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A THEORY FOR BROADBAND VARACTOR PARAMETRIC AMPLIFIERS

by

Walter H. Ku

Research Report No. PIBMRI-1091-62

for

Rome Air Development Center
Research and Technology Division
Air Force Systems Command
United States Air Force
Griffiss Air Force Base, New York
Contract No. AF-30(602)-2213
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FOREWORD

The work reported herein was sponsored by the Rome Air Development Center, Air Force Systems Command, Griffiss Air Force Base, New York, under Contract No. AF-30(602)-2213.

ABSTRACT

This thesis is concerned with the development of a general and rigorous broadbanding theory for varactor parametric amplifiers. Fundamental gain-bandwidth limitations of a varactor parametric amplifier are obtained which are independent of the equalizer. Results obtained in this theory lead to the design and synthesis of broadband varactor parametric amplifiers.

The circuit considered in this thesis is that of linear variable capacitors embedded in an arbitrary, passive, lossless environment in which the signal frequency and one of the sideband frequencies can exist. In the course of developing a broadbanding theory, some fundamental inherent properties of a parametric device are revealed. The first is that the varactor, when embedded in an arbitrary, passive environment, presents an impedance which can be characterized by a positive function rather than a positive-real function. The second and more important inherent property is that, due to the frequency-coupling action of the variable capacitor, the scattering coefficient at the varactor port is quadratic in nature. A broadbanding theory has been obtained, taking into account both of these properties. Some realizability conditions are presented which are indicative of the inherent properties of a parametric device.

A finite gain-bandwidth restriction exists for any varactor device because of the presence of its associated parasitic elements. The limitation on the transducer power gain function for an optimum nonreciprocal three-port equalizer using a single varactor is derived, from which we obtain the maximum flat gain attainable over a prescribed band of frequencies. Furthermore, optimum equalizers may be designed to yield a modified form of Butterworth or Tchebycheff transducer power gain responses of arbitrary order. These responses may be chosen to approach the maximum gain-bandwidth as closely as desired. Some of these results are generalized for the case of n-varactor diodes embedded in a passive, lossless environment.

Another part of this thesis deals with the stability of systems incorporating n-varactors embedded in an arbitrary, linear, stable, time-invariant environment. The

stability behavior must be considered in order for the varactor to be useful as a device for linear-controlled amplification. A stability criterion for the three-frequency mode of operation is obtained when we invoke the ideal sideband-termination assumption. The validity of the usual truncation technique as applied to stability considerations is discussed.

Without imposing the ideal sideband-termination assumption, the stability is governed by a linear three-term recursion relation and the derived infinite determinant is of the special type associated with continued fractions. A real-frequency stability condition is established for the undriven response of n-varactor devices coupled through linear, stable, time-invariant n-ports. This stability criterion leads to easily obtained bounds on the pumping ratios of the time-variable elements. Finally, a representation theorem on the steady-state response of a driven system is presented.

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CHAPTER I

INTRODUCTION

With the recent advent of semiconductor diodes the design of parametric devices has become a practical reality of great importance and has stimulated the network theorist to investigate the fundamental capabilities of these parametric devices. Past attention has been focused mainly on the physical aspects of the device and performance limitations of specific circuits which are not fundamental attributes of the device. This thesis is concerned with the development of a general and rigorous broadbanding theory for varactor parametric amplifiers. Fundamental gain-bandwidth limitations of a varactor parametric amplifier are obtained which are independent of the associated circuitry. Results obtained in this theory lead to the design and synthesis of broadband varactor parametric amplifiers.

The circuit considered in this work is that of linear variable capacitors embedded in an arbitrary, passive, lossless environment in which the signal frequency and one of the sideband frequencies can exist. We are concerned with a linearized theory of a parametric device within the scope of small signal analysis. This theory reveals some rather fundamental properties of a parametric device that are in marked contrast with a tunnel diode device, although both are two-terminal negative resistance devices. The first property is that the varactor, when embedded in a passive environment, presents an impedance which can be characterized by a positive function rather than a positive-real function. The second inherent property of these parametric devices is the quadratic nature of the expression of the scattering coefficient at the diode port. This is due to the interaction between the signal and the sideband circuits through the mixing or frequency-coupling action of the variable capcitor. This is also indicative of the fact that the broadbanding of a parametric device involves essentially the problem of the design of a broadband equalizer to match a load impedance when the load impedance is a function of the characteristic of the equalizer under consideration. A broadbanding theory has been obtained, taking into account both of these properties, and, furthermore, some important realizability conditions on the transducer power gain function are delineated.

A finite gain-bandwidth restriction exists for any varactor device because of the presence of its associated parasitic elements. The limitation on the transducer power gain function for an optimum nonreciprocal three-port equalizer is derived, from which we obtain the maximum flat gain using a single varactor attainable over a prescribed band of frequencies. Furthermore, optimum equalizers may be designed to yield a modified form of Butterworth or Tchebycheff transducer power gain responses of arbitrary order. These responses may be chosen to approach the maximum gain-bandwidth as closely as desired. These results are generalized for the case of n-varactor diodes embedded in a passive lossless environment, which constitutes a special case of the traveling wave or distributed parametric device.

The parametric amplifier has become an important and useful device largely because of its low-noise properties. The low-noise performance of the varactor arises from the relatively small dissipation. The noise performance of a dissipationless parametric amplifier is well-known. When the spreading resistance is not negligible, there is an ultimate limit on the noise performance. This is briefly discussed.

In order for the varactor diode to be useful as a device for linear controlled amplification, the stability question must be carefully investigated. We first show that, within the framework of small-signal analysis, stability is delineated by the roots of a transcendental relation derived from the infinite Hill determinant. When we assume the pump to be purely sinusoidal, a three-term recursion relation is obtained and the infinite determinant is of the special type associated with continued fractions. We then show that if we impose the ideal sideband-terminations assumption, stability is implied if the equation $Z_3(p) + z_D(p) = 0$ has no root in the entire closed right-half p-plane, where $Z_3(p)$ is an arbitrary positive-real function and $z_D(p)$ is the varactor diode impedance. The validity of using this as the stability criterion is discussed.

Using an energy argument, a stability test for systems incorporating n varactors embedded in an arbitrary, linear, stable, time-invariant environment is established, in which the ideal sideband-termination assumption is not imposed. The stability criterion obtained leads to easily obtained bounds on the pumping ratios of the time-variable elements. Finally, we prove that for a linear variable system, whose parameters are periodic functions of period T driven with the excitation e^{pt}, Re $p \geq 0$, the steady-state response must be of the form $e^{pt} V(t)$, where $V(t)$ is a periodic function of period T, provided that the undriven system is stable.

A point contact or junction diode, when back-biased, has an equivalent circuit[1,2] shown in Fig. 1(a). It is composed of a nonlinear capacitance, a constant resistance R_s, and an inductance L_o in series. Physically, the nonlinear capacitance is the depletion layer or barrier capacity which is voltage sensitive, L_o is the lead inductance, and R_s is the base or spreading resistance of bulk semiconductor. The voltage sensitive barrier resistance, which shunts the barrier capacitance, is so large in the back-biased condition that it can be neglected. The value of the barrier capacitance depends on the voltage (or the charge) across the capacitance; typically, it varies as the square root or cubic root of the voltage across it.

Within the scope of the small-signal analysis, the pumped varactor appears to the small signals as a linear time-variable capacitor as shown in Fig. 1(b). The series resistance of the varactor is assumed to be small and is neglected in our broadbanding work. Similarly, if the frequency band of operation is not too high, the parasitic inductance need not be included and the simplified model of Fig. 1(c) suffices. A network model which includes the parasitic inductance L_o is treated in Section V.6 to determine its effect on the gain-bandwidth performance.

CHAPTER II

FUNDAMENTAL RELATIONS OF NONLINEAR REACTANCE AMPLIFIERS

II.1 Introduction

Manley and Rowe[3] have derived some general energy relations which govern the behavior of energy flow in circuits containing nonlinear capacitors and inductors. These important relations form the basis of the theory of parametric amplifiers and other related devices. The Manley-Rowe relations, which essentially set a weighted sum of real powers at various frequencies entering a nonlinear reactance to zero, are remarkable in that they are independent of the shape of the nonlinear characteristic and of the power levels at the various frequencies. The only assumption that has been made in the derivation is that the nonlinear characteristic is single-valued.

The usefulness of the Manley-Rowe power relations can be illustrated by using the ideal sideband-terminations assumption, i.e., we assume that all sidebands, except a few of interest, are suppressed. This leads to the so-called "inverting" and "non-inverting" parametric devices. The plausibility of achieving parametric amplification in the "inverting" device is demonstrated.

The small-signal analysis is the primary tool employed in this investigation, in that we assume that the amplitudes of the signal and all the generated sidebands are small compared to the pump. Hence, we consider only the sidebands generated at frequencies $\omega_1 + n\omega_o$ instead of $\omega_{m,n} = m\omega_1 + n\omega_o$. Although both circuits containing nonlinear reactance and linear time-variable reactance will be capable of frequency conversion, they are quite different as far as mathematical analysis is concerned. By using the linear time-variable model, all the analytical techniques applicable to linear systems are available.

II.2 Manley-Rowe Formulas

A nonlinear lossless capacitor is, by definition, a two-terminal device in which charge q and voltage v are related through an expression of the form

$$q = f(v) \quad . \tag{2-1}$$

We shall exclude hysteresis and so require $f(v)$ to be a single-valued function, but otherwise its shape is arbitrary.

Now assume that signals at two incommensurate frequencies, ω_1 and ω_o, are applied by signal generators to the nonlinear capacitor. In general, all of the frequencies $\omega_{mn} = m\omega_1 + n\omega_o$ will be present in the circuit and in the nonlinear capacitor, where m and n take on all integral values. Hence, the charge q flowing into the nonlinear capacitor can be written as a double Fourier series in the following form:

$$q = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} Q_{m,n} e^{j(m\omega_1 + n\omega_o)t} \quad . \tag{2-2}$$

Since q is real,

$$Q_{m,n} = \overline{Q}_{-m,-n} \quad . \tag{2-3}$$

Taking the total derivative of Eq. (2-2) with respect to time, we obtain the current i flowing into the nonlinear capacitor.

$$i = \frac{dq}{dt} = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} I_{m,n} e^{j(m\omega_1 + n\omega_o)t} \quad , \tag{2-4}$$

where

$$I_{m,n} = j(m\omega_1 + n\omega_o) Q_{m,n} = \overline{I}_{-m,-n} \quad . \tag{2-5}$$

Since the nonlinear characteristic of Eq. (2-1) is assumed to be single-valued, the voltage v across the nonlinear capacitor must consist of the same frequency components as the charge q. Thus v may also be represented as a double Fourier series

$$v = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} V_{m,n} e^{j(m\omega_1 + n\omega_o)t} , \tag{2-6}$$

where

$$V_{m,n} = \bar{V}_{-m,-n} . \tag{2-7}$$

The average power flowing into the nonlinear capacitor at the frequencies $\pm|m\omega_1 + n\omega_o|$ is

$$W_{m,n} = 2 \operatorname{Re} V_{m,n} \bar{I}_{m,n} = -2(m\omega_1 + n\omega_o) \operatorname{Re}(jV_{m,n} \bar{Q}_{m,n}) . \tag{2-8}$$

Now, since the nonlinear capacitor is assumed lossless, we have from conservation of energy

$$\sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} W_{m,n} = 0 . \tag{2-8}$$

Equation (2-8) may be rewritten by multiplying and dividing each term by its corresponding frequency:

$$\sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} (m\omega_1 + n\omega_o) \frac{W_{m,n}}{(m\omega_1 + n\omega_o)} = 0 . \tag{2-9}$$

Splitting into two terms, and writing the appropriate range for the summations, we have

$$\omega_1 \sum_{m=0}^{\infty} \sum_{n=-\infty}^{\infty} \frac{m W_{mn}}{m\omega_1 + n\omega_o} + \omega_o \sum_{m=-\infty}^{\infty} \sum_{n=0}^{\infty} \frac{n W_{mn}}{m\omega_1 + n\omega_o} = 0 . \tag{2-10}$$

It has been shown[4,5] that each of the two terms of Eq. (2-10) may be separately set equal to zero, i.e.,

$$\sum_{m=0}^{\infty} \sum_{n=-\infty}^{\infty} \frac{m W_{mn}}{m\omega_1 + n\omega_o} = 0 \quad , \tag{2-11}$$

and

$$\sum_{m=-\infty}^{\infty} \sum_{n=0}^{\infty} \frac{n W_{mn}}{m\omega_1 + n\omega_o} = 0 \quad . \tag{2-12}$$

These are the energy relations of Manley and Rowe. The two Manley-Rowe relations are not independent of the law of conservation of energy; in fact, multiplication of Eq. (2-11) by ω_1, of Eq. (2-12) by ω_o, and addition give Eq. (2-8). The result was originally proved for a nonlinear capacitor, but the extension to a nonlinear inductor is obvious.

Furthermore, Manley-Rowe relations can be extended to apply to any combinations of nonlinear reactors and linear, lossless, time-invariant networks. First, because of the linearity and time invariance of the latter networks, no frequency coupling can take place, i.e., no power can be transferred from one frequency to another in these networks. In addition, since they are lossless, they cannot change the relation between average power at frequency $m\omega_1 + n\omega_o$ at the terminals of the nonlinear reactors.

II.3 Inverting and Non-Inverting Parametric Devices

The usefulness of the Manley-Rowe power relations can now be illustrated by considering two devices in which power flow at only three frequencies of major importance is allowed. This corresponds to assuming that the nonlinear element is terminated in an ideal filter that suppresses the current through or the voltages across the nonlinear element at all but the significant frequencies. Let the signal generator at $\omega_1 = \omega_s$ represent a signal source, and the generator at ω_o the local oscillator or pump source. Consider first the case where power flow is allowed at the idler or difference frequency,

$\omega_i = \omega_o - \omega_s$, then the summations of Eqs.(2-11) and (2-12) reduce to

$$\frac{W_o}{\omega_o} + \frac{W_i}{\omega_i} = 0 \quad , \tag{2.13}$$

$$\frac{W_s}{\omega_s} - \frac{W_i}{\omega_i} = 0 \quad . \tag{2.14}$$

We are supplying powers at ω_o, hence W_o is positive. The fact that W_s and W_i may be simultaneously negative predicts that infinite gain is possible and that such a device is potentially unstable at both ω_s and ω_i when embedded in a linear, passive environment. This capability manifests itself through the fact that the real part of the input immitance at the terminals of the varactor is negative for both the signal and idler frequencies. Equations (2-13) and (2-14) are often used to demonstrate the plausibility of parametric amplification in this so-called "inverting" device. It is to be noted from Eq.(2-14) that idler frequency dissipation is required for the desired amplification at the signal frequency. This is another fundamental property of this device which can be deduced from the Manley-Rowe relations.

Next, we consider the case where the capacitor is terminated so that power flows only at frequencies $\omega_l = \omega_s$, ω_o, and $\omega_u = \omega_o + \omega_s$, the sum frequency. Then this so-called "non-inverting" device obeys the relations

$$\frac{W_s}{\omega_s} + \frac{W_u}{\omega_u} = 0 \quad , \tag{2-15}$$

$$\frac{W_o}{\omega_o} + \frac{W_u}{\omega_u} = 0 \quad . \tag{2-16}$$

These relations predict that the device is unconditionally stable when embedded in an arbitrary linear passive environment. Furthermore, the maximum transducer power gain that can be obtained with such a device is precisely given by the ratio of ω_u/ω_s

when used as a frequency up-converter.

The choice of the names "inverting" and "non-inverting" in the two cases discussed is based on what the devices do to the signal spectrum, as may be seen by considering not a single frequency but a band of signal frequencies. The distribution of the frequency spectrums is illustrated in Fig. 2. Of the two cases, the inverting parametric device is of major interest and will be our main concern in the subsequent work.

Another significant point needs to be indicated here. It is important to note that the artifice of restricting the analysis to a particular finite set of desired frequencies imposes severe limitations on the validity of the results obtained for a specific application. For example, when this assumption of ideal sideband-terminations is applied in the study of the stability behavior of the inverting device, the validity of the results must be carefully examined.

II.4 Small-Signal Analysis

Although the Manley-Rowe relations provide a basis for deducing some general properties of nonlinear reactance circuits, they are of course no substitute for a detailed circuit analysis of a particular nonlinear device. Several approximate characterizations of nonlinear elements lend themselves to standard analytical techniques and give results which are satisfactory in most practical applications. Most of these approximate methods involve some form of small-signal analysis. We shall assume that the signals and all the generated sidebands are small in amplitude compared to the pump.[6] Hence, these small signals see the pumped varactor essentially as a linear time-variable capacitance $C(t)$ at fundamental frequency ω_o, instead of a nonlinear element. In the linear mode we are considering, all the analytical techniques applicable to linear systems such as the principle of superposition, Fourier analysis, and the Floquet theory[7] are available. But, on the other hand, we can only predict small-signal properties, not the saturation effects.

Within the framework of small-signal theory, the pump source and the nonlinear capacitor can immediately be replaced by a linear time-variable capacitor $C(t)$. Once $C(t)$ is specified, the actual pumping circuit used is of no more interest and will not be discussed further in this work. The linear time-variable capacitor, $C(t)$, relates the charge $q(t)$ on its positive plate to the voltage $v(t)$ across its plates as follows:

$$q(t) = C(t)\, v(t) \quad . \tag{2-17}$$

The pumped varactor operating in the linear mode is illustrated in Fig. 3. In most parametric applications the varactor is pumped periodically, so that in a small-signal theory the effect of the pumping is to produce a time-variable capacitor for which the following Fourier series applies:

$$C(t) = \sum_{r=-\infty}^{\infty} C_r \, e^{j r \omega_o t} \quad , \tag{2-18}$$

where

$$C_r = \overline{C}_{-r} \tag{2-19}$$

because $C(t)$ is a real function of time.

Now we consider the situation where the parametric element is embedded in an arbitrary, linear, time-invariant environment (Fig. 4). Suppose the embedding network is lumped, then the equilibrium state is givterned by an ordinary differential equation with periodic coefficients of period $T = \omega_o/2\pi$. According to Floquet theory, we can always find initial conditions yielding a solution of the form

$$v(t) = e^{\mu t} \psi(t) \quad , \tag{2-20}$$

where $\psi(t)$ is periodic with period $T = \omega_o/2\pi$ and μ is any one of the associated Floquet exponents. A real-frequency stability test has been established for the undriven response of varactor devices coupled through linear, stable, time-invariant n-ports.[8] This will be presented in Chapter VIII where we shall treat the stability question more extensively. For the present purpose, we shall assume that the stability of the undriven system has been established (in the sense that all the Floquet exponents have non-positive real parts); then it is proved in Section VIII.4 that the steady-state response of the system driven with e^{pt}, Re $p \geq 0$, is of the form

$$v(t) = e^{pt} V(t) \quad , \qquad (2\text{-}21)$$

where $V(t)$ is periodic with period $T = \omega_o / 2\pi$.

Developing the corresponding periodic function $V(t)$ into Fourier series, we can write

$$V(t) = \sum_{k=-\infty}^{\infty} V_k \, e^{jk\omega_o t} \quad . \qquad (2\text{-}22)$$

From Eq. (2-17), and using $i(t) = -dq/dt$,

$$i(t) = -\frac{d}{dt}\left[C(t)\, V(t)\, e^{pt}\right] \quad , \qquad (2\text{-}23)$$

or

$$i(t) = -\frac{d}{dt}\left[\sum_{r,k} C_r V_k \, e^{\left[p + j(r+k)\omega_o\right]t}\right] \quad ,$$

$$= -\sum_{r,k} C_r V_k \left[p + j(r+k)\omega_o\right] e^{\left[p + j(r+k)\omega_o\right]t} \quad . \qquad (2\text{-}24)$$

Let $r + k = n$, and

$$i(t) = -\sum_{n=-\infty}^{\infty} I_n \, e^{(p + jn\omega_o)t} \quad ; \qquad (2\text{-}25)$$

we obtain from Eq. (2-24) coupled with Eq. (2-25)

$$i(t) = -\sum_{n,r=-\infty}^{\infty} C_r V_{n-r} (p + jn\omega_o) \, e^{(p + jn\omega_o)t} \quad , \qquad (2\text{-}26)$$

and

$$I_n = (p + jn\omega_o) \sum_{r=-\infty}^{\infty} C_r V_{n-r} \quad , \tag{2-27}$$

where $n = 0, \pm 1, \pm 2, \ldots, \ldots$. Hence, we have succeeded in expressing I_n in terms of the coefficients C_n and V_n. For $n = 0$, $n = -1$, and $n = 1$, we have, respectively,

$$I_o = p \sum_{r=-\infty}^{\infty} C_r V_{-r} \quad ,$$

$$I_{-1} = (p - j\omega_o) \sum_{r=-\infty}^{\infty} C_r V_{-1-r} \quad , \tag{2-28}$$

and

$$I_{+1} = (p + j\omega_o) \sum_{r=-\infty}^{\infty} C_r V_{+1-r} \quad .$$

A special but important case is that of a weakly-pumped capacitor:

$$C(t) = C_o + 2C_1 \cos(\omega_o t) \quad . \tag{2-29}$$

The quantity

$$\rho = \frac{2C_1}{C_o} \tag{2-30}$$

is known as the pumping ratio. Typical values of ρ range from 0.2 to 0.5. For this case, Eq. (2-27) simplifies to

$$\frac{I_n}{(p + jn\omega_o)} = C_1 V_{n-1} + C_o V_n + C_{-1} V_{n+1} \quad , \tag{2-31}$$

$$C_{-1} = C_1 \quad ,$$

in which the current generated at each frequency, I_n, is determined solely by the voltages at that frequency and the two frequencies removed by the pump frequency ω_o. Equation (2-31) is in the form of a three-term recursion relation which characterizes the case of a weakly-pumped varactor where all the sideband frequencies $\omega_s + n\omega_o$ are taken into account.

II.5 Assumption of Ideal Sideband-Terminations and Derivation of Varactor Diode Impedance

The immediate task here is to apply the results obtained in the previous section to derive the expressions for the varactor diode impedance assuming ideal sideband-terminations for all sideband frequencies except a few of major importance. First, let us consider the case in which all $V_n = 0$ except $n = 0, -1$ and $+1$, but without imposing the weakly-pumped varactor or sinusoidal pump condition. For the present application, we shall consider the driver to be $\exp(j\omega_s t)$ rather than $\exp(pt)$. From Eqs. (2-27),

$$
\begin{aligned}
I_{-1} &= (j\omega_s - j\omega_o)\, C_o V_{-1} + (j\omega_s - j\omega_o)\, C_{-1} V_o + (j\omega_s - j\omega_o)\, C_{-2} V_1 , \\
I_o &= (j\omega_s)\, C_1 V_{-1} + (j\omega_s)\, C_o V_o + (j\omega_s)\, C_{-1} V_1 , \qquad (2\text{-}32) \\
I_{+1} &= (j\omega_s + j\omega_o)\, C_2 V_{-1} + (j\omega_s + j\omega_o)\, C_1 V_o + (j\omega_s + j\omega_o)\, C_o V_1 ,
\end{aligned}
$$

where $C_1 = C_{-1}$ and $\overline{C}_2 = C_{-2}$. Changing the notation from V_o to V_s, V_{-1} to $\overline{V}_i$, V_{+1} to V_u, and similarly with the corresponding I's, we obtain

$$
\begin{bmatrix} \overline{I}_i \\ I_s \\ I_u \end{bmatrix} = \begin{bmatrix} -j\omega_i C_o & -j\omega_i C_{-1} & -j\omega_i C_{-2} \\ j\omega_s C_1 & j\omega_s C_o & j\omega_s C_{-1} \\ j\omega_u C_2 & j\omega_u C_1 & j\omega_u C_o \end{bmatrix} \begin{bmatrix} \overline{V}_i \\ V_s \\ V_u \end{bmatrix} \qquad (2\text{-}33)
$$

where $\omega_i = \omega_o - \omega_s$ is the idler frequency and $\omega_u = \omega_o + \omega_s$ is the upper-sideband frequency. The matrix equation (2-33) essentially corresponds to the small-signal varactor diode admittance matrix given by Rowe in Reference 6.

The diode impedance z_D is shown in Fig. 6, in which the d.c. capacitance contribution C_o is shown explicitly shunting the equalizer. The diode admittance at the signal frequency is related to the diode impedance at the idler and the upper-sideband frequencies by the following equation:

$$y_D(j\omega_s) = \omega_s C_1 \left[\frac{j2\omega_i\omega_u(\mathrm{Re}\, C_2)\, z_D(j\omega_u)\, \bar{z}_D(j\omega_i) + C_1\left[\omega_i \bar{z}_D(j\omega_i) - \omega_u z_D(j\omega_u)\right]}{1 - \omega_i\omega_u |C_2|^2\, \bar{z}_D(j\omega_i)\, z_D(j\omega_u)} \right] \quad (2\text{-}34)$$

Since the network is excited by a source at ω_s, the following equalities hold:

$$Y_3(j\omega_i) = -y_D(j\omega_i) \quad , \quad (2\text{-}35)$$

$$Y_3(j\omega_u) = -y_D(j\omega_u) \quad , \quad (2\text{-}36)$$

where Y_3 is the admittance seen looking into the equalizer at the diode port 3 (see Fig. 6). Substituting into Eq.(2-34) gives

$$y_D(j\omega_s) = \omega_s C_1 \left[\frac{j2\omega_i\omega_u(\mathrm{Re}\, C_2)\, Z_3(j\omega_u)\, \bar{Z}_3(j\omega_i) + C_1\left[\omega_u Z_3(j\omega_u) - \omega_i \bar{Z}_3(j\omega_i)\right]}{1 - \omega_i\omega_u |C_2|^2\, \bar{Z}_3(j\omega_i)\, Z_3(j\omega_u)} \right] . \quad (2\text{-}37)$$

In Eq.(2-37), the diode impedance is expressed as a function of the impedances looking into the equalizer at the idler and upper-sideband frequencies. Note that the presence of terms in C_2 means that the effects of both the pump and its first harmonic on the small signals are included.

For a weakly-pumped varactor defined by Eq.(2-29), $C_2 \equiv 0$; then Eq.(2-37) reduces to

$$y_D(j\omega_s) = \omega_s C_1^2 \left[\omega_u Z_3(j\omega_u) - \omega_i \bar{Z}_3(j\omega_i)\right] \quad . \tag{2-38}$$

For the inverting device, the voltage at ω_u is suppressed to give $Z_3(j\omega_u) = 0$. Then Eq.(2-37) becomes

$$y_D(j\omega_s) = -\omega_s \omega_i C_1^2 \bar{Z}_3(j\omega_i)$$

or

$$z_D(j\omega_s) = -\frac{1}{\omega_s \omega_i C_1^2 \bar{Z}_3(j\omega_i)} \quad . \tag{2-39}$$

Since

$$\text{Re}\, z_D(j\omega_s) = -\frac{1}{\omega_s(\omega_o - \omega_s) C_1^2} \text{Re}\, Y_3(j\omega_i) \quad , \tag{2-40}$$

it is clear that the varactor diode impedance will exhibit a negative real part within the frequency band of interest. Finally, for the non-inverting device, the voltage at ω_i is suppressed to give $Z_3(j\omega_i) = 0$. Now Eq.(2-37) becomes

$$y_D(j\omega_s) = +\omega_s \omega_u C_1^2 Z_3(j\omega_u) \quad . \tag{2-41}$$

CHAPTER III

STEADY-STATE TRANSDUCER POWER GAIN FUNCTION AND PRELIMINARY RESULTS

III.1 Preliminary Notation

Let A be an arbitrary matrix. Then A', $\overline{A}$, A^*, A^{-1} and $\det(A)$ denote, respectively, the transpose, the complex conjugate, the complex conjugate transpose, the inverse, and the determinant of A. Column vectors are denoted by $\underset{\sim}{a}$, $\underset{\sim}{b}$, $\underset{\sim}{V}$, $\underset{\sim}{I}$, etc., or in the alternative form $\underset{\sim}{x} = (x_1, x_2, \ldots, x_n)'$ whenever it is desirable to exhibit the components explicitly. The matrices 1_n, $\underset{\sim}{0}_n$ and $0_{m,n}$ represent, in the same order, the $n \times n$ identity matrix, the n-dimensional zero column-vector and the $m \times n$ zero matrix. A diagonal matrix with diagonal elements $u_1, u_2, \ldots, u_n$ is written as $A = \text{diag}\left[u_1, u_2, \ldots, u_n\right]$. For a hermitian matrix $A = A^*$, $A > 0$ means that A is the matrix of a non-negative quadratic form. If A is positive definite, $A > 0$, and if A is semi-positive definite, $A \geq 0$.

A matrix $A(p)$ is said to be real if $\overline{A}(p) = A(\overline{p})$. In particular, $\overline{A}(j\omega) = A(-j\omega)$ for all real ω. Since we will be dealing with functions which are not necessarily real for real p throughout this work, we must introduce an operation that generalizes the notion of "replacing p by $-p$." Let $A(p)$ be an arbitrary rational (or meromorphic) matrix. Then

$$A_*(p) \equiv A^*(-\overline{p}) \quad .$$

Note that for $p = j\omega$, $A_*(j\omega) = A^*(j\omega)$. Also, $A_{**}(p) = A(p)$ and $(AB)_* = B_* A_*$. An $n \times n$ matrix $A(p)$ is said to be paraconjugate hermitian if $A^*(p) = A(-\overline{p})$, and paraconjugate unitary if $A_*(p)\,A(p) = 1_n$. On $p = j\omega$, the two conditions reduce to $A^*(j\omega) = A(j\omega)$ and $A^*(j\omega)\,A(j\omega) = 1_n$, respectively, which are the usual criteria for hermitian and unitary matrices.

A scalar function $a(p)$ satisfying $a_*(p) \equiv \bar{a}(-\bar{p}) = a(p)$ is called paraconjugate. If $a(p)$ is real it is actually even. A rational function $b(p)$ satisfying $b_*(p)\, b(p) = 1$ is called a Blaschke product, i.e., an all-pass factor. A regular Blaschke product is analytic in $\text{Re } p > 0$. Since $b_*(p)\, b(p) = 1$ implies that $|b(j\omega)|^2 = 1$, any Blaschke product is automatically analytic on $p = j\omega$. A regular Blaschke product $b(p)$ may always be represented in the form

$$b(p) = e^{j\theta} \prod_{r=1}^{n} \frac{p - p_r}{p + \bar{p}_r} \quad , \tag{3-1}$$

where $\text{Re } p_r > 0$, $r = 1, 2, \ldots, n$. If $b(p)$ is real, $b(p)\, b(-p) = 1$ and $\theta = 0$ or π.

Finally, the norm of a matrix A, $||A||$, is defined to be the positive square root of the largest eigenvalue of $A^* A$.

III.2 Scattering Formulation with Complex and Frequency Dependent Normalization

It is well known that a passive n-port N can always be described in terms of its $n \times n$ scattering matrix $S(p)$ normalized to a set of real and positive port numbers.[9] Recently, Youla[10,11] described a method for defining on the real-frequency axis the scattering matrix of an n-port normalized to n arbitrary non-Foster positive functions $z_1(j\omega)$, $z_2(j\omega)$, $\ldots$, $z_n(j\omega)$. More precisely, it was shown that to any passive lumped n-port N, and any prescribed set of functions with positive real parts over the frequency range of interest, it is possible to assign an $n \times n$ "normalized" scattering matrix, $S(p)$, having all the properties of a scattering matrix normalized to a set of positive port numbers.

The augmented n-port N_A corresponding to the linear, time-invariant n-port N is derived from N by inserting $z_k(j\omega)$ in the corresponding kth port. This is depicted schematically in Fig. 5, where we assume that the impedances $z_1(j\omega)$, $z_2(j\omega)$, $\ldots$, $z_n(j\omega)$ have positive real parts over the frequency band W, i.e.,

$$\text{Real } z_k(j\omega) = r_k(\omega) > 0 \;, \qquad \omega \in W \;, \qquad (k = 1, 2, \ldots, n) \;. \tag{3-2}$$

The normalized incident wave amplitudes, $a_1, a_2, \ldots, a_n$, and reflected wave amplitudes, $b_1, b_2, \ldots, b_n$, impinging on the n-port N are defined as follows:

$$2\sqrt{r_k}\; a_k = V_k + z_k I_k \;, \tag{3-3}$$

and

$$2\sqrt{r_k}\; b_k = V_k - \bar{z}_k I_k \;, \tag{3-4}$$

$$(k = 1, 2, \ldots, n) \;.$$

All square roots have been chosen positive. In matrix form,

$$2 R^{1/2} \underset{\sim}{a} = \underset{\sim}{V} + Z \underset{\sim}{I} \;, \tag{3-5}$$

and

$$2 R^{1/2} \underset{\sim}{b} = \underset{\sim}{V} - \bar{Z} \underset{\sim}{I} \;, \tag{3-6}$$

where

$$Z = \text{diag}\left[z_1, z_2, \ldots, z_n\right] \;,$$

$$R = \text{diag}\left[r_1, r_2, \ldots, r_n\right] \;,$$

$$R^{\pm 1/2} = \text{diag}\left[r_1^{\pm 1/2}, r_2^{\pm 1/2}, \ldots, r_n^{\pm 1/2}\right] \;,$$

$$\underset{\sim}{a} = (a_1, a_2, \ldots, a_n)' \;, \tag{3-7}$$

$$\underset{\sim}{b} = (b_1, b_2, \ldots, b_n)' \;,$$

$$\underset{\sim}{V} = (V_1, V_2, \ldots, V_n)' \;,$$

and

$$\underset{\sim}{I} = (I_1, I_2, \ldots, I_n)' \;.$$

The $n \times n$ scattering matrix $S(j\omega)$ of N, normalized with respect to a set of prescribed functions $z_1(j\omega)$, $z_2(j\omega)$, ..., $z_n(j\omega)$, is defined by means of the linear matrix equation

$$\underset{\sim}{b} = S \underset{\sim}{a} \quad . \tag{3-8}$$

It has been shown[10] that the normalization procedure defined by Eqs.(3-3) and (3-4) has succeeded in preserving all the important properties possessed by a scattering matrix normalized to real positive port numbers. Hence, if N is lossless over W, the "normalized" $n \times n$ scattering matrix $S(j\omega)$ is unitary for $\omega \in W$, i.e., $1_n - S^*(j\omega)\, S(j\omega) = 0_n$. From (3-8) and the unitary character of S, we have

$$\underset{\sim}{a} = S^*(j\omega) \underset{\sim}{b} \quad . \tag{3-9}$$

Now suppose that port k is terminated in z_k, then $V_k = -z_k I_k$ and, from Eq. (3-3), $a_k = 0$. Also, if port k is termination in $-\bar{z}_k(j\omega)$ instead of $z_k(j\omega)$, then $V_k = \bar{z}_k I_k$ and, from Eq.(3-4), $b_k = 0$. In short, closing a port on its respective normalization impedance obliterates the corresponding incident wave, whereas termination in the negative complex conjugate of the normalization impedance obliterates the reflected wave. In the first case the port is said to be "matched" and in the second to be "paraconjugate matched."

If port k is energized through a generator E_g with internal impedance $z_k(j\omega)$, and all other ports are closed on their respective normalization impedances, then

$$a_k = \frac{E_g}{2\sqrt{r_k(j\omega)}} \quad , \tag{3-10}$$

and

$$a_\ell = 0 \quad , \qquad \ell \neq k \quad . \tag{3-11}$$

Hence, from Eq.(3-8),

$$b_\ell = s_{\ell k}(j\omega)\, a_k \ , \qquad (\ell = 1, 2, \ldots, n)$$

$$= s_{\ell k}(j\omega) \frac{E_g}{2\sqrt{r_k(j\omega)}} \ . \qquad (3\text{-}12)$$

Equation (3-11) coupled with (3-4) yields

$$b_\ell = -\sqrt{r_\ell(j\omega)}\; I_\ell \ , \qquad \ell \neq k \ , \qquad (3\text{-}13)$$

and therefore (see (3-12))

$$G_{k\ell}(\omega) = \frac{r_\ell(j\omega)\,|I_\ell|^2}{\dfrac{|E_g|^2}{4\, r_k(j\omega)}} = |s_{\ell k}(j\omega)|^2 \ , \qquad \ell \neq k \ , \qquad (3\text{-}14)$$

the transducer power gain from port k to port ℓ. In other words, the transducer power gain from port k to port r under "matched" conditions is measured precisely by the square of the magnitude of the transfer coefficient s_{rk}. To determine $s_{kk}(j\omega)$, let $Z_k(j\omega)$ represent the impedance seen looking into port k under matched terminations. Then $V_k = Z_k I_k$ and the division of Eq. (3-4) by (3-3) yields, with the aid of (3-12),

$$\frac{b_k}{a_k} = s_{kk}(j\omega) = \frac{V_k - \bar{z}_k I_k}{V_k + z_k I_k} = \frac{Z_k - \bar{z}_k}{Z_k + z_k} \ . \qquad (3\text{-}15)$$

If the n-port N possesses an impedance matrix, Z_N, then it is possible to express S in terms of Z_N and Z. Using $\underset{\sim}{V} = Z_N \underset{\sim}{I}$, we have

$$2R^{1/2}\,\underset{\sim}{a} = (Z_N + Z)\,\underset{\sim}{I} \ ,$$

$$2R^{1/2}\,\underset{\sim}{b} = (Z_N - \bar{Z})\,\underset{\sim}{I} \ .$$

Therefore,

$$2\,R^{1/2}\,S\,\underset{\sim}{a} = R^{1/2}\,S\,R^{-1/2}\,(Z_N + Z)\,\underset{\sim}{I} = (Z_N - \bar{Z})\,\underset{\sim}{I} \quad ,$$

and

$$S = R^{-1/2}\,(Z_N - \bar{Z})\,(Z_N + Z)^{-1}\,R^{1/2} \quad . \tag{3-16}$$

III.3 Derivation of the Steady-State Transducer Power Gain Function

The normalization procedure discussed in the preceding section will now be utilized to derive an expression for the maximum transducer power gain attainable with a single varactor device embedded in a lossless environment. The general structure under consideration is shown schematically in Fig. 6, in which the varactor diode is placed across an arbitrary, passive, lossless 3-port in which the signal and idler frequencies can exist. C_o and C_1 denote the first two coefficients of an equivalent linear variable capacitor defined by Eq. (2-29). The constant capacitance contribution is shown explicitly shunting the equalizer. Hence, the 3-port N is not an arbitrary lossless 3-port but must be subject to the restriction that the diode port 3 is shunted by a capacitance of value not less than C_o. The diode impedance at signal frequency is given by Eq. (2-39).

Let $S(j\omega_s)$ be the scattering matrix of N normalized to $z_1 = R_g$ at port 1, $z_2 = R_L$ at port 2, and

$$z_3(j\omega_s) = -\,\bar{z}_D(j\omega_s) = \frac{1}{\omega_s\,\omega_i\,C_1^2\,Z_3(j\omega_i)} \tag{3-17}$$

at port 3. Note that Re $z_3(j\omega_s)$ is positive over the frequency band of interest. Using the normalization technique defined by Eqs. (3-3) and (3-4), S is unitary and Eq. (3-9) holds. From the termination conditions at each of the ports, we have $a_2 = 0$, $b_3 = 0$, and $a_1 = E_g/2\sqrt{R_g}$. Therefore,

$$\frac{E_g}{2\sqrt{R_g}} = a_1 = \bar{s}_{11} b_1 + \bar{s}_{21} b_2 \quad , \tag{3-18}$$

$$0 = \bar{s}_{12} b_1 + \bar{s}_{22} b_2 \quad , \tag{3-19}$$

$$a_3 = \bar{s}_{13} b_1 + \bar{s}_{23} b_2 \quad . \tag{3-20}$$

By Eqs.(3-18) and (3-19),

$$b_2 = -\frac{\bar{s}_{12} a_1}{\bar{\Delta}} \quad , \tag{3-21}$$

where

$$\Delta = s_{11} s_{22} - s_{12} s_{21} \quad . \tag{3-22}$$

Furthermore, using the unitary condition, it can be shown[12] that $|\Delta(j\omega)| = |s_{33}(j\omega)|$.

The steady-state transducer power gain, $G_t(\omega_s)$, is by definition the ratio of the average power absorbed by the load R_L to the maximum available power from the generator. From Eq.(3-21) and $a_2 = 0$,

$$R_L |I_2|^2 = |b_2|^2 - |a_2|^2 = \left|\frac{s_{12}}{s_{33}}\right|^2 |a_1|^2 \quad . \tag{3-23}$$

Therefore,

$$G_t(\omega_s) = \frac{R_L |I_2|^2}{\frac{|E_g|^2}{4 R_g}} = \left|\frac{s_{12}(j\omega_s)}{s_{33}(j\omega_s)}\right|^2 \quad . \tag{3-24}$$

The unitary condition, $S^*(j\omega) S(j\omega) = 1_3$, implies that $|s_{12}(j\omega)| \le 1$. Consequently,

$$G_t(\omega_s) \le \frac{1}{|s_{33}(j\omega_s)|^2} \equiv G(\omega_s) \quad . \tag{3-25}$$

Applying (3-15) and (3-17), $s_{33}(j\omega_s)$ can be expressed as

$$s_{33}(j\omega_s) = \frac{Z_3(j\omega_s) - \bar{z}_3(j\omega_s)}{Z_3(j\omega_s) + z_3(j\omega_s)} = \frac{Z_3(j\omega_s) + z_D(j\omega_s)}{Z_3(j\omega_s) - \bar{z}_D(j\omega_s)} , \quad (3\text{-}26)$$

$$s_{33}(j\omega_s) = \frac{\omega_s \omega_i C_1^2 - Y_3(j\omega_s)\bar{Y}_3(j\omega_i)}{\omega_s \omega_i C_1^2 + Y_3(j\omega_s) Y_3(j\omega_i)} . \quad (3\text{-}27)$$

This expression demonstrates clearly the quadratic nature of the scattering coefficient at the diode port, i.e., s_{33} is quadratic in Y_3 whereas in the case of equalization of passive load or tunnel diode the corresponding expression is linear in Y_3. This is an inherent property of a parametric device. The amplifying capability of the varactor parametric amplifier can be attributed directly to the frequency coupling mechanism inherent in the device. The quadratic nature is due to the interaction between the signal and the sideband circuits through the variable capacitor. It indicates that the broadbanding of a parametric device is analogous to the problem of the design of a broadband equalizer to match a load impedance when the load impedance itself is a function of the characteristic of the equalizer under consideration. It indicates in general the complexity of this particular broadbanding problem.

Substituting into (3-25),

$$G_t(\omega_s) \leq \left| \frac{Z_3(j\omega_s) - \bar{z}_D(j\omega_s)}{Z_3(j\omega_s) + z_D(j\omega_s)} \right|^2 \equiv G(\omega_s) , \quad (3\text{-}28)$$

and, finally,

$$G_t(\omega_s) \leq \left| \frac{\omega_s \omega_i C_1^2 + Y_3(j\omega_s) Y_3(j\omega_i)}{\omega_s \omega_i C_1^2 - Y_3(j\omega_s)\bar{Y}_3(j\omega_i)} \right|^2 \equiv G(\omega_s) . \quad (3\text{-}29)$$

Relation (3-28) actually embodies the solution to the broadbanding problem for both the tunnel diode and varactor with the loss and parasitic inductance taken into account. A systematic treatment on the broadband tunnel diode amplifiers has been published by Youla and Smilen.[12,13] It can be concluded that the ultimate power gain for both the tunnel diode and varactor devices is completely delimited by the equalizer impedance Z_3 facing the respective negative impedance element. For the varactor case, however, the ultimate power gain is delimited not only by the equalizer impedance at the signal frequency of interest, but also by this same impedance at the idler frequency. This is exhibited explicitly by (3-29).

III.4 Reciprocal and Nonreciprocal Equalizations

To actually realize the gain in (3-29), it is necessary to use an equalizer incorporating a 3-port circulator since the $|s_{12}|^2$ must be made unity, irrespective of $|s_{33}|^2$. The nonreciprocal structure is shown schematically in Fig. 7. The circulator is used to connect the generator, the varactor and the load. A lossless reciprocal 2-port equalizer is placed between the circulator and the varactor, and a lossless filter is also shown (in dotted lines) which may be used if frequency shaping at the high frequencies is desired.

Now we shall consider briefly the reciprocal equalization case. If N is restricted to be reciprocal, S must be symmetric and we can no longer adjust $|s_{12}(j\omega)|^2$ and $|s_{33}(j\omega)|^2$ independently by using a circulator as the isolating element. In this case it can be shown[12] that

$$|s_{12}(j\omega_s)| \leq \frac{1 + |s_{33}(j\omega_s)|}{2} , \tag{3-30}$$

and hence

$$G_t(\omega_s) = \left| \frac{s_{12}(j\omega_s)}{s_{33}(j\omega_s)} \right|^2 \leq \frac{1}{4} \left\{ 1 + \left| \frac{Z_3(j\omega_s) - \bar{z}_D(j\omega_s)}{Z_3(j\omega_s) + z_D(j\omega_s)} \right| \right\}^2 . \tag{3-31}$$

Consequently, for the same Z_3, optimum, lossless, nonreciprocal equalization yields at most 6 db more gain over optimum, lossless, reciprocal equalization. This fundamental result was originally derived for the tunnel diode case[12] and was found to be a general property applicable to any 3-port equalizer when one of its ports is paraconjugate matched. From this point on, we will not discuss the case of reciprocal equalization any further, because without the facility of using the circulator as the isolating element, $|s_{12}|$ and $|s_{33}|$ cannot be adjusted independently and the situation is quite complicated. We shall consider only the nonreciprocal amplifier in which the ultimate transducer power gain is realized, i.e., $G_t(\omega_s) = G(\omega_s)$, and would be used interchangeably in our subsequent work.

CHAPTER IV

A BROADBANDING THEORY FOR VARACTOR PARAMETRIC AMPLIFIER

IV.1 Introduction

The broadbanding problem for a prescribed passive load and a resistive generator was treated by Bode[14] and Fano.[15] A new broadband matching theory has since been developed by Youla,[16] based on the principle of complex normalization. The new theory is applicable to the problem of designing an active equalizer incorporating an active impedance, which is a prescribed function of frequency, to achieve a preassigned transducer power gain working between a resistive generator and resistive load. For the varactor parametric amplifier, the varactor diode impedance is not prescribed in the sense that it depends directly on the impedance looking into the equalizer at the diode port. In essence, the equalizer and the active load cannot be treated independently because they interact with each other through the frequency-coupling action in the varactor. We present in this section a broadbanding technique which takes this effect into full account.

When the load, passive or active, is prescribed, the transmission zeros of the load impose certain inherent restrictions which are reflection onto the transducer power gain function. In the varactor problem, since the load and the equalizer are interrelated, we cannot talk meaningfully about transmission zeros of the load; instead, we define the transmission zeros of the device as a whole which characterize the equalizer as well as the varactor load. It is shown that these transmission zeros are specified from the preassigned transducer power gain function and that they again impose restrictions which are reflected onto the transducer power gain function.

In Section IV.5, the broadbanding technique is applied to the case where $\omega_s \omega_i \cong \omega_{so} \omega_{io}$ and $Y_3(j\omega_s) \cong \overline{Y}_3(j\omega_i)$, and an illustrative design example is completely worked out starting from the prescribed transducer power gain function (Appendix A). A more exact theory applicable to this case is presented in Chapter V. The purpose here is to demonstrate that the general broadbanding theory does not only present a solution in principle. In fact, for the general case, this is at present the only technique at our disposal. Moreover, although only the d.c. capacitance C_o is considered in this chapter, the broadbanding theory presented is valid for varactors with additional parasitic elements.

IV.2 Analytic Extension of the Steady-State Transducer Power Gain Function

In order to proceed with the broadbanding, it is necessary to extend the steady-state transducer power gain expression to the entire p-plane. There are various ways of extending the variables ω_s and ω_i, since the only constraint to be satisfied is that at real frequency

$$\omega_s + \omega_i = \omega_o \quad , \tag{4-1}$$

where the pump frequency is prescribed. Two possible schemes for the extension are: a) $j\omega_s$ and $j\omega_i$ are extended into p and $j\omega_o - p$, respectively; and b) $j\omega_s$ and $j\omega_i$ are extended into $j\omega_{so} + p$ and $j\omega_{io} - p$, respectively. In the first case, the extension of $Z_3(j\omega_s)$, where Z_3 is the impedance looking into the equalizer at the diode port, is a positive-real function and that of $\overline{Z}_3(j\omega_i)$ is a positive function (in the sense of Baum[17]). In the second case, both extensions of $Z_3(j\omega_s)$ and $\overline{Z}_3(j\omega_i)$ are positive functions.

Using the first scheme, the diode impedance at the signal frequency becomes

$$z_D(p) = - \frac{1}{p(p_o - p)_* \, C_1^2 \, Z_{3*}(p_o - p)} \quad , \tag{4-2}$$

where $p_o \equiv j\omega_o$ and $f_*(p) \equiv \overline{f}(-\overline{p})$ for any rational function $f(p)$. It can easily be shown that $Z_{3*}(p_o - p) = \overline{Z}_3(p_o + \overline{p})$ is a positive function. It should be brought to attention that the impedance of a varactor embedded in a passive medium will, in general, not be a real function of p due to the frequency coupling inherent in the parametric devices. It follows that $G(\omega)$ is in general not an even function of ω.

The steady-state scattering formalism described in Section III.2 can be extended to the entire p-plane. In short, the complex normalization technique can be extended to the entire p-plane by defining $a_k(p)$, $b_k(p)$, and $S(p)$ normalized to any prescribed set of rational, non-Foster, positive functions $z_1(p)$, $z_2(p)$, ..., $z_n(p)$ such as to preserve the rationality, the unitary property of a lossless n-port, and the analyticity of $S(p)$ in the closed right-half p-plane. This is summarized as follows:

$$r_k(p) \equiv \frac{z_k(p) + z_{k*}(p)}{2} \quad ,$$

$$r_k(p) = h_{k*}(p)\, h_k(p) \quad , \qquad (4\text{-}3)$$

$$2\, h_k(p)\, a_k(p) = V_k(p) + z_k(p)\, I_k(p) \quad ,$$

$$2\, h_{k*}(p)\, b_k(p) = V_k(p) - z_{k*}(p)\, I_k(p) \quad ,$$

$$\underset{\sim}{b}(p) = S(p)\, \underset{\sim}{a}(p) \quad ,$$

where $r_k(p)$ is said to be paraconjugate, i.e., $r_{k*}(p) = r_k(p)$, and the factorization of $r_k(p)$ is such that $h_k(p)$ and $h_k(p)/h_{k*}(p)$ are analytic for $\mathrm{Re}\, p > 0$. For matched termination at port k, $a_k(p) = 0$, and for paraconjugate termination, $b_k(p) = 0$. And

$$s_{33}(p) = \left[\frac{h_3(p)}{h_{3*}(p)}\right] \left[\frac{Z_3(p) - z_{3*}(p)}{Z_3(p) + z_3(p)}\right] \qquad (4\text{-}4)$$

will be analytic in $\mathrm{Re}\, p \geq 0$ provided the normalization impedances are non-Foster positive functions.

For the varactor device, we shall retain the paraconjugate termination condition at the diode port, i.e.,

$$z_3(p) = -z_{D*}(p) = \frac{1}{p(p_o - p)_* C_1^2 Z_3(p_o - p)} , \tag{4-5}$$

but now $z_3(p)$ is not a positive function. Hence, the analyticity of $s_{33}(p)$ in $\operatorname{Re} p \geq 0$ will not be satisfied, although the factorization of $r_3(p)$ is still valid. We then modify the normalization scheme to

$$\begin{aligned} 2 h_k(p) a_k(p) &= V_k(p) + z_k(p) I_k(p) , \\ 2 h_k(p) b_k(p) &= V_k(p) - z_{k*}(p) I_k(p) . \end{aligned} \tag{4-6}$$

Hence

$$s_{33}(p) = \frac{Z_3(p) + z_D(p)}{Z_3(p) - z_{D*}(p)} = \frac{p(p_o - p)_* C_1^2 - Y_3(p) Y_{3*}(p_o - p)}{p(p_o - p)_* C_1^2 + Y_3(p) Y_3(p_o - p)} , \tag{4-7}$$

which is the analytical continuation of Eq. (3-27). As expected, $s_{33}(p)$ is not regular in the right-half p-plane. The r.h.p. poles of $s_{33}(p)$ are due to the r.h.p. zeros of $Z_3(p) - z_{D*}(p)$ or $p(p_o - p)_* C_1^2 + Y_3(p) Y_3(p_o - p)$. Note that if $p = p_r$ is a zero of $p(p_o - p)_* C_1^2 + Y_3(p) Y_3(p_o - p)$, then $p = -p_r + j\omega_o$ is also a zero. $s_{33}(p)$ may have poles on the $j\omega$-axis (which will lead to a zero of $G(\omega)$), since

$$\omega_r(\omega_o - \omega_r) C_1^2 + Y_3(j\omega_r) Y_3(j\omega_o - j\omega_r) = 0 . \tag{4-8}$$

may have a solution for $\omega_r(\omega_o - \omega_r) \leq 0$.

Stability consideration restricts that

$$Z_3(p) + z_D(p) \neq 0 \tag{4-9}$$

or

$$p(p_o - p)_* C_1^2 - Y_3(p)\, Y_{3*}(p_o - p) \neq 0 \qquad (4\text{-}10)$$

for p in the closed r.h.p. Consequently, the only permissible r.h.p. zeros of $s_{33}(p)$ are those due to the r.h.p. poles of $Y_3(p_o - p)$. Evidently, if $p = p_k$, $\mathrm{Re}\, p_k < 0$ is a zero of $p(p_o - p)_* C_1^2 - Y_3(p)\, Y_{3*}(p_o - p)$, then $p = \bar{p}_k + j\omega_o$ is also a zero. Fig. 8 shows some typical locations for the zeros and poles of $s_{33}(p)$.

Denote the strict r.h.p. poles of $s_{33}(p)$ by p_r and define the rational function $b_o(p)$, which is not real for real p, by

$$b_o(p) = \prod_{r=1}^{n} \frac{p - p_r}{p + \bar{p}_r}, \qquad \mathrm{Re}\, p_r > 0, \qquad r = 1, 2, \ldots, n . \qquad (4\text{-}11)$$

Then $b_o(p)$ is analytic in $\mathrm{Re}\, p \geq 0$ and $b_o(p)\, b_{o*}(p) = 1$. Thus

$$s(p) \equiv b_o(p)\, s_{33}(p) = b_o(p) \left[\frac{p(p_o - p)_* C_1^2 - Y_3(p)\, Y_{3*}(p_o - p)}{p(p_o - p)_* C_1^2 + Y_3(p)\, Y_3(p_o - p)} \right] \qquad (4\text{-}12)$$

is a rational function which is analytic in the strict r.h.p., having the same real frequency magnitude as $s_{33}(p)$. The transducer power gain $G(p)$ is given by

$$G(p) = \left[\frac{p(p_o - p)_* C_1^2 + Y_3(p)\, Y_3(p_o - p)}{p(p_o - p)_* C_1^2 - Y_3(p)\, Y_{3*}(p_o - p)} \right] \left[\frac{p(p_o - p)_* C_1^2 + Y_{3*}(p)\, Y_{3*}(p_o - p)}{p(p_o - p)_* C_1^2 - Y_{3*}(p)\, Y_3(p_o - p)} \right] \qquad (4\text{-}13)$$

and

$$G(p) = \frac{1}{s_{33}(p)\, s_{33*}(p)} = \frac{1}{s(p)\, s_*(p)}, \qquad (4\text{-}14)$$

valid in the entire p-plane.

IV.3 Realizability Conditions and Factorizations

We have shown that the transducer power gain of a lossless, nonreciprocal, 3-port equalizer operating between a resistive generator and load with the third port terminated by the varactor is completely determined by the real-frequency magnitude of the rational function

$$s_{33}(p) = \frac{Z_3(p) + z_D(p)}{Z_3(p) - z_{D*}(p)} = \frac{p(p_o - p)_* C_1^2 - Y_3(p)\, Y_{3*}(p_o - p)}{p(p_o - p)_* C_1^2 + Y_3(p)\, Y_3(p_o - p)} , \qquad (4\text{-}15)$$

since $G(\omega) = 1/|s_{33}(j\omega)|^2$. The purpose of this section is to delineate the realizability conditions of a prescribed transducer power gain function for this varactor parametric amplifier.

For $p = j\omega$,

$$|s_{33}(j\omega)|^2 = 1 - \frac{4\omega(\omega_o - \omega)\, C_1^2 \ \mathrm{Re}\left[Y_3(j\omega)\right] \ \mathrm{Re}\left[Y_3(j\omega_o - j\omega)\right]}{\left|\omega(\omega_o - \omega)\, C_1^2 + Y_3(j\omega)\, Y_3(j\omega_o - j\omega)\right|^2} . \qquad (4\text{-}16)$$

Because of the positive character of $Y_3(p)$ and $Y_{3*}(p_o - p)$,

$$\begin{aligned} |s_{33}(j\omega)| &\leq 1 , \quad \text{when} \quad \omega(\omega_o - \omega) \geq 0 , \\ |s_{33}(j\omega)| &\geq 1 , \quad \text{when} \quad \omega(\omega_o - \omega) \leq 0 . \end{aligned} \qquad (4\text{-}17)$$

This stems from the fact that $\mathrm{Re}\, z_3(j\omega) \geq 0$ for $\omega(\omega_o - \omega) \geq 0$ and $\mathrm{Re}\, z_3(j\omega) \leq 0$ for $\omega(\omega_o - \omega) \leq 0$ (see Eq. (3-17)). Physically, when the signal frequency equals or exceeds the pump, there will be no gain. Consequently,

$$\begin{aligned} G(\omega) &\geq 1 , \quad \text{when} \quad \omega(\omega_o - \omega) \geq 0 , \\ G(\omega) &< 1 , \quad \text{when} \quad \omega(\omega_o - \omega) \leq 0 . \end{aligned} \qquad (4\text{-}18)$$

Furthermore, since $G(\omega) = G(\omega_o - \omega)$, the prescribed transducer power gain function must be symmetrical with respect to $\omega_o/2$. Fig. 9 shows explicitly the permissible regions for the magnitude $G(\omega)$. Transition points at $\omega = 0$ and $\omega = \omega_o$ and a possible shape satisfying the symmetry condition are also presented.

From Eq. (4-13), we obtain

$$G(p) = \frac{1}{s_{33}(p)\, s_{33*}(p)} = \left[\frac{p(p_o - p)_* C_1^2 + Y_3(p)\, Y_3(p_o - p)}{p(p_o - p)_* C_1^2 - Y_3(p)\, Y_{3*}(p_o - p)}\right]\left[\frac{p(p_o - p)_* C_1^2 + Y_{3*}(p)\, Y_{3*}(p_o - p)}{p(p_o - p)_* C_1^2 - Y_{3*}(p)\, Y_3(p_o - p)}\right] \tag{4-19}$$

$$G(p) = 1 + \left\{\frac{p(p_o - p)_* C_1^2 \left[Y_3(p) + Y_{3*}(p)\right]\left[Y_3(p_o - p) + Y_{3*}(p_o - p)\right]}{\left[p(p_o - p)_* C_1^2 - Y_3(p)\, Y_{3*}(p_o - p)\right]\left[p(p_o - p)_* C_1^2 - Y_{3*}(p)\, Y_3(p_o - p)\right]}\right\} \tag{4-20}$$

$$1 - \frac{1}{G(p)} = \left\{\frac{p(p_o - p)_* C_1^2 \left[Y_3(p) + Y_{3*}(p)\right]\left[Y_3(p_o - p) + Y_{3*}(p_o - p)\right]}{\left[p(p_o - p)_* C_1^2 + Y_3(p)\, Y_3(p_o - p)\right]\left[p(p_o - p)_* C_1^2 + Y_{3*}(p)\, Y_{3*}(p_o - p)\right]}\right\}. \tag{4-21}$$

The above expressions infer immediately that

$$G(p) = G_*(p) \quad , \tag{4-22}$$

and

$$G(p) = G_*(p_o - p) \quad . \tag{4-23}$$

Furthermore, it can be shown from Eq. (4-21) that, for $p = \sigma + j\omega_o/2$,

$$1 - \frac{1}{G(\sigma + \frac{j\omega_o}{2})} = \frac{\left[\sigma^2 + (\frac{\omega_o}{2})^2\right] C_1^2 \left|Y_3(\sigma + \frac{j\omega_o}{2}) + \overline{Y}_3(-\sigma + \frac{j\omega_o}{2})\right|^2}{\left|\left[\sigma^2 + (\frac{\omega_o}{2})^2\right] C_1^2 + Y_3(\sigma + \frac{j\omega_o}{2})\, Y_3(-\sigma + \frac{j\omega_o}{2})\right|^2} \tag{4-24}$$

Therefore,

$$1 - \frac{1}{G(\sigma + \frac{j\omega_o}{2})} \geq 0 \qquad (4\text{-}25)$$

or, alternatively,

$$\left[G(\sigma + \frac{j\omega_o}{2})\right]^{-1} \leq 1 \quad , \qquad (4\text{-}26)$$

for all real σ, and zeros of $1 - [1/G(p)]$ on the $p = \sigma + (j\omega_o/2)$ axis must be of even multiplicity.

To sum up, we have shown that a realizable $G(\omega)$ must satisfy the following conditions:

a) $G(\omega)$ rational;

b) $G(\omega)$ real function of ω;

c) $G(\omega)$ symmetrical with respect to $\omega_o/2$;

d) $G(\omega) \geq 0$ for all real ω;

e) $G(\omega) \geq 1$ for $\omega(\omega_o - \omega) \geq 0$,
$G(\omega) \leq 1$ for $\omega(\omega_o - \omega) \leq 0$;

f) $[G(\sigma + j\omega_o/2)]^{-1} \leq 1$ for all real σ.

From b), $G(j\omega) = \overline{G}(j\omega)$; hence, $G(p) = G_*(p)$, i.e., $G(p) = G_*(p)$ if and only if $G(\omega)$ is a real function of ω. From the symmetry condition c), $G(j\omega + (j\omega_o/2)) = G(-j\omega + (j\omega_o/2)) = \overline{G}(-j\omega + (j\omega_o/2))$, or $G(j\omega) = \overline{G}(-j\omega + (j\omega_o/2))$. By analytic continuation, $G(p) = G_*(p_o - p)$; hence, $G(p) = G_*(p_o - p)$ if and only if $G(\omega)$ is symmetrical with respect to $\omega_o/2$.

Since $G(p)$ satisfies $G_*(p) \equiv \overline{G}(-\overline{p}) = G(p)$, $G(p)$ is paraconjugate. From d), any zero of $G(p)$ on the real-frequency axis must be of even multiplicity. By factoring its numerator and denominator polynomials into products of Hurwitz and anti-Hurwitz factors, it is possible to write

$$G(p) = \frac{1}{s_o(p)\, s_{o*}(p)} \quad , \tag{4-27}$$

where $s_o(p)$ is rational and analytic in $\mathrm{Re}\ p \geq 0$ and $s_o^{-1}(p)$ is analytic in $\mathrm{Re}\ p > 0$. The factorized function $s_o(p)$ is uniquely determined up to a constant scalar multiple $e^{j\theta}$. The most general rational solution $s(p)$ of the equation $G(p) = 1/s(p)\, s_*(p)$ is given by

$$s(p) = b(p)\, s_o(p) \quad , \tag{4-28}$$

$b(p)$ being a Blaschke product; $b(p)$ is regular if and only if $s(p)$ is analytic in $\mathrm{Re}\ p > 0$.

Using similar reasoning, we can show from conditions c) and f) that $1 - [1/G(p)]$ admits a factorization

$$1 - \frac{1}{G(p)} = \eta_o(p)\, \eta_{o*}(p_o - p) \quad , \tag{4-29}$$

where $\eta_o(p)$ is rational and analytic together with its inverse in the half plane below the axis $p = \sigma + j\omega_o/2$. The factorized function $\eta_o(p)$ is uniquely determined up to a constant scalar multiple $e^{j\phi}$ and the most general rational solution $\eta(p)$ of the equation $1 - [1/G(p)] = \eta(p)\, \eta_*(p_o - p)$ is given by

$$\eta(p) = d(p)\, \eta_o(p) \quad , \tag{4-30}$$

$d(p)$ being a modified Blaschke product defined by $d(p)\, d_*(p_o - p) = 1$. Moreover, $d(p)$ may always be represented in the form

$$d(p) = e^{j\theta} \prod_{r=1}^{n} \frac{p - p_r}{p + (-p_o - \bar{p}_r)} \quad , \tag{4-31}$$

and $|d(\sigma + j\omega_o/2)|^2 = 1$; hence, $d(p)$ is automatically analytic on the entire $p = \sigma + j\omega_o/2$ axis.

IV.4 Broadbanding and Inverse Theorem for the General Case

We can solve for $Z_3(p)$ from the expression (4-12) for $s(p)$ as

$$Z_3(p) = \frac{b_o(p)\left[Y_3(p_o - p) + Y_{3*}(p_o - p)\right]}{p(p_o - p)_* C_1^2 \left[b_o(p) - s(p)\right]} - \frac{Y_3(p_o - p)}{p(p_o - p)_* C_1^2} . \quad (4\text{-}32)$$

Now, the above expression for varactor diode corresponds to the passive load case[16] with $z_\ell(p)$ substituted by $Y_3(p_o - p)/p(p_o - p)_* C_1^2$. In the passive load $z_\ell(p)$ is prescribed, whereas here it is directly related to the equalizer impedance to be designed. Hence, instead of using the above expression, we must provide other means to solve the synthesis or inverse problem.

To solve for $Y_3(p)$ in terms of the transducer power gain function $G(p) = 1/s_{33}(p)\, s_{33*}(p) = 1/s(p)\, s_*(p)$, we proceed from Eq. (4-12) as follows:

$$b_o(p) - s(p) = b_o(p) \left\{ \frac{Y_3(p)\left[Y_3(p_o - p) + Y_{3*}(p_o - p)\right]}{p(p_o - p)_* C_1^2 + Y_3(p)\, Y_3(p_o - p)} \right\} , \quad (4\text{-}33)$$

$$b_{o*}(p_o - p) - s_*(p_o - p) = b_{o*}(p_o - p) \left\{ \frac{Y_{3*}(p_o - p)\left[Y_3(p) + Y_{3*}(p)\right]}{p(p_o - p)_* C_1^2 + Y_{3*}(p)\, Y_{3*}(p_o - p)} \right\} \quad (4\text{-}34)$$

and, when multiplying Eqs. (4-33) and (4-34), and dividing by $1 - \left[1/G(p)\right] = 1 - s(p)\, s_*(p)$ of Eq. (4-21), we have

$$\frac{Y_3(p)\, Y_{3*}(p_o - p)}{p(p_o - p)_* C_1^2} = \frac{\left[b_o(p) - s(p)\right]\left[b_{o*}(p_o - p) - s_*(p_o - p)\right]}{b_o(p)\, b_{o*}(p_o - p)\left[1 - s(p)\, s_*(p)\right]} . \quad (4\text{-}35)$$

Now,

$$1 - \frac{1}{G(p)} = 1 - s(p)\, s_*(p) = \eta(p)\, \eta_*(p_o - p) \quad , \tag{4-36}$$

where, from (4-28) and (4-30), $s(p) = b(p)\, s_o(p)$ and $\eta(p) = d(p)\, \eta_o(p)$. $s(p)$ is defined by Eq. (4-12) and $\eta(p)$ is given by

$$\eta(p) = b_o(p) \left\{ \frac{p\, C_1 \left[Y_3(p_o - p) + Y_{3*}(p_o - p) \right]}{p(p_o - p)_* C_1^2 + Y_3(p)\, Y_3(p_o - p)} \right\} \quad . \tag{4-37}$$

Therefore,

$$Y_3(p) = p\, C_1 \frac{\left[b_o(p) - s(p) \right]}{\eta(p)} \quad . \tag{4-38}$$

Equations (4-35) and (4-38) serve as the starting point of the broadbanding theory to be presented.

First of all, we can ascertain quite easily that $Y_3(p)$ is real for real p if and only if

$$\frac{b_o(p) - s(p)}{b_{o*}(-p) - s_*(-p)} = \frac{\eta(p)}{\eta_*(-p)} \quad , \tag{4-39}$$

and that $\mathrm{Re}\, Y_3(j\omega) \geq 0$ for all real ω if and only if

$$j\omega \left\{ \left[b_o(j\omega) - s(j\omega) \right] \overline{\eta}(j\omega) - \left[b_o(j\omega) - \overline{s}(j\omega) \right] \eta(j\omega) \right\} \geq 0 \quad , \tag{4-40}$$

for all real ω.

From Eq. (4-7), and using $s(p) = b_o(p)\, s_{33}(p)$,

$$b_o(p) - s(p) = b_o(p) \left\{ \frac{-\left[z_D(p) + z_{D*}(p) \right]}{Z_3(p) - z_{D*}(p)} \right\} \quad , \tag{4-41}$$

we see that every pole of $z_{D*}(p)$ on $\text{Re } p = 0$ and every zero of $\left[z_D(p) + z_{D*}(p)\right]$ in $\text{Re } p \geq 0$ must be a zero of $b_o(p) - s(p)$. It will be shown later that $b_o(p) - s(p)$ has only one intrinsic zero at infinity due to the presence of the d.c. capacitance C_o at the diode port, all the other zeros being characteristic of the combined effects of the equalizer and the varactor diode load. Similarly, from Eq.(4-37), zeros of $\eta(p)$ in $\text{Re } p \geq 0$ are restricted.

Define a function $\lambda(p)$ by

$$\lambda(p) \equiv b_o(p) \frac{\left[Y_3(p_o - p) + Y_{3*}(p_o - p)\right]}{p(p_o - p)_* C_1^2 + Y_3(p) Y_3(p_o - p)} , \qquad (4\text{-}42)$$

and denote any zero of $\lambda(p)$ in $\text{Re } p \geq 0$, $p_\alpha = \sigma_\alpha + j\omega_\alpha$, $\sigma_\alpha \geq 0$, of multiplicity k_α, as a transmission zero of the device of order k_α. Then each zero of transmission imposes certain restrictions on $s(p)$ and $\eta(p)$. It will be shown that the restrictions on $s(p)$ and $\eta(p)$ at the transmission zero at infinity, which in turn is reflected on $G(p)$, lead to the gain-bandwidth limitation. At present, we can conclude that the multiplicity of transmission zero at infinity is at least one, due to the shunt capacitance C_o.

In essence, we are given a prescribed transducer power gain function satisfying the realizability conditions we have presented earlier; we wish to delineate further conditions such that $Y_3(p)$ given by Eq.(4-38) would be positive-real with the prescribed residue at infinity. Although the function $\lambda(p)$ is not known from $G(p)$, all the information on the location and multiplicity of the transmission zeros, p_α, is specified from the prescribed transducer power gain function (see Eq.(4-21)). It should be noted that $b_o(p)$ as defined previously is the regular Blaschke product formed with the strict r.h.p. poles of $s_{33}(p)$. But now $b_o(p)$ as defined originally cannot be obtained from the prescribed data; instead, we shall specify $b_o(p)$ to be formed from the poles of $G(p)$ in the upper right quadrant bounded by the $p = j\omega$ and $p = \sigma + j\omega_o/2$ axis. Then $b_o(p)$ is uniquely determined, but it is, in general, not the Blaschke product that renders $s_{33}(p)$ analytic in $\text{Re } p > 0$.

Each zero of transmission imposes certain restrictions on $s(p)$ and $\eta(p)$ which are reflected onto $G(\omega)$. To formulate these restrictions quantitatively, let us represent the power series expansions of $b_o(p)$, $s(p)$, $\eta(p)$ and $\lambda(p)$ about p_α by

$$b_o(p) = \sum_{i=0}^{\infty} B_i^\alpha (p - p_\alpha)^i ,$$

$$s(p) = \sum_{i=0}^{\infty} A_i^\alpha (p - p_\alpha)^i ,$$

$$\eta(p) = \sum_{i=0}^{\infty} N_i^\alpha (p - p_\alpha)^i , \qquad (4\text{-}43)$$

and

$$\lambda(p) = \sum_{i=0}^{\infty} \lambda_i^\alpha (p - p_\alpha)^i ,$$

where $|p_\alpha|$ is finite. In addition, represent

$$s_o(p) = \sum_{i=0}^{\infty} a_i^\alpha (p - p_\alpha)^i ,$$

$$b(p) = \sum_{i=0}^{\infty} b_i^\alpha (p - p_\alpha)^i ,$$

$$\eta_o(p) = \sum_{i=0}^{\infty} n_i^\alpha (p - p_\alpha)^i , \qquad (4\text{-}44)$$

and

$$d(p) = \sum_{i=0}^{\infty} d_i^\alpha (p - p_\alpha)^i .$$

For $|p_\alpha| = \infty$, $(p - p_\alpha)$ is to be replaced by $1/p$ and coefficients are A_i^∞, B_i^∞, N_i^∞, etc. From $s(p) = b(p)\, s_o(p)$ and $\eta(p) = d(p)\, \eta_o(p)$, then A_i^α and N_i^α can be represented as Cauchy products as follows:

$$A_i^\alpha = \sum_{r=0}^{i} a_r^\alpha b_{i-r}^\alpha \quad , \tag{4-45}$$

and

$$N_i^\alpha = \sum_{r=0}^{i} n_r^\alpha d_{i-r}^\alpha \quad . \tag{4-46}$$

From the prescribed $G(\omega)$, B_i^α, a_i^α and n_i^α are known, and b_i^α and d_i^α are to be determined using the restrictions on the coefficients to be derived.

We now formulate the restrictions according to the location of the transmission zeros:

1. Transmission zeros in the strict r.h.p. of order k_α, $\sigma_\alpha > 0$ (due to the strict r.h.p. zeros of $\left[Y_3(p_o - p) + Y_{3*}(p_o - p)\right]$ of multiplicity k_α):

$$\sum_{i=0}^{\infty} B_i^\alpha (p - p_\alpha)^i - \sum_{i=0}^{\infty} A_i^\alpha (p - p_\alpha)^i = Y_3(p_\alpha) \sum_{i=0}^{\infty} \lambda_i^\alpha (p - p_\alpha)^i \quad ,$$

$$\sum_{i=0}^{\infty} N_i^\alpha (p - p_\alpha)^i = p_\alpha C_1 \sum_{i=0}^{\infty} \lambda_i^\alpha (p - p_\alpha)^i \quad ,$$

$$\lambda_i^\alpha = 0 \quad , \qquad i = 0, 1, \ldots, k_\alpha - 1 \quad .$$

Therefore,

$$\begin{aligned} B_i^\alpha &= A_i^\alpha \quad , \qquad i = 0, 1, \ldots, k_\alpha - 1 \quad , \\ N_i^\alpha &= 0 \quad , \qquad i = 0, 1, \ldots, k_\alpha - 1 \quad . \end{aligned} \tag{4-47}$$

2. Transmission zeros at $p = j\omega_\alpha$ of order k_α, $|\omega_\alpha| < \infty$.

(a) If $Y_3(p)$ has a zero at $p = j\omega_\alpha$, then

$$\sum_{i=0}^{\infty} B_i^\alpha (p - j\omega_\alpha)^i - \sum_{i=0}^{\infty} A_i^\alpha (p - j\omega_\alpha)^i = \left[\beta'(\omega_\alpha)(p - j\omega_\alpha)\right] \sum_{i=0}^{\infty} \lambda_i^\alpha (p - j\omega_\alpha)^i \quad ,$$

$$\sum_{i=0}^{\infty} N_i^\alpha (p - j\omega_\alpha)^i = (j\omega_\alpha)\, C_1 \sum_{i=0}^{\infty} \lambda_i^\alpha (p - j\omega_\alpha)^i \quad ,$$

$$\lambda_i^\alpha = 0 \quad , \qquad i = 0, 1, \ldots, k_\alpha - 1 \quad ,$$

where $\lim_{p \to j\omega_\alpha} Y_3(p)/(p - j\omega_\alpha) = \beta'(\omega_\alpha)$. Therefore,

$$\begin{aligned} B_i^\alpha &= A_i^\alpha \quad , \qquad i = 0, 1, \ldots, k_\alpha \quad , \\ N_i^\alpha &= 0 \quad , \qquad i = 0, 1, \ldots, k_\alpha - 1 \quad ; \end{aligned} \tag{4-48}$$

in addition, by dividing

$$B_{k_\alpha+1}^\alpha - A_{k_\alpha+1}^\alpha = \beta'(\omega_\alpha)\, \lambda_{k_\alpha}^\alpha \quad ,$$

$$N_{k_\alpha}^\alpha = j\omega_\alpha\, C_1\, \lambda_{k_\alpha}^\alpha \quad ,$$

one obtains

$$j\omega_\alpha \left[\frac{B_{k_\alpha+1}^\alpha - A_{k_\alpha+1}^\alpha}{N_{k_\alpha}^\alpha} \right] = \frac{\beta'(\omega_\alpha)}{C_1} > 0 \quad . \tag{4-49}$$

(b) If $Y_3(p)$ has neither a pole nor a zero at $p = j\omega_\alpha$, then

$$B_i^\alpha = A_i^\alpha \quad , \qquad i = 0, 1, \dots, k_\alpha - 1 \quad ,$$
$$N_i^\alpha = 0 \quad , \qquad i = 0, 1, \dots, k_\alpha - 1 \quad . \tag{4-50}$$

(c) If $Y_3(p)$ has a pole at $j\omega_\alpha$, then

$$\sum_{i=0}^{\infty} B_i^\alpha (p - j\omega_\alpha)^i - \sum_{i=0}^{\infty} A_i^\alpha (p - j\omega_\alpha)^i = \left[\frac{\alpha(\omega_\alpha)}{p - j\omega_\alpha}\right] \sum_{i=0}^{\infty} \lambda_i^\alpha (p - j\omega_\alpha)^i \quad ,$$

$$\sum_{i=0}^{\infty} N_i^\alpha (p - j\omega_\alpha)^i = (j\omega_\alpha)\, C_1 \sum_{i=0}^{\infty} \lambda_i^\alpha (p - j\omega_\alpha)^i \quad ,$$

$$\lambda_i^\alpha = 0 \quad , \qquad i = 0, 1, \dots, k_\alpha - 1 \quad ,$$

where $\alpha(\omega_\alpha)$ is the residue of $Y_3(p)$ at the pole. Therefore,

$$B_i^\alpha = A_i^\alpha \quad , \qquad i = 0, 1, \dots, k_\alpha - 2 \quad ,$$
$$N_i^\alpha = 0 \quad , \qquad i = 0, 1, \dots, k_\alpha - 1 \quad ; \tag{4-51}$$

in addition, by dividing

$$B_{k_\alpha - 1}^\alpha - A_{k_\alpha - 1}^\alpha = \alpha(\omega_\alpha)\, \lambda_{k_\alpha}^\alpha \quad ,$$

$$N_{k_\alpha}^\alpha = (j\omega_\alpha)\, C_1\, \lambda_k^\alpha \quad ,$$

one obtains

$$j\omega_\alpha \left[\frac{B^\alpha_{k_\alpha} - A^\alpha_{k_\alpha - 1}}{N^\alpha_{k_\alpha}} \right] = \frac{\alpha(\omega_\alpha)}{C_1} > 0 \quad . \tag{4-52}$$

From the prescribed transducer power gain function, we would not know, a priori, which of the above cases will apply. It can be shown, by examining conditions (4-48), (4-49), (4-50), (4-51) and (4-52), that the following restrictions

$$B^\alpha_i = A^\alpha_i \quad , \qquad N^\alpha_i = 0 \quad ,$$

and (4-53)

$$j\omega_\alpha \left[\frac{B^\alpha_{k_\alpha - 1} - A^\alpha_{k_\alpha - 1}}{N^\alpha_{k_\alpha}} \right] \geq 0$$

will suffice.

3. Transmission zero at infinity of order k_∞ :

$$\sum_{i=0}^{\infty} B^\infty_i \left(\frac{1}{p}\right)^i - \sum_{i=0}^{\infty} A^\infty_i \left(\frac{1}{p}\right)^i = p\,C_o \sum_{i=0}^{\infty} \lambda^\infty_i \left(\frac{1}{p}\right)^i \quad ,$$

$$\sum_{i=0}^{\infty} N^\infty_i \left(\frac{1}{p}\right)^i = p\,C_1 \sum_{i=0}^{\infty} \lambda^\infty_i \left(\frac{1}{p}\right)^i \quad ,$$

$$\lambda^\infty_i = 0 \quad , \qquad i = 0, 1, \ldots, k_\infty - 1 \quad .$$

Therefore,

$$B^\infty_i = A^\infty_i \quad , \qquad i = 0, 1, \ldots, k_\infty - 2 \quad ,$$

$$N^\infty_i = 0 \quad , \qquad i = 0, 1, \ldots, k_\infty - 2 \quad ; \tag{4-54}$$

in addition, by dividing

$$B^{\infty}_{k_{\infty}-1} - A^{\infty}_{k_{\infty}-1} = C_o \lambda^{\infty}_{k_{\infty}} ,$$

$$N^{\infty}_{k_{\infty}-1} = C_1 \lambda^{\infty}_{k_{\infty}} ,$$

one obtains

$$\frac{B^{\infty}_{k_{\infty}-1} - A^{\infty}_{k_{\infty}-1}}{N^{\infty}_{k_{\infty}-1}} = \frac{C}{C_1} \geq \frac{C_o}{C_1} , \qquad (4\text{-}55)$$

where $C \geq C_o$.

Conditions (4-47), (4-53) and (4-54), when coupled with relations (4-45) and (4-46), can be used to determine $b(p)$ and $d(p)$. Thus, by using the above information, we can obtain the gain-bandwidth restriction from the condition (4-55) and also, at the same time, completely solve the inverse problem starting from the prescribed transducer power gain function. In summary, the necessary and sufficient conditions are presented in Theorems 1 and 2.

Theorem 1:

Let $z_D(p) = -1/p(p_o - p)_* C_1^2 Z_{3*}(p_o - p)$ be the impedance of a varactor diode in an inverting parametric device, where $p_o \equiv j\omega_o$ is the pump frequency and $Y_3(p)$ is the input admittance looking into the equalizer at signal frequency at the diode port with a d.c. capacitance C_o shunting the port; and let $b_o(p)$ be a regular all-pass formed with half of the right half plane poles of $1/G(p)$ in the upper right quadrant bounded by the $p = j\omega$ and $p = \sigma + (j\omega_o/2)$ axis. Then Eqs.(4-33) and (4-37) define a rational positive-real function $Y_3(p)$ with residue at infinity not less than the prescribed C_o if and only if:

1) $s(p)$ and $\eta(p)$ are rational functions of p satisfying the following conditions:

(a) $\quad |s(j\omega)| \leq 1 \quad$ for $\quad \omega(\omega_o - \omega) \geq 0$,

$\quad |s(j\omega)| \geq 1 \quad$ for $\quad \omega(\omega_o - \omega) \leq 0$;

(b)
$$\frac{b_o(p) - s(p)}{b_{o*}(-p) - s_*(-p)} = \frac{\eta(p)}{\eta_*(-p)} \quad ;$$

(c)
$$j\omega \left\{ \left[b_o(j\omega) - s(j\omega) \right] \bar{\eta}(j\omega) - \left[\bar{b}_o(j\omega) - \bar{s}(j\omega) \right] \eta(j\omega) \right\} \geq 0$$

for all real ω.

2) Any strict right half-plane transmission zero, $p_\alpha = \sigma_\alpha + j\omega_\alpha$, $\sigma_\alpha > 0$, of multiplicity k_α must be a zero of both $b_o(p) - s(p)$ and $\eta(p)$ of at least order k_α; i.e.,

$$B_i^\alpha = A_i^\alpha \quad , \qquad (i = 0, 1, \ldots, k_\alpha - 1) \quad ,$$

$$N_i^\alpha = 0 \quad , \qquad (i = 0, 1, \ldots, k_\alpha - 1) \quad .$$

3) Any finite $j\omega$ transmission zero, $p_\alpha = j\omega_\alpha$, $|\omega_\alpha| < \infty$, of multiplicity k_α must be a zero of both $b_o(p) - s(p)$ and $\eta(p)$ of at least order k_α; i.e.,

$$B_i^\alpha = A_i^\alpha \quad , \qquad (i = 0, 1, \ldots, k_\alpha - 2) \quad ,$$

$$N_i^\alpha = 0 \quad , \qquad (i = 0, 1, \ldots, k_\alpha - 1) \quad ,$$

and, in addition,

$$j\omega_\alpha \left[\frac{B_{k_\alpha - 1}^\alpha - A_{k_\alpha - 1}^\alpha}{N_{k_\alpha}^\alpha} \right] \geq 0 \quad .$$

4) The transmission zero at $p = \infty$ of multiplicity k_∞ must be a zero of both $b_o(p) - s(p)$ and $\eta(p)$ of at least order $k_\infty - 1$, i.e.,

$$B_i^\infty = A_i^\infty \;, \qquad (i = 0, 1, \ldots, k_\infty - 2) \;,$$

$$N_i^\infty = 0 \;, \qquad (i = 0, 1, \ldots, k_\infty - 2) \;,$$

and, furthermore,

$$\frac{B^\infty_{k_\infty - 1} - A^\infty_{k_\infty - 1}}{N^\infty_{k_\infty - 1}} = \frac{C}{C_1} \geq \frac{C_o}{C_1} \;,$$

where $C > C_o$.

Proof: Necessity: already shown.

Sufficiency: Using Eqs.(4-33) and (4-37), we have

$$Y_3(p) = p\, C_1 \,\frac{b_o(p) - s(p)}{\eta(p)} \;. \tag{4-38}$$

From condition 1), we know that $Y_3(p)$ is rational. It follows from 1)(b) that $Y_3(p) = \overline{Y}(\bar{p})$; thus the reality of $Y_3(p)$ is established. And from 1)(c) we see that Eq. (4-38) always defines a $Y_3(p)$ with non-negative real part on $p = j\omega$.

Conditions 2), 3) and 4) guarantee that $Y_3(p)$ is analytic in Re $p > 0$ and has, at most, simple poles on the finite $j\omega$-axis with a non-negative residue, and a simple pole at $p = \infty$ with at least the prescribed residue C_o.

Theorem 2:

The real rational function of frequency $G(\omega)$, which is symmetrical with respect to $\omega_o/2$, is realizable as the transducer power gain of an inverting parametric device depicted in Fig. 7 if and only if:

1) (a) $G(\omega) \geq 0$ for all real ω,

(b) $G(\omega) \geq 1$ for $\omega(\omega_o - \omega) \geq 0$,

$G(\omega) \leq 1$ for $\omega(\omega_o - \omega) \leq 0$,

(c) $\left[G(\sigma + \frac{p_o}{2})\right]^{-1} \leq 1$ for all real σ;

2) The function $1/G(p)$ admits a factorization of the form

$$\frac{1}{G(p)} = s(p)\, s_*(p) \quad ,$$

and the function $1 - |1/G(p)|$ admits a factorization of the form

$$1 - \frac{1}{G(p)} = \eta(p)\, \eta_*(p_o - p)$$

in which $s(p)$ and $\eta(p)$ satisfy conditions 1) - 4) of Theorem 1.

Proof: Necessity: already shown.

Sufficiency: Given conditions 1) and 2), from Theorem 1 the function $Y_3(p)$ as defined by (4-38) is positive-real with a residue at infinity not less than C_o. According to Darlington's classical theorem, $Z_3(p) = Y_3^{-1}(p)$ may be synthesized as the input impedance of a lossless 2-port terminated in a resistor.

IV.5 Broadbanding and Inverse Theorem for the Case $\omega_s \omega_i \cong \omega_{so} \omega_{io}$ and $Y_3(j\omega_s) \cong \overline{Y}_3(j\omega_i)$

In this section we shall treat the case where the assumptions that $\omega_s \omega_i \cong \omega_{so} \omega_{io}$ and $Y_3(j\omega_s) \cong \overline{Y}_3(j\omega_i)$ hold. A more complete treatment of this case is presented in Chapter V, where the exact gain-bandwidth limitation is obtained directly from the prescribed gain function without resorting to the factorization technique essential for the general case. The purpose here is to demonstrate that the broadbanding theory described in the previous section is not just a solution of the inverse problem in principle, but actually can be employed to determine the gain-bandwidth limitation and to obtain the synthesis for broadband parametric amplifiers. Since the procedure for this case is analogous to that of the general case, we shall summarize the pertinent results here.

Invoking both the assumptions, $\omega_s \omega_i \cong \omega_{so} \omega_{io}$ and $Y_3(j\omega_s) \cong \overline{Y}_3(j\omega_i)$, $|s_{33}(j\omega)|^2 \leq 1$ and $G(\omega^2) = 1/|s_{33}(j\omega)|^2$ is an even function of ω. Letting $Y = Y_3/\sqrt{\omega_{so}\omega_{io}}\, C_1$,

$$s_{33}(p) = \frac{\omega_{so}\omega_{io} C_1^2 - Y_3^2(p)}{\omega_{so}\omega_{io} C_1^2 + Y_3(p) Y_{3*}(p)} = \frac{1 - Y^2(p)}{1 + Y(p) Y_*(p)} , \tag{4-56}$$

where $Y_*(p) \equiv Y(-p)$ since $Y(p)$ is positive-real. From stability considerations, $Y(p) \neq 1$ for $\text{Re } p \geq 0$; hence, the only permissible r.h.p. zeros of $s_{33}(p)$ are due to the r.h.p. poles of $Y_*(p)$. Moreover, we note that $1 + Y(p) Y_*(p) \neq 0$ on the entire $p = j\omega$ axis. The strict r.h.p. poles of $s_{33}(p)$ are due to the strict r.h.p. zeros of the even function $1 + Y(p) Y_*(p)$.

Denote the strict r.h.p. poles of $s_{33}(p)$ by $p = p_r$, $r = 1, 2, \ldots, n$, and define the regular Blaschke product (real for real p)

$$b_o(p) = \prod_{r=1}^{n} \frac{p - p_r}{p + \overline{p}_r} . \tag{4-57}$$

Thus

$$s(p) \equiv b_o(p)\, s_{33}(p) = b_o(p) \left[\frac{1 - Y^2(p)}{1 + Y(p)\, Y_*(p)}\right] \tag{4-58}$$

is a real rational passive scattering coefficient. Furthermore,

$$1 - \frac{1}{G(-p^2)} = 1 - s(p)\, s_*(p) = \left[\frac{Y(p) + Y_*(p)}{1 + Y(p)\, Y_*(p)}\right]^2 \tag{4-59}$$

and

$$b_o(p) - s(p) = b_o(p)\, Y(p) \left\{\frac{[Y(p) + Y_*(p)]}{1 + Y(p)\, Y_*(p)}\right\} = Y(p) \left\{b_o(p) \sqrt{1 - \frac{1}{G(-p^2)}}\right\} = Y(p) \left\{b_o(p)\, \lambda(p)\right\} \tag{4-60}$$

where $\lambda(p)$ is defined by

$$\lambda(p) \equiv \sqrt{1 - \frac{1}{G(-p^2)}} \quad . \tag{4-61}$$

Note that every zero of $\lambda(p)$ in $\text{Re}\, p \geq 0$ must be a zero of $b_o(p) - s(p)$. Denote any zero of $\lambda(p)$ in $\text{Re}\, p \geq 0$, $p_\alpha = \sigma_\alpha + j\omega_\alpha$, $\sigma_\alpha \geq 0$, of multiplicity k_α as a transmission zero of the device of order k_α. It should be noted from Eq. (4-59) that $1 - [1/G(-p^2)]$ is a perfect square of an even function of p; the square root operation corresponds to the factorization of $1 - [1/G(p)] = \eta(p)\, \eta_*(p_o - p)$ in the general case. In this special case, in addition, the function $\lambda(p)$ can be obtained from the prescribed $G(p)$.

As in the general case, we can only consider the zeros of transmission of the device since the equalizer and the varactor diode (which takes the place of the load in the passive matching problem) are completely tied together. All the transmission zeros are due to the combined effects of the equalizer and the varactor diode except the transmission zero at infinity where the multiplicity is at least one due to the presence of the shunt capacitance C_o. In contrast with the general case, now $b_o(p)$ is uniquely determined up to a plus or minus sign from the prescribed $G(\omega^2)$; $b_o(p)$ being a real regular Blaschke product formed from the strict r.h.p. poles of $1/G(-p^2)$ with half of their respective multiplicities.

Represent the power series expansion of $b_o(p)$, $s(p)$, $\lambda(p)$, $s_o(p)$ and $b(p)$ in the neighborhood of the zero of transmission, $p_\alpha = \sigma_\alpha + j\omega_\alpha$, as in Eqs.(4-43) and (4-44). From $s(p) = b(p)\, s_o(p)$, where $s_o(p)$ is the minimum factorization of $1/G(-p^2)$, the coefficients A_i^α of $s(p)$ can be expressed in terms of

$$A_i^\alpha = \sum_{r=0}^{i} a_r^\alpha b_{i-r}^\alpha \quad . \qquad (4\text{-}62)$$

From the prescribed $G(\omega^2)$, λ_i^α, B_i^α and a_i^α are known, and b_i^α would be determined using the conditions on the coefficients to be derived.

The restrictions are summarized as follows:

1. Transmission zeros in the strict r.h.p. of order k_α, $\sigma_\alpha > 0$.

$$B_i^\alpha = A_i^\alpha \ , \qquad i = 0, 1, \ldots, k_\alpha \quad . \qquad (4\text{-}63)$$

2. Transmission zeros at $p = j\omega_\alpha$ of order k_α, $|\omega_\alpha| < \infty$.

(a) If $Y(p)$ has a zero at $p = j\omega_\alpha$, then

$$B_i^\alpha = A_i^\alpha \ , \qquad i = 0, 1, \ldots, k_\alpha \ ,$$

$$\frac{B_{k_\alpha+1}^\alpha - A_{k_\alpha+1}^\alpha}{\lambda_{k_\alpha}^\alpha} > 0 \quad . \qquad (4\text{-}64)$$

(b) If $Y(p)$ has neither a pole nor a zero at $p = j\omega_\alpha$, then

$$B_i^\alpha = A_i^\alpha \ , \qquad i = 0, 1, \ldots, k_\alpha - 1 \quad . \qquad (4\text{-}65)$$

(c) If $Y(p)$ has a pole at $p = j\omega_\alpha$, then

$$B_i^\alpha = A_i^\alpha \quad , \qquad i = 0, 1, \ldots, k_\alpha - 2 \quad ,$$

$$\frac{B_{k_\alpha - 1}^\alpha - A_{k_\alpha - 1}^\alpha}{\lambda_{k_\alpha}^\alpha} = \alpha(\omega_\alpha) > 0 \quad . \tag{4-66}$$

From the prescribed $G(\omega^2)$, we would not know which of the above three cases will apply. It can be shown, however, that the restrictions

$$B_i^\alpha = A_i^\alpha \quad , \qquad i = 0, 1, \ldots, k_\alpha - 2 \quad ,$$

$$\frac{B_{k_\alpha - 1}^\alpha - A_{k_\alpha - 1}^\alpha}{\lambda_{k_\alpha}^\alpha} \geq 0 \tag{4-67}$$

will suffice.

3. Transmission zero at infinity of order k_∞.

$$B_i^\infty = A_i^\infty \quad , \qquad i = 0, 1, \ldots, k_\infty - 2 \quad , \tag{4-68}$$

$$\frac{B_{k_\infty - 1}^\infty - A_{k_\infty - 1}^\infty}{\lambda_{k_\infty}^\infty} = \frac{C}{\sqrt{\omega_{so}\,\omega_{io}}\;C_1} \geq \frac{C_o}{\sqrt{\omega_{so}\,\omega_{io}}\;C_1} \quad . \tag{4-69}$$

Conditions (4-63), (4-67) and (4-68), when coupled with the relation (4-62), can be used to determine $b(p)$. Thus, using the above information, we can obtain the gain-bandwidth restriction from (4-69) and, at the same time, complete the synthesis from the prescribed $G(\omega^2)$. The results are summarized in Theorems 3 and 4 and an illustrative example is completely worked out in Appendix A.

<u>Theorem 3:</u> (Assumptions: $\omega_s \omega_i \cong \omega_{so}\omega_{io}$ and $Y_3(j\omega_s) \cong \overline{Y}_3(j\omega_i)$)

Let $z_D(p) = -1/\omega_{so}\omega_{io} C_1^2 Z_3(p)$ be the impedance of a varactor diode in an inverting parametric device, where $Y_3(p)$ is the input admittance looking into the equalizer at the diode port with a d.c. capacitance C_o shunting the port; and let $b_o(p)$ be a real, rational, regular Blaschke product formed with the strict right half-plane poles of $1/G(-p^2)$ with half of their respective multiplicity. Then Eq. (4-60) defines a rational positive-real function $Y_3(p)$ with residue at infinity not less than the prescribed C_o if and only if

1) (a) $s(p)$ is a real, rational, passive scattering coefficient,

(b) $1 - s(p)\, s_*(p) = f^2(p)$, $f(p)$ real rational;

2) Any strict right half-plane transmission zero (strict right half-plane zero of $\lambda(p)$), $p_\alpha = \sigma_\alpha + j\omega_\alpha$, $\sigma_\alpha > 0$, of multiplicity k_α must be a zero of $b_o(p) - s(p)$ of at least order k_α, i.e.,

$$B_i^\alpha = A_i^\alpha \ , \qquad i = 0, 1, \ldots, k_\alpha - 1 \ ;$$

3) Any finite $j\omega$ transmission zero, $p_\alpha = j\omega_\alpha$, $|\omega_\alpha| < \infty$, of multiplicity k_α must be a zero of $b_o(p) - s(p)$ of at least order $k_\alpha - 1$, i.e.,

$$B_i^\alpha = A_i^\alpha \ , \qquad i = 0, 1, \ldots, k_\alpha - 2$$

and, in addition,

$$\frac{B_{k_\alpha - 1}^\alpha - A_{k_\alpha - 1}^\alpha}{\lambda_{k_\alpha}^\alpha} \geq 0 \ ;$$

4) The transmission zero at infinity of multiplicity k_∞, where k_∞ is even and $k_\infty - 2$ equals the order of zero of $Y(p) + Y_*(p)$ since $|Y(\infty)| = \infty$, must be a zero of

$b_o(p) - s(p)$ of at least order $k_\infty - 1$, i.e.,

$$B_i^\infty = A_i^\infty , \qquad i = 0, 1, \ldots, k_\infty - 2 ,$$

and, furthermore,

$$\frac{B^\infty_{k_\infty - 1} - A^\infty_{k_\infty - 1}}{\lambda^\infty_{k_\infty}} = \frac{C}{\sqrt{\omega_{so}\,\omega_{io}}\; C_1} \geq \frac{C_o}{\sqrt{\omega_{so}\,\omega_{io}}\; C_1} .$$

Proof: Necessity: already shown.

Sufficiency: From Eq. (4-60), solve for $Y(p)$ as follows:

$$Y(p) = \frac{b_o(p) - s(p)}{b_o(p)\sqrt{1 - s(p)\, s_*(p)}} . \qquad (4\text{-}70)$$

For sufficiency, we need to prove that conditions 1) - 4) imply that $Y(p)$ is positive-real with residue at infinity not less than $C_o/\sqrt{\omega_{so}\,\omega_{io}}\; C_1$. From condition 1), $Y(p)$ is a real and rational function of p. From Eq. (4-70), it follows that

$$Y(p) + Y_*(p) = \frac{b_o(p) - s(p)}{b_o(p)\sqrt{1 - s(p)\, s_*(p)}} + \frac{b_{o*}(p) - s_*(p)}{b_{o*}(p)\sqrt{1 - s(p)\, s_*(p)}}$$

$$= \frac{2 - s(p)\, b_{o*}(p) - s_*(p)\, b_o(p)}{\sqrt{1 - s(p)\, s_*(p)}} ,$$

since $b_o(p)\, b_{o*}(p) = 1$. Therefore, from condition 1)(a),

$$\operatorname{Re} Y(j\omega) = \frac{1 - \frac{1}{2}\left[s(j\omega)\, \bar{b}_o(j\omega) + \bar{s}(j\omega)\, b_o(j\omega)\right]}{\sqrt{1 - |s(j\omega)|^2}} \geq 0 .$$

Hence, Eq. (4-70) always defines a $Y(p)$ with a non-negative real part on the $j\omega$-axis.

It follows from 2) that $Y(p)$ is analytic at every strict right half-plane transmission zero. At a $j\omega$ transmission zero, it follows from 3) that $Y(p)$ has at most a simple finite $j\omega$ pole with a non-negative residue. And, finally, from 4) it follows that $Y(p)$ has a simple pole at $p = \infty$ with residue greater than or equal to the prescribed value $C_o/\sqrt{\omega_{so}\omega_{io}}\ C_1$.

To sum up, it follows from 1) - 4) that $Y_3(p)$ has a non-negative real part on $p = j\omega$, is analytic in $\mathrm{Re}\ p > 0$ and has at most simple finite $j\omega$ poles with a non-negative residue, and has a simple pole at $p = \infty$ with at least the prescribed residue C_o.

<u>Theorem 4:</u> (Assumptions: $\omega_s\omega_i \cong \omega_{so}\omega_{io}$ and $Y_3(j\omega_s) \cong \overline{Y}_3(j\omega_i)$)

The real, rational, even function of frequency $G(\omega^2)$ is realizable as the transducer power gain of a varactor inverting device represented by Fig. 7 if and only if

1) (a) $G(\omega^2) \geq 1$ for all real ω,

(b) $1 - [1/G(-p^2)]$ is a perfect square of a real, rational, even function of p;

2) The function $1/G(-p^2)$ admits a factorization of the form

$$\frac{1}{G(-p^2)} = s(p)\, s_*(p) \quad ,$$

where $s(p)$ satisfies conditions 1) - 4) of Theorem 1.

Proof: For the proof of Theorem 4, refer to the proof of Theorem 2 in Section IV.4.

CHAPTER V

FUNDAMENTAL GAIN-BANDWIDTH LIMITATIONS OF OPTIMUM SINGLE-VARACTOR PARAMETRIC AMPLIFIERS

V.1 Introduction

It is generally known[18] that by using more complicated coupling networks, wider bandwidths can be achieved than the conventional single-tuned parametric amplifier. Recently, Matthaei[19] described a procedure for achieving a varactor amplifier with large bandwidth by designing wide-band signal and idler "equal-ripple" bandpass filters around a given varactor. More recently, Aron[20] and Kuh[21] have presented some results on the fundamental gain-bandwidth limitations of a varactor parametric amplifier. They have found that by using a high-gain approximation the parametric amplifier may be treated with complete analogy to the tunnel diode.

The high-gain approximation used in References 20 and 21 is essentially an idealization which assume that $Y(p) = Y_3(p)/\sqrt{\omega_{so}\omega_{io}}\ C_1 \cong 1$. Thus, the scattering coefficient at the diode port is given by

$$s_{33}(p) \cong 2\left[\frac{1 - Y(p)}{1 + Y(p)}\right] ,$$

instead of Eq. (5-24), which is quadratic in $Y(p)$. With this approximation the problem reduces to that of equalizing a tunnel diode-like device. Although both tunnel diode and varactor are two-terminal active devices, the mechanism of achieving amplification is inherently different. For varactor, amplification is obtained through frequency coupling. As a result, an inherent property of a varactor device is the quadratic nature of the expression of the scattering coefficient at the diode port. A direct consequence of this property is that the usual Butterworth or Tchebycheff response is not realizable as a transducer power gain function of the varactor amplifier.

In Section V.2 we obtain an optimum gain-bandwidth upper-bound for the case $\omega_s \omega_i \cong \omega_{so} \omega_{io}$. This is a theoretical limitation which may be used as an estimate of the optimum gain-bandwidth which cannot be exceeded. In Section V.3 we derive an exact optimum gain-bandwidth limitation when the additional assumption $Y_3(j\omega_s) \cong \overline{Y}_3(j\omega_i)$ is invoked. For an ideal flat response, the optimum gain-bandwidth limitation is derived as

$$G_{T\ MAX} = \cosh^2 \left[\frac{\pi \sqrt{\omega_{so}\omega_{io}}\ C_1}{\omega_c C_o} \right] .$$

Finally, modified forms of Butterworth and Tchebycheff response function are applied, both of which approach the flat response over the prescribed band as the order of these responses is increased.

The assumption that $Y_3(j\omega_s) \cong \overline{Y}_3(j\omega_i)$ can be realized only approximately by using a low-pass to multiple band-pass frequency transformation defined[22] as follows:

$$p = \frac{(\lambda^2 + \omega_{so}^2)(\lambda^2 + \omega_{io}^2)}{\lambda(\lambda^2 + \omega_\infty^2)} ,$$

where ω_{so} and ω_{io} are the respective signal and idler band center frequencies, and

$$\omega_\infty^2 \equiv \frac{\omega_{so}^2 + \omega_{io}^2}{2} ,$$

and p and λ denote the low-pass and band-pass variables, respectively.

V.2 Derivation of Optimum Gain-Bandwidth Upper-Bound for the Case $\omega_s \omega_i \cong \omega_{so} \omega_{io}$

For the case $\omega_s \omega_i \cong \omega_{so} \omega_{io}$, the scattering coefficient at the diode port simplifies to

$$s_{33}(p) = \frac{\omega_{so} \omega_{io} C_1^2 - Y_3(p) Y_{3*}(p_o - p)}{\omega_{so} \omega_{io} C_1^2 + Y_3(p) Y_3(p_o - p)} = \frac{1 - Y(p) Y_*(p_o - p)}{1 + Y(p) Y(p_o - p)} , \quad (5\text{-}1)$$

where $Y(p) \equiv Y_3(p)/\sqrt{\omega_{so} \omega_{io}}\, C_1$. Stability consideration restricts that $Z_3(p) + z_D(p) \neq 0$ or $1 - Y(p) Y_*(p_o - p) \neq 0$ for $\text{Re } p \geq 0$. Let

$$\tilde{s}(p) \equiv \frac{1 - Y(p)}{1 + Y(p)} , \qquad Y(p) \equiv \frac{1 - \tilde{s}(p)}{1 + \tilde{s}(p)} ; \quad (5\text{-}2)$$

then

$$G(p) = \frac{1}{s_{33}(p) s_{33*}(p)} = \left[\frac{1 + \tilde{s}(p) \tilde{s}(p_o - p)}{\tilde{s}(p) + \tilde{s}_*(p_o - p)} \right] \left[\frac{1 + \tilde{s}_*(p) \tilde{s}_*(p_o - p)}{\tilde{s}_*(p) + \tilde{s}(p_o - p)} \right] . \quad (5\text{-}3)$$

It can be shown that $1 + \tilde{s}(p) \tilde{s}(p_o - p)$ has no zero or pole on the entire $p = j\omega$ axis and its zeros and poles exhibit the following symmetry: If $p = p_\nu$ is a zero or pole of $1 + \tilde{s}(p) \tilde{s}(p_o - p)$, then $p = -p_\nu + p_o$ is also a zero or pole. Hence, $1 + \tilde{s}(p) \tilde{s}(p_o - p)$ admits the factorization in the form $f(p) f(p_o - p)$, where $f(p)$ is restricted to be analytic together with its inverse in r.h.p., and the minimum factorization $f(p)$ is unique up to a constant multiple $e^{j\theta}$. Therefore,

$$G(p) = \left[\frac{f(p) f(p_o - p)}{\tilde{s}(p) + \tilde{s}_*(p_o - p)} \right] \left[\frac{f_*(p) f_*(p_o - p)}{\tilde{s}_*(p) + \tilde{s}(p_o - p)} \right] \equiv g(p) g_*(p) . \quad (5\text{-}4)$$

In the final factorization of $G(p)$ into $g(p) g_*(p)$, there is a unique choice for $g(p)$ when the restrictions $g(p)$ and $g^{-1}(p)$, both analytic in the strict r.h.p., are invoked. Noting that zeros of $\tilde{s}(p) + \tilde{s}_*(p_o - p)$ are all in the strict l.h.p. from stability consideration, we can conclude that

$$g(p) = \frac{f(p)\, f_*(p_o - p)}{\frac{1}{2}\left[\tilde{s}(p) + \tilde{s}_*(p_o - p)\right]} \quad . \tag{5-5}$$

Now we proceed by relating the behavior of the factorized functions f(p) and g(p) about $p = \infty$ with that of $\tilde{s}(p)$. From the fact that about $p = \infty$,

$$Y_3(p) = p\,C + y_3(p) \quad , \tag{5-6}$$

where $C \geq C_o$ and $y_3(p)$ is an arbitrary positive-real function which is finite at $p = \infty$ (for convenience, take $C = C_o$), the expansion of $\tilde{s}(p) = 1 - Y(p)/1 + Y(p) = \sqrt{\omega_{so}\,\omega_{io}}\; C_1 - Y_3(p)/\sqrt{\omega_{so}\,\omega_{io}}\; C_1 + Y_3(p)$ about infinity is given by

$$\tilde{s}(p) = -1 + \frac{A_1}{p} + \frac{A_2}{p^2} + 0(\frac{1}{p^3}) \quad , \tag{5-7}$$

where

$$A_1 = \frac{2\sqrt{\omega_{so}\,\omega_{io}}\; C_1}{C_o} \quad , \tag{5-8}$$

and

$$A_2 = -A_1\left[\frac{A_1}{2} + \frac{y_3(\infty)}{C_o}\right] \quad . \tag{5-9}$$

Let the expansion of f(p) about $p = \infty$ be denoted by

$$f(p) = B_o + \frac{B_1}{p} + \frac{B_2}{p^2} + 0(\frac{1}{p^3}) \quad . \tag{5-10}$$

Since the expansion of $f(p)\, f(p_o - p)$ about $p = \infty$ must agree with that of $\frac{1}{2}\left[1 + \tilde{s}(p)\, \tilde{s}(p_o - p)\right]$, the following relations are obtained by equating coefficients of like powers of $1/p$:

$$B_o^2 = 1 \quad ,$$

$$B_1^2 + B_o B_1 p_o - 2 B_o B_2 = \frac{1}{2}(A_1^2 + 2A_2 - A_1 p_o) \quad . \tag{5-11}$$

From Eqs.(5-5), (5-7) and (5-10), we have

$$g(p) = \frac{f(p)\, f_*(p_o - p)}{\frac{1}{2}\left[\tilde{s}(p) + \tilde{s}_*(p_o - p)\right]}$$

$$= (-1)\left\{\frac{B_o^2 + \frac{1}{p}\left[2B_o(\mathrm{Re}\, B_1)\right] + \frac{1}{p^3}\left[|B_1|^2 + 2B_o(\mathrm{Re}\, B_2) + B_o\overline{B}_1 p_o\right] + \ldots}{1 - \frac{1}{p}(A_1) - \frac{1}{p^2}(A_2 + \frac{1}{2}A_1 p_o) + \ldots}\right\} , \tag{5-12}$$

and the expansion of $g(p)$ about $p = \infty$ is given by

$$g(p) = -1\left\{1 + \frac{1}{p}\left[A_1 + 2B_o(\mathrm{Re}\, B_1)\right] + 0(\frac{1}{p^2})\right\} , \tag{5-13}$$

where, according to (5-11), $B_o = \pm 1$.

The function $f(p)$, as defined by the minimum factorization of the equation $\frac{1}{2}\left[1 + \tilde{s}(p)\,\tilde{s}(p_o - p)\right] = f(p)\, f(p_o - p)$, is a bounded scattering coefficient which is unitary at $p = \infty$. It is shown in Theorem 6 (Appendix B) that if B_o is real, then B_1 must also be real and, more importantly, the product $B_o B_1$ must be of negative sign. Imposing the condition that $B_o B_1 < 0$, the residue of $g(p)$ at $p = \infty$ is given by

$$A_1 + 2B_o B_1 = A_1 - 2|B_1| < A_1 \quad , \tag{5-14}$$

where $A_1 = 2\sqrt{\omega_{so}\,\omega_{io}}\; C_1/C_o$ and is independent of the equalizer. Note that $G(\omega) = |g(j\omega)|^2$ and $g(p)$ is without zero or pole in the entire right-half p-plane. Using calculus of residue, we obtain

$$\int_{-\infty}^{+\infty} \ell n\, G(\omega)\, d\omega = 2\pi \left[A_1 + B_o(B_1 + \overline{B}_1)\right]$$

$$= 2\pi \left[A_1 - 2\left|B_1\right|\right]$$

$$< 4\pi \left[\frac{\sqrt{\omega_{so}\,\omega_{io}}\; C_1}{C_o}\right] , \qquad (5\text{-}15)$$

which is an optimum gain-bandwidth upper-bound. The conclusion to be drawn from (5-15) is that the optimum gain-bandwidth is bounded by a quantity that is completely independent of the equalizer. As far as practical utilization is concerned, Eq.(5-15) can give us some estimate of the optimum gain-bandwidth which cannot be exceeded but, on the other hand, we have shown in Theorem 6 that $\left|B_1\right| \neq 0$, i.e., the gain-bandwidth upper-bound given in (5-15) cannot be attained.

If we impose the additional assumption that $Y_3(j\omega_s) \cong \overline{Y}_3(j\omega_i)$, then the transducer power gain function is an even function of ω and (5-15) becomes

$$\int_{o}^{\infty} \ell n\, G(\omega^2)\, d\omega = \pi \left[A_1 - 2\left|B_1\right|\right] < 2\pi \left[\frac{\sqrt{\omega_{so}\,\omega_{io}}\; C_1}{C_o}\right] . \qquad (5\text{-}16)$$

It is interesting to note that the result (5-16) was originally proved in our study for equalizers of low complexity by evaluating the r.h.p. poles and the permissible zeros of

$$\hat{s}_{33}(p) = s_{33}(p)\left[\frac{Y_*(p)}{Y(p)}\right] = \left[\frac{1 - Y^2(p)}{1 + Y(p)\,Y_*(p)}\right]\left[\frac{Y_*(p)}{Y(p)}\right] . \qquad (5\text{-}17)$$

Let

$$Y(p) = \frac{a_n p^n + a_{n-1} p^{n-1} + \ldots + a_1 p + a_o}{b_{n-1} p^n + b_{n-2} p^{n-2} + \ldots + b_1 p + b_o} , \qquad (5\text{-}18)$$

and let $\sum_{r=1}^{n}(p_r)$ and $\sum_{k=1}^{n}(\hat{p}_k)$ denote, respectively, the sum of the r.h.p. poles and permissible r.h.p. zeros of $\hat{s}_{33}(p)$; then it can be shown that

$$\int_0^\infty \ell n\, G(\omega^2)\, d\omega = 2\pi \left[\sum_{r=1}^{n}(p_r) - \sum_{k=1}^{n}(\hat{p}_k) \right] , \tag{5-19}$$

where $\sum(p_r)$ and $\sum(\hat{p}_k)$ can be expressed as functions of the coefficients a's and b's of $Y(p)$. By imposing the positive-reality and stability restriction on these coefficients, we were able to prove for $n = 1, 2, 3$ that

$$\sum_{r=1}^{n}(p_r) - \sum_{r=1}^{n}(\hat{p}_k) < \frac{\sqrt{\omega_{so}\,\omega_{io}}\; C_1}{C_o} . \tag{5-20}$$

Clearly, this method cannot be generalized for arbitrary n since it involves the finding of the sum of the r.h.p. roots of an even polynomial of degree $2n$ in terms of its coefficients. The approach we used in proving the optimum gain-bandwidth upper-bound completely circumvents this difficulty and the result holds for arbitrary equalizers of any degree of complexity.

V.3 Exact Optimum Gain-Bandwidth Limitation for the Case $\omega_s\,\omega_i \cong \omega_{so}\,\omega_{io}$ and $Y_3(j\omega_s) \cong \overline{Y}_3(j\omega_i)$

We shall now assume that $\omega_s\,\omega_i \cong \omega_{so}\,\omega_{io}$ where ω_{so} and ω_{io} are the respective band center frequencies and that $Y_3(j\omega_s) \cong \overline{Y}_3(j\omega_i)$, which can be realized approximately by using a low-pass to multiple band-pass frequency transformation discussed in Section V.1. With these two assumptions, the diode impedance at the signal frequency becomes

$$z_D(j\omega_s) = -\frac{1}{\omega_{so}\,\omega_{io}\,C_1^2\,Z_3(j\omega_s)} , \tag{5-21}$$

and the scattering coefficient at the diode port simplifies to

$$s_{33}(j\omega_s) = \frac{\omega_{so}\,\omega_{io}\,C_1^2 - \left[Y_3(j\omega_s)\right]^2}{\omega_{so}\,\omega_{io}\,C_1^2 + \left|Y_3(j\omega_s)\right|^2} = \frac{1 - \left[Y(j\omega_s)\right]^2}{1 + \left|Y(j\omega_s)\right|^2} \quad , \tag{5-22}$$

where $Y \equiv Y_3/\sqrt{\omega_{so}\omega_{io}}\;C_1$. Now, the real part of the normalizing impedance at port 3, $z_3(j\omega_s) = -\bar{z}_D(j\omega_s)$, is non-negative on the entire $p = j\omega_s$ axis. In addition, it is clear that

$$|s_{33}(j\omega_s)|^2 = 1 - \frac{4\left[\mathrm{Re}\;Y(j\omega_s)\right]^2}{\left[1 + |Y(j\omega_s)|^2\right]^2} \leq 1 \quad ; \tag{5-23}$$

hence, $G \equiv 1/|s_{33}|^2$ is an even function of ω_s and its magnitude is greater than or equal to unity for all real ω_s.

To proceed with the broadbanding problem, we consider the analytic extension of the steady-state expressions. We have

$$s_{33}(p) = \frac{1 - \left[Y(p)\right]^2}{1 + Y(p)\,Y_*(p)} \quad , \tag{5-24}$$

and

$$G(-p^2) \equiv \frac{1}{s_{33}(p)\,s_{33*}(p)} = \frac{\left[1 + Y(p)\,Y_*(p)\right]^2}{\left[1 - \left[Y(p)\right]^2\right]\left[1 - \left[Y_*(p)\right]^2\right]} \quad , \tag{5-25}$$

where $f_*(p) \equiv f(-p)$ for any arbitrary, real, rational function $f(p)$.

Furthermore, a realizable transducer power gain function $G(-p^2)$ must be of such a form that $1 - \left[1/G(-p^2)\right]$ is a perfect square of a real, rational, even function of p since

$$1 - \frac{1}{G(-p)^2} = \left[\frac{Y(p) + Y_*(p)}{1 + Y(p)\,Y_*(p)}\right]^2 \quad .$$

Thus

$$\sqrt{1 - \frac{1}{G(-p^2)}} = \pm \left[\frac{Y(p) + Y_*(p)}{1 + Y(p)\, Y_*(p)} \right] .$$

Taking the positive sign to avoid ambiguity, we have

$$\frac{1 - \sqrt{1 - \frac{1}{G(-p^2)}}}{1 + \sqrt{1 - \frac{1}{G(-p^2)}}} = \left[\frac{1 - Y(p)}{1 + Y(p)} \right] \left[\frac{1 - Y_*(p)}{1 + Y_*(p)} \right] = \tilde{s}(p)\, \tilde{s}_*(p) . \tag{5-26}$$

The factorization is unique up to a plus or minus sign if we specify that $\tilde{s}(p)$ is analytic together with its inverse in the strict right-half p-plane. Invoking the stability requirement, $Y(p) \neq 1$ in the entire closed right-half p-plane, we can identify that

$$\tilde{s}(p) = \frac{1 - Y(p)}{1 + Y(p)} = \frac{\sqrt{\omega_{so}\, \omega_{io}}\; C_1 - Y_3(p)}{\sqrt{\omega_{so}\, \omega_{io}}\; C_1 + Y_3(p)} . \tag{5-27}$$

The expansion of $\tilde{s}(p)$ about $p = \infty$ is

$$\tilde{s}(p) = -1 + \frac{A_1}{p} + 0\left(\frac{1}{p^2}\right) , \tag{5-28}$$

where

$$A_1 = \frac{2\sqrt{\omega_{so}\, \omega_{io}}\; C_1}{C} , \qquad (C \geq C_o) ,$$

and is independent of the equalizer. From (5-26),

$$\frac{1 - \sqrt{1 - \frac{1}{G(\omega^2)}}}{1 + \sqrt{1 - \frac{1}{G(\omega^2)}}} = |\tilde{s}(j\omega)|^2 , \tag{5-29}$$

and $\tilde{s}(p)$ is a minimum-phase passive scattering coefficient. Using calculus of residue, we obtain the integral relation

$$\int_0^\infty \ln \left[\frac{1 + \sqrt{1 - \frac{1}{G(\omega^2)}}}{1 - \sqrt{1 - \frac{1}{G(\omega^2)}}} \right] d\omega = \pi(A_1) = \pi \left[\frac{2\sqrt{\omega_{so}\omega_{io}}\, C_1}{C} \right] \leq \pi \left[\frac{2\sqrt{\omega_{so}\omega_{io}}\, C_1}{C_o} \right], \tag{5-30}$$

which is the fundamental gain-bandwidth restriction independent of the equalizer.

We are now in a position to state the necessary and sufficient conditions on the realizability of a transducer power gain function of a varactor parametric amplifier for the case $\omega_s \omega_i \cong \omega_{so}\omega_{io}$ and $Y_3(j\omega_s) \cong \overline{Y}_3(j\omega_i)$.

Theorem 5:

Assuming $\omega_s \omega_i \cong \omega_{so}\omega_{io}$ and $Y_3(j\omega_s) \cong \overline{Y}_3(j\omega_i)$, the real, rational, even function of frequency $G(\omega^2)$ is realizable as the transducer power gain of a varactor inverting device if and only if:

1) (a) $G(\omega^2) \geq 1$ for all real ω,

(b) $1 - \frac{1}{G(-p^2)}$ is a perfect square of a real, rational, even function of p,

2) $$\int_0^\infty \ln \left[\frac{1 + \sqrt{1 - \frac{1}{G(\omega^2)}}}{1 - \sqrt{1 - \frac{1}{G(\omega^2)}}} \right] d\omega \leq \pi \left[\frac{2\sqrt{\omega_{so}\omega_{io}}\, C_1}{C_o} \right].$$

Proof: Necessity: already shown.

Sufficiency: Give 1) and 2), then $1 - \sqrt{1 - 1/G(-p^2)} \Big/ 1 + \sqrt{1 - 1/G(-p^2)}$ admits a factorization of the form $\tilde{s}(p)\,\tilde{s}_*(p)$ where $\tilde{s}(p)$ is a minimum scattering coefficient with residue at infinity less than or equal to $(2\sqrt{\omega_{so}\omega_{io}}\ C_1/C_o)$. Therefore, Y(p) is positive-real with residue at infinity not less than the prescribed $C_o/\sqrt{\omega_{so}\omega_{io}}\ C_1$.

It is interesting to note that any realizable $G(\omega^2)$ satisfying the integral restriction 2) of Theorem 5 will automatically satisfy the optimum upper-bound limitation embodied in Eq. (5-16),

$$\int_0^\infty \ell n\, G(\omega^2)\, d\omega < \pi \left[\frac{2\sqrt{\omega_{so}\omega_{io}}\ C_1}{C_o}\right] , \qquad (5\text{-}16)$$

since

$$G(\omega^2) < \left[\frac{1 + \sqrt{1 - 1/G(\omega^2)}}{1 - \sqrt{1 - 1/G(\omega^2)}}\right]$$

for all real ω. This is as it should be since the optimum upper-bound limitation is a necessary condition for realizability.

For the ideal flat response shape, we choose $G(\omega^2)$ = a constant G_T over the band $0 < \omega < \omega_c$. Then, irrespective of its behavior outside this band of frequencies,

$$\ell n \left[\frac{1 + \sqrt{1 - 1/G_T}}{1 - \sqrt{1 - 1/G_T}}\right] \cdot \omega_c \leq \pi \left[\frac{2\sqrt{\omega_{so}\omega_{io}}\ C_1}{C_o}\right] , \qquad (5\text{-}31)$$

or

$$\left[\frac{1 + \sqrt{1 - 1/G_T}}{1 - \sqrt{1 - 1/G_T}}\right] \leq \exp\left[\frac{\pi 2\sqrt{\omega_{so}\omega_{io}}\ C_1}{\omega_c C_o}\right] . \qquad (5\text{-}32)$$

Therefore,

$$G_T \leq G_{T\,MAX} = \cosh^2\left[\frac{\pi\sqrt{\omega_{so}\omega_{io}}\ C_1}{\omega_c C_o}\right] , \qquad (5\text{-}33)$$

which is the optimum gain-bandwidth relation for the ideal flat response. Also,

$$\omega_c = \frac{\pi}{\cosh^{-1}\sqrt{G_{T\ MAX}}} \left[\frac{\sqrt{\omega_{so}\,\omega_{io}}\ C_1}{C_o} \right] , \tag{5-34}$$

where ω_c is the low-pass bandwidth. Let $\rho = 2\,C_1/C_o$ be the pumping ratio; (5-33) is equivalent to

$$G_{T\ MAX} = \cosh^2 \left[\frac{\pi \sqrt{\omega_{so}\,\omega_{io}}}{\omega_c} \ \frac{\rho}{2} \right] . \tag{5-33b}$$

V.4 Modified Butterworth Response

We shall now consider the modified Butterworth response prescribed by

$$G(\hat{\omega}^2, n) = \frac{\hat{\omega}^{4n} + \hat{\omega}^{2n}(K_n + 1) + \left[(K_n + 1)^2/4\right]}{\hat{\omega}^{4n} + \hat{\omega}^{2n}(K_n + 1) + (K_n)} , \tag{5-35}$$

where $K_n \geq 1$ and $\hat{\omega} \equiv \omega/\omega_c$. The d.c. gain is given by

$$G(0) = \frac{(K_n + 1)^2}{4K_n} . \tag{5-36}$$

It can readily be shown that this characteristic (5-35) satisfies condition 1) of Theorem 5. Let w_c denote the true 3 db bandwidth, then w_c and the normalization frequency ω_c are related via the formula

$$\frac{w_c}{\omega_c} = \left\{ \frac{1}{2}(K_n + 1) \left[\frac{K_n - 1}{\sqrt{K_n^2 - 6K_n + 1}} - 1 \right] \right\}^{1/2n} . \tag{5-37}$$

From Eq.(5-35),

$$G(-\hat{p}^2, n) = \frac{\hat{p}^{4n} + (-1)^n \hat{p}^{2n}(K_n + 1) + \frac{(K_n + 1)^2}{4}}{\hat{p}^{4n} + (-1)^n \hat{p}^{2n}(K_n + 1) + (K_n)}$$

$$= \frac{\left[\hat{p}^{2n} + (-1)^n \frac{(K_n + 1)}{2}\right]^2}{\left[\hat{p}^{2n} + (-1)^n (K_n)\right]\left[\hat{p}^{2n} + (-1)^n\right]} , \qquad (5\text{-}38)$$

$$1 - \frac{1}{G(-\hat{p}^2, n)} = \left[\frac{(K_n - 1)/2}{\hat{p}^{2n} + (-1)^n \left(\frac{K_n + 1}{2}\right)}\right]^2 , \qquad (5\text{-}39)$$

and

$$\sqrt{1 - \frac{1}{G(-\hat{p}^2, n)}} = (-1)^n \left[\frac{(K_n - 1)/2}{\hat{p}^{2n} + (-1)^n \left(\frac{K_n + 1}{2}\right)}\right] , \qquad (5\text{-}40)$$

where the factor $(-1)^n$ is added so that at $\hat{p} = 0$, $\sqrt{1 - 1/G(-\hat{p}^2, n)}$ will be non-negative for all n. Therefore,

$$\frac{1 - \sqrt{1 - 1/G(-\hat{p}^2, n)}}{1 + \sqrt{1 - 1/G(-\hat{p}^2, n)}} = \tilde{s}(p)\, \tilde{s}_*(p)\Big|_n = \frac{\hat{p}^{2n} + (-1)^n}{\hat{p}^{2n} + (-1)^n K_n} . \qquad (5\text{-}41)$$

The factorization for $\tilde{s}(p)$ is given by

$$\tilde{s}(p) = - \frac{(\hat{p}^n + a_{n-1} \hat{p}^{n-1} + \ldots + a_1 \hat{p} + a_o)}{K_n^{1/2}(K_n^{-1/2} \hat{p}^n + a'_{n-1} \hat{p}^{n-1} + \ldots + a'_1 \hat{p} + 1)} , \qquad (5\text{-}42)$$

where the a_i are the coefficients of the Butterworth polynomial of order n and $a'_i = a_i/(K_n)^{i/2n}$ (Reference 23).

Expanding $\tilde{s}(p)$ about $p = \infty$ gives

$$\tilde{s}(p) = -1 + \frac{(K_n^{1/2n} - 1)\, a_{n-1}\, \omega_c}{p} + 0(\frac{1}{p^2}) \ , \qquad (5\text{-}43)$$

$$a_{n-1} = \frac{1}{\sin \frac{\pi}{2n}} \ .$$

Equating the first two coefficients of Eq. (5-43) with that of

$$\tilde{s}(p) = -1 + \frac{A_1}{p} + 0(\frac{1}{p^2}) \ , \qquad (5\text{-}44)$$

$$A_1 = \frac{2\sqrt{\omega_{so}\,\omega_{io}}\; C_1}{C} \ , \qquad \text{with} \quad C \geq C_o \ ,$$

we obtain for $C = C_o$

$$(K_n^{1/2n} - 1) = (\sin \frac{\pi}{2n}) \frac{2\sqrt{\omega_{so}\,\omega_{io}}\; C_1}{\omega_c\, C_o} \ . \qquad (5\text{-}45)$$

The solution for K_n is

$$K_n = \left[1 + \frac{2(\sin \frac{\pi}{2n})\sqrt{\omega_{so}\,\omega_{io}}\; C_1}{\omega_c\, C_o} \right]^{2n} \ . \qquad (5\text{-}46)$$

Substituting the relation between K_n and the d.c. gain $G(0)$ and the relation between ω_c and the true 3 db bandwidth w_c, Eqs. (5-36) and (5-37), we obtain the gain-bandwidth restriction for the modified Butterworth response as follows:

$$w_c = \left\{ \frac{2\left[\sqrt{\omega_{so}\,\omega_{io}}\; C_1/C_o\right]\left[\sin\frac{\pi}{2n}\right]}{(K_n^{1/2n} - 1)} \right\} \left\{ \frac{(K_n + 1)}{2} \left[(K_n - 1)\sqrt{\frac{1}{(K_n^2 - 6K_n + 1)}} - 1 \right] \right\}^{1/2n} \tag{5-47}$$

where

$$K_n = \left[2\,G(0) - 1\right] + 2\sqrt{G^2(0) - G(0)} \quad .$$

As $n \rightarrow \infty$, let $\omega_c \cong w_c$; then

$$\lim_{n \rightarrow \infty} K_n = e^{2\pi\sqrt{\omega_{so}\,\omega_{io}}\; C_1/\omega_c C_o} \quad , \tag{5-48}$$

From Eqs. (5-36) and (5-48),

$$\lim_{n \rightarrow \infty} G(0) = \frac{\left[e^{2\pi\sqrt{\omega_{so}\,\omega_{io}}\; C_1/\omega_c C_o} + 1\right]^2}{4\, e^{2\pi\sqrt{\omega_{so}\,\omega_{io}}\; C_1/\omega_c C_o}} \quad . \tag{5-49}$$

Therefore,

$$\lim_{n \rightarrow \infty} G(0) = \left\{ \frac{1}{2}\left[e^{\pi\sqrt{\omega_{so}\,\omega_{io}}\; C_1/\omega_c C_o} + e^{-\pi\sqrt{\omega_{so}\,\omega_{io}}\; C_1/\omega_c C_o} \right] \right\}^2$$

$$= \cosh^2\left[\pi\sqrt{\omega_{so}\,\omega_{io}}\; C_1/\omega_c C_o\right] = G_{T\,MAX} \quad . \tag{5-50}$$

Thus we can conclude that in the limit $n \rightarrow \infty$, the gain-bandwidth limitation for the modified Butterworth response reduces to the optimum gain-bandwidth relation for the ideal flat response.

V.5 Modified Tchebycheff Response

We next consider the modified Tchebycheff response prescribed by

$$G(\hat{\omega}^2, n) = \frac{\epsilon^4 T_n^4(\hat{\omega}) + \epsilon^2 T_n^2(\hat{\omega})(K_n + 1) + \frac{(K_n + 1)^2}{4}}{\epsilon^4 T_n^4(\hat{\omega}) + \epsilon^2 T_n^2(\hat{\omega})(K_n + 1) + (K_n)} \quad , \qquad (5\text{-}51)$$

where $K_n \geq 1$, $\hat{\omega} \equiv \omega/\omega_c$, ϵ is the ripple parameter, and $T_n(\hat{\omega})$ is the nth order Tchebycheff polynomial[22]

$$\begin{aligned} T_n(\hat{\omega}) &= \cos(n \cos^{-1} \hat{\omega}) \quad , \qquad 0 \leq \hat{\omega} \leq 1 \quad , \\ &= \cosh(n \cosh^{-1} \hat{\omega}) \quad , \qquad \hat{\omega} \geq 1 \quad . \end{aligned} \qquad (5\text{-}52)$$

The d.c. gain is given by

$$\begin{aligned} G(0) &= \frac{(K_n + 1)^2}{4 K_n} \quad , \qquad \text{(for } n \text{ odd)}, \\ G(0) &= \frac{\left[\left(\frac{K_n + 1}{2}\right) + \epsilon^2\right]^2}{(K_n + \epsilon^2)(1 + \epsilon^2)} \quad , \qquad \text{(for } n \text{ even)}. \end{aligned} \qquad (5\text{-}53)$$

For small ripple ϵ, the difference between $G(0)$ for n odd and even is negligible. It can readily be shown that the modified Tchebycheff response given by Eq. (5-51) satisfies condition 1) of Theorem 5. Again, the normalization frequency, ω_c, can be related to the true bandwidth, w_c.

From Eq. (5-51),

$$G(-\hat{p}^2, n) = \frac{\epsilon^4 T_n^4(\hat{p}) + \epsilon^2 T_n^2(\hat{p})(K_n + 1) + \frac{(K_n + 1)^2}{4}}{\epsilon^4 T_n^4(\hat{p}) + \epsilon^2 T_n^2(\hat{p})(K_n + 1) + (K_n)}$$

$$= \frac{\left[\epsilon^2 T_n^2(\hat{p}) + \frac{(K_n + 1)}{2}\right]^2}{\left[\epsilon^2 T_n^2(\hat{p}) + (K_n)\right]\left[\epsilon^2 T_n^2(\hat{p}) + (1)\right]} , \qquad (5\text{-}54)$$

$$1 - \frac{1}{G(-\hat{p}^2, n)} = \left[\frac{(K_n + 1)/2}{\epsilon^2 T_n^2(\hat{p}) + \left[(K_n + 1)/2\right]}\right]^2 , \qquad (5\text{-}55)$$

and

$$\sqrt{1 - \frac{1}{G(-\hat{p}^2, n)}} = \frac{(K_n - 1)/2}{\epsilon^2 T_n^2(\hat{p}) + \left[(K_n + 1)/2\right]} . \qquad (5\text{-}56)$$

Therefore,

$$\frac{1 - \sqrt{1 - 1/G(-\hat{p}^2, n)}}{1 + \sqrt{1 - 1/G(-\hat{p}^2, n)}} = \tilde{s}(p)\, \tilde{s}_*(p)\Big|_n = \frac{\epsilon^2 T_n^2(\hat{p}) + 1}{\epsilon^2 T_n^2(\hat{p}) + K_n} , \qquad (5\text{-}57)$$

or

$$\tilde{s}(p)\, \tilde{s}_*(p)\Big|_n = \frac{T_n^2(\hat{p}) + (1/\epsilon^2)}{T_n^2(\hat{p}) + (1/\hat{\epsilon}^2)} , \qquad (5\text{-}58)$$

where $\hat{\epsilon}^2 \equiv \epsilon^2/K_n$. The factorization for $\tilde{s}(p)$ is given by

$$\tilde{s}(p) = - \frac{(\hat{p}^n + b_{n-1}\hat{p}^{n-1} + \ldots + b_1 \hat{p} + b_o)}{(\hat{p}^n + b'_{n-1}\hat{p}^{n-1} + \ldots + b'_1 \hat{p} + b_o)} ,$$

where the b_i are the coefficients of the Tchebycheff polynomial for a ripple parameter ϵ and the b_i' are the corresponding ones for the ripple factor $\epsilon/\sqrt{K_n}$.[23]

Expanding $\tilde{s}(p)$ about $p = \infty$ gives

$$\tilde{s}(p) = -1 + \frac{(b_{n-1}' - b_{n-1})\,\omega_c}{p} + 0\left(\frac{1}{p^2}\right) , \qquad (5\text{-}59)$$

where

$$b_{n-1} = \frac{\sinh\left[\frac{1}{n}\sinh^{-1}\frac{1}{\epsilon}\right]}{\sin\frac{\pi}{2n}} , \qquad (5\text{-}60)$$

and

$$b_{n-1}' = \frac{\sinh\left[\frac{1}{n}\sinh^{-1}\frac{\sqrt{K_n}}{\epsilon}\right]}{\sin\frac{\pi}{2n}} . \qquad (5\text{-}61)$$

Using the same procedure as in the modified Butterworth case, we obtain by equating coefficients

$$\sinh\left[\frac{1}{n}\sinh^{-1}\frac{\sqrt{K_n}}{\epsilon}\right] - \sinh\left[\frac{1}{n}\sinh^{-1}\frac{1}{\epsilon}\right] = \left[\sin\frac{\pi}{2n}\right]\frac{2\sqrt{\omega_{so}\omega_{io}}\,C_1}{\omega_c\,C_o} . \qquad (5\text{-}62)$$

The solution for K_n is

$$K_n = \epsilon^2\sinh^2\left\{n\sinh^{-1}\left[\frac{2\sqrt{\omega_{so}\,\omega_{io}}\,C_1}{\omega_c\,C_o}\left(\sin\frac{\pi}{2n}\right) + \sinh\left(\frac{1}{n}\sinh^{-1}\frac{1}{\epsilon}\right)\right]\right\} . \qquad (5\text{-}63)$$

The above relation, coupled with Eq. (5-53) and the relation between ω_c and the true bandwidth w_c, gives the gain-bandwidth restriction for the modified Tchebycheff response.

As $\epsilon \rightarrow 0$ and $n \rightarrow \infty$,

$$\lim_{\substack{\epsilon \to 0 \\ n \to \infty}} K_n = e^{2\pi \sqrt{\omega_{so}\omega_{io}}\, C_1/\omega_c C_o} , \tag{5-64}$$

which is the same as that of the modified Butterworth case ($n \to \infty$) and reduces to the gain-bandwidth relation for the ideal flat response.

V.6 Additional Bandwidth Limitation Imposed by Series Parasitic Inductance

The analysis for the optimum nonreciprocal varactor amplifier using a single varactor for the case $\omega_s \omega_i \cong \omega_{so}$ and $Y_3(j\omega_s) \cong \overline{Y}_3(j\omega_i)$ was presented in Section V.3. We shall now consider the same amplifier configuration using a more exact varactor equivalent circuit which includes its series parasitic inductance L_o (Fig. 10).

Recalling from Section V.3,

$$\frac{1 - \sqrt{1 - 1/G(-p^2)}}{1 + \sqrt{1 - 1/G(-p^2)}} = \left[\frac{1 - Y(p)}{1 + Y(p)}\right] \left[\frac{1 - Y_*(p)}{1 + Y_*(p)}\right] = \tilde{s}(p)\, \tilde{s}_*(p) , \tag{5-26}$$

and

$$\tilde{s}(p) = \frac{1 - Y(p)}{1 + Y(p)} = \frac{\sqrt{\omega_{so}\,\omega_{io}}\; C_1 - Y_3(p)}{\sqrt{\omega_{so}\,\omega_{io}}\; C_1 + Y_3(p)} , \tag{5-27}$$

where $\tilde{s}(p)$ is a minimum-phase passive scattering coefficient. At high frequencies, the behavior of $Y_3(p)$ can be expressed as

$$Y_3(p) = p\, C_o + \frac{1}{p L + \hat{z}_3(p)} , \tag{5-65}$$

where $L \geq L_o$ and $\hat{z}_3(p)$ is an arbitrary positive-real function, finite at $p = \infty$. Then it can be shown that the expansion of $\tilde{s}(p)$ about $p = \infty$ is

$$\tilde{s}(p) = -1 + \frac{A_1}{p} + \frac{A_2}{p^2} + \frac{A_3}{p^3} + 0\left(\frac{1}{p^4}\right) , \tag{5-66}$$

where

$$A_1 = 2\sqrt{\omega_{so}\,\omega_{io}}\; C_1/C_o , \qquad A_2 = 2\,\omega_{so}\,\omega_{io}\; C_1^2/C_o^2 , \tag{5-67}$$

and

$$A_3 = \left[2\sqrt{\omega_{so}\,\omega_{io}}\; C_1/C_o^3\, L\right]\left[\omega_{so}\,\omega_{io}\, C_1^2\, L - C_o\right] . \tag{5-68}$$

Thus,

$$\ell n\, \frac{1}{\tilde{s}(p)} = j\pi + \frac{1}{p}\left[\frac{2\sqrt{\omega_{so}\omega_{io}}\; C_1}{C_o}\right] - \frac{1}{p^3}\left\{\frac{2}{3}\left[\frac{\sqrt{\omega_{so}\omega_{io}}\; C_1}{C_o^3\, L}\right]\left[3\, C_o - \omega_{so}\omega_{io} C_1^2\, L\right]\right\} + 0\left(\frac{1}{p^4}\right) \tag{5-69}$$

Since

$$\left[\frac{1 + \sqrt{1 - 1/G(\omega^2)}}{1 - \sqrt{1 - 1/G(\omega^2)}}\right] = \frac{1}{|\tilde{s}(j\omega)|^2} , \tag{5-70}$$

we can obtain two simultaneous integral restrictions as follows:

$$\int_0^\infty \ell n \left[\frac{1 + \sqrt{1 - 1/G(\omega^2)}}{1 - \sqrt{1 - 1/G(\omega^2)}}\right] d\omega = \pi \left[\frac{2\sqrt{\omega_{so}\,\omega_{io}}\; C_1}{C_o}\right] , \tag{5-71}$$

$$\int_0^\infty \omega^2\, \ell n \left[\frac{1 + \sqrt{1 - 1/G(\omega^2)}}{1 - \sqrt{1 - 1/G(\omega^2)}}\right] d\omega = \pi \left[\frac{2\sqrt{\omega_{so}\omega_{io}}\; C_1\,(3\, C_o - \omega_{so}\omega_{io}\, C_1^2\, L)}{3\, L\, C_o^3}\right] . \tag{5-72}$$

For the ideal flat response shape, we choose $G(\omega^2) = G_T$, a constant, over the band $0 < \omega < \omega_c$. Then, irrespective of its behavior outside this band of frequencies,

$$\ln\left[\frac{1+\sqrt{1-1/G_T}}{1-\sqrt{1-1/G_T}}\right]\cdot\omega_c \leq \pi\left[\frac{2\sqrt{\omega_{so}\,\omega_{io}}\;C_1}{C_o}\right], \tag{5-73}$$

and

$$\ln\left[\frac{1+\sqrt{1-1/G_T}}{1-\sqrt{1-1/G_T}}\right]\cdot\omega_c^3 \leq \pi\left[\frac{2\sqrt{\omega_{so}\,\omega_{io}}\;C_1\,(3\,C_o-\omega_{so}\,\omega_{io}\,C_1^2\,L)}{L\,C_o^3}\right]. \tag{5-74}$$

Therefore,

$$G_{T\,MAX} = \cosh^2\left[\frac{\pi\sqrt{\omega_{so}\,\omega_{io}}\;C_1}{\omega_c\,C_o}\right], \tag{5-75}$$

and

$$G_{T\,MAX} = \cosh^2\left[\frac{\pi\sqrt{\omega_{so}\,\omega_{io}}\;C_1\,(3\,C_o-\omega_{so}\,\omega_{io}\,C_1^2\,L)}{\omega_c^3\,L\,C_o^3}\right]. \tag{5-76}$$

The additional restriction, (5-74), which is due to the parasitic inductance, may be rewritten as

$$\ln\left[\frac{1+\sqrt{1-1/G_T}}{1-\sqrt{1-1/G_T}}\right]\cdot\omega_c \leq \left\{\pi\left[\frac{2\sqrt{\omega_{so}\,\omega_{io}}\;C_1}{C_o}\right]\right\}\left\{\frac{3\,C_o-\omega_{so}\,\omega_{io}\,C_1^2\,L}{\omega_c^2\,L\,C_o^2}\right\}, \tag{5-77}$$

from which we can infer that for $\omega_c^2 \geq \left\{3\,C_o-\omega_{so}\,\omega_{io}\,C_1^2\,L\right\}/\left\{L\,C_o^2\right\}$, (5-74) is the more restrictive bandwidth limitation. In other words, the presence of the series parasitic inductance serves to limit the bandwidth over which the optimum gain-bandwidth can be attained. This has also been found to be the case in the tunnel diode amplifiers.[13]

Furthermore, from the additional integral restriction (5-72), we can infer that

$$\frac{3\,C_o}{\omega_{so}\,\omega_{io}\,C_1^2} > L \geq L_o \;, \qquad (5\text{-}78)$$

since the integral in (5-72) is constrained to be positive from the fact that $G(\omega^2) \geq 1$ (and $G(\omega^2) \not\equiv 1$) for all real ω. The usefulness of condition (5-78) as a stability criterion for a dissipationless varactor is questionable, however, since in this analysis we have assumed not only the ideal-sideband terminations but also the conditions that $\omega_s \omega_i \cong \omega_{so} \omega_{io}$ and $Y_3(j\omega_s) \cong \overline{Y}_3(j\omega_i)$.

CHAPTER VI

MULTIPLE VARACTOR PARAMETRIC AMPLIFIERS

VI.1 Parametric Amplifiers with n Varactor Diodes

Before embarking on the actual treatment of parametric amplifiers incorporating n varactor diodes, we shall describe the complex normalization technique for a scattering matrix of a passive n-port normalized to a full $n \times n$ matrix. This is relevant because of a unique characteristic of an n-varactor system; namely, although they are uncoupled physically, their electrical behavior appears as a completely coupled system.

In Section III.2, the scattering matrix of an n-port N normalized to n uncoupled impedances was described. Now we shall generalize this technique to define a scattering matrix for N normalized to a full $n \times n$ matrix Z. It has been shown[10] that the same technique carried over, provided

$$R(j\omega) = \frac{Z(j\omega) + Z^*(j\omega)}{2} > 0 \quad , \qquad \omega \in W \quad ; \tag{6-1}$$

then $R(j\omega)$ possesses a unique hermitian, positive definite square root $R^{1/2}$. Equations (3-3) and (3-4) are replaced by

$$2\, R^{1/2}\, \underset{\sim}{a} = \underset{\sim}{V} + Z\, \underset{\sim}{I} \quad , \tag{6-2}$$

$$2\, R^{1/2}\, \underset{\sim}{b} = \underset{\sim}{V} - Z^*\, \underset{\sim}{I} \quad . \tag{6-3}$$

Finally,

$$S = R^{-1/2} (Z_N - Z^*)(Z_N + Z)^{-1} R^{1/2} \quad , \tag{6-4}$$

which corresponds to Eq. (3-16) for the case $Z = \text{diag}\left[z_1, z_2, \ldots, z_n\right]$.

The amplifier we are investigating now is that employing n varactor diodes embedded in an arbitrary lossless nonreciprocal (n + 2)-port N. Suppose the n linear variable capacitors are characterized by

$$
\begin{aligned}
C_1(t) &= C_{01} + 2C_{11} \cos(\omega_o t + \theta_1) \\
C_2(t) &= C_{02} + 2C_{12} \cos(\omega_o t + \theta_2) \\
&\;\;\vdots \qquad\quad \vdots \\
C_n(t) &= C_{on} + 2C_{1n} \cos(\omega_o t + \theta_n)
\end{aligned}
\tag{6-5}
$$

Fig. 11 is then a schematic of a parametric amplifier using n varactor diodes.

Let us first consider the case where $\theta_r = 0$, $r = 1, 2, \ldots, n$. Then the inverting device is characterized by

$$I_{sr} = -j\omega_s C_{or} V_{sr} - j\omega_s C_{1r} \overline{V}_{ir} , \tag{6-6}$$

$$\overline{I}_{ir} = j\omega_i C_{1r} V_{sr} + j\omega_i C_{or} \overline{V}_{ir} , \tag{6-7}$$

$r = 1, 2, \ldots, n$. Let

$$\underset{\sim}{I}_s = (I_{s1}, I_{s2}, \ldots, I_{sn})' \tag{6-8}$$

$$\underset{\sim}{V}_s = (V_{s1}, V_{s2}, \ldots, V_{sn})' \tag{6-9}$$

$$\underset{\sim}{\overline{I}}_i = (\overline{I}_{i1}, \overline{I}_{i2}, \ldots, \overline{I}_{in})' \tag{6-10}$$

$$\underset{\sim}{\overline{V}}_i = (\overline{V}_{i1}, \overline{V}_{i2}, \ldots, \overline{V}_{in})' \; ; \tag{6-11}$$

then

$$\underset{\sim}{I}_s = -j\omega_s C_o \underset{\sim}{V}_s - j\omega_s C_1 \overline{\underset{\sim}{V}}_i \qquad (6\text{-}12)$$

$$\overline{\underset{\sim}{I}}_i = j\omega_i C_1 \underset{\sim}{V}_s + j\omega_i C_o \overline{\underset{\sim}{V}}_i \ , \qquad (6\text{-}13)$$

where

$$C_o = \text{diag}\left[C_{01}, C_{02}, \ldots, C_{on}\right] \equiv \mathcal{S}_o^{-1} \qquad (6\text{-}14)$$

$$C_1 = \text{diag}\left[C_{11}, C_{12}, \ldots, C_{1n}\right] \equiv \mathcal{S}_1^{-1} \ . \qquad (6\text{-}15)$$

Using

$$-\underset{\sim}{I}_s = \left[Y_D(j\omega_s) + j\omega_s C_o\right] \underset{\sim}{V}_s \qquad (6\text{-}16)$$

$$\overline{\underset{\sim}{I}}_i = \left[\overline{Y}_N(j\omega_i) + j\omega_i C_o\right] \overline{\underset{\sim}{V}}_i \ , \qquad (6\text{-}17)$$

it can be shown that

$$Y_D(j\omega_s) = -\omega_s \omega_i C_1 \overline{Z}_N(j\omega_i) C_1 \qquad (6\text{-}18)$$

or

$$Z_D(j\omega_s) = -\frac{1}{\omega_s \omega_i} \mathcal{S}_1 \overline{Y}_N(j\omega_i) \mathcal{S}_1 \ . \qquad (6\text{-}19)$$

Y_N is the $n \times n$ admittance matrix of an n-port $\hat{N}$ facing the n varactors (see Fig. 12). The expression for $Z_D(j\omega_s)$ demonstrates explicitly a unique property of a parametric device incorporating n time-variable elements; namely, even though the n varactors appear physically as n uncoupled elements, as far as their electrical properties are concerned, they behave as if there is coupling between every pair of varactors. In essence, $Z_D(j\omega_s)$ is no longer diagonal, having taken into account the loading effect of all the varactors.

Let S be the scattering matrix of the (n + 2)-port N (Fig. 12) normalized to R_g at port 1, R_L at port 2, and a full $n \times n$ matrix

$$Z(j\omega_s) = -\overline{Z}_D(j\omega_s) = \frac{1}{\omega_s \omega_i} \mathcal{J}_1 Y_N(j\omega_i) \mathcal{J}_1 \quad , \tag{6-20}$$

at ports 3 to $n + 2$, the paraconjugate match condition at the varactor diode ports. Denote S by

$$S = \left[\begin{array}{cc|c} s_{11} & s_{12} & \\ & & S_2 \\ s_{21} & s_{22} & \\ \hline & & \\ \multicolumn{2}{c|}{S_3} & S_N \end{array}\right] \quad , \tag{6-21}$$

where S_N is the scattering matrix of the n-port $\hat{N}$ as shown in Fig. 12. Since N is lossless, S is unitary, i.e., $S^* S = 1_{n+2}$; it can be shown that

$$G_t(\omega_s) = \left| \frac{s_{12}(j\omega_s)}{\det S_N(j\omega_s)} \right|^2 \quad . \tag{6-22}$$

Hence

$$G_t(\omega_s) \leq \frac{1}{\left|\det S_N(j\omega_s)\right|^2} \equiv G(\omega_s) \quad . \tag{6-23}$$

From Eq. (6-4), $S_N(j\omega_s)$ is given by

$$\begin{aligned} S_N(j\omega_s) &= R^{-1/2}(j\omega_s) \left[Z_N(j\omega_s) - Z^*(j\omega_s)\right] \left[Z_N(j\omega_s) + Z(j\omega_s)\right]^{-1} R^{1/2}(j\omega_s) \\ &= R^{-1/2}(j\omega_s) \left[Z_N(j\omega_s) - \frac{1}{\omega_s \omega_i} \mathcal{J}_1 Y_N^*(j\omega_i) \mathcal{J}_1\right] \left[Z_N(j\omega_s) + \frac{1}{\omega_s \omega_i} \mathcal{J}_1 Y_N(j\omega_i) \mathcal{J}_1\right]^{-1} \\ &\qquad R^{1/2}(j\omega_s) \quad . \end{aligned} \tag{6-24}$$

It follows that

$$\det S_N(j\omega_s) = \frac{\det \left[\omega_s \omega_i - Y_N(j\omega_s)\, \mathcal{A}_1\, Y_N^*(j\omega_i)\, \mathcal{A}_1\right]}{\det \left[\omega_s \omega_i + Y_N(j\omega_s)\, \mathcal{A}_1\, Y_N(j\omega_i)\, \mathcal{A}_1\right]} \quad . \tag{6-25}$$

For simplicity, consider the n varactors to be identical, i.e., $\mathcal{A}_1 = C_1 1_n$ and $\mathcal{A}_o = C_o 1_n$; then Eq. (6-19) simplifies to

$$Z_D(j\omega_s) = -\frac{1}{\omega_s \omega_i C_1^2} \overline{Y}_N(j\omega_i) \quad , \tag{6-26}$$

and Eq. (6-25) becomes

$$\det S_N(j\omega_s) = \frac{\det \left[1_n - (1/\omega_s \omega_i C_1^2)\, Y_N(j\omega_s)\, Y_N^*(j\omega_i)\right]}{\det \left[1_n - (1/\omega_s \omega_i C_1^2)\, Y_N(j\omega_s)\, Y_N(j\omega_i)\right]} \quad . \tag{6-27}$$

Now we shall impose the assumptions that $\omega_s \omega_i \cong \omega_{so} \omega_{io}$ and $Y_N(j\omega_s) \cong Y_N^*(j\omega_i)$. The latter is the generalization of the assumption $Y(j\omega_s) \cong \overline{Y}(j\omega_i)$ we have employed in Chapter V for the single varactor case. Equation (6-27) then becomes

$$\det S_N(j\omega_s) = \frac{\det \left[\omega_{so} \omega_{io} C_1^2 - Y_N^2(j\omega_s)\right]}{\det \left[\omega_{so} \omega_{io} C_1^2 + Y_N(j\omega_s)\, Y_N^*(j\omega_s)\right]} \quad , \tag{6-28}$$

and

$$G(\omega_s^2) \equiv \frac{1}{\left|\det S_N(j\omega_s)\right|^2} = \left| \frac{\det \left[\omega_{so} \omega_{io} C_1^2 + Y_N(j\omega_s)\, Y_N^*(j\omega_s)\right]}{\det \left[\omega_{so} \omega_{io} C_1^2 - Y_N^2(j\omega_s)\right]} \right|^2 \quad . \tag{6-29}$$

The analytic continuation of $G(\omega_s^2)$ is given by

$$G(-p^2) = \frac{1}{\left[\det S_N(p)\right]\left[\det S_{N*}(p)\right]} = \frac{\det^2 \left[\omega_{so} \omega_{io} C_1^2 + Y_N(p)\, Y_{N*}(p)\right]}{\det \left[\omega_{so} \omega_{io} C_1^2 - Y_N^2(p)\right]\left[\omega_{so} \omega_{io} C_1^2 - Y_{N*}^2(p)\right]} \quad . \tag{6-30}$$

Clearly,

$$1 - \frac{1}{G(-p^2)} = \left[\frac{\det \left[Y_N(p) + Y_{N*}(p)\right] \sqrt{\omega_{so}\,\omega_{io}}\; C_1}{\det \left[\omega_{so}\,\omega_{io}\, C_1^2 + Y_N(p)\, Y_{N*}(p)\right]} \right]^2 . \tag{6-31}$$

Thus,

$$\sqrt{1 - \frac{1}{G(-p^2)}} = \frac{\det \left[Y_N(p) + Y_{N*}(p)\right] \sqrt{\omega_{so}\,\omega_{io}}\; C_1}{\det \left[\omega_{so}\,\omega_{io}\, C_1^2 + Y_N(p)\, Y_{N*}(p)\right]} , \tag{6-32}$$

where the positive sign is chosen. Then

$$\frac{1 - \sqrt{1 - 1/G(-p^2)}}{1 + \sqrt{1 - 1/G(-p^2)}} = \frac{\det \left[\sqrt{\omega_{so}\,\omega_{io}}\; C_1 - Y_N(p)\right] \left[\sqrt{\omega_{so}\,\omega_{io}}\; C_1 - Y_{N*}(p)\right]}{\det \left[\sqrt{\omega_{so}\,\omega_{io}}\; C_1 + Y_N(p)\right] \left[\sqrt{\omega_{so}\,\omega_{io}}\; C_1 + Y_{N*}(p)\right]}$$

$$= \det \left[\sqrt{\omega_{so}\,\omega_{io}}\; C_1 - Y_N(p)\right] \left[\sqrt{\omega_{so}\,\omega_{io}}\; C_1 + Y_N(p)\right]^{-1} \left[\sqrt{\omega_{so}\,\omega_{io}}\; C_1 - Y_{N*}(p)\right]$$

$$\left[\sqrt{\omega_{so}\,\omega_{io}}\; C_1 + Y_{N*}(p)\right]^{-1} . \tag{6-33}$$

Defining

$$\tilde{S}_N(p) = \left[\sqrt{\omega_{so}\,\omega_{io}}\; C_1 - Y_N(p)\right] \left[\sqrt{\omega_{so}\,\omega_{io}}\; C_1 - Y_N(p)\right]^{-1} , \tag{6-34}$$

Equation (6-34) can be written as

$$\frac{1 - \sqrt{1 - 1/G(-p^2)}}{1 + \sqrt{1 - 1/G(-p^2)}} = \left[\det \tilde{S}_N(p)\right] \left[\det \tilde{S}_{N*}(p)\right] . \tag{6-35}$$

But $\tilde{S}_N$ can be interpreted as the scattering matrix of an n-port, each of whose ports must be shunted by a capacitance at least equal to C_o. Therefore, det $\tilde{S}_4$ must be subjected to the integral restriction $(C \geq C_o)$

$$\int_o^\infty \ell n \frac{1}{\left|\det \tilde{S}_N(j\omega)\right|} d\omega = \frac{2\pi n \sqrt{\omega_{so}\,\omega_{io}}\, C_1}{C} . \quad (6\text{-}36)$$

Combining Eqs.(6-35) and (6-36) we obtain

$$\int_o^\infty \ell n \left[\frac{1 + \sqrt{1 - 1/G(\omega^2)}}{1 - \sqrt{1 - 1/G(\omega^2)}}\right] d\omega \leq \frac{2\pi n \sqrt{\omega_{so}\,\omega_{io}}\, C_1}{C_o} , \quad (6\text{-}37)$$

which is the fundamental gain-bandwidth restriction independent of the equalizer.

For the ideal flat response shape, we choose $G(\omega^2)$ = a constant $(G_T)_n$ over the band $0 < \omega < \omega_c$. Then, irrespective of its behavior outside this band,

$$\left[\frac{1 + \sqrt{1 - 1/(G_T)_n}}{1 - \sqrt{1 - 1/(G_T)_n}}\right] \leq \exp\left[\frac{2\pi n \sqrt{\omega_{so}\,\omega_{io}}\, C_1}{C_o}\right] . \quad (6\text{-}38)$$

Therefore,

$$(G_T)_n \leq (G_{T\,MAX})_n = \cosh^2\left[\frac{n\pi \sqrt{\omega_{so}\,\omega_{io}}\, C_1}{\omega_c C_o}\right] , \quad (6\text{-}39)$$

which is the optimum gain-bandwidth relation for the ideal flat response with n identical varactors. For n varactors with parameters C_{or} and C_{1r}, $r = 1, 2, \ldots, n$, a straightforward extension of our analysis shows that

$$(G_T)_n \leq (G_{T\,MAX})_n = \cosh^2 \left[\frac{\pi \sqrt{\omega_{so}\,\omega_{io}}}{\omega_c} \sum_{r=1}^{n} \frac{C_{1r}}{C_{or}} \right] . \tag{6-40}$$

Equation (6-39) or Eq. (6-40) can be realized by cascading n optimum nonreciprocal amplifiers of the type shown in Fig. 7. The circulators used in such a cascaded amplifier provide the isolation between the various stages using single varactors.

VI.2 Extension to Traveling-Wave Parametric Amplifiers

The properties of traveling-wave and distributed parametric amplifiers employing discrete nonlinear reactances have been studied extensively.[24,25,26] But several important questions remain to be answered. For example, it is desirable to find the optimum pumping phase to give maximum stable transducer power gain for the arbitrary distributed parametric structure. Previous papers have all imposed the synchronous phase condition; when this is satisfied, there is no guarantee that the maximum negative resistance is obtained because this is dependent on the product $V_s V_i$.[25] Furthermore, no definite bandwidth estimation is available since most analysis work has been done on the steady-state behavior only. In this section we shall merely extend some of the results of the previous section and present some preliminary results from which further work can be continued to answer some of the outstanding questions.

From Eq. (6-4), we can obtain

$$\begin{aligned} \det(S_N) &= \det(Z_N - Z^*)(Z_N + Z)^{-1} \\ &= \det(Z_N + Z)^{-1}(Z_N - Z^*) \end{aligned} . \tag{6-41}$$

By defining

$$\phi \equiv 1 - (Z_N + Z)^{-1}(Z_N - Z^*) , \tag{6-42}$$

it is clear that

$$\phi = 2(Z_N + Z)^{-1} R \quad , \tag{6-43}$$

where

$$R = \frac{1}{2}(Z + Z^*) \quad .$$

Let the eigenvalues of ϕ be denoted by λ_r and $u_r = \lambda_r^{-1}$, $(r = 1, 2, \ldots, n)$; then

$$2(Z_N + Z)^{-1} R \underset{\sim}{x} = \lambda \underset{\sim}{x} \quad , \tag{6-44}$$

$$u R \underset{\sim}{x} = \frac{1}{2}(Z_N + Z) \underset{\sim}{x} \quad , \tag{6-45}$$

and by pre-multiplying both sides of (6-45) by $\underset{\sim}{x}^*$ and adding its conjugate, we obtain

$$2 \operatorname{Re} u = 1 + \frac{\underset{\sim}{x}^* R_{N_1} \underset{\sim}{x}}{\underset{\sim}{x}^* R \underset{\sim}{x}} \quad , \tag{6-46}$$

where $R_{N_1} \equiv R_N(\omega_s)$.

Equations (6-19) and (6-20) are to be modified, respectively, to

$$Z_D(j\omega_s) = -\frac{1}{\omega_s \omega_i} \mathscr{S}_1 \overline{Y}_N(j\omega_i) \overline{\mathscr{S}}_1 \quad , \tag{6-47}$$

$$Z(j\omega_s) = -\overline{Z}_D(j\omega_s) = \frac{1}{\omega_s \omega_i} \mathscr{S}_1 Y_N(j\omega_i) \overline{\mathscr{S}}_1 \quad , \tag{6-48}$$

where $\mathscr{S}_1 = C_1^{-1}$ is now interpreted with

$$C_1 = \operatorname{diag}\left[C_{11} e^{-j\theta_1}, C_{12} e^{-j\theta_2}, \ldots, C_{1n} e^{-j\theta_n}\right] \quad . \tag{6-49}$$

Thus

$$R = \frac{1}{\omega_s \omega_i} \mathcal{A}_1 G_{N_2} \bar{\mathcal{A}}_1 \quad , \tag{6-50}$$

where $G_{N_2} \equiv G_N(\omega_i)$. Substituting into Eq.(6-47),

$$2 \operatorname{Re} u = 1 + \frac{\underset{\sim}{x}^* R_{N_1} \underset{\sim}{x}}{\frac{1}{\omega_s \omega_i} \underset{\sim}{x}^* \mathcal{A}_1 G_{N_2} \bar{\mathcal{A}}_1 \underset{\sim}{x}} \quad , \tag{6-51}$$

or, for $\underset{\sim}{y} \equiv \bar{\mathcal{A}}_1 \underset{\sim}{x}$,

$$2 \operatorname{Re} u = 1 + \frac{\underset{\sim}{x}^* R_{N_1} \underset{\sim}{x}}{\frac{1}{\omega_s \omega_i} \underset{\sim}{y}^* G_{N_2} \underset{\sim}{y}} \quad . \tag{6-52}$$

Therefore,

$$\operatorname{Re} u \geq \frac{1}{2} \quad , \tag{6-53}$$

i.e., all the eigenvalues of ϕ must have positive real parts.

Denote the eigenvalues of $(Z_N + Z)^{-1}(Z_N - Z^*)$ by γ_r, $r = 1, 2, \ldots, n$; then, since $\lambda_r = u_r^{-1}$ are the eigenvalues of $\phi \equiv 1 - (Z_N + Z)^{-1}(Z_N - Z^*)$,

$$\lambda_r = 1 - \gamma_r \quad , \qquad r = 1, 2, \ldots, n \quad . \tag{6-54}$$

It is well known that

$$\det(\phi) = \prod_{r=1}^{n} \lambda_r \quad . \tag{6-55}$$

Hence

$$\det(S_N) = \det (Z_N + Z)^{-1}(Z_N - Z^*)$$

$$= \prod_{r=1}^{n} (\gamma_r) = \prod_{r=1}^{n} (1 - \lambda_r) = \prod_{r=1}^{n} \left(1 - \frac{1}{u_r}\right) \quad . \tag{6-56}$$

It follows that

$$\left|\det(S_N)\right|^2 = \prod_{r=1}^{n}\left[1 - (2\,\mathrm{Re}\,u_r)\left|\lambda_r\right|^2 + \left|\lambda_r\right|^2\right] \quad . \tag{6-57}$$

From Eq.(6-45),

$$\lambda_r = \frac{\underset{\sim}{x}_r^{*} R\, \underset{\sim}{x}_r}{\underset{\sim}{x}_r^{*} \frac{1}{2}(Z_{N_1} + Z)\, \underset{\sim}{x}_r} \tag{6-58}$$

and

$$\left|\lambda_r\right|^2 = \frac{\left[(1/\omega_s\omega_i)\, \underset{\sim}{x}_r^{*} \mathscr{J}_1 G_{N_2} \bar{\mathscr{J}}_1 \underset{\sim}{x}_r\right]^2}{\left|\underset{\sim}{x}_r^{*} \frac{1}{2}(Z_{N_1} + Z)\, \underset{\sim}{x}_r\right|^2} \quad . \tag{6-59}$$

Substituting Eqs.(6-52) and (6-59) into Eq.(6-57),

$$\left|\det(S_N)\right|^2 = \prod_{r=1}^{n}\left[1 - \frac{4(\underset{\sim}{x}_r^{*} R_{N_1} \underset{\sim}{x}_r)\left[(1/\omega_s\omega_i)\, \underset{\sim}{y}_r^{*} G_{N_2} \underset{\sim}{y}_r\right]}{\left|\underset{\sim}{x}_r^{*} Z_{N_1} \underset{\sim}{x}_r + \frac{1}{\omega_s\omega_i} \underset{\sim}{y}_r^{*} Y_{N_2} \underset{\sim}{y}_r\right|^2}\right] , \tag{6-60}$$

and finally

$$\left|\det(S_N)\right|^2 = \prod_{r=1}^{n}\left|\frac{\underset{\sim}{x}_r^{*} Z_{N_1} \underset{\sim}{x}_r - \frac{1}{\omega_s\omega_i} \underset{\sim}{y}_r^{*} Y_{N_2} \underset{\sim}{y}_r}{\underset{\sim}{x}_r^{*} Z_{N_1} \underset{\sim}{x}_r + \frac{1}{\omega_s\omega_i} \underset{\sim}{y}_r^{*} Y_{N_2} \underset{\sim}{y}_r}\right|^2 , \tag{6-61}$$

where $Z_{N_1} \equiv Z_N(j\omega_s)$, $Y_{N_2} \equiv Y_N(j\omega_i)$ and $\underset{\sim}{y} \equiv \bar{\mathscr{J}}_1 \underset{\sim}{x}$. Expression (6-61) contains the solution of the problem of finding the optimum pumping phase condition to give maximum stable single-frequency transducer power gain. It involves essentially a study of the extremal properties of this eigenvalue problem.

Another difficulty manifests itself through the power gain expression

$$G_t = \frac{|s_{12}|^2}{|\det S_N|^2} \quad , \tag{6-62}$$

where $|s_{12}|^2$ can be interpreted as the backward gain when the diode ports are terminated in $-\overline{Z}_D(j\omega_s)$, a full $n \times n$ matrix, and in general $|s_{12}|^2$ depends on the pumping phase. Although we have succeeded in showing that $G_t/|s_{12}|^2$ is related to the equalizer impedance Z_N only, the obstacle here is our inability to delineate $|s_{12}|^2$ more closely. A limited answer lies in the generalization of Eq.(3-31); i.e.,

$$G_t = \left| \frac{s_{12}}{\det(S_N)} \right|^2 \leq \frac{1}{4} \left[1 + \frac{1}{|\det S_N|} \right]^2 \quad , \tag{6-63}$$

where for the optimum case the equality holds, but for most practical amplifiers only the inequality applies. In any case, the transducer power gain is bounded by the quantity

$$\frac{1}{4} \left\{ 1 + \prod_{r=1}^{n} \left| \frac{\underset{\sim}{x}_r^* Z_{N_1} \underset{\sim}{x}_r + \frac{1}{\omega_s \omega_i} \underset{\sim}{y}_r^* Y_{N_2} \underset{\sim}{y}_r}{\underset{\sim}{x}_r^* Z_{N_1} \underset{\sim}{x}_r - \frac{1}{\omega_s \omega_i} \underset{\sim}{y}_r^* Y_{N_2} \underset{\sim}{y}_r} \right| \right\}^2 \quad . \tag{6-64}$$

CHAPTER VII

NOISE PERFORMANCE OF A VARACTOR PARAMETRIC AMPLIFIER

The parametric amplifier has become an important and useful device largely because of the fact that it is capable of low-noise amplification, since ideally a pure reactance does not contribute thermal noise to the circuit. In practice, unavoidable loss will accompany the nonlinear reactance and the nonlinear element may further introduce additional non-thermal noise. In the solid-state version of the parametric amplifier, using a back-biased semiconductor diode, Uhlir[2] has proved that shot noise is negligibly small and only thermal noise appears to be of any significance.

It has been shown by Penfield[27] that the series resistance R_s, which is the spreading resistance of the p-n junction, provides a fundamental limit on the noise performance of the parametric amplifier, and that this ultimate limit is achieved at a finite pump frequency. In an earlier work, Heffner and Wade[28] presented a noise theory in which the varactor loss is neglected. They concluded that for the lowest noise figure, the pump frequency should be as high as possible.

A characteristic of the parametric device which sets it apart from conventional devices is that the noise processes come from both the signal and the idler sources. In a simplified analysis, it may be assumed that the only source of noise is thermal noise generated in the varactor loss at each frequency and in the dissipative part of the idler termination. Since the series resistance R_s appears physically in series with the variable element, it is more convenient to work with a series equivalent circuit and use the corresponding ideal sideband-termination assumption in which currents at all sideband frequencies are suppressed except at the desired frequencies.

Then, using the same procedure as in Section II.5, the small-signal impedance matrix, including the series resistance, is given by

$$\begin{bmatrix} \overline{V}_i \\ \\ V_s \end{bmatrix} = \begin{bmatrix} (R_s - \frac{S_o}{j\omega_i}) & (\frac{S_1}{j\omega_s}) \\ \\ (-\frac{S_1}{j\omega_i}) & (R_s + \frac{S_o}{j\omega_s}) \end{bmatrix} \begin{bmatrix} \overline{I}_i \\ \\ I_s \end{bmatrix} . \qquad (7\text{-}1)$$

Then, analogous to Eq. (2-39), we have

$$z_D(j\omega_s) = - \frac{S_1^2}{\omega_s \omega_i \overline{Z}_3(j\omega_i)} , \qquad (7\text{-}2)$$

where the d.c. elastance S_1 and R_s are incorporated into the equalizer. Now let $\hat{z}_D$ be the impedance of the varactor diode, including S_1 and R_s, and $\hat{Z}_3$ be the equalizer impedance, which is a completely arbitrary positive-real function; then

$$\hat{z}_D(j\omega_s) = z_D(j\omega_s) + R_s + \frac{S_1}{j\omega_s} , \qquad (7\text{-}3)$$

$$Z_3(j\omega_i) = \hat{Z}_3(j\omega_i) + R_s + \frac{S_1}{j\omega_i} , \qquad (7\text{-}4)$$

and Eq. (7-2) can be written as

$$\hat{z}_D(j\omega_s) = R_s + \frac{S_o}{j\omega_s} - \frac{S_1^2}{\omega_s \omega_i \left[\overline{\hat{Z}}_3(j\omega_i) + R_s - \frac{S_o}{j\omega_i} \right]} . \qquad (7\text{-}5)$$

It is known that the customary noise figure F is not a good criterion by which to judge amplifiers of low gain, because the noise of the succeeding stage is important. Haus and Adler[29] have introduced the concept of the "noise measure" M_e by the definition

$$M_e = \frac{(F_e - 1)}{(1 - 1/G_e)} , \qquad (7-6)$$

where F_e is the exchangeable noise figure and G_e is the exchangeable gain. It was proved by Penfield[30] that for a single negative resistance M_e is given by

$$M_e = \frac{-P_e}{k\, T_o\, \Delta f} , \qquad (7-7)$$

where P_e is the exchangeable noise power of the negative resistance. Assuming that the only source of noise is thermal noise generated in the varactor diode at ω_s and ω_i and in the dissipative part of the idler termination, Penfield[27] gave an expression for the noise measure M_e as follows:

$$M_e = \frac{T_D}{T_o}\,\frac{R_s}{(-R)} + \left[\frac{\omega_s}{\omega_i}\right] \frac{R_s\, T_D + R_i\, T_i}{T_o\,(R_s + R_i)} \left[\frac{R_s - R}{-R}\right] , \qquad (7-8)$$

where

$$R \equiv \mathrm{Re}\left[\hat{z}_D(j\omega_s)\right] , \qquad R_i \equiv \mathrm{Re}\left[\hat{Z}_3(j\omega_i)\right] ,$$

and T_D and T_i are the absolute temperatures of the varactor and the idler termination, respectively.

For a dissipationless varactor, $R_s = 0$, M_e reduces to

$$M_e = \left[\frac{\omega_s}{\omega_i}\right] \frac{T_i}{T_o} , \qquad (7-9)$$

which is the noise performance of a lossless varactor parametric amplifier and agrees with the result of Heffner and Wade. For a lossy varactor, it has been shown that by tuning the idler and choosing an optimum pump frequency an ultimate limit of the noise performance is achieved.

CHAPTER VIII

STABILITY OF LINEAR TIME-VARIABLE SYSTEMS

VIII.1 Introduction

In this chapter we shall show that within the framework of small-signal analysis and purely sinusoidal pump a linear three-term recursion relation is obtained. Then the stability is delineated by the roots of a transcendental relation derived from the infinite Hill determinant which is of the special type associated with continued fractions. By imposing the ideal sideband-terminations assumption, we consider the three- and four-frequency modes, the stability of which is implied if the equation $Z_3(p) + z_D(p) = 0$ has no root in the closed right-half p-plane.

It should be noted that most of our work on broadbanding is based on the ideal sidebandband-termination assumption, in which we assume that all the sideband frequencies are suppressed, except a few of direct interest. This is obviously not a physically realizable assumption. Nevertheless, this assumption has been used in most of the existing work in this field. The justification for imposing this assumption usually cited is that practical parametric amplifiers have been successfully built based on this model. It seems reasonable to assert that, for a stable system, predictions on the gain and bandwidth based on the three-frequency mode of operation would be qualitatively substantiated, at least on a first-order basis; i.e., the effects of the neglected sideband frequencies on the gain-bandwidth performance of a parametric amplifier may be incorporated using a perturbation technique. A similar assertion, however, cannot be made for the stability behavior. It is clear that the assumption of a three-frequency model may lead to serious difficulty as far as the stability of the system is concerned. Furthermore, if a system with an infinite number of frequencies is known as stable, there is no assurance that this will imply that a truncated version of the same system, consisting of a finite number of frequencies, will be stable.

In this chapter we shall consider systems consisting of varactors embedded in a lumped but otherwise arbitrary linear time-invariant environment. Then the equilibrium state is governed by an ordinary total differential equation with periodic coefficients. We can apply the Floquet theory which provides an existence theorem on the form of the solutions of the equation, but which of itself furnishes no information on the nature of the characteristic or Floquet exponents. Stability is implied if all the Floquet exponents have non-positive real parts. No general, easily applicable, necessary and sufficient conditions for stability are available. However, recently, a real-frequency stahility test was established by Youla[8] for the undriven response of varactor-like devices coupled through a linear, stable, time-invariant n-port.

For a linear variable circuit whose parameters are periodic functions of time of period T driven by the excitation e^{pt}, a statement can be made concerning the steady-state response of the driven system. In Section VIII.4 we shall prove that the steady-state response is a function of the form $e^{pt}\,V(t)$, where $V(t)$ is a periodic function of period T, provided the undriven system is stable.

VIII.2 Linear Three-Term Recursion Relations

We shall now consider the linear three-term recursion relations by first stating some of the conditions implicitly assumed in Section II.4. The schematic of a system consisting of n linear variable capacitors embedded in a lumped, linear, time-invariant environment is shown in Fig. 13. The current $\underset{\sim}{i}(t)$ at the terminals of n-port N can be expressed by

$$\underset{\sim}{i}(t) = -\frac{d}{dt}\left[C(t)\,\underset{\sim}{v}(t)\right] \quad , \qquad (8\text{-}1)$$

where

$$\underset{\sim}{v}(t) = \left[v_1(t),\ v_2(t),\ \ldots,\ v_n(t)\right]' \quad , \qquad (8\text{-}2)$$

$$\underset{\sim}{i}(t) = \left[i_1(t),\ i_2(t),\ \ldots,\ i_n(t)\right]' \quad , \qquad (8\text{-}3)$$

and

$$C(t) = \text{diag}\left[C_1(t),\ C_2(t),\ \ldots,\ C_n(t)\right] \quad , \tag{8-4}$$

$$C(t) = C(t+T) \quad . \tag{8-5}$$

Since the n-port N is assumed to be lumped, the equilibrium state of the system is governed by an ordinary total differential equation with periodic coefficients of period $T = \omega_o/2\pi$. Then, according to the Floquet theory, every solution of the equation has the form

$$\underset{\sim}{v}(t) = e^{\mu t}\, \underset{\sim}{\psi}(t) \quad , \tag{8-6}$$

where $\underset{\sim}{\psi}(t)$ is periodic of period T. In addition, the lumped, linear, time-invariant N can be described by an input-output relation

$$\underset{\sim}{v}(t) = \underset{\sim}{v}_o(t) + \int_o^t W(\tau)\, \underset{\sim}{i}(t-\tau)\, d\tau \quad , \qquad t \geq 0 \quad , \tag{8-7}$$

where $W(\tau)$ is the $n \times n$ matrix weighting function of n-port N and $\underset{\sim}{v}_o(t)$ takes into account the initial conditions.

Developing the periodic functions $C(t)$, $\underset{\sim}{\psi}(t)$ and $C(t)\underset{\sim}{\psi}(t)$ into Fourier series, we have (see Eq. (2-18))

$$C(t) = \sum_{k=-\infty}^{\infty} C_k\, e^{jk\omega_o t} \quad , \tag{8-8}$$

$$\underset{\sim}{\psi}(t) = \sum_{k=-\infty}^{\infty} \underset{\sim}{b}_k\, e^{jk\omega_o t} \quad , \tag{8-9}$$

and

$$C(t)\,\underset{\sim}{\psi}(t) = \sum_{k=\infty}^{\infty} \underset{\sim}{d}_k\, e^{jk\omega_o t} \quad , \tag{8-10}$$

where $\omega_o = 2\pi T$, and it is to be noted that C_k a:e now matrix coefficients instead of scalars as in Eq.(2-18). The vector coefficients $\underset{\sim}{d}_k$ can be expressed in terms of $\underset{\sim}{b}_k$ and C_k as

$$\underset{\sim}{d}_r = \sum_{k=-\infty}^{\infty} C_k \underset{\sim}{b}_{r-k} , \quad (r = 0, \pm 1, \pm 2, \ldots) . \tag{8-11}$$

It has been shown[8] that, by assuming that the n-port N satisfies both the exponentially stable and derivative-stable conditions, it is possible to obtain the following vector expression relating $\underset{\sim}{b}_r$ and $\underset{\sim}{d}_r$:

$$\underset{\sim}{b}_r = (\mu + jr\omega_o) Z(\mu + jr\omega_o) \underset{\sim}{d}_r , \tag{8-12}$$

where Z is the impedance matrix of N. The significant fact to be noted here is that it is necessary for N to be derivative-stable, i.e., the derivative of $W(t, \tau)$ with respect to τ should be integrable for all t, in addition to the exponentially stable requirement.

By substituting Eq.(8-11) into Eq.(8-12), a system of an infinite set of equations for $\underset{\sim}{b}_r$ is obtained. For the system to be compatible, the associated Hill determinant must vanish. The stability characteristic of this system is delineated by the roots of a transcendental equation derived from this infinite Hill determinant.[31]

Assume the varactors are weakly pumped (or, equivalently, assume the pump is purely sinusoidal); then $C_1(t), C_2(t), \ldots, C_n(t)$ are as given by Eqs.(6-5), and C_o and C_1 are defined by Eqs.(6-14) and (6-49). Substituting into Eq.(8-12) coupled with Eq.(8-11) yields the linear three-term recursion relations

$$\underset{\sim}{d}_r = C_1 \underset{\sim}{b}_{r-1} + C_o \underset{\sim}{b}_r + \overline{C}_1 \underset{\sim}{b}_{r+1} \tag{8-13}$$

and

$$\underset{\sim}{b}_{r+1} + \Delta_r \underset{\sim}{b}_r + \underset{\sim}{b}_{r-1} = \underset{\sim}{0}_r , \quad (r = 0, \pm 1, \pm 2, \ldots) , \tag{8-14}$$

where

$$\Delta_r = 2\rho^{-1}\left[1_n - \frac{1}{(\mu + jr\omega_o)} \mathcal{S}_o Y(\mu + jr\omega_o)\right] , \tag{8-15}$$

$$\rho = \text{diag}\left[\rho_1 e^{j\theta_1}, \rho_2 e^{j\theta_2}, \ldots, \rho_n e^{j\theta_n}\right] . \tag{8-16}$$

For simplicity, consider a single weakly-pumped varactor

$$C(t) = C_o + 2C_1 \cos(\omega_o t) , \tag{8-17}$$

where $\rho = 2C_1/C_o$ is the pumping ratio. Then the three-term recursion relations are

$$d_r = C_1 b_{r-1} + C_o b_r + C_1 b_{r+1} \tag{8-18}$$

and

$$b_{r+1} + \Delta_r b_r + b_{r-1} = 0 , \quad (r = 0, \pm 1, \pm 2, \ldots) , \tag{8-19}$$

where

$$\Delta_r = \frac{2}{\rho}\left[1 - \frac{1}{(\mu + jr\omega_o) C_o z(\mu + jr\omega_o)}\right] . \tag{8-20}$$

Equation (8-19) is the linear three-term recursion relation which defines the stability behavior of a single-varactor parametric device in which all the sideband frequencies have been taken into consideration.

VIII.3 Truncation Technique for an Infinite System

Stability is implied if all the Floquet exponents have non-positive real parts, i.e., Re $\mu \leq 0$. The Floquet exponent μ is any solution of the infinite determinant associated with the system represented by (8-19)

$$b_{r+1} + \Delta_r b_r + b_{r-1} = 0 \quad , \qquad (r = 0, \pm 1, \pm 2, \ldots) \quad , \tag{8-19}$$

where

$$\Delta_r = \frac{2}{\rho}\left[1 - \frac{1}{(\mu + jr\omega_o) C_o z(\mu + jr\omega_o)}\right] \quad . \tag{8-20}$$

For the three-frequency mode of operation, which is the primary basis for most of our broadbanding work, all the sidebands are suppressed except the signal and the idler; that is,

$$b_r = 0 \quad \text{for all } r \text{ except } r = 0 \text{ and } r = -1 \quad . \tag{8-21}$$

Then we obtain a system of two equations as follows:

$$\text{for} \quad r = 0, \qquad \Delta_o b_o + b_{-1} = 0 \quad , \tag{8-22}$$

$$\text{for} \quad r = -1, \qquad b_o + \Delta_{-1} b_{-1} = 0 \quad . \tag{8-23}$$

In order for it to be compatible,

$$\det \begin{bmatrix} \Delta_o & 1 \\ 1 & \Delta_{-1} \end{bmatrix} = 0 \quad , \tag{8-24}$$

or

$$\Delta_o \Delta_{-1} = 1 \quad . \tag{8-25}$$

Substituting (8-20) into Eq.(8-25), we have

$$(C_o/C_1)^2 \frac{1}{\mu(\mu - j\omega_o) C_o^2} \left[\mu C_o - y(p)\right] \left[(\mu - j\omega_o) C_o - y(\mu - j\omega_o)\right] = 1 \quad . \tag{8-26}$$

Since $Y_3(\mu) = y(\mu) - \mu C_o$ and $Y_3(\mu - j\omega_o) = y(\mu - j\omega_o) - (\mu - j\omega_o) C_o$, where $Y_3 = Z_3^{-1}$ and Z_3 is the equalizer impedance facing the variable part of the varactor diode, Eq. (8-26) can be written in the form

$$Y_3(\mu)\, Y_3(\mu - j\omega_o) = \mu(\mu - j\omega_o)\, C_1^2 \quad . \tag{8-27}$$

Therefore, for stability, all the solutions of Eq. (8-27) must be such that Re $\mu \leq 0$. We can state the stability criterion for the three-frequency model alternatively, as follows: For stability, the equation

$$Z_3(p) + z_D(p) = 0 \tag{8-28}$$

must not have a solution in the entire closed right-half p-plane if we wish to include stability at the real-frequency boundary.

We now consider the four-frequency mode of operation in which we allow only the signal, the idler and the upper-sideband frequencies to exist. Let $b_r = 0$ for all r except $r = 0$, $r = -1$, and $r = +1$; then it follows from (8-20) that

$$\det \begin{bmatrix} \Delta_1 & 1 & 0 \\ 1 & \Delta_o & 1 \\ 0 & 1 & \Delta_{-1} \end{bmatrix} = 0 \tag{8-29}$$

or

$$\Delta_1 \Delta_o \Delta_{-1} - \Delta_1 - \Delta_{-1} = 0 \quad . \tag{8-30}$$

In terms of Y_3, Eq. (8-31) becomes

$$\mu(\mu - j\omega_o)\, C_1^2\, Z_3(\mu)\, Z_3(\mu - j\omega_o) + \mu(\mu + j\omega_o)\, C_1^2\, Z_3(\mu)\, Z_3(\mu + j\omega_o) = 1 \quad . \tag{8-31}$$

Since the diode impedance for a weakly-pumped varactor operating in a four-frequency mode is given by Eq. (2-38) as

$$z_D(\mu) = -\frac{1}{\mu(\mu - j\omega_o)C_1^2 Z_3(\mu - j\omega_o) + \mu(\mu + j\omega_o)C_1^2 Z_3(\mu + j\omega_o)} ; \qquad (8\text{-}32)$$

thus the stability can again be stated in terms of the solutions of the equation $Z_3(p) + z_D(p) = 0$.

Clearly, the stability criteria for the three-frequency and the four-frequency modes can be completely different. In this case the presence of upper-sideband frequency tends to reduce the negative resistance; hence it might have a stabilizing effect on the characteristic of the three-frequency model. This picture is misleading because if other neglected sideband frequencies are present, having mutually inverted spectrums, then additional negative resistances would be introduced into the system. It seems clear that for stability the complete system, with all its sidebands, must be considered.

The crux of the problem is to determine μ from the infinite determinantal equation which is, in general, transcendental. For systems governed by the linear three-term recursion relation,

$$b_{r+1} + \Delta_r b_r + b_{r-1} = 0 , \qquad (r = 0, \pm 1, \pm 2, \ldots) , \qquad (8\text{-}33)$$

the infinite determinant is of the special type associated with continued fractions. To develop this, first let

$$D_r \equiv \left(\frac{\rho}{2}\right) \Delta_r = \left[1 - \frac{1}{(\mu + jr\omega_o) C_o \, z(\mu + jr\omega_o)}\right] ; \qquad (8\text{-}34)$$

then

$$\left(\frac{\rho}{2}\right) b_{r+1} + D_r b_r + \left(\frac{\rho}{2}\right) b_{r-1} = 0 , \qquad (r = 0, \pm 1, \pm 2, \ldots) . \qquad (8\text{-}35)$$

The relative voltage amplitudes can be shown to be given by

$$\frac{b_r}{b_{r-1}} = \cfrac{-(\rho/2)}{D_r - \cfrac{(\rho/2)^2}{D_{r+1} - \cfrac{(\rho/2)^2}{D_{r+2} - \cdots}}} \quad , \qquad r \geq 1 \quad , \tag{8-36}$$

$$\frac{b_r}{b_{r+1}} = \cfrac{-(\rho/2)}{D_r - \cfrac{(\rho/2)^2}{D_{r-1} - \cfrac{(\rho/2)^2}{D_{r-2} - \cdots}}} \quad , \qquad r \leq -1 \quad . \tag{8-37}$$

Furthermore, in order for the system (8-35) to be compatible, expression (8-37) must equal the inverse of (8-36). Thus the infinite determinantal equation is completely equivalent to (let $r = 0$)

$$D_o - (\rho/2)^2 \left[\cfrac{1}{D_1 - \cfrac{(\rho/2)^2}{D_2 - \cfrac{(\rho/2)^2}{D_3 - \cdots}}} + \cfrac{1}{D_{-1} - \cfrac{(\rho/2)^2}{D_{-2} - \cfrac{(\rho/2)^2}{D_{-3} - \cdots}}} \right] = 0 \quad , \tag{8-38}$$

where the pumping ratio ρ is explicitly exhibited. Expression (8-38) is suitable for computation.

Let $r = 1$ in (8-36) and $r = -2$ in (8-37), and substitute these results into (8-38); it can be expressed equivalently as

$$D_o - (\rho/2)^2 \left[\frac{1}{D_{-1} + (\rho/2)(b_{-2}/b_{-1})} \right] + (\rho/2)(b_1/b_o) = 0 \quad , \tag{8-39}$$

or

$$\left[D_o + (\rho/2)(b_1/b_o)\right]\left[D_{-1} + (\rho/2)(b_{-2}/b_{-1})\right] = (\rho/2)^2 \quad . \qquad (8\text{-}40)$$

From the theory of linear three-term recursion relations, it can be shown[32] that

$$|b_1/b_o| \leq 1 \qquad (8\text{-}41a)$$

and

$$|b_{-2}/b_{-1}| \leq 1 \quad . \qquad (8\text{-}41b)$$

Thus the second term in each of the brackets in (8-40) would be bounded by $(\rho/2)$, where ρ is the pumping ratio. Using the above results, it can be concluded that the error introduced by a truncation technique became small as the number of terms used was increased. Unfortunately, we cannot deduce more specific conclusions.

It seems clear that the validity of the usual technique of truncating the infinite set of equations with a finite set and considering the solution of the finite set as an approximation to that of the infinite set must be carefully examined as applied to stability. Desoer[33] has presented an iterative method to estimate the effect of the infinite number of neglected frequency terms on the amplitudes. But, at the present at least, there is no known method to estimate the effect of the neglected terms on the eigenvalues of an infinite system.

VIII.4 A Stability Test and the Steady-State Response of a Driven System

Using an energy argument, a stability test is established[8] for the undriven response of varactor-like devices coupled through a linear, stable, time-invariant n-port. Although the result was originally proved under more general conditions, it will be used here for a lumped system in which the weakly-pumped varactors are uncoupled. First, we discuss in more detail the assumptions imposed on the n-port N in order to derive the fundamental relation (8-12).

The first assumption is that N is exponentially stable; i.e., for any choice of initial conditions

$$\lim_{t \to \infty} e^{-\epsilon t} \, || \underset{\sim}{v}_o(t) || = 0 \tag{8-42}$$

for all $\epsilon > 0$ and the weighting function is integrable

$$\int_0^{\infty} ||W(\tau)|| \, d\tau < \infty \quad . \tag{8-43}$$

It is to be noted that the exponential stability assumption is weaker than the usual stability definition which states that $\underset{\sim}{i}$ uniformly bounded for $t \geq 0$ implies that $\underset{\sim}{v}$ uniformly bounded for $t \geq 0$; hence, it actually requires that there exist a constant ℓ such that $||\underset{\sim}{v}_o(t)|| < \ell$ for $t \geq 0$. If we impose the condition that $\lim_{t \to \infty} ||\underset{\sim}{v}_o(t)|| = 0$, instead of (8-42), then the Floquet exponents cannot lie on the boundary. The second assumption, which is usually overlooked, is that N is also derivative-stable; i.e.,

$$\int_0^{\infty} \left|\left| \frac{d\,W(\tau)}{d\tau} \right|\right| d\tau < \infty \quad . \tag{8-44}$$

Using the relation (8-12) and an energy argument, it is then shown[8] that a sufficient real-frequency criterion for stability of an uncoupled weakly-pumped n-varactor device is given by

$$||\omega \, Z(j\omega)||_{MAX} \cdot C_M < 1 \tag{8-45}$$

where

$$C_M = \text{largest} \left\{ C_{01}(1+\rho_1),\ C_{02}(1+\rho_2),\ \ldots,\ C_{on}(1+\rho_n) \right\} \quad . \tag{8-46}$$

For a single weakly-pumped varactor, the criterion

$$|\omega \, z(j\omega)|_{MAX} < \frac{1}{C_o(1+\rho)} \tag{8-47}$$

ensures stability. Thus the stability criterion leads to easily obtained bounds on the pumping ratios of the time-variable elements.

Since the Floquet theory does not apply to a driven system, we shall now present a result which relates the steady-state response of a driven system consisting of a linear time-variable capacitor embedded in a linear time-invariant structure driven with the excitation e^{pt} to the response of the undriven system. More precisely, we wish to prove the following statement: Given a linear, lumped, variable system whose parameters are periodic functions of time. If this system is driven with the excitation e^{pt}, Re $p \geq 0$; then the steady-state response must be of the form

$$e^{pt} V(t) \quad ,$$

where $V(t)$ is periodic, having the same period as the time-varying parameters of the system, provided that the undriven system is stable in the sense that all the Floquet exponents have non-positive real parts.

To proceed with the proof of this statement, we first note that it is well known that any nth order linear differential equation can be transformed into an $n \times n$ system; thus, let us consider a system of equations of the type

$$\frac{dX}{dt} = A(t) X + e^{pt} 1_n \quad , \tag{8-48}$$

where

$$A(t) = A(t + T) \quad , \tag{8-49}$$

and X is an $n \times n$ matrix and $e^{pt} 1_n$ is the driver. Denote X_h as the matrix solution of the homogeneous system

$$\frac{dX_h}{dt} = A(t) X_h(t) \quad , \tag{8-50}$$

and let $X_h(t) = X_o(t)$ be a fundamental set of solutions, i.e., $\det \left(X_o(t)\right) \neq 0$; then the most general solution of the homogeneous system is given by

$$X_h(t) = X_o(t)\, B \quad , \tag{8-51}$$

where B is a constant matrix. Choosing the normalization $X_o(0) = 1_n$, it then follows from (8-49) that the most general solution is

$$X_h(t) = X_o(t)\, X_o(T) \quad . \tag{8-52}$$

According to Floquet theory, a fundamental matrix of the periodic lumped system (8-50) can always be represented as a product of a periodic matrix with the same period and a solution matrix for a system with constant coefficients. Thus $X_o(t)$ must be of the form

$$X_o(t) = P(t)\, e^{tK} \quad , \tag{8-53}$$

where $P(t) = P(t + T)$ and $P(t)$ is non-singular since $X_h(t)$ is a fundamental matrix.

To link the behavior of the solutions of (8-48) and (8-50), we shall use Volterra-type integral equations, and express the solution of (8-48) as solutions of the following integral equation[34] involving solutions of (8-50):

$$X(t) = X_o(t)\, B + \int_o^t X_o(t)\, X_o^{-1}(\tau)\, e^{p\tau}\, d\tau \quad . \tag{8-54}$$

Since the undriven system is assumed to be stable, the term $X_o(t)\, B$ will vanish as $t \rightarrow \infty$. Let $B = 0_n$; then a particular solution of (8-48) is given by

$$X(t) = \int_o^t X_o(t)\, X_o^{-1}(\tau)\, e^{p\tau}\, d\tau \quad . \tag{8-55}$$

Alternatively, it can be verified by direct substitution that

$$X_c(t) = \int_{-\infty}^{t} X_o(t)\, X_o^{-1}(\tau)\, e^{p\tau}\, d\tau \tag{8-56}$$

is also a particular solution. Substituting (8-53), (8-56) becomes

$$X_c(t) = P(t) \int_{-\infty}^{t} e^{(t-\tau)K}\, P^{-1}(\tau)\, e^{p\tau}\, d\tau \quad . \tag{8-57}$$

Now let $t - \tau = u$; we have

$$X_c(t) = e^{pt} \left\{ P(t) \int_{0}^{\infty} e^{uK}\, P^{-1}(t-u)\, e^{-pu}\, du \right\} \quad . \tag{8-58}$$

By hypothesis, the undriven system is stable, i.e., the eigenvalues of K are non-positive, and, noting the fact that $\mathrm{Re}\, p \geq 0$, the integral is convergent. Furthermore, by inspection, the matrix expression inside the bracket is periodic with period T. Therefore, we can conclude that the steady-state response of the driven system must be of the form

$$X_c(t) = e^{pt}\, V(t, p) \quad , \tag{8-59}$$

where $V(t, p)$ is periodic with period T and is a function of p. This result constitutes a representation theorem for the steady-state response of a driven system with an exponential driver e^{pt}.

CHAPTER IX

SUMMARY

Parametric devices utilizing back-biased semiconductor diodes exhibit several inherent properties which are not shared with other active devices. To cite a well-known property, the noise processes in a parametric device derive not only from the signal source but also from the sideband sources. In the course of developing a broadbanding theory for the parametric amplifier, some fundamental and perhaps unique properties are revealed. The first property is that the varactor, when embedded in an arbitrary passive environment, presents an impedance which can be characterized by a positive function rather than a positive-real function. The second and more important inherent property of these parametric devices is the quadratic nature of the expression of the scattering coefficient at the varactor diode port. This is due primarily to the interaction between the signal and the sideband circuits through the frequency-coupling action of the variable capacitor. In essence, the broadbanding of a parametric device is somewhat analogous to the problem of the design of a broadband equalizer to match a load impedance when the load impedance itself is a function of the characteristic of the equalizer under consideration!

It can be shown that an immediate consequence of the latter property is that the usual Butterworth or Tchebycheff response is not realizable as a transducer power gain function of the varactor parametric amplifier. It is known that by using a high-gain approximation, the broadbanding problem can be simplified so that the above-mentioned quadratic effect will not manifest itself. However, this approximation is not imposed in our theory; our point of view is that, besides the obvious theoretical reason for treating the broadbanding problem in its full generality, practically it might not be desirable to design a parametric amplifier with too high a gain.

A major part of this thesis is concerned with the development of a general and rigorous broadbanding theory for the varactor parametric amplifier. A general theory is embodied in Theorems 1 and 2 in which the prescribed transducer power gain function must satisfy certain restrictions derived in part from the parasitic elements of the varactor. Some realizability conditions on the transducer power gain functions are obtained which are indicative of the inherent properties of the parametric device. In general, two separate factorization processes must be performed which might lead to computational difficulty as the order of complexity of the system increases. This broadbanding technique is applied to the case where we assume $\omega_s \omega_i \cong \omega_{so} \omega_{io}$ and $Y_3(j\omega_s) \cong \overline{Y}_3(j\omega_i)$, and an illustrative design example is completely worked out starting from the prescribed transducer power gain function. The results agree with that obtained by an exact but more restrictive theory. For the general case, the theory in Chapter IV is the only available technique at our disposal.

In Chapter V, we consider cases in which certain assumptions can be made. Fundamental gain-bandwidth limitations are obtained which are independent of the associated circuitry. These fundamental limitations arise because of the presence of the associated parasitic elements. For the case $\omega_s \omega_i \cong \omega_{so} \omega_{io}$, an optimum gain-bandwidth upper-bound is derived using a property of an arbitrary bounded scattering coefficient which is unitary at any point on the $j\omega$-axis. The exact property is proved in Appendix B. This theoretical limitation is useful as an estimate of the optimum gain-bandwidth. When the additional assumption that $Y_3(j\omega_s) \cong \overline{Y}_3(j\omega_i)$ is invoked, an exact optimum gain-bandwidth theory is obtained. Synthesis procedures are developed for any physically realizable transducer power gain function. Modified Butterworth and Tchebycheff responses of arbitrary order are defined and realized using these synthesis procedures. It was proved that these modified responses are capable of achieving the optimum gain-bandwidth performance as the order is increased.

The properties of parametric amplifiers incorporating n varactor diodes are discussed in Chapter VI. Although we have n physically uncoupled varactors, it is shown that as far as their electrical behavior is concerned they appear as a completely coupled system. Some results of Chapter V are generalized for multiple varactor amplifiers, and some extensions are made to treat the traveling-wave version.

The noise performance of a varactor device is treated briefly, using the concept of the noise measure. The series spreading resistance R_s is included in the analysis to show its effect on the noise performance.

In Chapter VIII, the stability of systems incorporating n varactors embedded in an arbitrary, linear, stable, time-invariant environment is discussed. We first proved that the stability criterion that $Z_3(p) + z_D(p) = 0$ has no solution in the closed right-half p-plane is valid, provided the ideal sideband-terminations assumption leading to the three-frequency mode is invoked. The validity of the usual truncation technique as applied to stability considerations is discussed, but no definitive answer to this question is given. A real-frequency condition on the stability of the undriven system is available. Finally, we presented a representation theorem on the steady-state response of a driven system with an exponential driver. In conclusion, however, the crucial problem concerning the validity of the truncation technique remains unsolved and constitutes a major obstacle in the understanding of the theory of parametric devices. Another point worth considering is that since most work is confined to the linear mode, information concerning stationary states can be obtained only by treating the parametric device in a nonlinear mode of operation.

APPENDIX A

This appendix contains an example illustrating the broadbanding technique embodied in Theorems 3 and 4 in Chapter IV for the case $\omega_s \omega_i \cong \omega_{so} \omega_{io}$ and $Y_3(j\omega_s) \cong \overline{Y}_3(j\omega_i)$.

Prescribe the transducer power gain function as

$$G(\hat{\omega}^2) = \frac{\hat{\omega}^8 + \hat{\omega}^4(K_2 + 1) + [(K_2 + 1)/2]^2}{\hat{\omega}^8 + \hat{\omega}^4(K_2 + 1) + (K_2)} , \tag{A-1}$$

where $\hat{\omega} \equiv \omega/\omega_c$, $K_2 \geq 1$ and $G(0) = (K_2 + 1)^2/4K_2$. Thus

$$G(-\hat{p}^2) = \frac{\hat{p}^8 + \hat{p}^4(K_2 + 1) + [(K_2 + 1)/2]^2}{\hat{p}^8 + \hat{p}^4(K_2 + 1) + (K_2)} . \tag{A-2}$$

Then we can uniquely determine from the prescribed $G(-\hat{p}^2)$ the following:

$$s_o(\hat{p}) = \frac{\left[\hat{p}^2 + \hat{p}(K_2)^{1/4}\sqrt{2} + (K_2)^{1/2}\right]\left[\hat{p}^2 + \hat{p}\sqrt{2} + 1\right]}{\left[\hat{p}^2 + \hat{p}\left[(K_2 + 1)/2\right]^{1/4}\sqrt{2} + \left[(K_2 + 1)/2\right]^{1/2}\right]^2} , \tag{A-3}$$

$$b_o(\hat{p}) = \frac{\left[\hat{p}^2 - \hat{p}\left[(K_2 + 1)/2\right]^{1/4}\sqrt{2} + \left[(K_2 + 1)/2\right]^{1/2}\right]}{\left[\hat{p}^2 + \hat{p}\left[(K_2 + 1)/2\right]^{1/4}\sqrt{2} + \left[(K_2 + 1)/2\right]^{1/2}\right]} , \tag{A-4}$$

$$\lambda(\hat{p}) \equiv \sqrt{1 - \frac{1}{G(-\hat{p}^2)}} = \frac{\left[(K_2 - 1)/2\right]}{\left[\hat{p}^4 + \left[(K_2 + 1)/2\right]\right]} , \tag{A-5}$$

and there is a transmission zero at infinity of multiplicity four; i.e., $p_\alpha = \infty$, $k_\infty = 4$.

The coefficients of the power series expansions of $b_o(p)$ and $a_o(p)$ about the transmission zero at infinity are given, respectively, by

$$\begin{aligned} B_o^\infty &= 1 \\ B_1^\infty &= -\left[(K_2 + 1)/2\right]^{1/4}(2)(\sqrt{2}) \\ B_2^\infty &= \left[(K_2 + 1)/2\right]^{1/2}(4) \\ B_3^\infty &= -\left[(K_2 + 1)/2\right]^{3/4}(2)(\sqrt{2}) \quad , \end{aligned} \tag{A-6}$$

and

$$\begin{aligned} a_o^\infty &= 1 \\ a_1^\infty &= \left[(K_2^{1/4} + 1) - \left[(K_2 + 1)/2\right]^{1/4}(2)\right](\sqrt{2}) \\ a_2^\infty &= \left[(K_2^{1/4} + 1) - \left[(K_2 + 1)/2\right]^{1/4}(2)\right]^2 \end{aligned} \tag{A-7}$$

$$\begin{aligned} a_3^\infty = \Big[& K_2^{1/4}(K_2^{1/4} + 1)(\sqrt{2}) + (K_2^{1/4} + 1)\left[(K_2 + 1)/2\right]^{1/2}(4)(\sqrt{2}) \\ & - \left[(K_2 + 1)/2\right]^{1/4}\left[(K_2 + 1)/2\right]^{1/2}(2)(\sqrt{2}) - (K_2^{1/4} + 1)^2\left[(K_2 + 1)/2\right]^{1/4}(2)(\sqrt{2})\Big]. \end{aligned}$$

Imposing the conditions (4-68) and (4-69) on the coefficients, and using (4-62), we have

$$\begin{aligned} B_o^\infty &= A_o^\infty = a_o^\infty b_o^\infty \\ B_1^\infty &= A_1^\infty = a_o^\infty b_1^\infty + a_1^\infty b_o^\infty \\ B_2^\infty &= A_2^\infty = a_o^\infty b_2^\infty + a_1^\infty b_1^\infty + a_2^\infty b_o^\infty \quad , \end{aligned} \tag{A-8}$$

and

$$\frac{B_3^\infty - A_3^\infty}{\lambda_4^\infty} \geq \frac{C_o}{\sqrt{\omega_{so}\,\omega_{io}}\; C_1} \quad , \tag{A-9}$$

where

$$\lambda_4^\infty = (K_2 - 1)/2 \quad .$$

Substituting (A-6) and (A-7) into (A-8),

$$\begin{aligned} b_o^\infty &= 1 \\ b_1^\infty &= -(K_2^{1/4} + 1)\,(\sqrt{2}\,) \\ b_2^\infty &= (K_2^{1/4} + 1)^2 \quad , \end{aligned} \tag{A-10}$$

and

$$b(p) = \frac{\hat{p} - (K_2^{1/4} + 1)\,(\sqrt{2}/2)}{\hat{p} + (K_2^{1/4} + 1)\,(\sqrt{2}/2)} \quad . \tag{A-11}$$

Then, by substituting

$$\begin{aligned} A_3^\infty &= a_o^\infty b_3^\infty + a_1^\infty b_2^\infty + a_2^\infty b_1^\infty + a_3^\infty b_o^\infty \\ b_3^\infty &= -(K_2^{1/4} + 1)^3\,(\sqrt{2}/2) \\ \lambda_4^\infty &= (K_2 - 1)/4 \\ B_3^\infty &= -\left[(K_2 + 1)/2\right]^{3/4}(2)\,(\sqrt{2}\,) \end{aligned} \tag{A-12}$$

into condition (A-9), the gain-bandwidth restriction becomes

$$\frac{\sqrt{2}}{(K_2^{1/4} - 1)} \omega_c \geq \frac{C_o}{\sqrt{\omega_{so} \omega_{io}}\, C_1} , \qquad \text{(A-13)}$$

where K_2 and the d.c. gain $G(0)$ are related by $G(0) = (K_2 + 1)^2/4K_2$; the normalizing frequency ω_c and the true 3 db bandwidth are related by

$$\frac{w_c}{\omega_c} = \left\{ \left[(K_2 + 1)/2\right] \left[(K_2 - 1)\sqrt{\frac{1}{K_2^2 - 6K_2 + 1}} - 1\right] \right\}^{1/4} . \qquad \text{(A-14)}$$

Finally, from Eq.(4-70), $Y_3(\hat{p})$ is obtained as

$$Y_3(\hat{p}) = \sqrt{\omega_{so} \omega_{io}}\, C_1 Y(\hat{p}) = \sqrt{\omega_{so} \omega_{io}}\, C_1 \left[\frac{b_o(\hat{p}) - s(\hat{p})}{b_o(\hat{p})\sqrt{1 - 1/G(-\hat{p}^2)}} \right]$$

$$= \sqrt{\omega_{so} \omega_{io}}\, C_1 \left\{ \frac{\hat{p}^2 + \hat{p}\left[\frac{1}{2}(K_2^{1/4} + 1)\right] + \left[\frac{1}{2}(K_2^{1/2} + 1)\right]}{\hat{p}\left[\frac{1}{2}(K_2^{1/4} - 1)\right] + \left[\frac{1}{2}(K_2^{1/4} - 1)\right]} \right\} . \qquad \text{(A-15)}$$

APPENDIX B

The following theorem is used in the derivation of the optimum gain-bandwidth upper-bound in Section V.2.

Theorem 6:

Let $s(p)$ be an arbitrary bounded scattering coefficient (not necessarily real for real p) which is unitary at any point on the $j\omega$-axis, $p = j\omega_k$ ($|\omega_k|$ may be infinite), but $|s(p)| \not\equiv 1$; then the product $\beta_o\beta_1$ must be negative, where β_o and β_1 are the first two coefficients of the power series expansion of $s(p)$ in the neighborhood of $p = j\omega_k$.

Proof: By hypothesis, $s(p)$ is a bounded scattering coefficient; i.e., $s(p)$ is analytic and $|s(p)| \leq 1$ in the closed right-half p-plane.

Define a function $z(p)$ as

$$z(p) = 1 - s(p) \quad . \tag{B-1}$$

It is clear that $z(p)$ is a positive function. Since $s(p)$ is assumed to be unitary at $p = j\omega_k$, $s(p)$ exhibits the form

$$s(j\omega_k) = e^{j\theta_k} \quad , \tag{B-2}$$

at $p = j\omega_k$. It follows that

$$e^{-j\theta_k} s(p) = 1 \quad \text{at} \quad p = j\omega_k \quad . \tag{B-3}$$

By defining $\hat{z}(p)$ as

$$\hat{z}(p) = 1 - e^{-j\theta_k} s(p) \quad , \tag{B-4}$$

$\hat{z}(p)$ is again a positive function, but now it has a simple zero at $p = j\omega_k$.

First, suppose $|\omega_k| < \infty$; the power series expansion of $\hat{z}(p)$ about $p = j\omega_k$ is given by

$$\hat{z}(p) = \beta(p - j\omega_k) + \cdots \quad , \qquad \beta > 0 \quad . \tag{B-5}$$

Since $\hat{z}(p)$ is a positive function, the coefficient of the $(p - j\omega_k)$ term, β, must be positive. Note that β cannot equal zero because in that case $z(p)$ would be identically zero and $|s(p)| \equiv 1$.

Expansion of $s(p)$ about $p = j\omega_k$ gives

$$s(p) = e^{j\theta_k} + \beta_1(p - j\omega_k) + \cdots \quad , \tag{B-6}$$

where $\beta_o \equiv e^{j\theta_k}$. Substituting into Eq.(B-4) and equating coefficients of the $(p - j\omega_k)$ term, we obtain

$$\beta = -e^{-j\theta_k}\beta_1 > 0 \quad . \tag{B-7}$$

But, from $e^{-j\theta_k} s(j\omega_k) = 1$, we have $e^{-j\theta_k} = \overline{s}(j\omega_k) = \overline{\beta}_o$. Therefore,

$$-\overline{s}(j\omega_k)\beta_1 = -\overline{\beta}_o\beta_1 > 0 \quad , \tag{B-8}$$

or

$$\overline{\beta}_o\beta_1 < 0 \quad . \tag{B-9}$$

Incidentally, this also reveals the interesting fact that if β_o is real, then β_1 must also be real, even though the $s(p)$ we are considering is not necessarily a real function of p. For $|\omega_k| = \infty$, replace $(p - j\omega_k)$ by p^{-1} and the same proof holds.

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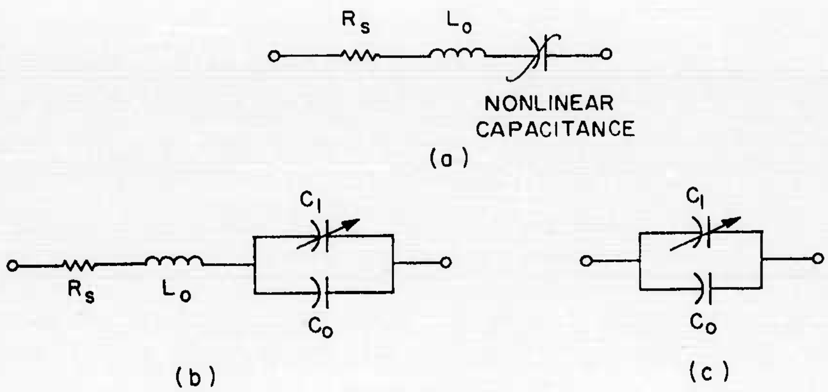


Fig. 1 Equivalent circuits for a varactor diode

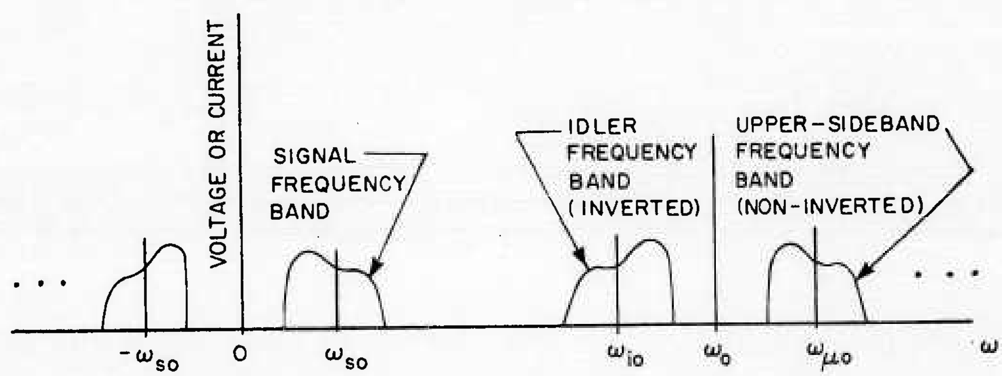


Fig. 2 Distribution of the frequency spectrums illustrating the meaning of "inverting" and "non-inverting" devices.

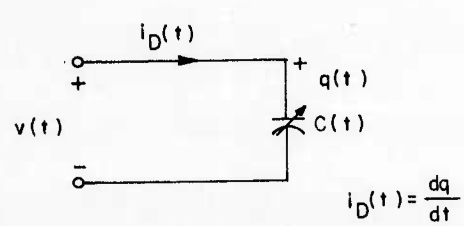


Fig. 3 Pumped varactor represented as a linear time-variable capacitor.

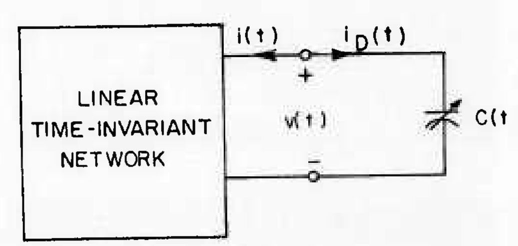
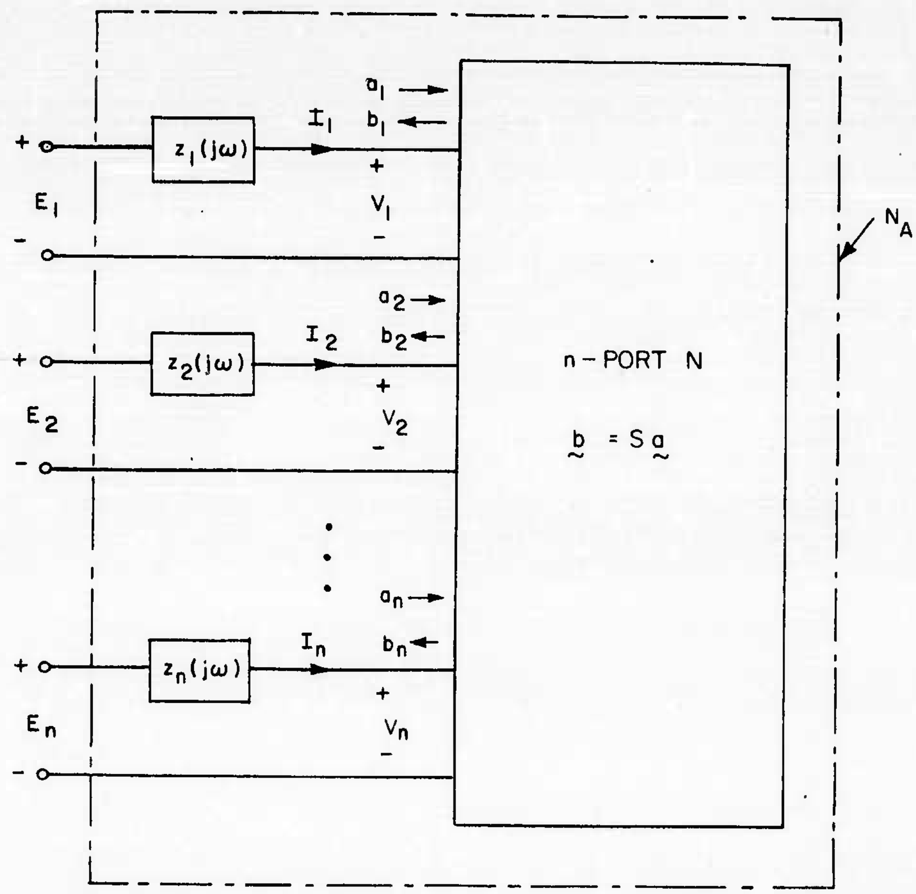


Fig. 4 Embedding of a parametric element in a linear, time-invariant environment.



$\underset{\sim}{a} = (a_1, a_2, \ldots, a_n)'$; $\underset{\sim}{b} = (b_1, b_2, \ldots, b_n)'$

$\underset{\sim}{V} = (V_1, V_2, \ldots, V_n)'$; $\underset{\sim}{I} = (I_1, I_2, \ldots, I_n)'$

Fig. 5 Schematic of an augmented n-port N_A corresponding to the n-port N .

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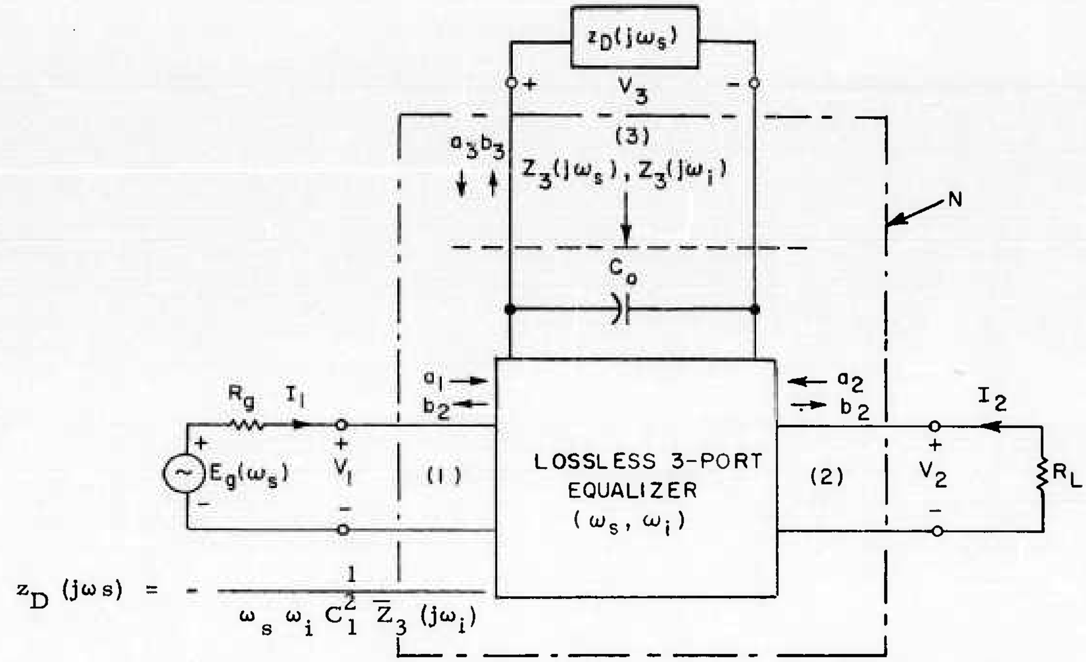

$$z_D(j\omega s) = - \frac{1}{\omega_s \omega_i C_1^2 \overline{Z}_3(j\omega_i)}$$

Fig. 6 Schematic of a single varactor diode embedded in an arbitrary lossless 3-port in which ω_s and ω_i can exist.

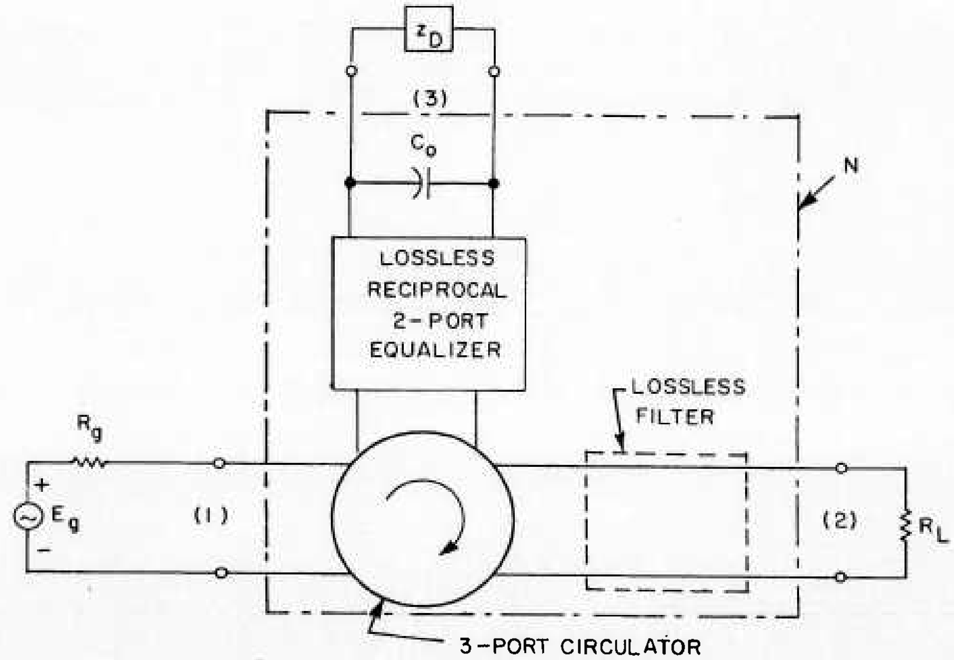

Fig. 7 Schematic of a nonreciprocal parametric amplifier with a single varactor.

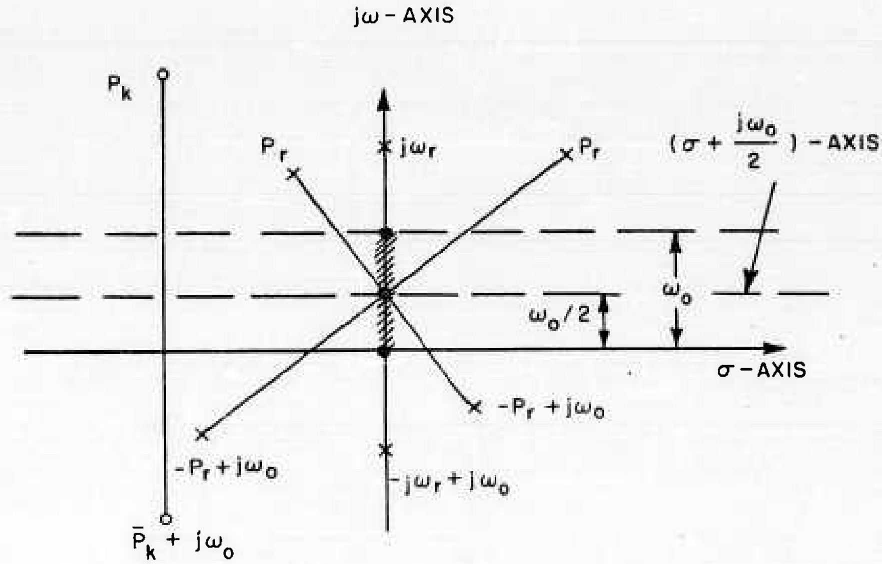

Fig. 8 Symmetry of the poles and zeros of $s_{33}(p)$.

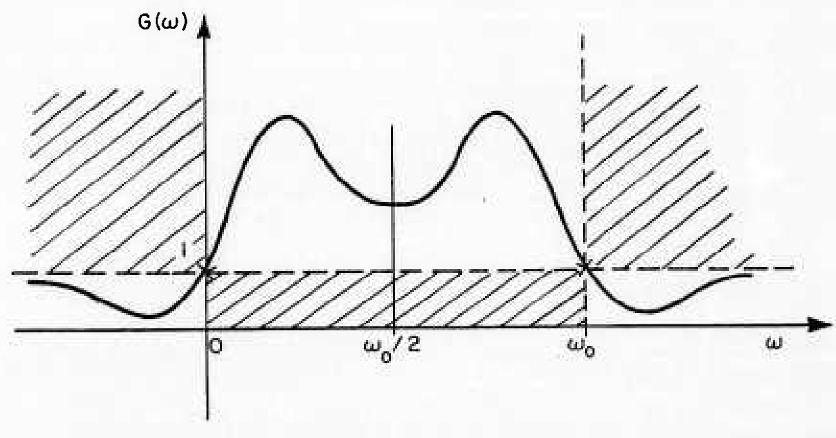

Fig. 9 Permissible region for G (ω) and a possible shape satisfying symmetry condition about $\omega_{o/2}$.

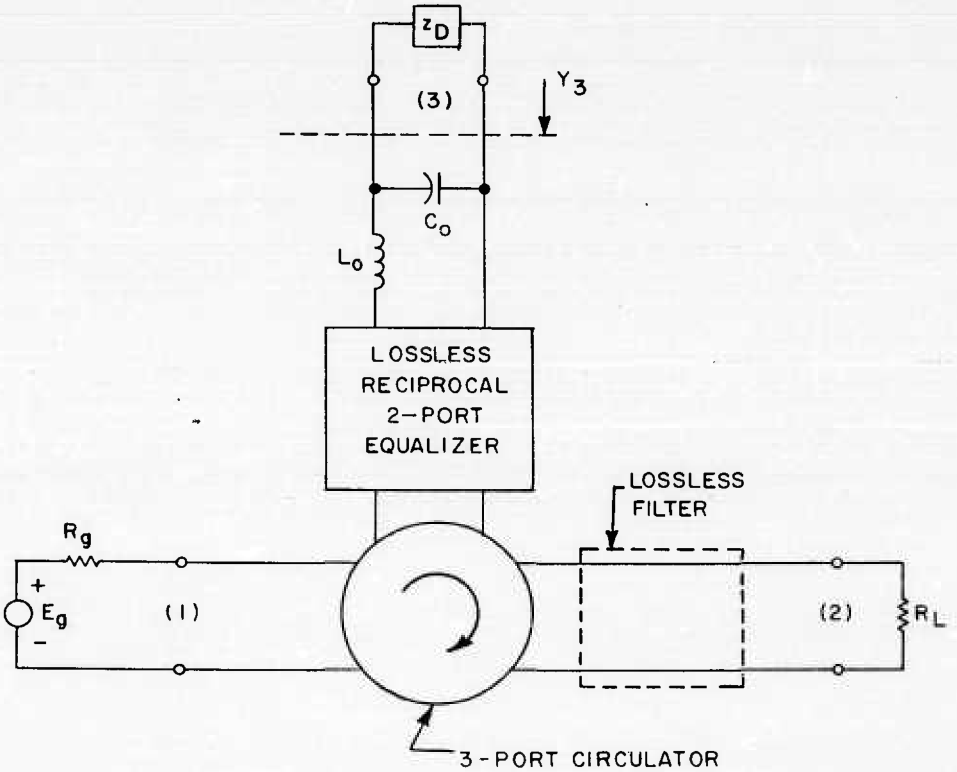

Fig. 10 Nonreciprocal Parametric Amplifier for varactor with series parasitic inductance.

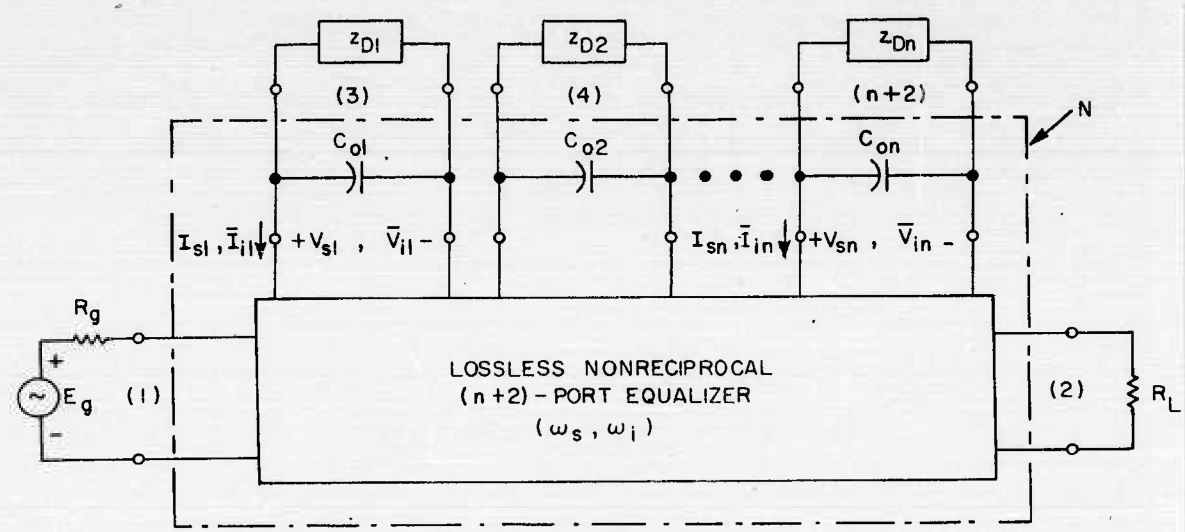

Fig. 11 Schematic of a parametric amplifier using n varactors.

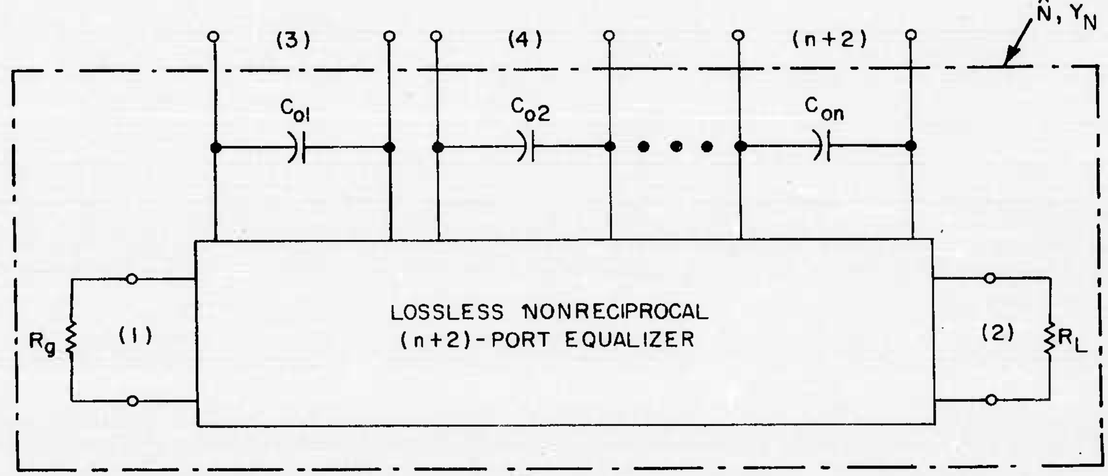

Fig. 12 Schematic showing the meaning of Y_n, the n x n admittance matrix facing the n varactors.

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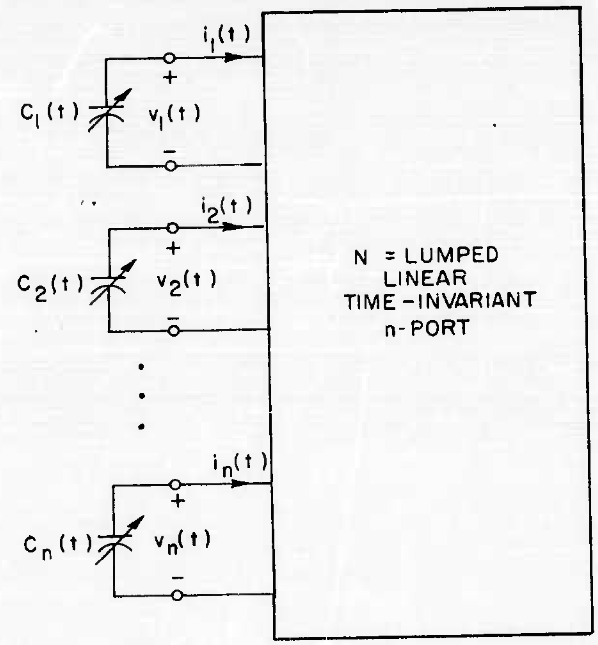

Fig. 13 Schematic of n varactors embedded in a linear, time-invariant environment.

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