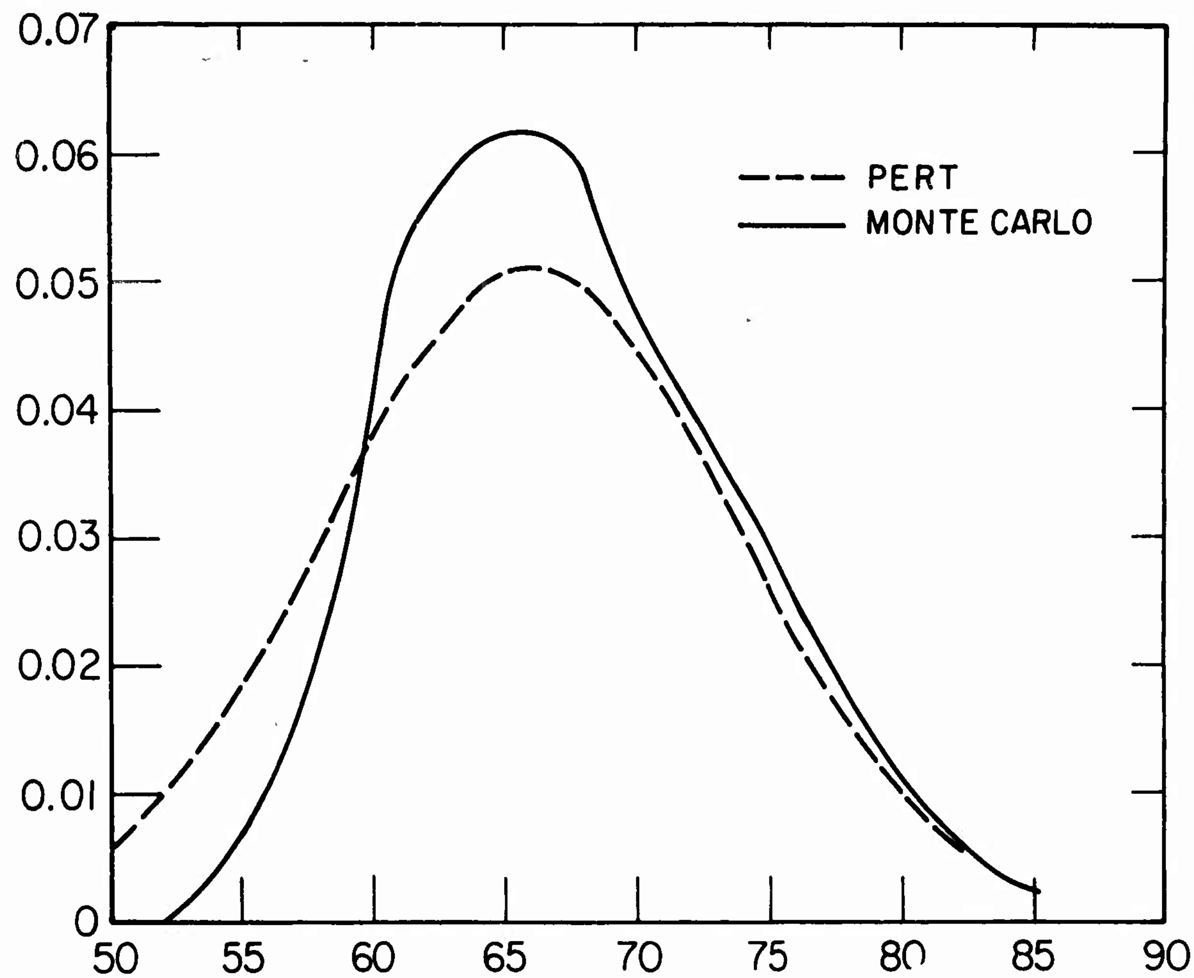


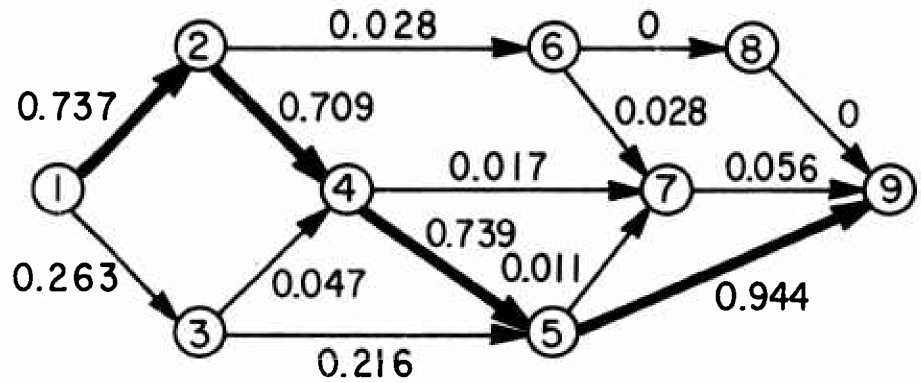

FIGURE A-10




PROBABILITY DENSITY CURVES

FIGURE A-11

EXAMPLE USING BETA DISTRIBUTION




| | PERT | SIMULATION |
|---|---|---|
| $\mu_P$ | 66 | 67.00±0.13 |
| $\sigma_P^2$ | 60.27 | 42.39±2 |

FIGURE A-12

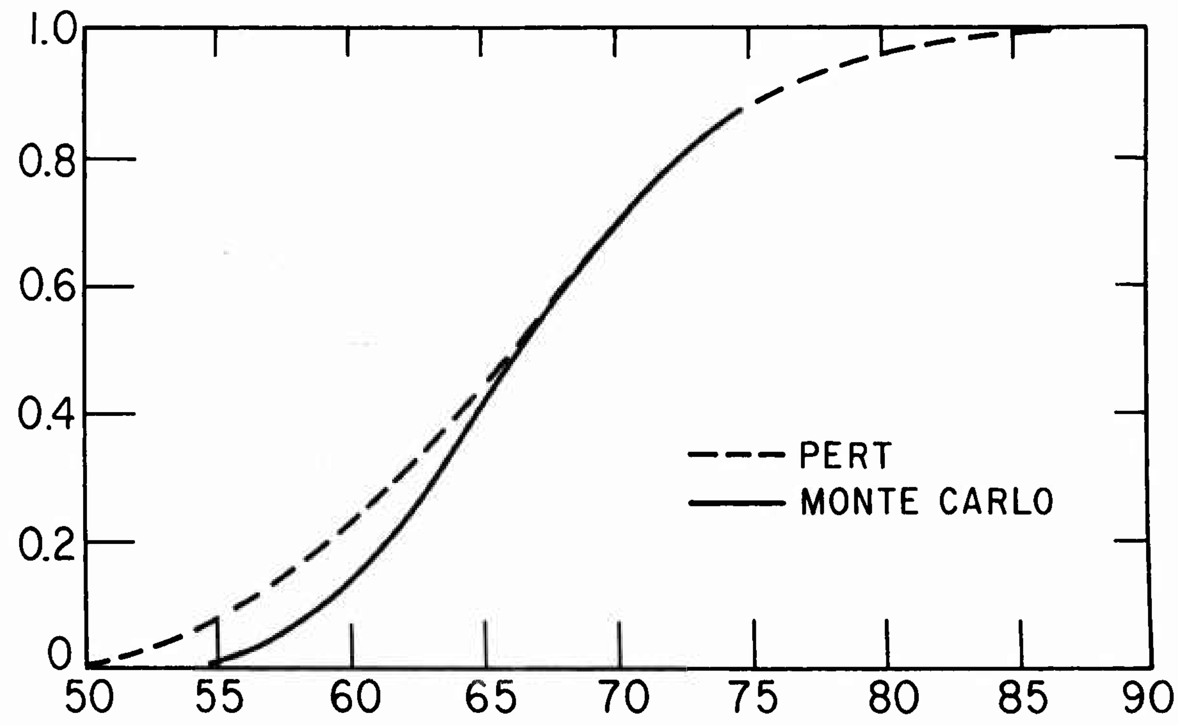
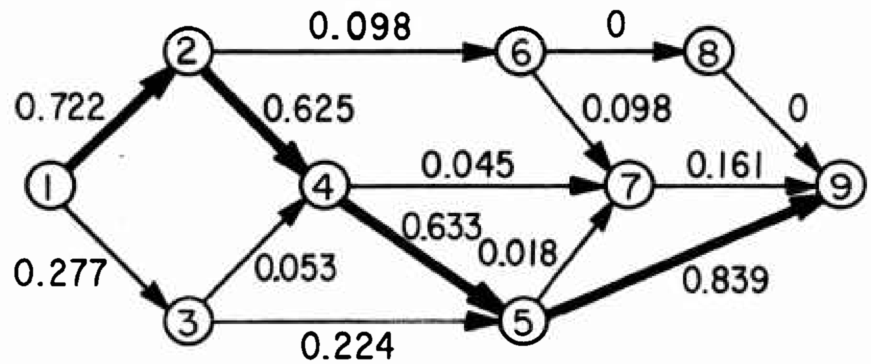



FIGURE A-13

EXAMPLE USING TRIANGULAR DISTRIBUTION




| | PERT | SIMULATION |
|---|---|---|
| $\mu_P$ | 69 | 71.69 ± 0.13 |
| $\sigma_P^2$ | 92.77 | 58.94 ± 3 |

FIGURE A-14

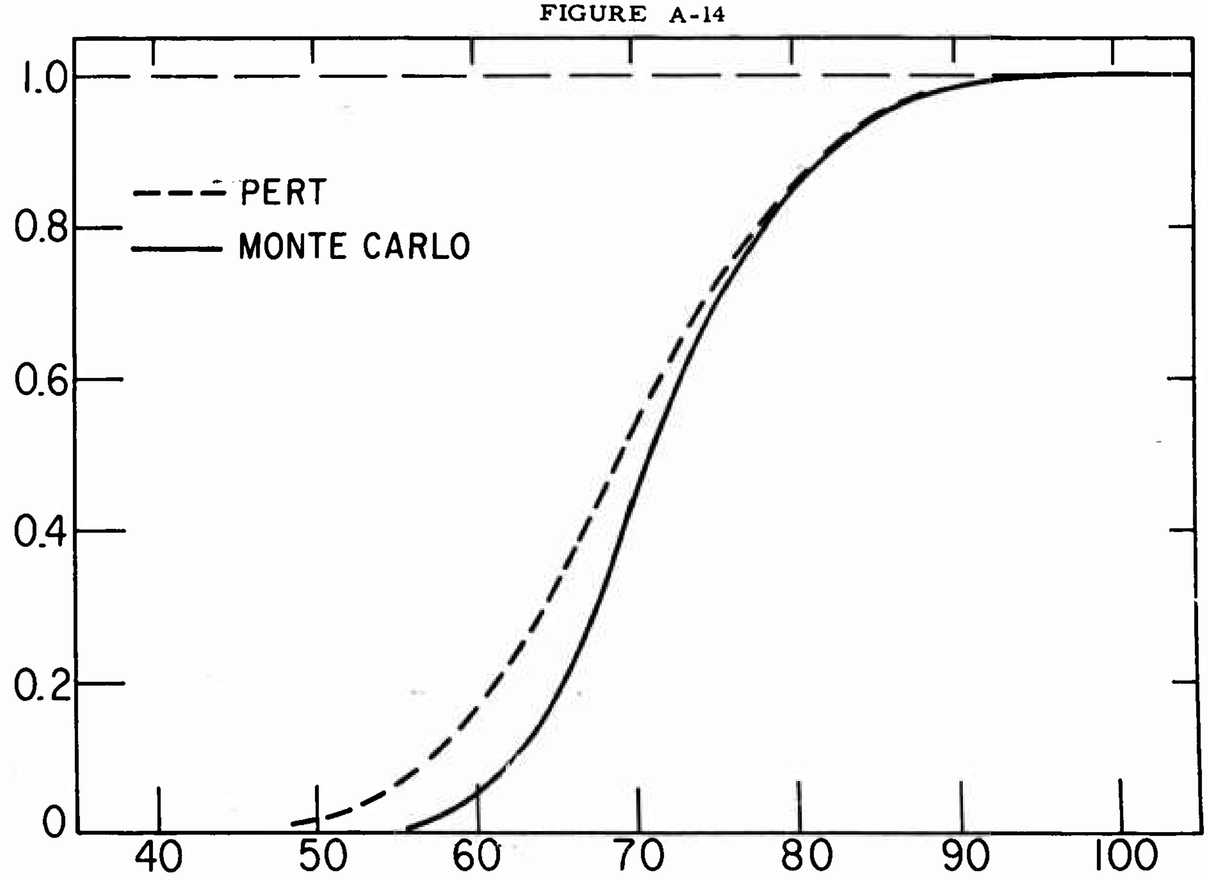
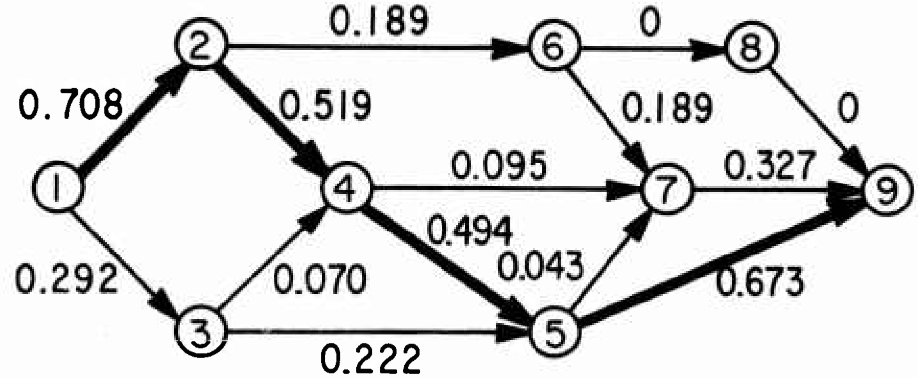



FIGURE A-15

## EXAMPLE USING UNIFORM DISTRIBUTION




| | PERT | SIMULATION |
|---|---|---|
| $\mu_P$ | 72 | 77.61 ± 0.2 |
| $\sigma_P^2$ | 172 | 92.34 ± 5 |

FIGURE A-16

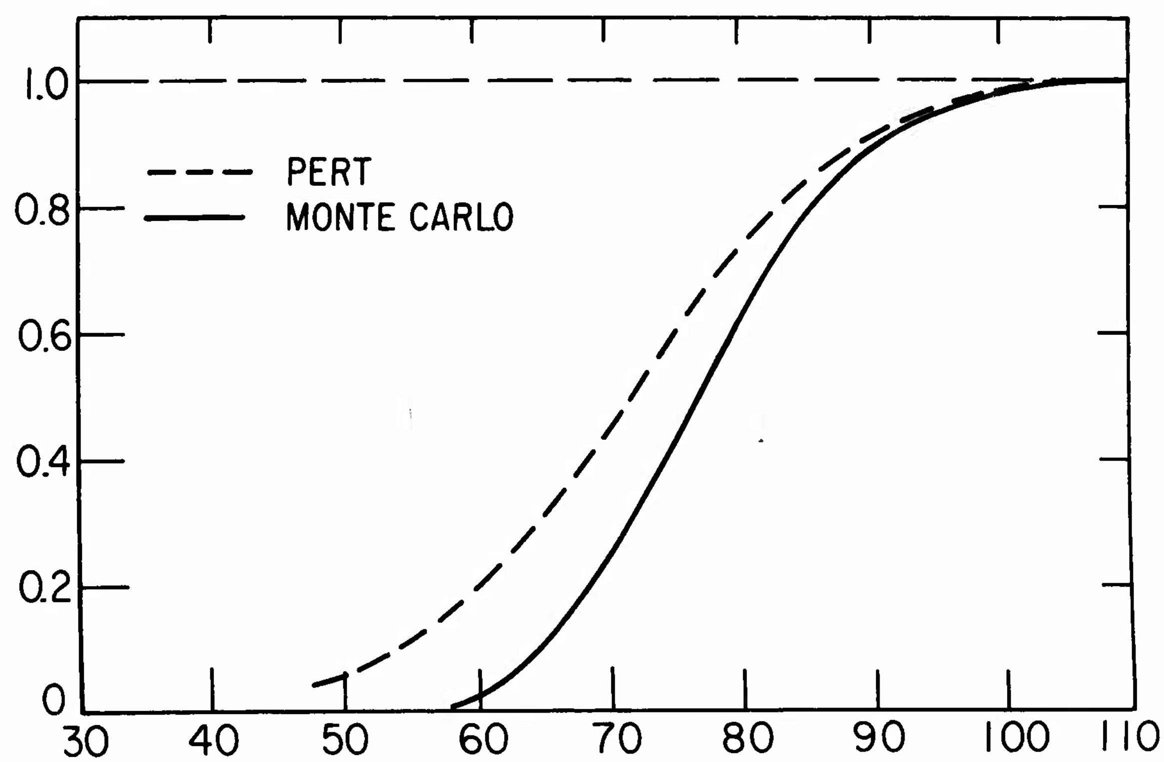



FIGURE A-17

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