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A CONTRIBUTION TO THE THEORY OF ISOTROPIC LOCKING

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Department of Aeronautics and Astronautics
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Stanford, California

A CONTRIBUTION TO THE THEORY OF ISOTROPIC LOCKING

by

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LIST OF SYMBOLS

a,b,d	constants
c, $\bar{c}$	wave speed
3e	dilatation
f, f_1, f_2, f_3, g	functions
h	depth of the beam
k	elastic-locking surface (radial coordinate)
p	pressure
q	non-dimensional wave front
r	radial coordinate
r_2	elastic-locking interface
s	stress dilatation or hydrostatic stress
t, $\bar{t}$	time
u,v,w	displacement
$\dot{u},\dot{v},\dot{w}$	particle velocity
u_0	strain energy of dilatation
$u_{0\ell}$	locked part of the strain energy of dilatation
u'	strain energy of distortion
u'_ℓ	locked part of the strain energy of distortion
x,y,z	cartesian coordinates
x_ℓ, y_ℓ	elastic-locking interface (cartesian coordinate)

A	cross-sectional area
$C, C_1 \ldots$	constants
D	flexural rigidity
E	modulus of elasticity
G	shear modulus
I	moment of inertia
J_1, J_2, J_3	invariants of strain tensor

LIST OF SYMBOLS (Cont'd)

K	bulk modulus
L	length
M	bending moment
R	external radius of the disk
R_2	non-dimensional elastic-locking interface
T, $\bar{T}$	non-dimensional time
U	strain energy
V	volume
α,β	constants
δ	Kronecker's delta
∇^2	Laplacian operator
$\epsilon,\epsilon_x\cdots$	strain
$\bar{\epsilon},\bar{\epsilon}_x$	strain in the locked region
ϵ_l	locking strain
θ	angular coordinate
λ_e	Lamè's constant
λ	material constant in locked region
ν	Poisson's ratio
ρ	density
σ,σ_x	normal stress
$\bar{\sigma},\bar{\sigma}_x$	normal stress in locked region
τ,τ_{xy}	shearing stress
$\bar{\tau},\bar{\tau}_{xy}$	shearing stress in locked region
ϕ	locking function
ψ	wave function
ω	angular velocity

I. INTRODUCTION

I.1 The Concept of Locking

Solids exhibit wide and significant variations in physical properties. Thus it is extremely difficult to characterize the mechanical behaviour of solids by a single equation or a single set of equations which can be used in practical analysis to determine stresses, strains, displacements and other related quantities. Therefore an engineer is compelled to devise 'ideal materials' such as the perfectly elastic solid, the perfectly plastic body etc. The idealization depends on the particular problem to be solved and the accuracy desired in the final solution. The locking material considered in this report is one of the 'ideal materials'. The phenomenon of locking which characterizes this ideal material can be visualized by considering the following model.

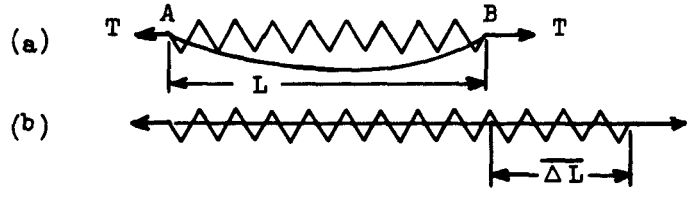


Fig. 1.1.

A spring AB of spring constant k_s has a flexible wire attached firmly to its ends. The cross-sectional area A and the modulus of elasticity E of the wire are such that the quantity EA/L is very large compared with the spring constant k_s of the spring. If a tensile force T is applied to the ends of the spring, the spring elongates by a certain amount ΔL. As T is increased ΔL also increases. For a certain value of $\Delta L = \overline{\Delta L}$ the wire becomes taut as indicated in Fig. 1.1b. After the elongation has attained the value $\overline{\Delta L}$, any attempt to elongate the spring involves the elongation of both the spring and the wire together. Thus a considerably larger force is necessary to produce

a given amount of elongation than is necessary to produce the same amount of elongation when the displacement is less than $\overline{\Delta L}$. The variation of T with ΔL is then as shown in Fig. 1.2

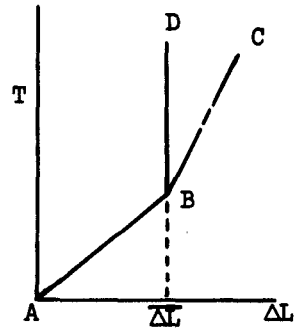


Fig. 1.2.

The system is said to be locked when $\Delta L = \overline{\Delta L}$. If the wire were infinitely rigid, the T - ΔL curve would be the curve ABD instead of ABC. This type of locking is called ideal locking.

A granular material consisting of grains of various sizes constitutes another example of a locking material. A simple model can be constructed if the material is made of grains of two sizes as indicated in Fig. 1.3.

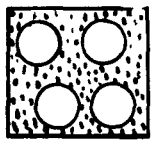


Fig. 1.3.

The larger grains are assumed to be vary hard to deform compared with the smaller grains. If the model is compressed by a uniform pressure p the volume decreases. The volumetric strain is denoted by 3e. For a certain value of $3e = 3e_l$ the model assumes the form as indicated in Fig. 1.4.

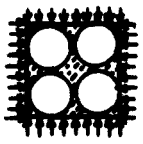


Fig. 1.4.

It can be seen from the figure that the volume can be decreased only very slightly beyond this state. However, shear strain can be produced by the application of a shear force. (Fig. 1.5)

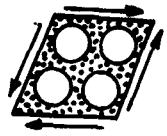
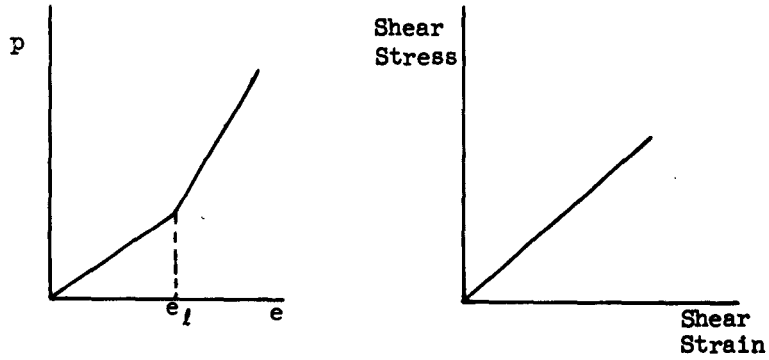


Fig. 1.5.

The stress-strain diagram is then as indicated in Fig. 1.6.

Fig. 1.6.



The material is said to be locked when $e = e_l$.

I.2 Application of the Idealization

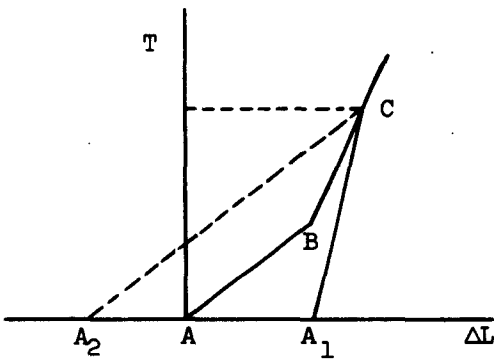
In many cases, granular soil which behaves very similar to the second example considered in section I.1 can be idealized as a locking material. Another example of locking material would be rubber. It is a common experience that it takes very little effort to stretch a rubber band upto a certain amount. With increasing deformation it becomes harder to stretch the rubber. The material resists greater load with little deformation. This behaviour fits very well with the concept of locking as described earlier. Stress problems in rubberlike materials can be treated as problems in locking media.

These are two examples in which the material can be idealized as a locking material. However this is not an exhaustive list.

I.3 Loading and Unloading, Conservative Property

The spring-wire model will again be considered in this section to discuss the loading-unloading properties of locking materials. If a tensile force T is gradually applied to the ends of the spring, the spring elongates. The loading process can be represented by the line ABC in the T - ΔL diagram (Fig. 1.7). If the tensile force T is now gradually reduced to zero, the unloading curve in the T - ΔL diagram is CBA provided the spring and the wire are still elastic; i.e., the unloading curve follows the loading curve in the reverse

Fig. 1.7.



direction, i.e., the system is conservative. Systems which follow a curve like $ABCA_1$ can easily be invented, but unloading along a curve like CA_2 would be in conflict with the basic laws of mechanics, since it would involve the creation of mechanical energy. In this report only conservative materials will be considered.

I.4 Locking Condition, Locking Surface

In the case of a spring model one can easily state that locking takes place when the elongation ΔL has attained a value $\overline{\Delta L}$ or the strain has reached a value $\overline{\Delta L}/L$. Similarly, in the case of the granular material locking takes place when the bulk strain 3e has become equal to $3e_l$. Therefore, in general, the condition of locking depends on the state of strain in the body. In an isotropic material the locking condition should be independent of the particular choice of the coordinate system. It therefore depends on the three invariants J_1, J_2, J_3, of the strain tensor which are defined in the following way in terms of principal strains ϵ_1, ϵ_2, ϵ_3:

$$\left.\begin{aligned} J_1 &= \tfrac{1}{3}(\epsilon_1 + \epsilon_2 + \epsilon_3) \\ J_2 &= \tfrac{1}{6}[(\epsilon_1 - \epsilon_2)^2 + (\epsilon_2 - \epsilon_3)^2 + (\epsilon_3 - \epsilon_1)^2] \\ J_3 &= \epsilon_1\epsilon_2\epsilon_3 \end{aligned}\right\} \qquad (1.1)$$

From Eq. (1.1), the locking condition

$$\phi(J_1, J_2, J_3) = 0 \qquad (1.2)$$

can also be expressed in terms of the principal strains

$$\phi(\epsilon_1, \epsilon_2, \epsilon_3) = 0 \qquad (1.3)$$

This equation represents a surface in the principal strain space which is called the locking surface. The locking surface can also be defined in a general six-dimensional strain space.

The "locking function" ϕ is defined in such a way that the strain state of the body is elastic if $\phi < 0$. Then in ideal locking ϕ is always less than or equal to zero, i.e., $\phi = 0$ in the locked regions of the body while $\phi < 0$ in regions that are still elastic. However in a non-ideally locking body ϕ can be greater than zero.

I.5 Volumetric and Distortional Locking

The locking condition can depend on the three invariants J_1, J_2 and J_3 of the strain tensor. However, in isotropic bodies (under small deformations) the shear stress does not produce any bulk strain and the hydrostatic stress does not produce any shear strain. Thus two different types of locking can be defined in an isotropic body.

(i) Volumetric locking which depends on the invariant J_1 of the strain dilatation.

(ii) Distortional locking which depends on the invariants J_2', J_3' of the strain deviator.

Again, due to isotropy, the stress and strain deviators are related by Hooke's law in a volumetrically locked region, while hydrostatic stress and bulk strain are related by Hooke's law in distortionally locked regions. However, some regions could be locked both volumetrically and distortionally. In the latter case there is no further change in strain if both the volumetric and distortional locking are ideal. The possible types of locking and the stress-strain relationship in locked regions will be studied in the next chapter.

I.6 Uniqueness

Locking materials are conservative. Therefore, a unique solution for an equilibrium stress problem or a dynamic response problem can be expected under appropriate boundary and initial conditions. These will be studied in the next chapter.

II. GENERAL THEORY

II.1 Strain Energy in Locking Materials

Strain energy per unit volume in a locking material is assumed to be a unique function of strains. This statement can be easily justified in the case of a non-ideally locking body in which stresses depend uniquely on the strains as discussed in section I.3. The statement can also be justified in the case of an ideally locking body where the stresses can increase or decrease by indefinite amounts while the strains remain constant because no work is done during such changes.

Further, if the assumption of small deformations is made, the dilatation $3e$ in an isotropic locking body depends reversibly and uniquely on the hydrostatic stress s. Also each component of the strain deviator ϵ'_{ij} depends reversibly and uniquely on the corresponding component of the stress deviator. The functional relationship is the same for all pairs of stress and strain deviators, i.e.,

$$\left.\begin{aligned} e &= \frac{1}{3}\ (\epsilon_{11} + \epsilon_{22} + \epsilon_{33}) = e(s) \\ s &= \frac{1}{3}\ (\sigma_{11} + \sigma_{22} + \sigma_{33}) = s(e) \\ \epsilon'_{ij} &= (\epsilon_{ij} - \delta_{ij}e) = f(\sigma'_{ij}) \\ \sigma'_{ij} &= (\sigma_{ij} - \delta_{ij}s) = f^{-1}(\epsilon'_{ij}) \end{aligned}\right\} \qquad (2.1)$$

where δ_{ij} is the Kronecker delta.

Certain simplification in the expression for strain energy can be achieved by using the preceding set of equations.

The increment of strain energy per unit volume can be written as

$$du = \sum_{i,j} \sigma_{ij}\, d\epsilon_{ij}$$

$$= \sum_{i,j} (s\delta_{ij} + \sigma'_{ij})(de\delta_{ij} + d\epsilon'_{ij}) \qquad i,j = 1,2,3$$

Because

$$\left.\begin{aligned} &\sum \sigma'_{ii} = 0 \quad , \qquad \sum d\epsilon'_{ii} = 0 \\ &du = 3sde + \sum_{i,j} \sigma'_{ij} d\epsilon'_{ij} \end{aligned}\right\} \qquad (2.2)$$

From Eqs. (2.1) and (2.2), one can see that the strain energy can be split into two parts u_0 and u'. The part u_0 depends on the dilatation e while the part u' depends on the strain deviator. Furthermore, the strain energy is independent of the choice of the coordinate system in an isotropic body. Therefore, u' must be a function of the invariants of the strain deviator, J'_2 and J'_3.

Thus one can see that

$$du_0 = \frac{du_0}{de}\, de = 3sde \qquad (2.3a)$$

$$du' = \frac{\partial u'}{\partial J'_2}\, dJ'_2 + \frac{\partial u'}{\partial J'_3}\, dJ'_3 = \sum_{i,j} \sigma'_{ij} d\epsilon'_{ij} \qquad i,j = 1,2,3 \qquad (2.3b)$$

II.2 Relationship between the Components of Stress and Strain Tensors in Terms of Strain Energy Function

Principal stresses (σ_1, σ_2, σ_3) and principal strains (ϵ_1, ϵ_2, ϵ_3) will be used in the further analysis. Then the strain dilatation is

$$e = \frac{1}{3}(\epsilon_1 + \epsilon_2 + \epsilon_3)$$

and the strain deviator is

$$\epsilon'_{ij} = \begin{bmatrix} \epsilon'_1 & - & - \\ - & \epsilon'_2 & - \\ - & - & \epsilon'_3 \end{bmatrix} = \begin{bmatrix} \epsilon_1 - e & - & - \\ - & \epsilon_2 - e & - \\ - & - & \epsilon_3 - e \end{bmatrix}$$

The three components of the strain deviator are not independent of each other. One of them can be expressed in terms of the other two:

$$\epsilon'_3 = -(\epsilon'_1 + \epsilon'_2)$$

Similar relationships for principal stresses yield the following result:

$$\sigma'_3 = -(\sigma'_1 + \sigma'_2) \tag{2.4}$$

Therefore

$$du' = \sigma'_1 d\epsilon'_1 + \sigma'_2 d\epsilon'_2 + \sigma'_3 d\epsilon'_3$$

$$= (2\sigma'_1 + \sigma'_2)d\epsilon'_1 + (\sigma'_1 + 2\sigma'_2)d\epsilon'_2 \tag{2.5}$$

Further,

$$J'_2 = \frac{1}{2}(\epsilon'^2_1 + \epsilon'^2_2 + \epsilon'^2_3) = J'_2(\epsilon'_1, \epsilon'_2)$$

$$J'_3 = \epsilon'_1\epsilon'_2\epsilon'_3 = J'_3(\epsilon'_1, \epsilon'_2)$$

Hence

$$dJ'_2 = \frac{\partial J'_2}{\partial \epsilon'_1} d\epsilon'_1 + \frac{\partial J'_2}{\partial \epsilon'_2} d\epsilon'_2$$

$$dJ'_3 = \frac{\partial J'_3}{\partial \epsilon'_1} d\epsilon'_1 + \frac{\partial J'_3}{\partial \epsilon'_2} d\epsilon'_2$$

Then

$$du' = \left(\frac{\partial u'}{\partial J_2'}\frac{\partial J_2'}{\partial \epsilon_1'} + \frac{\partial u'}{\partial J_3'}\frac{\partial J_3'}{\partial \epsilon_1'}\right) d\epsilon_1'$$

$$+ \left(\frac{\partial u'}{\partial J_2'}\frac{\partial J_2'}{\partial \epsilon_2'} + \frac{\partial u'}{\partial J_3'}\frac{\partial J_3'}{\partial \epsilon_2'}\right) d\epsilon_2' \qquad (2.6)$$

Comparing Eqs. (2.5) and (2.6) one can see that

$$2\sigma_1' + \sigma_2' = \frac{\partial u'}{\partial J_2'}\frac{\partial J_2'}{\partial \epsilon_1'} + \frac{\partial u'}{\partial J_3'}\frac{\partial J_3'}{\partial \epsilon_1'}$$

$$\sigma_1' + 2\sigma_2' = \frac{\partial u'}{\partial J_2'}\frac{\partial J_2'}{\partial \epsilon_2'} + \frac{\partial u'}{\partial J_3'}\frac{\partial J_3'}{\partial \epsilon_2'}$$

The expressions for J_2' and J_3' in terms of the components of strain deviator are

$$\left.\begin{aligned} J_2' &= \epsilon_1'^2 + \epsilon_2'^2 + \epsilon_1'\epsilon_2' \\ J_3' &= -\epsilon_1'^2\epsilon_2' - \epsilon_1'\epsilon_2'^2 \end{aligned}\right\} \qquad (2.6a)$$

Therefore

$$\left.\begin{aligned} 2\sigma_1' + \sigma_2' &= (2\epsilon_1' + \epsilon_2')\left(\frac{\partial u}{\partial J_2'} - \epsilon_2'\frac{\partial u'}{\partial J_3'}\right) \\ \sigma_1' + 2\sigma_2' &= (\epsilon_1' + 2\epsilon_2')\left(\frac{\partial u'}{\partial J_2'} - \epsilon_1'\frac{\partial u'}{\partial J_3'}\right) \end{aligned}\right\} \qquad (2.7)$$

One can solve for σ_1', σ_2' and hence σ_3' from Eqs. (2.7) and (2.4). Thus

$$\sigma_1' = \epsilon_1'\frac{\partial u'}{\partial J_2'} + \frac{1}{3}\left(\epsilon_1'^2 - 2\epsilon_1'\epsilon_2' - 2\epsilon_2'^2\right)\frac{\partial u'}{\partial J_3'} \qquad (2.8a)$$

$$\sigma_2' = \epsilon_2' \frac{\partial u'}{\partial J_2'} + \frac{1}{3} \left(-2\epsilon_1'^2 - 2\epsilon_1'\epsilon_2' + \epsilon_2'^2\right) \frac{\partial u'}{\partial J_3'} \tag{2.8b}$$

$$\sigma_3' = \epsilon_3' \frac{\partial u'}{\partial J_2'} + \frac{1}{3} \left(\epsilon_1'^2 + 4\epsilon_1'\epsilon_2' + \epsilon_2'^2\right) \frac{\partial u'}{\partial J_3'} \tag{2.8c}$$

Also, from Eq. (2.3a)

$$s = \frac{1}{3} \frac{du_0}{de} \tag{2.9}$$

Then the components of the stress tensor can be calculated from the following relationships:

$$\left.\begin{aligned} \sigma_1 &= \sigma_1' + s \\ \sigma_2 &= \sigma_2' + s \\ \sigma_3 &= \sigma_3' + s \end{aligned}\right\} \tag{2.10}$$

Equations (2.8)-(2.10) show how the stress-strain law depends on the relationships expressing the strain energy in terms of strains, i.e., the strain energy function. The stress-strain law can then be established if the strain energy functions $u(e)$ and the $u'(J_2', J_3')$ are known. Therefore, the investigation in the next few sections will be to determine the strain energy functions in locking materials.

II.3 Strain Energy Function in Volumetric and Distortional Locking

As discussed in Chapter I, a material locks when a certain function of principal strains called locking function $\phi(\epsilon_1, \epsilon_2, \epsilon_3)$ reaches a certain given value. One can define the function ϕ such that for $\phi < 0$ the material is elastic, and that $\phi = 0$ represents the state of incipient locking. Then in an ideally locking material $\phi > 0$ is impossible. However ϕ could be greater than zero in a non-ideally locking body.

In general the strain energy per unit volume u_0 and the strain energy of distortion per unit volume u' in a locked region can be written as

$$\left.\begin{aligned} u_0 &= \frac{9}{2} Ke^2 + u_{0\ell} \\ u' &= 2GJ_2' + u_\ell' \end{aligned}\right\} \qquad (2.11)$$

In these equations $(9/2)Ke^2$ and $2GJ_2'$ would be the strain energy of dilation and distortion if the material were deforming as an elastic material. They are called elastic parts of the strain energy. $u_{0\ell}(e)$ and $u_\ell'(J_2', J_3')$ are the excess of strain energy over the elastic values. These are called locked parts of the strain energy. In further analysis, u_ℓ' will be assumed to be a function of J_2' only.

As mentioned in Section I.5, stress and strain deviators are related by Hooke's law in a volumetrically locked region, while hydrostatic stress and dilation are related by Hooke's law in a distortionally locked region. Then $u_{0\ell}$ is zero in a distortionally locked region while u_ℓ' is zero in a volumetrically locked region.

Similarly, the stress tensor in a locked region can be split into elastic and locked parts:

$$\left.\begin{aligned} s &= 3Ke + s_\ell \\ \sigma_{ij}' &= 2G\epsilon_{ij}' + \sigma_{ij\ell}' \end{aligned}\right\} \qquad (2.12)$$

$3Ke$ and $2G\epsilon_{ij}'$ are the elastic parts of the stress tensor which would be the stresses if the material were deforming elastically. s_ℓ and $\sigma_{ij\ell}'$ are the excess of stresses over the elastic values in the locked regions. These are called the locked stress components. s_ℓ is zero in a distortionally locked region while $\sigma_{ij\ell}'$ is zero in a volumetrically locked region.

Now, let us consider a body in a locked state. The case of non-ideal volumetric locking will be studied. Ideal volumetric locking is a limiting case of non-ideal locking.

In the volumetrically locked state $\phi \geq 0$. Stresses and strain energy are given by Eqs. (2.11) and (2.12) with $\sigma'_{ijl} = u'_l = 0$. If the strain state of the body in the locked region changes such that ϕ becomes equal to zero, the locked region again becomes elastic wherever ϕ has become equal to zero. Then the expressions for the stresses and strain energy should reduce to the elastic values as ϕ tends to zero, i.e.,

$$\left.\begin{aligned} &\lim_{\phi \to 0^+} u_{0l}(e) = 0 \\ &\lim_{\phi \to 0^+} s_l(e) = 0 \end{aligned}\right\} \qquad (2.13)$$

0^+ is used to indicate that the limits are taken from the locked state.

u_{0l} is a continuous function of one variable, e. Further, u_{0l} must tend to zero as the locking function ϕ tends to zero. Then e must be a function of ϕ. Also, as explained in section I.5 the volumetric locking function ϕ depends only on the invariant $J_1 = e$. Thus

$$e = e(\phi)$$

$$\phi = \phi(e)$$

In these equations it has been implicitly assumed that e and ϕ are interchangeable, i.e.,

$$\frac{d\phi}{de} \neq 0 \qquad (2.14)$$

If e is replaced by ϕ in Eq. (2.13) one obtains

$$\lim_{\phi \to 0^+} u_{0l}(\phi) = 0$$

If it is assumed that u_{0l} allows a power series representation for $\phi \geq 0$, one can write

$$\begin{aligned} &u_{0l} = \alpha_1\phi + \alpha_2\phi^2 + \ldots \\ &\phi = \phi(e) \geq 0 \end{aligned} \qquad (2.15)$$

where

$\alpha_1, \alpha_2, \ldots$ are constants.

The result can be summarized in the following way. Volumetric locking takes place when $\phi = \phi(e) \geq 0$. In a volumetrically locked region distortion continues to follow Hooke's law while the dilation must be computed from the formulae that apply in the locked regions:

$$\left.\begin{aligned} u_0 &= \frac{9}{2} K e^2 + u_{0\ell} \\ u' &= 2GJ_2' \\ u_{0\ell} &= \alpha_1 \phi + \alpha_2 \phi^2 + \ldots \\ \phi &= \phi(e) \geq 0 , \quad \frac{d\phi}{de} \neq 0 \end{aligned}\right\} \tag{2.16}$$

Similarly, by assuming that u_ℓ' is a function of J_2' only, and that the distortional locking function ϕ is a function of J_2' only, one can derive the following equations applicable in a distortionally locked region:

$$\left.\begin{aligned} u_0 &= \frac{9}{2} K e^2 \\ u' &= 2GJ_2' + u_\ell' \\ u_\ell' &= \beta_1 \phi + \beta_2 \phi^2 + \ldots \\ \phi &= \phi(J_2') \geq 0 , \quad \frac{d\phi}{dJ_2'} = 0 \\ \lim_{\phi \to 0^+} & \quad u_\ell' = 0 \\ \lim_{\phi \to 0^+} & \quad \sigma_{ij\ell}' = 0 \end{aligned}\right\} \tag{2.17}$$

In concluding one can summarize the achievements in this section as follows. It has been possible to express the strain energy as a function of e, J_2' and the locking function ϕ, in volumetric as well as distortional locking. (See Eqs. 2.16, 2.17) Thus if the locking function for the material is determined by experimental or other methods, the strain energy function can be determined. Once the strain energy function is known the stress-strain relationship in the locked region can be established from the formulae derived in section II.2.

In further analysis it will be assumed that the locking function is known and the stress-strain relationship in distortional and volumetric locking will be discussed.

II.4. Stress-Strain Relationship in Distortional Locking

From the discussion in section II.3, one can write the expression for the strain energy function in a distortionally locking material in the following way. For $\phi(J_2') \leq 0$

$$\left.\begin{aligned} u_0 &= \frac{9}{2} K e^2 \\ u' &= 2GJ_2' \\ \text{For } \phi(J'_2) &\geq 0 \\ u_0 &= \frac{9}{2} K e^2 \\ u' &= 2GJ_2' + \beta_1\phi + \beta_2\phi^2 + \ldots \end{aligned}\right\} \qquad (2.18)$$

Then from Eqs. (2.8)-(2.10) the stress-strain relationship can be obtained. For $\phi \leq 0$ the strain energy function and Eqs. (2.8)-(2.10) give us the well-known Hooke's law. However for $\phi \geq 0$ the following stress-strain relationship can be obtained using Eqs. (2.8)-(2.10) and (2.18).

$$\sigma_1' = 2G\epsilon_1' + \beta_1 \frac{d\phi}{dJ_2'} \epsilon_1' + 2\beta_2\phi \frac{d\phi}{dJ_2'} \epsilon_1' + \cdots$$

$$\sigma_2' = 2G\epsilon_2' + \beta_1 \frac{d\phi}{dJ_2'} \epsilon_2' + 2\beta_2\phi \frac{d\phi}{dJ_2'} \epsilon_2' + \cdots$$

$$\sigma_3' = 2G\epsilon_3' + \beta_1 \frac{d\phi}{dJ_2'} \epsilon_3' + 2\beta_2\phi \frac{d\phi}{dJ_2'} \epsilon_3' + \cdots$$

$$s = 3Ke$$

Comparing these equations with Eqs. (2.12)

$$\sigma_{1\ell}' = \beta_1 \frac{d\phi}{dJ_2'} \epsilon_1' + 2\beta_2\phi \frac{d\phi}{dJ_2'} \epsilon_1' + \cdots$$

$$\sigma_{2\ell}' = \beta_1 \frac{d\phi}{dJ_2'} \epsilon_2' + 2\beta_2\phi \frac{d\phi}{dJ_2'} \epsilon_2' + \cdots$$

$$\sigma_{3\ell}' = \beta_1 \frac{d\phi}{dJ_2'} \epsilon_3' + 2\beta_2\phi \frac{d\phi}{dJ_2'} \epsilon_3' + \cdots$$

$$s_\ell = 0$$

However, from Eqs. (2.17)

$$\lim_{\phi \to 0^+} (\sigma_{1\ell}', \sigma_{2\ell}', \sigma_{3\ell}') = 0$$

Therefore

$$\beta_1 = 0$$

because (see Eqs. 2.17)

$$\frac{d\phi}{dJ_2'} \neq 0$$

and ϵ_1', ϵ_2', ϵ_3' can not all be zero. Then

$$\sigma_{1\ell}' = 2\beta_2\phi \frac{d\phi}{dJ_2'} \epsilon_1' + O(\phi^2) \qquad (2.19a)$$

$$\sigma'_{2l} = 2\beta_2\phi \frac{d\phi}{dJ'_2} \epsilon'_2 + O(\phi^2) \tag{2.19b}$$

$$\sigma'_{3l} = 2\beta_2\phi \frac{d\phi}{dJ'_2} \epsilon'_3 + O(\phi^2) \tag{2.19c}$$

The stress-strain relationship in a distortionally locked region can then be expressed by the following relationships

$$\left.\begin{aligned} \sigma_1 &= \sigma'_1 + s = 3Ke + 2G\epsilon'_1 + 2\beta_2\phi \frac{d\phi}{dJ'_2} \epsilon'_1 + O(\phi^2) \\ \sigma_2 &= \sigma'_2 + s = 3Ke + 2G\epsilon'_2 + 2\beta_2\phi \frac{d\phi}{dJ'_2} \epsilon'_2 + O(\phi^2) \\ \sigma_3 &= \sigma'_3 + s = 3Ke + 2G\epsilon'_3 + 2\beta_2\phi \frac{d\phi}{dJ'_2} \epsilon'_3 + O(\phi^2) \end{aligned}\right\} \tag{2.19d}$$

The constants $\beta_1 = 0, \beta_2, \ldots$ depend on the particular material.

Thus it has been possible to express the stress-strain relationship in a distortionally locked region in the form of Eq. (2.19d) which contain infinite series on the right-hand sides. However these equations can be simplified by introducing the concept of stress increments. This will be discussed in the next section.

II.5. Concept of Stress Increment

Let us consider a point of the body where $\phi(J'_2)$ has just attained the value zero. Then the point is in a state of incipient locking. The stresses and strains at the point in the state of incipient locking are denoted with a suffix I , e.g., σ_{1I}, etc. Suppose the value of ϕ at the point increases by an arbitrary small positive quantity ϕ_f, the stress and strain at the point increase from the value at the state of incipient locking. These stresses and strains are denoted with a suffix f. By using the Eq. (2.19d), one can then write the following equations

$$\sigma_{1I} = 3Ke_I + 2G\epsilon'_{1I}$$

$$\sigma_{1f} = 3Ke_f + 2G\epsilon'_{1f} + 2\beta_2\phi_f \frac{d\phi}{dJ'_2} \epsilon'_{1f} + O(\phi_f^2)$$

i.e.,

$$\sigma_{1f} - \sigma_{1I} = 3K(e_f - e_I) + 2G(\epsilon'_{1f} - \epsilon'_{1I}) + 2\beta_2 \phi_f \frac{d\phi}{dJ'_2} \epsilon'_{1f} + O(\phi_f^2) \qquad (2.20)$$

The increments $\sigma_{1f} - \sigma_{1I}$, $\epsilon'_{1f} - \epsilon'_{1I}$, $e_f - e_I$ are now denoted by the symbols $\Delta\sigma_1$, $\Delta\epsilon_1$, Δe. Further $\phi_f = \phi_f - \phi_I$ is denoted by the symbol $\Delta\phi$. Then Eq. (2.20) can be written as

$$\Delta\sigma_1 = 3K\Delta e + 2G\Delta\epsilon'_1 + 2\beta_2\Delta\phi \frac{d\phi}{dJ'_2} \epsilon'_1 + O[(\Delta\phi)^2]$$

Similarly

$$\Delta\sigma_2 = 3K\Delta e + 2G\Delta\epsilon'_2 + 2\beta_2\Delta\phi \frac{d\phi}{dJ'_2} \epsilon'_2 + O[(\Delta\phi)^2]$$

$$\Delta\sigma_3 = 3K\Delta e + 2G\Delta\epsilon'_3 + 2\beta_2\Delta\phi \frac{d\phi}{dJ'_2} \epsilon'_3 + O[(\Delta\phi)^2]$$

The quantities $\Delta\sigma_1$, $\Delta\sigma_2$ and $\Delta\sigma_3$ are the stresses at the point in excess of the stresses at the point at the state of incipient locking. They can be split into two parts in the following way:

$$\Delta\sigma_1 = \Delta\sigma_{1e} + \Delta\sigma_{1l}$$

$$\Delta\sigma_2 = \Delta\sigma_{2e} + \Delta\sigma_{2l}$$

$$\Delta\sigma_3 = \Delta\sigma_{3e} + \Delta\sigma_{3l}$$

where

$$\Delta\sigma_{1e} = 3K\Delta e + 2G\Delta\epsilon'_1$$

$$\Delta\sigma_{2e} = 3K\Delta e + 2G\Delta\epsilon'_2$$

$$\Delta\sigma_{3e} = 3K\Delta e + 2G\Delta\epsilon'_3$$

are the stress increments calculated from Hooke's law and

$$\Delta\sigma_{1\ell} = 2\beta_2\Delta\phi \frac{d\phi}{dJ_2'} \epsilon_1' + O(\Delta\phi)^2$$

$$\Delta\sigma_{2\ell} = 2\beta_2\Delta\phi \frac{d\phi}{dJ_2'} \epsilon_2' + O(\Delta\phi)^2$$

$$\Delta\sigma_{3\ell} = 2\beta_2\Delta\phi \frac{d\phi}{dJ_2'} \epsilon_3' + O(\Delta\phi)^2$$

are the excess of the locked stresses over the elastic values. If the limiting values are considered as $\Delta\phi$ tends to zero in these equations, one obtains the following result.

$$\left.\begin{aligned} d\sigma_{1\ell} &= 2\beta_2 d\phi \left(\frac{d\phi}{dJ_2'}\right) \epsilon_1' \\ d\sigma_{2\ell} &= 2\beta_2 d\phi \left(\frac{d\phi}{dJ_2'}\right) \epsilon_2' \\ d\sigma_{3\ell} &= 2\beta_2 d\phi \left(\frac{d\phi}{dJ_2'}\right) \epsilon_3' \end{aligned}\right\} \qquad (2.21)$$

Similarly

$$\left.\begin{aligned} d\sigma_{1e} &= 3Kde + 2Gd\epsilon_1' \\ d\sigma_{2e} &= 3Kde + 2Gd\epsilon_2' \\ d\sigma_{3e} &= 3Kde + 2Gd\epsilon_3' \end{aligned}\right\} \qquad (2.22)$$

The quantities $d\sigma_{1\ell}$, $d\sigma_{2\ell}$, $d\sigma_{3\ell}$ are called locked stress increments. $d\sigma_{1e}$, $d\sigma_{2e}$, $d\sigma_{3e}$ are called elastic stress increments. The total stress increments are then given by the following equations:

$$\left.\begin{aligned} d\sigma_1 &= d\sigma_{1e} + d\sigma_{1\ell} \\ d\sigma_2 &= d\sigma_{2e} + d\sigma_{2\ell} \\ d\sigma_3 &= d\sigma_{3e} + d\sigma_{3\ell} \end{aligned}\right\} \qquad (2.23)$$

Thus, using the concept of stress increments, it has been possible to obtain the simple relationships (2.21)-(2.23) between the stress increments, strain and strain increments. These increments have been calculated at the locking surface, i.e., the surface $\phi = 0$ in the principal strain space. These relationships are sufficient to discuss some interesting geometrical properites pertaining to the locking surface and to the problems of ideal distortional locking. These will be investigated in the next few sections.

II.6. Direction of the Locked Stress Increment Relative to the Locking Surface

As discussed in section II.4, distortional locking takes place when

$$\phi(J_2') = 0$$

Now, J_2' is a function of ϵ_1', ϵ_2' and these in turn depend on $\epsilon_1, \epsilon_2, \epsilon_3$. We may therefore write

$$\phi(J_2') = \phi[J_2'(\epsilon_1, \epsilon_2, \epsilon_3)]$$

and may form partial derivitives like

$$\frac{\partial \phi}{\partial \epsilon_1} = \frac{d\phi}{dJ_2'} \frac{\partial J_2'}{\partial \epsilon_1}$$

The definition of J_2' as given in section II.2, may be rewritten in the form

$$J_2'(\epsilon_1, \epsilon_2, \epsilon_3) = \frac{1}{18} [(2\epsilon_1 - \epsilon_2 - \epsilon_3)^2 + (2\epsilon_2 - \epsilon_1 - \epsilon_3)^3 + (2\epsilon_3 - \epsilon_1 - \epsilon_2)^2]$$

from which

$$\frac{\partial J_2'}{\partial \epsilon_1} = \frac{1}{3} (2\epsilon_1 - \epsilon_2 - \epsilon_3) = \epsilon_1'$$

and hence

$$\frac{\partial \phi}{\partial \epsilon_1} = \frac{d\phi}{dJ_2'} \epsilon_1'$$

Similarly

$$\frac{\partial \phi}{\partial \epsilon_2} = \frac{d\phi}{dJ_2'} \epsilon_2'$$

$$\frac{\partial \phi}{\partial \epsilon_3} = \frac{d\phi}{dJ_2'} \epsilon_3'$$

Then these equations and Eqs. (2.21) yield for the stress increments

$$d\sigma_{1\ell} = 2\beta_2 d\phi \frac{\partial \phi}{\partial \epsilon_1}$$

$$d\sigma_{2\ell} = 2\beta_2 d\phi \frac{\partial \phi}{\partial \epsilon_2}$$

$$d\sigma_{3\ell} = 2\beta_2 d\phi \frac{\partial \phi}{\partial \epsilon_3}$$

or

$$d\vec{\sigma}_\ell = 2\beta_2 d\phi \text{ grad } \phi \tag{2.24}$$

The vector grad ϕ is normal to the surface $\phi = 0$. Therefore it can be concluded that $d\vec{\sigma}_\ell$ is normal to the distortional locking surface. This is an interesting result which is useful in establishing the uniqueness of a solution in equilibrium stress problems or dynamic response problems in an ideal locking medium.

II.7. Magnitude of Stress Increment and the Locking Surface

The locking function ϕ_f corresponding to the state of strain $\vec{\epsilon}_f = (\epsilon_{1f}, \epsilon_{2f}, \epsilon_{3f})$ can be expressed in terms of the locking function corresponding to the state of strain $\vec{\epsilon}_I = (\epsilon_{1I}, \epsilon_{2I}, \epsilon_{3I})$ by the following power series expansion

$$\phi_f = \phi_I + \left(\frac{\partial \phi}{\partial \epsilon_1}\right)_{\vec{\epsilon} = \vec{\epsilon}_I} (\epsilon_{1f} - \epsilon_{1I}) + \left(\frac{\partial \phi}{\partial \epsilon_2}\right)_{\vec{\epsilon} = \vec{\epsilon}_I} (\epsilon_{2f} - \epsilon_{2I})$$

$$+ \left(\frac{\partial \phi}{\partial \epsilon_3}\right)_{\vec{\epsilon} = \vec{\epsilon}_I} (\epsilon_{3f} - \epsilon_{3I}) + \ldots \qquad (2.25)$$

If the state I corresponds to the state $\phi_I = 0$ (i.e., incipient locking), and $(\epsilon_{1f} - \epsilon_{1I}, \epsilon_{2f} - \epsilon_{2I}, \epsilon_{3f} - \epsilon_{3I})$ are replaced by $\Delta\vec{\epsilon} = (\Delta\epsilon_1, \Delta\epsilon_2, \Delta\epsilon_3)$, Eq. (2.25) can be written as

$$\phi_f = \phi_f - \phi_I = (\text{grad } \phi)_{\vec{\epsilon} = \vec{\epsilon}_I} \cdot \Delta\vec{\epsilon} + O(|\Delta\vec{\epsilon}|^2)$$

If one considers the limit as $\Delta\vec{\epsilon}$ tends to zero, one obtains

$$d\phi = (\text{grad } \phi)_{\vec{\epsilon} = \vec{\epsilon}_I} \cdot d\vec{\epsilon}$$

From Eqs. (2.24) the locked stress increment is then

$$d\vec{\sigma}_\ell = 2\beta_2[(\text{grad } \phi)_{\vec{\epsilon} = \vec{\epsilon}_I} \cdot d\vec{\epsilon}]\text{grad } \phi \qquad (2.26)$$

This equation can also be written as

$$d\vec{\sigma}_\ell = 2\beta_2 |\text{grad } \phi|^2_{\vec{\epsilon} = \vec{\epsilon}_I} (\vec{n} \cdot d\vec{\epsilon})\vec{n} \qquad (2.26a)$$

where $\vec{n}$ is the unit vector in the principal strain space normal to the locking surface at $\vec{\epsilon} = \vec{\epsilon}_I$. Then $\vec{n} \cdot d\vec{\epsilon}$ is the component of increment of strain normal to the locking surface. Therefore it can be concluded that the magnitude of $d\vec{\sigma}_\ell$ is proportional to the increment of strain normal to the locking surface.

The formula (2.26) will now be used in the next section to discuss ideal distortional locking.

II.8. Case of Ideal Distortional Locking

From Eq. (2.26) β_2 can be expressed as

$$\beta_2 = \frac{|d\vec{\sigma}_\ell|}{2(\text{grad } \phi \cdot d\vec{\epsilon})|\text{grad } \phi|}$$

i.e., β_2 is proportional to the ratio of the magnitude of the increment of the locked part of the stress to the increment of the strain normal to the locking surface. Further, the preceding equation is applicable to both ideal and non-ideal distortional locking materials when $\phi = 0$.

In case of the ideal locking material stresses $d\vec{\sigma}_\ell$ can increase while the end point of the strain vector remains on the locking surface, i.e.,

$$\text{grad } \phi \cdot d\vec{\epsilon} = 0$$

while $|d\vec{\sigma}_\ell|$ is not zero. Then β_2 tends to infinity. However the quantity $d\lambda = 2\beta_2[\text{grad } \phi \cdot d\vec{\epsilon}]$ can tend to a finite limit. The locked stress increment can then be written as

$$d\vec{\sigma}_\ell = d\bar{\lambda} \text{ grad } \phi \tag{2.27}$$

i.e.,

$$\left.\begin{aligned} d\sigma_{1\ell} &= d\bar{\lambda} \frac{\partial \phi}{\partial \epsilon_1} \\ d\sigma_{2\ell} &= d\bar{\lambda} \frac{\partial \phi}{\partial \epsilon_2} \\ d\sigma_{3\ell} &= d\bar{\lambda} \frac{\partial \phi}{\partial \epsilon_3} \end{aligned}\right\} \tag{2.27a}$$

Suppose $\phi(J_2')$ is assumed to be

$$\phi(J_2') = J_2' - \epsilon_\ell^2 \tag{2.28}$$

The increment of the locked part of the stress can be calculated from Eq. (2.27a):

$$d\sigma_{1\ell} = d\bar{\lambda}\epsilon'_1$$

$$d\sigma_{2\ell} = d\bar{\lambda}\epsilon'_2$$

$$d\sigma_{3\ell} = d\bar{\lambda}\epsilon'_3$$

Then the complete incremental stress-strain relationship corresponding to the locking function given by Eq. (2.28) is

$$\left.\begin{aligned} d\sigma_1 &= 3Kde + 2Gd\epsilon'_1 + d\bar{\lambda}\epsilon'_1 \\ d\sigma_2 &= 3Kde + 2Gd\epsilon'_2 + d\bar{\lambda}\epsilon'_2 \\ d\sigma_3 &= 3Kde + 2Gd\epsilon'_3 + d\bar{\lambda}\epsilon'_3 \end{aligned}\right\} \qquad (2.29)$$

where $d\bar{\lambda}$ is determined (as explained in section II.11) in the particular problem being solved.

Another simple locking condition is

$$\phi = \phi(J'_2) = \phi(\epsilon'_1,\epsilon'_2) = |\epsilon'_i - \epsilon'_j| - C = 0 \qquad (2.30)$$

$$i,j = 1,2,3$$

From physical considerations it is obvious that the quantity $|\epsilon'_i - \epsilon'_j|$ to be used in Eq. (2.30) corresponds to one of maximum difference between two principal strain deviators. This can also be interpreted as the condition of maximum shearing strain for locking. The locked stress increments in this case are

$$d\sigma_{1\ell} = d\bar{\lambda}\frac{\partial}{\partial\epsilon_1}|\epsilon'_i - \epsilon'_j|$$

$$d\sigma_{2\ell} = d\bar{\lambda}\frac{\partial}{\partial\epsilon_2}|\epsilon'_i - \epsilon'_j|$$

$$d\sigma_{3\ell} = d\bar{\lambda}\frac{\partial}{\partial\epsilon_3}|\epsilon'_i - \epsilon'_j|$$

The stress-strain incremental relationship similar to Eq. (2.29) can then be obtained.

II.9. Case of Volumetric Locking

In this section the stress-strain relationship in volumetrically locking bodies will be studied.

From the results of the section II.3 volumetric locking takes place when

$$\phi = \phi(e) = 0$$

and in the volumetrically locked region

$$u' = 2GJ_2'$$

$$u_0 = \frac{9}{2} Ke^2 + u_{0\ell}$$

where

$$u_{0\ell} = \alpha_1 \phi + \alpha_2 \phi^2 + \dots$$

Then from the formulae (2.8) and (2.9)

$$\sigma_1' = 2G\epsilon_1'$$

$$\sigma_2' = 2G\epsilon_2'$$

$$\sigma_3' = 2G\epsilon_3'$$

$$s = 3Ke + \frac{1}{3}\alpha_1 \frac{d\phi}{de} + \frac{2}{3}\alpha_2 \phi \frac{d\phi}{de} + \dots$$

But

$$\lim_{\phi \to 0^+} s = 3Ke$$

Therefore,

$$\alpha_1 = 0$$

because

$$\frac{d\phi}{de} \neq 0$$

Then

$$s = 3Ke + \frac{2}{3}\alpha_2\phi\frac{d\phi}{de} + \ldots$$

The stress-strain relationship in a volumetrically locked region is then

$$\left.\begin{aligned}\sigma_1 &= 2G\epsilon_1' + 3Ke + \frac{2}{3}\alpha_2\phi\frac{d\phi}{de} + \ldots\\ \sigma_2 &= 2G\epsilon_2' + 3Ke + \frac{2}{3}\alpha_2\phi\frac{d\phi}{de} + \ldots\\ \sigma_3 &= 2G\epsilon_3' + 3Ke + \frac{2}{3}\alpha_2\phi\frac{d\phi}{de} + \ldots\end{aligned}\right\} \qquad (2.31)$$

By following a procedure similar to that of section II.5, stress increments can be defined. Thus

$$\left.\begin{aligned}d\sigma_1 &= d\sigma_{1e} + d\sigma_{1\ell}\\ d\sigma_2 &= d\sigma_{2e} + d\sigma_{2\ell}\\ d\sigma_3 &= d\sigma_{3e} + d\sigma_{3\ell}\end{aligned}\right\} \qquad (2.32)$$

where

$$\left.\begin{aligned}d\sigma_{1e} &= 2Gd\epsilon_1' + 3Kde\\ d\sigma_{2e} &= 2Gd\epsilon_2' + 3Kde\\ d\sigma_{3e} &= 2Gd\epsilon_3' + 3Kde\end{aligned}\right\} \qquad (2.33)$$

and

$$d\sigma_{1\ell} = d\sigma_{2\ell} = d\sigma_{3\ell} = \frac{2}{3}\alpha_2 d\phi\frac{d\phi}{de} \qquad (2.33a)$$

As in the case of distortional locking, a volumetric locking surface in principal strain space can be defined in the following way.

$$\phi = \phi(e) = \phi(\epsilon_1, \epsilon_2, \epsilon_3) = 0$$

By following a procedure similar to that of section II.6, the normality of the locked stress increment to the volumetric locking surface can be established. Also the magnitude of the locked stress increment is proportional to the component of strain normal to the volumetric locking surface. This result is similar to that obtained in section II.7. The equation of locked stress increment is

$$d\vec{\sigma}_\ell = 2\alpha_2[(\text{grad } \phi)\cdot d\vec{\epsilon}] \text{ grad } \phi \tag{2.34}$$

where

$$\text{grad } \phi = \left(\frac{\partial \phi}{\partial \epsilon_1}, \frac{\partial \phi}{\partial \epsilon_2}, \frac{\partial \phi}{\partial \epsilon_3}\right)$$

II.10. Particular Cases of Volumetric Locking

A simple volumetric locking condition is

$$\phi = -(\epsilon_1 + \epsilon_2 + \epsilon_3) - \epsilon_\ell \tag{2.35}$$

where ϵ_ℓ is a positive constant. Here locking is assumed to be in compression only. Then from Eq. (2.33a)

$$d\vec{\sigma}_\ell = 6\alpha_2(de, de, de)$$

Therefore,

$$d\sigma_1 = 2Gd\epsilon_1' + (3K + 6\alpha_2)de$$

$$d\sigma_2 = 2Gd\epsilon_2' + (3K + 6\alpha_2)de$$

$$d\sigma_3 = 2Gd\epsilon_3' + (3K + 6\alpha_2)de$$

The hydrostatic part of the stress tensor is then

$$ds = (3K + 6\alpha_2)de$$

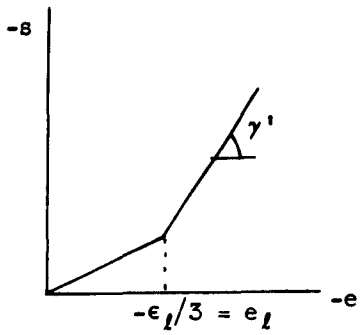
For a material with the stress-strain diagram of the type of Fig. 2.1, this equation can be integrated. Thus

$$s = (3K + 6\alpha_2)e + s_0$$

where

$$3K + 6\alpha_2 = \tan \gamma'$$

Fig. 2.1.



Also when $e = e_l$ the material is in incipient locking state. Then

$$s = -3Ke_l$$

hence

$$s_0 = -6\alpha_2 e_l$$

Then

$$s = (3K + 6\alpha_2)e - 6\alpha_2 e_l$$

or

$$s = \lambda(e - e_l) + 3Ke_l \qquad (2.36)$$

where

$$\lambda = 3K + 6\alpha_2$$

Equations (2.35) and (2.36) together with distortional elasticity complete the stress-strain relationship. In case of ideal locking

$$\alpha_2 \to \infty$$

and

$$(\text{grad } \phi)\cdot d\vec{\epsilon} = 0$$
$$\phi = 0$$

Then

$$d\sigma_{1l} = d\sigma_{2l} = d\sigma_{3l} = d\lambda$$

The quantity $d\lambda$ is to be determined in the particular problem being solved.

II.11. Problem of Equilibrium-Uniqueness

So far the investigations in the chapter have been directed towards studying stress-strain relationships in locking materials. In this section the formulation of an equilibrium stress problem and the question of uniqueness under prescribed boundary conditions will be investigated. Further discussion in the thesis will be restricted to ideal locking materials only.

Let us consider a body of locking material enclosed by a surface S. Let the surface traction $\vec{P}_I$ be prescribed over a part S_p of the surface S and the displacements $\vec{v}_I$ be prescribed over the remaining part $S - S_p = S_v$. Depending on the strain pattern the body will have locked regions V_ℓ and elastic regions V_e. In particular cases either V_ℓ or V_e could be zero. The stress distribution $\vec{\sigma}_I$, the strain pattern $\vec{\epsilon}_I$, and the displacements $\vec{u}_I$ in the body corresponding to the given boundary values should satisfy the equilibrium conditions, appropriate stress-strain relationship in the locked and elastic regions and kinematics. These equations when referred to a set of cartesian coordinates are:

(i) Equilibrium Conditions

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + X = 0$$

$$\frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} + Y = 0$$

$$\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} + Z = 0$$

(ii) Stress-Strain Relationship

(a) If $\phi < 0$, the elastic stress-strain relationship should be satisfied. They are (λ_e is Lamé's constant)

$$\sigma_x = 3\lambda_e e + 2G\epsilon_x$$

$$\sigma_y = 3\lambda_e e + 2G\epsilon_y$$

$$\sigma_z = 3\lambda_e e + 2G\epsilon_z$$

$$\tau_{xy} = 2G\epsilon_{xy}$$

$$\tau_{yz} = 2G\epsilon_{yz}$$

$$\tau_{zx} = 2G\epsilon_{zx}$$

(b) If $\phi = 0$, and the material is an ideal distortionally locking material, the stress-strain relationships are

$$d\sigma_x = 3\lambda_e de + 2Gd\epsilon_x + d\bar{\lambda}\frac{\partial\phi}{\partial\epsilon_x}$$

$$d\sigma_y = 3\lambda_e de + 2Gd\epsilon_y + d\bar{\lambda}\frac{\partial\phi}{\partial\epsilon_y}$$

$$d\sigma_z = 3\lambda_e de + 2Gd\epsilon_z + d\bar{\lambda}\frac{\partial\phi}{\partial\epsilon_z}$$

$$d\tau_{xy} = 2Gd\epsilon_{xy} + d\bar{\lambda}\frac{\partial\phi}{\partial\epsilon_{xy}}$$

$$d\tau_{yz} = 2Gd\epsilon_{yz} + d\bar{\lambda}\frac{\partial\phi}{\partial\epsilon_{yz}}$$

$$d\tau_{zx} = 2Gd\epsilon_{zx} + d\bar{\lambda}\frac{\partial\phi}{\partial\epsilon_{zx}}$$

and

$$\phi = 0$$

(c) If $\phi = 0$, and the material is an ideal volumetrically locking material, the stress-strain relationships are:

$$d\sigma_x = 3\lambda_e de + 2Gd\epsilon_x + d\lambda\frac{\partial\phi}{\partial\epsilon_x}$$

$$d\sigma_y = 3\lambda_e de + 2Gd\epsilon_y + d\lambda\frac{\partial\phi}{\partial\epsilon_y}$$

$$d\sigma_z = 3\lambda_e de + 2Gd\epsilon_z + d\lambda \frac{\partial \phi}{\partial \epsilon_z}$$

$$d\tau_{xy} = 2Gd\epsilon_{xy}$$

$$d\tau_{yz} = 2Gd\epsilon_{yz}$$

$$d\tau_{zx} = 2Gd\epsilon_{zx}$$

$$\phi = 0$$

(iii) Kinematics

$$\epsilon_x = \frac{\partial u}{\partial x}$$

$$\epsilon_y = \frac{\partial v}{\partial y}$$

$$\epsilon_z = \frac{\partial w}{\partial z}$$

$$\epsilon_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$$

$$\epsilon_{yz} = \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial x}\right)$$

$$\epsilon_{zx} = \frac{1}{2}\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)$$

From the sets (i), (ii), and (iii) of these equations, one can see that the equations and unknowns are well balanced in elastic and locked regions. In an ideally locked region six stress-strain relations, the locking condition, six kinematic relations, and three equilibrium equations are available to determine six stress components, six strain components, three components of displacements and the proportionality parameter $\bar{\lambda}$ or λ.

The preceding formulation of equilibrium stress problem is in cartesian coordinates. Similarly, the problem can be formulated in other coordinate systems. Now the question of uniqueness of the solution of a problem of equilibrium will be considered.

Statement of Uniqueness Theorem: Let a body made of locking material enclosed by a surface S be subjected to the surface traction $\vec{P}_I$ prescribed over a part S_p of the surface S. Let the displacements $\vec{v}_I$ be prescribed over the remaining part of the surface $S - S_p = S_v$. Further, let the stress distribution, the strain pattern, and the displacements in the body corresponding to these boundary values be $\vec{\sigma}_I$, $\vec{\epsilon}_I$, and $\vec{u}_I$. Now, suppose the boundary values are given increments $d\vec{P}$ and $d\vec{v}_I$, then a unique stress distribution $\vec{\sigma}_I + d\vec{\sigma}$ and a unique strain pattern $\vec{\epsilon}_I + d\vec{\epsilon}$ are assured in the body.

Proof: Uniqueness of $\vec{\sigma}_I + d\vec{\sigma}$, $\vec{\epsilon}_I + d\vec{\epsilon}$ is assured if the uniqueness of $d\vec{\sigma}$, $d\vec{\epsilon}$ is proved. Suppose the increments $d\vec{\sigma}$, $d\vec{\epsilon}$ are not unique, at least two sets of solutions

(i) $d\vec{\sigma}_a$, $d\vec{\epsilon}_a$, $d\vec{u}_a$

(ii) $d\vec{\sigma}_b$, $d\vec{\epsilon}_b$, $d\vec{u}_b$

are possible. Both solutions satisfy the same set of boundary values. Then the stress distribution

$$\left.\begin{aligned} d\vec{\sigma}_a - d\vec{\sigma}_b &= \sigma^* \\ \text{the strain pattern} \\ d\vec{\epsilon}_a - d\vec{\epsilon}_b &= \epsilon^* \\ \text{and the displacements} \\ d\vec{u}_a - d\vec{u}_b &= u^* \end{aligned}\right\} \qquad (2.37)$$

are the solutions of a problem satisfying the following boundary values.

(i) Surface traction on the boundary S_p is zero

(ii) Displacements on the boundary S_v are zero

Further, the body forces are assumed to be prescribed throughout the body. Then they are zero for the problem defined by * quantities.

The stresses $\vec{\sigma}^*$ are the solutions of a problem of equilibrium. Therefore, they should satisfy the equilibrium conditions. These conditions when referred to a set of cartesian coordinates can be written in the following form:

$$\frac{\partial}{\partial x}\sigma_x^* + \frac{\partial}{\partial y}\tau_{yx}^* + \frac{\partial}{\partial z}\tau_{zx}^* = 0 \qquad (2.38a)$$

$$\frac{\partial}{\partial x}\tau_{xy}^* + \frac{\partial}{\partial y}\sigma_y^* + \frac{\partial}{\partial z}\tau_{zy}^* = 0 \qquad (2.38b)$$

$$\frac{\partial}{\partial x}\tau_{xz}^* + \frac{\partial}{\partial y}\tau_{yz}^* + \frac{\partial}{\partial z}\sigma_z^* = 0 \qquad (2.38c)$$

Multiplying Eq. (2.38a) by u^*, (2.38b) by v^*, Eq. (2.38c) by w^*, adding and integrating the resulting expression over the volume, one obtains

$$\iiint_{\text{Volume}} \left[u^* \left(\frac{\partial \sigma_x^*}{\partial x} + \frac{\partial \tau_{yx}^*}{\partial y} + \frac{\partial \tau_{zx}^*}{\partial z} \right) + v^* \left(\frac{\partial \tau_{xy}^*}{\partial x} + \frac{\partial \sigma_y^*}{\partial y} + \frac{\partial \tau_{zy}^*}{\partial z} \right) + w^* \left(\frac{\partial \tau_{xz}^*}{\partial x} + \frac{\partial \tau_{yz}^*}{\partial y} + \frac{\partial \sigma_z^*}{\partial z} \right) \right] dxdydz = 0 \qquad (2.39)$$

Using the divergence theorem this equation can be simplified to the following form.

$$\iint_{\text{Surface}} \left[u^* \left(\ell\sigma_x^* + m\tau_{xy}^* + n\tau_{xz}^* \right) + v^* \left(\ell\tau_{xy}^* + m\sigma_y^* + n\tau_{yz}^* \right) + w^* \left(\ell\tau_{zx}^* + n\tau_{yz}^* + n\sigma_z^* \right) \right] ds$$

$$- \iiint_{\text{Volume}} \left[\sigma_x^* \epsilon_x^* + \sigma_y^* \epsilon_y^* + \sigma_z^* \epsilon_z^* + 2\tau_{xy}^* \epsilon_{xy}^* + 2\tau_{yz}^* \epsilon_{yz}^* + 2\tau_{zx}^* \epsilon_{zx}^* \right] dxdydz = 0 \qquad (2.40)$$

where ℓ, m, and n are the direction cosines, and $\epsilon^*_x \cdots \epsilon^*_{zx}$ are the components of the strain tensor corresponding to the stresses $\sigma^*_x \cdots \tau^*_{zx}$.

Because the boundary values for this problem are zero, Eq. (2.40) simplifies to

$$\iiint_{\text{Volume}} \vec{\sigma}^* \cdot \vec{\epsilon}^* \, dxdydz = 0 \tag{2.41}$$

In general depending on the strain state, the volume of the body is divided into elastic regions V_e and locked regions V_ℓ. Then, Eq. (2.41) becomes

$$\iiint_{V_e} \vec{\sigma}^* \cdot \vec{\epsilon}^* \, dxdydz + \iiint_{V_\ell} \vec{\sigma}^* \cdot \vec{\epsilon}^* \, dxdydz = 0 \tag{2.42}$$

The positive definiteness of the integrands of this equation will now be studied.

For practical convenience the integrand over the elastic regions V_e is denoted by I_e and the integrand over the locked region V_ℓ by I_ℓ. Because the stresses and strains are related by Hooke's law in the elastic regions

$$I_e \geq 0 \tag{2.43}$$

In this equation the equal sign applies only if all $\sigma d\epsilon$ are identically zero. Using Eqs. (2.37), the integrand I_ℓ can be written as

$$I_\ell = (d\vec{\sigma}_a - d\vec{\sigma}_b) \cdot (d\vec{\epsilon}_a - d\vec{\epsilon}_b) \tag{2.45}$$

The type of locking could be ideal or non-ideal. Only the case of ideal locking will be considered.

Ideal locking: In case of the ideal locking strain vectors $d\vec{\epsilon}_a$ and $d\vec{\epsilon}_b$ in the locked region must be along the locking surface (Fig. 2.2)

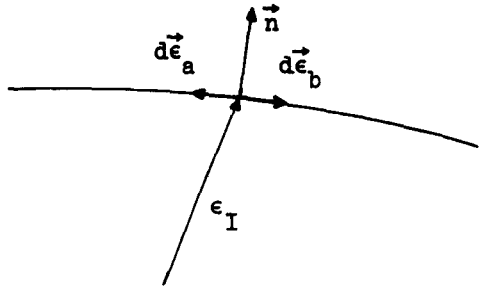


Fig. 2.2.

The stress increments in thelocked region $d\vec{\sigma}_a$ and $d\vec{\sigma}_b$ can be split into elastic and locked stress increments. Thus

$$d\vec{\sigma}_a = d\vec{\sigma}_{ae} + d\vec{\sigma}_{al}$$

$$d\vec{\sigma}_b = d\vec{\sigma}_{be} + d\vec{\sigma}_{bl}$$

Then the expression for I_l is

$$I_l = (d\vec{\sigma}_{ae} - d\vec{\sigma}_{be})\cdot(d\vec{\epsilon}_a - d\vec{\epsilon}_b) + (d\vec{\sigma}_{al} - d\vec{\sigma}_{bl})\cdot(d\vec{\epsilon}_a - d\vec{\epsilon}_b)$$

Then, I_l can be written as

$$I_l = I_1 + I_2$$

where

$$\left.\begin{aligned} I_1 &= (d\vec{\sigma}_{ae} - d\vec{\sigma}_{be})\cdot(d\vec{\epsilon}_a - d\vec{\epsilon}_b) \\ I_2 &= (d\vec{\sigma}_{al} - d\vec{\sigma}_{bl})\cdot(d\vec{\epsilon}_a - d\vec{\epsilon}_b) \end{aligned}\right\} \qquad (2.46)$$

The stress $(d\vec{\sigma}_{ae} - d\vec{\sigma}_{be})$ corresponding to $(d\vec{\epsilon}_a - d\vec{\epsilon}_b)$ can be found from Hooke's law. From the results of section II.5, $(d\vec{\sigma}_{al} - d\vec{\sigma}_{bl})$ is normal to the locking surface while (Fig. 2.2) $d\vec{\epsilon}_a - d\vec{\epsilon}_b$ is tangential to the locking surface. Then

$$I_2 = (d\vec{\sigma}_{a\ell} - d\vec{\sigma}_{b\ell})\cdot(d\vec{\epsilon}_a - d\vec{\epsilon}_b) = 0 \qquad (2.47)$$

From this equation and Eq. (2.46), Eq. (2.42) can be rewritten as

$$\iiint_{V_e} \vec{\sigma}^* \cdot \vec{\epsilon}^* \, dxdydz + \iiint_{V_\ell} \vec{\sigma}^*_e \cdot \vec{\epsilon}^* \, dxdydz = 0$$

where

$$\vec{\sigma}^*_e = d\vec{\sigma}_{ae} - d\vec{\sigma}_{be}$$

In V_e, $\vec{\sigma}^* \cdot \vec{\epsilon}^*$ (which when expanded in cartesian coordinates is $\sigma^*_x\epsilon^*_x + \sigma^*_y\epsilon^*_y + \sigma^*_z\epsilon^*_z + 2\tau^*_{xy}\epsilon^*_{xy} + 2\tau^*_{yz}\epsilon^*_{yz} + 2\tau^*_{xz}\epsilon^*_{xz}$) is twice the strain energy $2u_e$ per unit volume of the material† corresponding to the stress distribution $\vec{\sigma}^*$. Similarly $\vec{\sigma}^*_e \cdot \vec{\epsilon}^*$ represents twice the strain energy $2u_{\ell e}$ in V_ℓ because the contribution from the locked stresses are zero (see Eq. 2.47) and $\vec{\sigma}^*_e = d\sigma_{ae} - d\sigma_{be}$ is related to $\vec{\epsilon}^* = d\vec{\epsilon}_a - d\vec{\epsilon}_b$ by Hooke's law. Then the preceding equation becomes

$$\iiint_V 2u\,dxdydz = 0$$

where u is the strain energy per unit volume of the body. Then the integral can be zero only if u is identically zero in the body. Hence, only rigid body displacements u^*, v^*, w^* are possible. This means that

$$\vec{\sigma}^* \equiv 0$$

$$\vec{\epsilon}^* \equiv 0$$

throughout the body. Then

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†Ref. 2., p. 171.

$$d\vec{\sigma}_a = d\vec{\sigma}_b$$
$$d\vec{\epsilon}_a = d\vec{\epsilon}_b$$

That is, the increments $d\vec{\sigma}$, $d\vec{\epsilon}$, are unique.

Though the stresses $d\vec{\sigma}_\ell$ do not contribute towards any work, it should be noted that they are not arbitrary but are determined, through the locking parameter $d\lambda$, by the sets of equations (equilibrium, stress-strain relationship, locking condition and kinematics) as discussed earlier.

This concludes the proof of the uniqueness theorem as stated on page 32.

II.12. Limitations on Displacement Boundary Conditions in Ideal Locking

It has been proved in the last section that if a solution of an equilibrium boundary value problem exists, it is unique. In this section a particular case will be illustrated where the solution does not exist.

Consider a body made of an ideally locking material. Let only displacements be prescribed on the surface of the body. Let the distribution of these displacements have a fixed pattern on the surface and increase the magnitude of $\vec{v}$ gradually from zero. For a certain value of $\vec{v} = \vec{v}_I$ the body locks in some region. In general the complete body will not be locked at once. As $\vec{v}$ is further increased the locked region increases until at a certain value of $\vec{v} = \vec{v}_c$, the complete body is locked. This is called the 'completely locked state' of the body. In certain cases the body can change at once from the elastic to the completely locked state.

If the body obeys the following distortional locking condition

$$\phi(J_2') = J_2' - C = 0$$

the body becomes completely locked when

$$J_2' = C$$

throughout the body. Then, because the locking is ideal

$$\Delta J_2' = 0$$

throughout the body, i.e.,

$$\Delta\epsilon_1 = \Delta\epsilon_2 = \Delta\epsilon_3.$$

Now consider the strain energy u of the body.

$$u = u_0(e) + u'(J_2')$$

After the body is complete locked $u'(J_2')$ cannot be increased, because the locking is ideal. Therefore additional work can be done only by increasing $u_0(e)$, that is, the body is deforming without distortion. But, the distribution of displacements on the surface has a fixed pattern. Then the prescribed displacement should be such that the body is deformed without distortion. However, this is a contradiction, because such displacements will not produce distortional locking. Hence it can be concluded that no further increase of $\vec{v}$ is possible after the body is completely locked distortionally.

Similar results can be obtained for volumetric locking with the locking condition defined by the following equation

$$\phi(e) = 0$$

II.13. Dynamic Problem – Uniqueness

In the present section the formulation of a dynamic response problem and the uniqueness under certain boundary and initial conditions will be studied.

The formulation of the dynamic response problem is very similar to the formulation of the equilibrium problem, excepting that the equilibrium conditions are replaced by the equations of motion. The second problem under investigation, the uniqueness, will be considered in the following way.

Statement of Uniqueness Theorem (dynamic problem): Consider a body of a locking material enclosed by a surface S. Let the surface traction $\vec{P}(t)$ be prescribed over a part S_P of the surface S and the displacements $\vec{v}(t)$ over the remaining part $S - S_P = S_v$. Also let the stress distribution $\vec{\sigma}_I$, the strain pattern $\vec{\epsilon}_I$, the displacements $\vec{u}_I$ and velocity $\dot{\vec{u}}_I$ in the body be known at time $t = t_I$. Then during the interval $t = t_I$ and $t = t_I + dt$ unique increments $d\vec{\sigma}$ of $\vec{\sigma}_I$, $d\vec{\epsilon}$ of $\vec{\epsilon}_I$, and $d\dot{\vec{u}}$ of $\dot{\vec{u}}_I$ are assured.

Proof: If the solution is not unique, at least two solutions are possible.

(i) $d\vec{\sigma}_a$, $d\vec{\epsilon}_a$, $d\vec{u}_a$

(ii) $d\vec{\sigma}_b$, $d\vec{\epsilon}_b$, $d\vec{u}_b$

Both solutions correspond to the same set of boundary values. Therefore, the stress distribution

$$d\vec{\sigma}_a - d\vec{\sigma}_b = \vec{\sigma}^*$$

the strain pattern

$$d\vec{\epsilon}_a - d\vec{\epsilon}_b = \vec{\epsilon}^* \qquad (2.48)$$

and the displacements

$$d\vec{u}_a - d\vec{u}_b = \vec{u}^*$$

are the solutions of a problem satisfying the following boundary values.

(i) surface traction on S_P is zero

(ii) displacements on S_v are zero

Furthermore, body forces for the problem defined by Eq. (2.48) are zero because the body forces are assumed to be prescribed in the body.

$\vec{\sigma}^*$, $\vec{\epsilon}^*$, $\vec{u}^*$ are the solutions of a problem in dynamics. Therefore, they should satisfy the equations of motion, appropriate stress-strain relationships and kinematics. These equations are the same as those discussed in section II.12, excepting that the equations of motion replace the equations of equilibrium.

The equations of motion for the stress distribution $\vec{\sigma}^*$ and the displacements $\vec{u}^*$ are

$$\begin{aligned}
&\frac{\partial \sigma_x^*}{\partial x} + \frac{\partial \tau_{yx}^*}{\partial y} + \frac{\partial \tau_{zx}^*}{\partial z} = \rho \frac{\partial^2 u^*}{\partial t^2} \\
&\frac{\partial \tau_{xy}^*}{\partial y} + \frac{\partial \sigma_y^*}{\partial y} + \frac{\partial \tau_{zy}^*}{\partial z} = \rho \frac{\partial^2 v^*}{\partial t^2} \qquad (2.49) \\
&\frac{\partial \tau_{xz}^*}{\partial x} + \frac{\partial \tau_{yz}^*}{\partial y} + \frac{\partial \sigma_z^*}{\partial z} = \rho \frac{\partial^2 w^*}{\partial t^2}
\end{aligned}$$

By using these equations an integral of the following type can be written

$$\begin{aligned}
\Delta t \iiint_{\text{Volume}} &\left[\frac{\partial u^*}{\partial t} \left(\rho \frac{\partial^2 u^*}{\partial t^2} - \frac{\partial \sigma_x^*}{\partial x} - \frac{\partial \tau_{xy}^*}{\partial y} - \frac{\partial \tau_{xz}^*}{\partial z} \right) \right. \\
&+ \frac{\partial v^*}{\partial t} \left(\rho \frac{\partial^2 v^*}{\partial t^2} - \frac{\partial \tau_{xy}^*}{\partial x} - \frac{\partial \sigma_y^*}{\partial y} - \frac{\partial \tau_{yz}^*}{\partial z} \right) \\
&\left. + \frac{\partial w^*}{\partial t} \left(\frac{\partial^2 w^*}{\partial t^2} - \frac{\partial \tau_{xz}^*}{\partial x} - \frac{\partial \tau_{yz}^*}{\partial y} - \frac{\partial \sigma_z^*}{\partial z} \right) \right] \quad dxdydz = 0
\end{aligned}$$

that is,

$$\begin{aligned}
&\Delta t \cdot \frac{\partial}{\partial t} \iiint \frac{\rho}{2} \left[\left(\frac{\partial u^*}{\partial t} \right)^2 + \left(\frac{\partial v^*}{\partial t} \right)^2 + \left(\frac{\partial w^*}{\partial t} \right)^2 \right] dxdydz \\
&= \Delta t \iiint \left[\frac{\partial u^*}{\partial t} \left(\frac{\partial \sigma_x^*}{\partial x} + \frac{\partial \tau_{xy}^*}{\partial y} + \frac{\partial \tau_{xz}^*}{\partial z} \right) \right. \\
&\left. + \frac{\partial v^*}{\partial t} \left(\frac{\partial \tau_{xy}^*}{\partial x} + \frac{\partial \sigma_y^*}{\partial y} + \frac{\partial \tau_{zy}^*}{\partial z} \right) + \frac{\partial w^*}{\partial t} \left(\frac{\partial \tau_{xz}^*}{\partial x} + \frac{\partial \tau_{y3}^*}{\partial y} + \frac{\partial \sigma_z^*}{\partial z} \right) \right] dxdydz \qquad (2.50)
\end{aligned}$$

By using the divergence theorem and observing that the boundary values are zero, this equation simplifies to

$$\iiint_{\text{Volume}} (I_1 + I_2)dxdydz = 0 \tag{2.51}$$

where

$$I_1 = \Delta \cdot \frac{\partial}{\partial t} \frac{\rho}{2} \left[\left(\frac{\partial u^*}{\partial t}\right)^2 + \left(\frac{\partial v^*}{\partial t}\right)^2 + \left(\frac{\partial w^*}{\partial t}\right)^2 \right]$$

$$I_2 = \vec{\sigma}^* \cdot \frac{\partial}{\partial t} (\vec{\epsilon}^*)\Delta t$$

I_1 can be written as

$$I_1 = \left\{ \frac{\rho}{2} \left[\left(\frac{\partial u^*}{\partial t}\right)^2 + \left(\frac{\partial v^*}{\partial t}\right)^2 + \left(\frac{\partial w^*}{\partial t}\right)^2 \right]_{t=t_I+\Delta t} \right.$$

$$\left. - \frac{\rho}{2} \left[\left(\frac{\partial u^*}{\partial t}\right)^2 + \left(\frac{\partial v^*}{\partial t}\right)^2 + \left(\frac{\partial w^*}{\partial t}\right)^2 \right]_{t=t_I} \right\}$$

at $t = t_I$ all the increments u^*, v^*, w^* and their time derivatives are zero because we know the solution $\vec{\sigma}_I$, $\vec{\epsilon}_I$, $\vec{u}_I$ and $\dot{\vec{u}}_I$ at $t = t_I$. Then

$$I_1 \geq 0 \tag{2.52}$$

Now consider I_2. We write

$$I_2 = (d\vec{\sigma}_a - d\vec{\sigma}_b) \cdot \left[\frac{\partial}{\partial t} (d\vec{\epsilon}_a - d\vec{\epsilon}_b)\Delta t \right]$$

that is,

$$I_2 = (d\vec{\sigma}_a - d\vec{\sigma}_b) \cdot \left[\left(d\vec{\epsilon}_a - d\vec{\epsilon}_b\right)_{t=t_I+\Delta t} - \left(d\vec{\epsilon}_a - d\vec{\epsilon}_b\right)_{t=t_I} \right]$$

Further, at $t = t_I$

$$d\vec{\epsilon}_a = d\vec{\epsilon}_b = 0$$

Therefore,

$$I_2 = (d\vec{\sigma}_a - d\vec{\sigma}_b)\cdot(d\vec{\epsilon}_a - d\vec{\epsilon}_b)_{t_I+\Delta t} \tag{2.53}$$

The expression on the right-hand side of this equation has been discussed in section II.12. It has been shown that the integrand corresponds to twice the strain energy. By following a procedure similar to that of section II.12 it can be shown that Eq. (2.51) implies that only rigid body displacements, independent of time, are possible and hence the uniqueness of stresses and strains.

III. PROBLEMS OF EQUILIBRIUM VOLUMETRIC LOCKING

III.1 Plane Stress Problems in Volumetric Locking

Let us consider a thin plate in a state of plane stress as shown in Fig. 3.1. The conditions of equilibrium for the stresses are

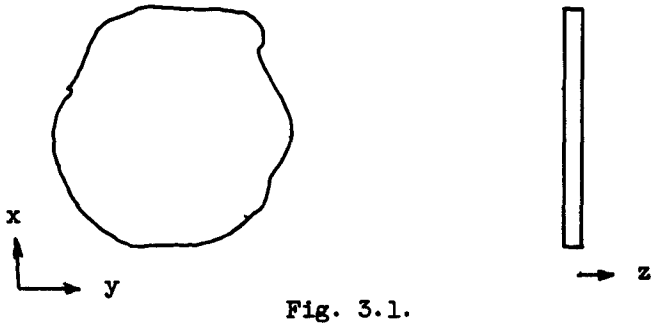

Fig. 3.1.

$$\left.\begin{aligned} \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} &= 0 \\ \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} &= 0 \end{aligned}\right\} \qquad (3.1)$$

These equations are identically satisfied if a stress function χ is defined such that

$$\sigma_x = \frac{\partial^2 \chi}{\partial y^2}, \quad \sigma_y = \frac{\partial^2 \chi}{\partial x^2}, \quad \tau_{xy} = -\frac{\partial^2 \chi}{\partial x \partial y} \qquad (3.2)$$

If volumetric locking occurs, stress and strain in the locked region are related in the following way.

$$\left.\begin{aligned} 2G(\epsilon_x - \epsilon_z) &= \sigma_x \\ 2G(\epsilon_y - \epsilon_z) &= \sigma_y \\ G\gamma_{xy} &= \tau_{xy} \\ \epsilon_x + \epsilon_y + \epsilon_z &= \epsilon_l \end{aligned}\right\} \qquad (3.3)$$

These equations can be simplified into a more useful form:

$$\left.\begin{aligned} \sigma_x &= 2G(2\epsilon_x + \epsilon_y - \epsilon_\ell) \\ \sigma_y &= 2G(\epsilon_x + 2\epsilon_y - \epsilon_\ell) \end{aligned}\right\} \tag{3.4}$$

$$\left.\begin{aligned} \epsilon_x &= \frac{1}{6G}(2\sigma_x - \sigma_y) + \frac{\epsilon_\ell}{3} \\ \epsilon_y &= \frac{1}{6G}(-\sigma_x + 2\sigma_y) + \frac{\epsilon_\ell}{3} \end{aligned}\right\} \tag{3.5}$$

Further, in order to assure single valued displacements in a simply connected body, the strains must satisfy the compatibility conditions. It is not possible to satisfy all the compatibility conditions if one assumes that the stresses σ_x, σ_y and τ_{xy} are functions of x and y only. It becomes necessary to assume that the stresses depend on z also. However, as in the case of elastic materials*, it can be shown that the dependence on z becomes negligable as the thickness of the plate becomes small. Then, the only compatibility condition to be satisfied by the strains is

$$\frac{\partial^2 \epsilon_x}{\partial y^2} + \frac{\partial^2 \epsilon_y}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y} \tag{3.6}$$

Then, by using Eqs. (3.1) and (3.4) this equation can be written in terms of stresses

$$\nabla^2(\sigma_x + \sigma_y) = 0 \tag{3.7}$$

where

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

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*Ref. 3, p. 241.

Then, this equation and Eq. (3.2) yield the following differential equation for X

$$\nabla^2\nabla^2 X = 0 \tag{3.8}$$

Thus, if the stress function X is obtained by solving this equation with appropriate boundary conditions, stresses, strains and hence displacements can be calculated.

The plane stress problem in an elastic locking material can now be stated in the following way. In general, a thin plate in a state of plane stress may contain locked and elastic regions. The stress function and hence the stresses can be obtained by solving Eq. (3.8). The solution in the elastic region can be determined by methods of solving plane stress problems in elastic bodies. Then the solution is complete if the stresses, strains and displacements in elastic and locked regions satisfy the appropriate boundary and interface conditions.

III.2 Pure Bending of Beams in Plane Stress

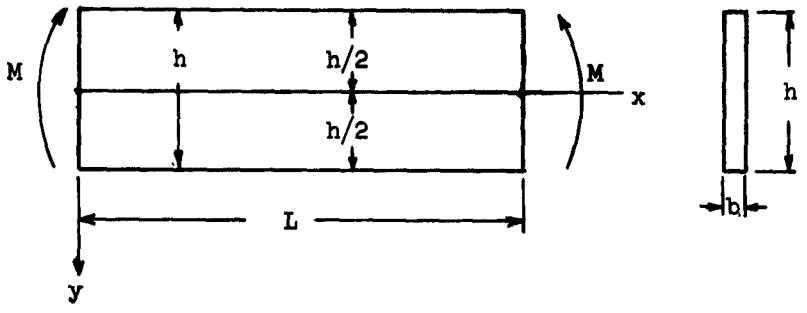
(i) Elastic Solution

We shall now study the stresses in a beam of narrow rectangular cross section made of ideal volumetric locking material subjected to pure bending moments M as shown in the Fig. 3.2. When the value of M is very small the corresponding strains are very small. The beam then deforms as an elastic beam. The stresses in such a beam are

$$\sigma_x = \frac{12My}{bh^3}$$

$$\sigma_y = \sigma_z = \tau_{yz} = \tau_{zx} = \tau_{xy} = 0$$

Fig. 3.2.



where h is the depth of the beam and b is the width of the beam. From the preceding expression for the stresses one can evaluate the dilatation:

$$\epsilon_x + \epsilon_y + \epsilon_z = \frac{1 - 2\nu}{E} \frac{12M}{bh^3} y$$

Because the beam is made of volumetrically locking material, locking takes place when

$$\epsilon_x + \epsilon_y + \epsilon_z = \pm \epsilon_\ell$$

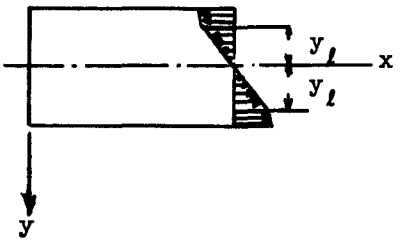
This locking condition assumes that the material can lock in compression as well as in tension. Then, the beam subjected to pure bending moments M locks at $y = \pm h/2$ when

$$M = M_\ell = \frac{6Eh^2b}{1 - 2\nu} \epsilon_\ell \qquad (3.9)$$

(ii) Problem after Incipient Locking

If M is increased beyond the limit given by Eq. (3.9), locked regions will develop near the upper and lower edges of the cross-section. There is an elastic core between these two locked regions. The interfaces between the elastic core and the locked regions are at $y = \pm y_\ell$ as shown in Fig. 3.3.

Fig. 3.3.



Because the locked regions are situated symmetrically with respect to the x-axis, the stress distribution is as shown in the figure and the neutral axis is the line $y = 0$. The stress function is governed by the differential equation

$$\nabla^2\nabla^2\chi = 0$$

in the elastic as well as in the locked regions. The boundary conditions for the problem will now be discussed.

Because the stress and displacement fields are antimetric with respect to the neutral plane $y = 0$, it suffices to restrict the further discussion to the region $y > 0$ only. Then, (Stresses, strains, and displacements in the locked region will be denoted with a bar over the letters, e.g., $\bar{\sigma}_x$) for $x = 0$ and $x = L$

$$\tau_{xy} = \bar{\tau}_{xy} = 0 \qquad (a)$$

$$\int_{-h/2}^{h/2} \sigma_x dy = 0 \qquad (b)$$

$$2b\int_{0}^{h/2} \sigma_x\, y\, dy = M \qquad (c)$$

for $y = h/2$

$$\bar{\sigma}_y = \bar{\tau}_{xy} = 0 \tag{d}$$

for $y = y_\ell$

$$u = \bar{u} \tag{e}$$

$$v = \bar{v} \tag{f}$$

$$\sigma_y = \bar{\sigma}_y \tag{g}$$

$$\tau_{xy} = \bar{\tau}_{xy} \tag{h}$$

$$\epsilon_x + \epsilon_y + \epsilon_z = \frac{1 - 2\nu}{E} \sigma_x = \epsilon_\ell \tag{i}$$

(3.10)

(iii) Solution in the Locked Regions

The stress function X in the locked region is assumed as

$$X = \frac{a}{6} y^3 - \frac{G\epsilon_\ell}{2} y^2 \tag{3.11}$$

The stresses in the locked region $y > 0$ are then given by the following expressions.

$$\left.\begin{aligned} \bar{\sigma}_x &= ay - G\epsilon_\ell \\ \bar{\sigma}_y &= 0 \\ \bar{\tau}_{xy} &= 0 \end{aligned}\right\} \tag{3.12}$$

The corresponding strains can be calculated from Eq. (3.5). They are

$$\left.\begin{aligned} \bar{\epsilon}_x &= \frac{ay}{3G} \\ \bar{\epsilon}_y &= -\frac{ay}{6G} + \frac{\epsilon_\ell}{2} \\ \bar{\gamma}_{xy} &= 0 \end{aligned}\right\} \tag{3.13}$$

These three equations can be integrated to obtain the displacements. Thus,

$$\left.\begin{aligned} \bar{u} &= \frac{axy}{3G} - \omega y + \bar{C}_1 \\ \bar{v} &= -\frac{ay^2}{12G} - \frac{ax^2}{6G} + \frac{\epsilon_l}{2} y + \omega x + \bar{C}_2 \end{aligned}\right\} \qquad (3.14)$$

(iv) Solution in the Elastic Region

In the elastic region the stress function is

$$\chi = \frac{d}{6} y^3$$

where d is a constant. From this the stresses in the elastic region can be calculated:

$$\left.\begin{aligned} \sigma_x &= dy \\ \sigma_y &= \tau_{xy} = 0 \end{aligned}\right\} \qquad (3.15)$$

The corresponding strains and displacements are given by the following equations:

$$\left.\begin{aligned} \epsilon_x &= \frac{yd}{E} \\ \epsilon_y &= \frac{\nu yd}{E} \\ \gamma_{xy} &= 0 \end{aligned}\right\} \qquad (3.16)$$

$$\left.\begin{aligned} u &= \frac{dxy}{E} - \omega_1 y + C_1 \\ v &= -\frac{\nu dy^2}{2E} - \frac{dx^2}{2E} + \omega_1 x + C_2 \end{aligned}\right\} \qquad (3.17)$$

(v) Matching the Boundary Conditions

The boundary conditions (3.10a,b,d,g,h) are automatically satisfied by the stresses as given in subsections (iii) and (iv). The conditions (3.10e,f) yield the following relationships:

$$\frac{d}{E} = \frac{a}{3G}$$

$$-\omega_1 y_\ell + C_1 = -\omega y_\ell + \bar{C}_1 \tag{3.18}$$

$$\frac{a y_\ell^2}{12G} + \frac{\epsilon_\ell}{2} y_\ell + \bar{C}_2 + \omega x = \frac{\nu y_\ell^2 d}{2E} + C_2 + \omega_1 x \tag{3.19}$$

Equation (3.19) can be satisfied only if $\omega = \omega_1$. Then

$$C_2 - \bar{C}_2 = \frac{a y_\ell^2}{6G}\left(\frac{1}{2} - \nu\right) + \frac{\epsilon_\ell y_\ell}{2}$$

$$C_1 = \bar{C}_1 \tag{3.20}$$

From this equation and Eq. (3.10i) we have

$$\left.\begin{aligned} b\,\frac{(1-2\nu)}{E}\, y_\ell = a y_\ell \frac{(1-2\nu)}{3G} &= \epsilon_\ell \\ C_2 - \bar{C}_2 &= \frac{3}{4}\epsilon_\ell y_\ell \end{aligned}\right\} \tag{3.21}$$

Now the only condition to be satisfied is (3.10c). This will be discussed in the next sub-section.

From Eqs. (3.18)-(3.20) one can observe that three of the five quantities $C_1, \bar{C}_1, C_2, \bar{C}_2, \omega$, can be arbitrarily chosen. These correspond to two rigid body displacements and a rigid body rotation permissible in a plane stress problem.

(vi) Relationship between Bending Moment and Curvature

From Eqs. (3.10), (3.12) and (3.15) the stresses in the locked and elastic regions can be expressed in the following way.

$$\left.\begin{aligned} \sigma_x &= \frac{E\epsilon_\ell}{1-2\nu}\ \frac{y}{y_\ell} \\ \bar{\sigma}_x &= \frac{3E\epsilon_\ell}{2(1+\nu)(1-2\nu)}\ \frac{y}{y_\ell} - \frac{E\epsilon_\ell}{2(1+\nu)} \end{aligned}\right\} \qquad (3.22)$$

By substituting these expressions in (3.10c) one obtains the bending moment

$$M = \frac{bE\epsilon_\ell}{6(1+\nu)}\left[y_\ell^2 + \frac{3}{4}\ \frac{1}{1-2\nu}\frac{h^3}{y_\ell} - \frac{3h^2}{4}\right] \qquad (3.23)$$

Strains in the elastic and locked regions can be obtained using Hooke's law and Eq. (3.5). Then ϵ_x in the elastic as well as the locked region is given by the following equation

$$\epsilon_x = \frac{\epsilon_\ell}{1-2\nu}\ \frac{y}{y_\ell} \qquad (3.24)$$

This equation shows that the plane sections before bending remain plane after bending. Also, from this equation, the curvature of beam is found to be

$$\kappa = \frac{\epsilon_\ell}{1-2\nu}\ \frac{1}{y_\ell}$$

Then the expression for the bending moment becomes

$$M = \frac{bE}{6(1+\nu)}\left[\frac{\epsilon_\ell^3}{(1-2\nu)^2}\ \frac{1}{\kappa^2} + \frac{3}{4}h^3\kappa - \frac{3}{4}\epsilon_\ell h^2\right] \qquad (3.25)$$

The variation of M with κ is shown in Fig. 3.4.

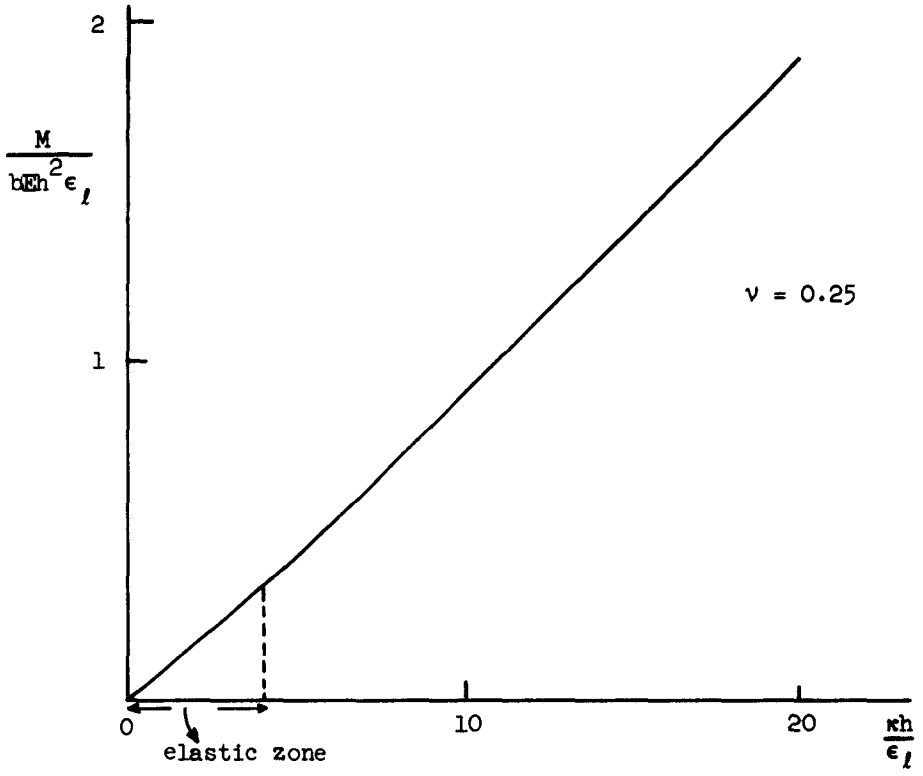


Fig. 3.4a.

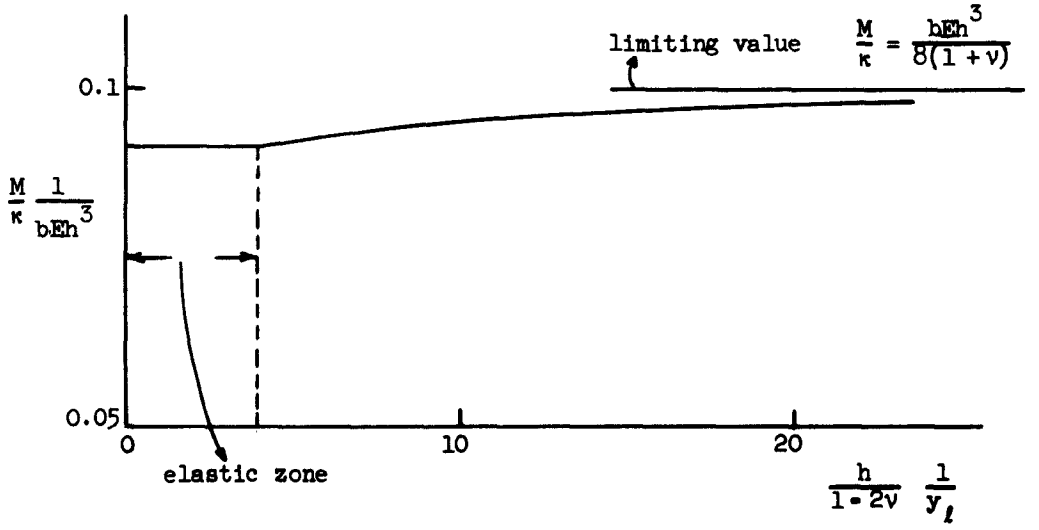


Fig. 3.4b.

The value of M/κ as y_l tends to zero can be evaluated from Eqs.(3.24) and (3.25). It is

$$\frac{M}{\kappa} = \frac{bEh^3}{8(1+\nu)} = EI_e \tag{3.26}$$

where I_e is defined as the effective moment of inertia. For a value of $\nu = 0.25$,

$$EI_e = \frac{Ebh^3}{10} \tag{3.27}$$

which is not an appreciable change from the elastic flexural rigidity $Ebh^3/12$. This can also be seen from Fig. 3.4b.

(vi) Application of the Results to the Solution of Beam Problems

As an application of the derived results, the problem of a beam subjected to a loading other than pure bending will be considered. In particular, the deflections of a simply supported beam AB under the action of a concentrated load P at mid span will be studied (Fig. 3.5)

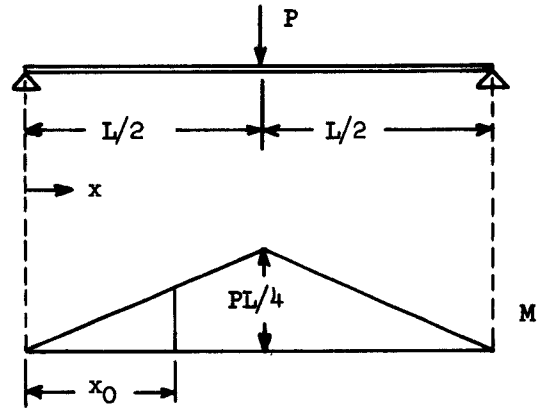


Fig. 3.5.

The bending moment diagram is shown in the figure. According to Eq. (3.9) the beam locks when

$$M = M_{\ell} = \frac{24Eh^2b}{1 - 2\nu} \epsilon_{\ell}$$

Then the beam begins to lock at $x = L/2$ when

$$P = P_{\ell} = \frac{96Eh^2b\epsilon_{\ell}}{(1 - 2\nu)L} \tag{3.28}$$

For $P > P_{\ell}$ there will be a locked region $x_0 < x < (L/2) + x_0$. At x_0 the bending moment is just equal to M_{ℓ}. Then

$$x_0 = \frac{48Eh^2b\epsilon_{\ell}}{(1 - 2\nu)P} \tag{3.29}$$

If one assumes that the moment-curvature relationship derived for pure bending can be used in this case, i.e., the change of curvature due to shearing forces is negligible, one has (see Eq. 3.23)

$$\left.\begin{aligned} &\frac{P}{2} x = EI\kappa \qquad 0 < x < x_0 \\ &\frac{P}{2} x = \frac{bE}{6(1 + \nu)} \left[\frac{\epsilon_{\ell}^3}{1 - 2\nu} \frac{1}{\kappa^2} + \frac{3}{4} h^3\kappa - \frac{3}{4} \epsilon_{\ell} h^2 \right] \\ &\qquad\qquad\qquad\qquad x_0 < x < \frac{L}{2} \end{aligned}\right\} \tag{3.30}$$

For small deflections κ can be expressed as the second derivative of the deflection w:

$$\kappa = - \frac{d^2w}{dx^2} \tag{3.31}$$

This equation and Eq. (3.30) can be used to determine w. However as explained in sub-section (vi), even in the extreme case where the complete cross-section of the beam is locked, the change of flexural rigidity from

the elastic value EI is very small. Then one can get upper and lower bounds for the deflection by considering the following two cases

(i) The complete beam is elastic

(ii) The region $0 < x < x_0$ is elastic while in the region $x_0 < x < (L/2)$ the beam is locked over the entire cross-section.

In case (i) the deflection is

$$w = -\frac{P}{12EI}\left(x^3 - \frac{3}{4}L^2x\right) \tag{3.32}$$

In case (ii) we have from Eq. (3.26)

$$\begin{aligned} EI\frac{d^2w}{dx^2} &= -\frac{P}{2}x && 0 < x < x_0 \\ EI_\ell\frac{d^2w}{dx^2} &= -\frac{P}{2}x && 0 < x < L/2 \end{aligned} \tag{3.33}$$

whence

$$\begin{aligned} w &= -\frac{P}{12EI}x^3 + C_1x + C_2 && 0 < x < x_0 \\ \bar{w} &= -\frac{P}{12EI_\ell}x^3 + C_3x + C_4 && x_0 < x < L/2 \end{aligned}$$

The following conditions can be used to determine C_1, C_2, C_3, C_4

$$\left.\begin{aligned} x = 0 \quad & w = 0 \\ x = \frac{L}{2} \quad & \frac{d\bar{w}}{dx} = 0 \\ x = x_0 \quad & w = \bar{w}, \quad \frac{dw}{dx} = \frac{d\bar{w}}{dx} \end{aligned}\right\} \tag{3.34}$$

Then

$$C_1 = -\frac{Px_0^2}{4E}\left(\frac{1}{I} - \frac{1}{I_\ell}\right) + \frac{P\ell^2}{16EI_\ell}$$

$$C_2 = 0$$

$$C_3 = -\frac{PL^2}{16EI_\ell}$$

$$C_4 = -\frac{Px_0^3}{3E}\left(\frac{1}{I} - \frac{1}{I_\ell}\right)$$

Then

$$w = -\frac{P}{E}\left\{\frac{x^3}{12I} + \left[\frac{x_0^2}{4}\left(\frac{1}{I} - \frac{1}{I_\ell}\right) - \frac{L^2}{16I_\ell}\right]x\right\} \qquad 0 < x < x_0$$

$$\bar{w} = -\frac{P}{E}\left\{\frac{x^3}{12I_\ell} - \left[\frac{L^2x}{16I_\ell} - \frac{x_0^3}{3}\left(\frac{1}{I} - \frac{1}{I_\ell}\right)\right]\right\} \qquad x_0 < x < \frac{L}{2} \tag{3.35}$$

In the extreme case when $x_0 = 0$

$$\bar{w} = -\frac{P}{12EI_\ell}\left(x^3 - \frac{3}{4}L^2x\right) \tag{3.36}$$

In the two extreme cases given by Eqs. (3.36) and (3.32) the maximum deflections are at $x = (L/2)$. The values are

$$w_{max} = \frac{PL^3}{48EI}$$

$$\bar{w}_{max} = \frac{PL^3}{48EI_\ell}$$

For $\nu = 0.25$, I/I_ℓ is given by Eq. (3.23). It is equal to 0.8333. Then

$$\frac{\bar{w}_{max}}{w_{max}} = 0.8333$$

The maximum deflection is reduced if the beam is made of locking material.

III.3 Rotating Disks

In this section stresses in a rotating disk (of constant thickness) made of ideally locking material will be studied. The present study is restricted to small deformation.

(i) Elastic Solution

When the angular velocity of the disk is very small the disk deforms as an elastic body. The stresses in such a solid disk are*

$$\left.\begin{aligned} \sigma_r &= \frac{3+\nu}{8}\rho\omega^2(R^2 - r^2) \\ \sigma_\theta &= \frac{3+\nu}{8}\rho\omega^2R^2 - \frac{1+3\nu}{8}\rho\omega^2r^2 \\ \sigma_z &= \tau_{r\theta} = \tau_{rz} = \tau_{\theta z} = 0 \end{aligned}\right\} \qquad (3.37)$$

where ρ is the mass per unit volume of the material, $2R$ is the diameter of the disk and r is the radial coordinate. From Eqs. (3.37) the dilatation is found to be

$$\begin{aligned} \epsilon_r + \epsilon_\theta + \epsilon_z &= \frac{1-2\nu}{E}(\sigma_r + \sigma_\theta) \\ &= \frac{1-2\nu}{4}\,\frac{\rho\omega^2}{E}\,[(3+\nu)R^2 - 2(1+\nu)r^2] \end{aligned} \qquad (3.38)$$

Its maximum occurs at $r = 0$ and is

$$(\epsilon_r + \epsilon_\theta + \epsilon_z)_{max} = \frac{3+\nu}{4}\,\rho\omega^2R^2\,\frac{1-2\nu}{E}$$

It reaches the locking limit ϵ_l when

$$\omega^2 = \omega_0^2 = \frac{4E\epsilon_l}{(3+\nu)\rho R^2(1-2\nu)} \qquad (3.39)$$

---

*Ref. 3, p. 70.

(ii) Problem after Initial Locking

If ω is increased beyond ω_0, the disk can no longer be analysed as an elastic disk. There will be a locked region $r < k$ and an elastic region $k < r < R$. The problem then reduces to that of computing the stresses, strains, and displacements in the elastic and locked regions such that they satisfy the equation of motion, the appropriate stress-strain relationship and kinematics as well as the boundary and interface conditions.

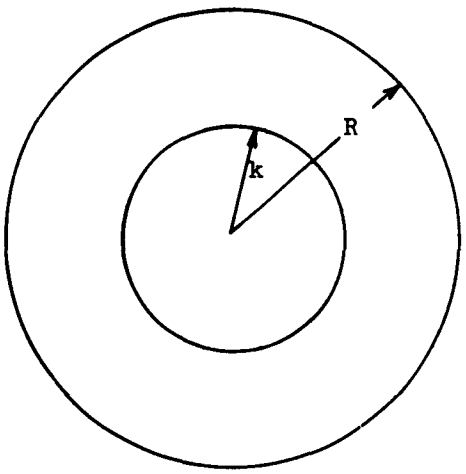


Fig. 3.6.

(iii) Expressions for Stresses, Strains and Displacements in the Locked Region

Under the assumption of plane stress and axisymmetry the quantities to be determined are the stresses $\bar{\sigma}_r, \bar{\sigma}_\theta$, strains $\bar{\epsilon}_r, \bar{\epsilon}_\theta, \bar{\epsilon}_z$ and the radial displacement $\bar{u}$.

Because the locking is ideal the following locking condition

$$\bar{\epsilon}_r + \bar{\epsilon}_\theta + \bar{\epsilon}_z = \frac{1 - 2\nu}{E} (\bar{\sigma}_r + \bar{\sigma}_\theta) = -\epsilon_l \qquad (3.40)$$

holds throughout the locked region. In the locked region, the stress and strain deviators are still related by Hooke's law, i.e.,

$$\left.\begin{aligned} 2G(\bar{\epsilon}_\theta - \bar{\epsilon}_z) &= \bar{\sigma}_\theta \\ 2G(\bar{\epsilon}_r - \bar{\epsilon}_z) &= \bar{\sigma}_r \end{aligned}\right\} \tag{3.41}$$

Other equations to be satisfied by the stresses, strains and displacements are the equation of motion and of kinematics, i.e.,

$$\left.\begin{aligned} \frac{d\bar{\sigma}_r}{dr} + \frac{\bar{\sigma}_r - \bar{\sigma}_\theta}{r} + \rho\omega^2 r &= 0 \\ \bar{\epsilon}_r &= \frac{d\bar{u}}{dr} \end{aligned}\right\} \tag{3.42}$$

$$\bar{\epsilon}_\theta = \frac{\bar{u}}{r} \tag{3.43}$$

Equations (3.40)-(3.43) are sufficient to determine the unknown quantities $\bar{\sigma}_r$, $\bar{\sigma}_\theta$, $\bar{\epsilon}_r$, $\bar{\epsilon}_\theta$, $\bar{\epsilon}_z$ and $\bar{u}$.

By solving (3.40) and (3.41) for stresses one obtains

$$\left.\begin{aligned} \bar{\sigma}_r &= 2G(2\bar{\epsilon}_r + \bar{\epsilon}_\theta - \epsilon_\ell) \\ \bar{\sigma}_\theta &= 2G(\bar{\epsilon}_r + 2\bar{\epsilon}_\theta - \epsilon_\ell) \end{aligned}\right\} \tag{3.44}$$

When this equation and Eq. (3.43) are substituted into the equation of motion a differential equation for u results.

$$\frac{d^2\bar{u}}{dr^2} + \frac{1}{r}\frac{d\bar{u}}{dr} - \frac{\bar{u}}{r^2} + \frac{\rho\omega^2 r(1+\nu)}{2E} = 0$$

It has the solution

$$\bar{u} = C_1 r + \frac{C_2}{r} - \frac{\rho\omega^2 r^3(1+\nu)}{16E} \tag{3.45}$$

Then, the stresses and strains are given by the following expressions:

$$\left.\begin{aligned} \bar{\epsilon}_r &= C_1 - \frac{C_2}{r^2} - \frac{3\rho\omega^2 r^2(1+\nu)}{16E} \\ \bar{\epsilon}_\theta &= C_1 + \frac{C_2}{r^2} - \frac{\rho\omega^2 r^2(1+\nu)}{16E} \end{aligned}\right\} \qquad (3.46)$$

$$\left.\begin{aligned} \bar{\sigma}_r &= \frac{E}{1+\nu}\left[3C_1 - \frac{C_2}{r^2} - \frac{7\rho\omega^2 r^2(1+\nu)}{16E} - \epsilon_\ell\right] \\ \bar{\sigma}_\theta &= \frac{E}{1+\nu}\left[3C_1 + \frac{C_2}{r^2} - \frac{5\rho\omega^2 r^2(1+\nu)}{16E} - \epsilon_\ell\right] \end{aligned}\right\} \qquad (3.47)$$

(iv) Stresses, Strains and Displacements in the Elastic Region

From the known solution* of the problem one can write

$$\left.\begin{aligned} u &= C_3 r + \frac{C_4}{r} - \frac{\rho\omega^2 r^3(1-\nu^2)}{8E} \\ \epsilon_r &= C_3 - \frac{C_4}{r^2} - \frac{3\rho\omega^2 r^2(1-\nu^2)}{8E} \\ \epsilon_\theta &= C_3 + \frac{C_4}{r^2} - \frac{\rho\omega^2 r^2(1-\nu^2)}{8E} \\ \sigma_r &= \frac{E}{1-\nu} C_3 - \frac{E}{1+\nu}\frac{C_4}{r^2} - \frac{\rho\omega^2 r^2(3+\nu)}{8} \\ \sigma_\theta &= \frac{E}{1-\nu} C_3 + \frac{E}{1+\nu}\frac{C_4}{r^2} - \frac{\rho\omega^2 r^2(1+3\nu)}{8} \end{aligned}\right\} \qquad (3.48)$$

(v) Boundary Conditions

The following boundary and interface conditions must be satisfied by the stresses, strains and displacements

*Ref. 3, p. 70

$$
\left.
\begin{array}{lll}
r = 0 & \bar{u} = 0 & (a) \\
r = k & u = \bar{u} & (b) \\
r = k & \sigma_r = \bar{\sigma}_r & (c) \\
r = k & \epsilon_r + \epsilon_\theta + \epsilon_z = \epsilon_\ell & (d) \\
r = R & \sigma_r = 0 & (e)
\end{array}
\right\} \qquad (3.49)
$$

Then, from Eqs. (3.45)-(3.48) and the boundary conditions (3.49) one can evaluate the constants C_1, C_2, C_3 and C_4 to obtain

$$
\left.
\begin{aligned}
C_1 &= \frac{\rho\omega^2 k^2}{4E}\left[(1-\nu^2) + (1+\nu)^2\frac{R^2}{k^2}\right] + \frac{1+\nu}{2(1-2\nu)}\,\epsilon_\ell\left(1+\frac{R^2}{k^2}\right) \\
&\qquad - \frac{\rho\omega^2(1+\nu)k^2}{8E}\left[(3+\nu)\frac{R^4}{k^4} + \frac{1-2\nu}{2}\right] \\
C_2 &= 0 \\
C_3 &= \frac{\rho\omega^2 k^2(1-\nu^2)}{4E} + \frac{1-\nu}{2(1-2\nu)}\,\epsilon_\ell \\
C_4 &= \frac{\rho\omega^2 k^2 R^2(1+\nu)^2}{4E} + \frac{1+\nu}{2(1-2\nu)}\,R^2\epsilon_\ell - \frac{\rho\omega^2 R^4(3+\nu)(1+\nu)}{8E}
\end{aligned}
\right\} \qquad (3.50)
$$

The unknown radius k can be obtained from the following equation (see Eq. 3.49c)

$$
\frac{k^4}{R^4} + \frac{4(1+\nu)}{1-2\nu}\frac{k^2}{R^2} - \frac{2(3+\nu)}{1-2\nu)} = -\frac{8E\epsilon_\ell}{\rho\omega^2}\,\frac{1}{(1-2\nu)^2}\,\frac{1}{R^2} \qquad (3.51)
$$

(vi) Jump in σ_θ

The variations of σ_r and σ_θ with r for $k/r = 0.5$ is shown in Fig. 3.7b. In Fig. 3.7a $\rho\omega^2R^2/E\epsilon_\ell$ has been plotted against k/r for several values of ν. The purpose of the figure is to study the effect of Poisson's ratio.

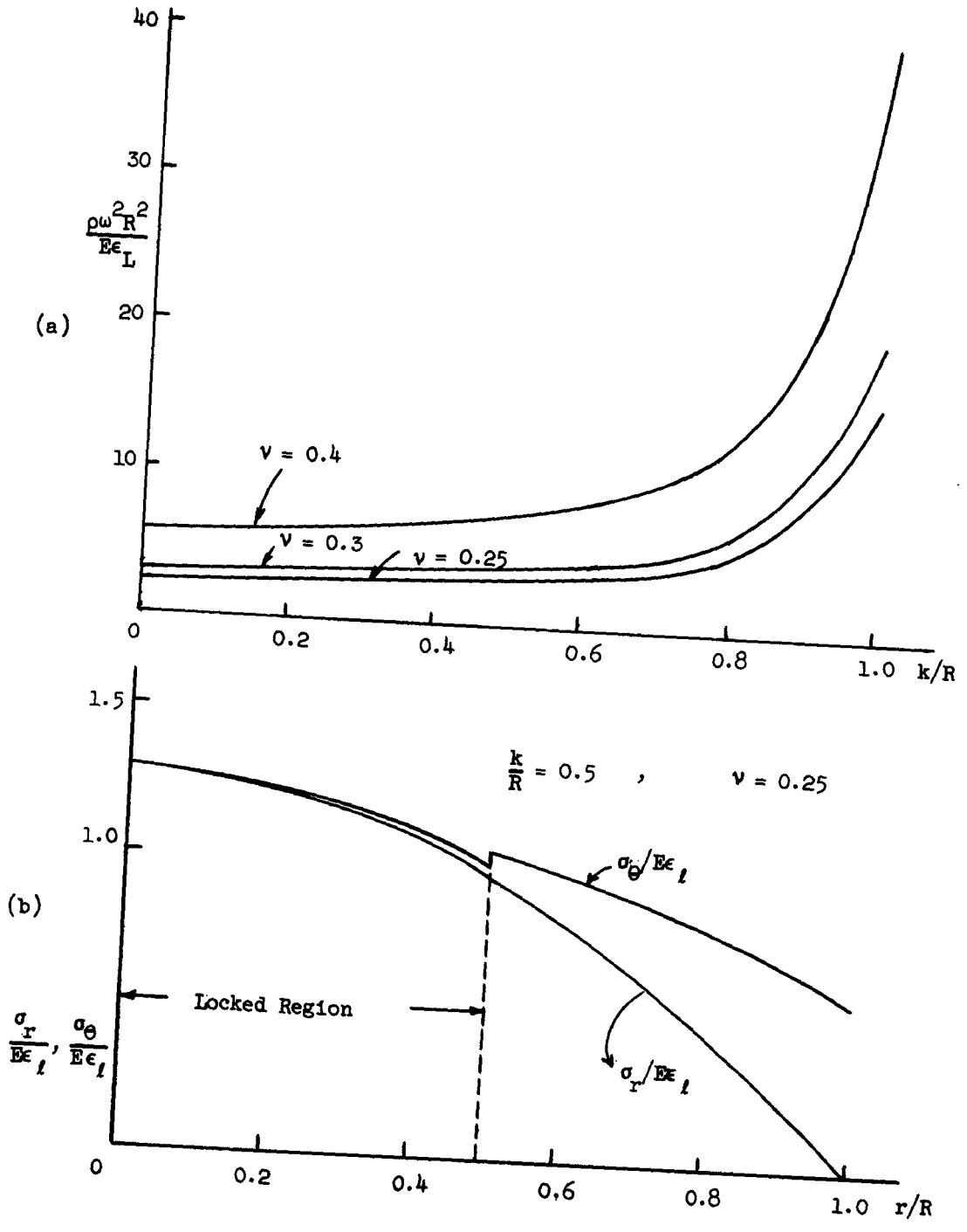

(a)
40
30
20
10
$\frac{\rho\omega^2R^2}{E\epsilon_L}$
$\nu = 0.4$
$\nu = 0.3$
$\nu = 0.25$
0
0.2
0.4
0.6
0.8
1.0 k/R
(b)
1.5
1.0
$\frac{k}{R} = 0.5$, $\nu = 0.25$
$\sigma_\theta/E\epsilon_\ell$
$\sigma_r/E\epsilon_\ell$
Locked Region
$\frac{\sigma_r}{E\epsilon_\ell}, \frac{\sigma_\theta}{E\epsilon_\ell}$
0
0.2
0.4
0.6
0.8
1.0 r/R

Fig. 3.7.

One can see that there is a jump in σ_θ at the elastic locking interface. This can be explained in the following way. At the interface we have satisfied the continuity of the displacement and the continuity of the radial stress, i.e., for $r = k$

$$u = \bar{u} \qquad \sigma_r = \bar{\sigma}_r \tag{3.52a,b}$$

From Eqs. (3.43), (3.44)

$$\bar{\sigma}_r = \frac{E}{1 + \nu} \left(2 \frac{\partial \bar{u}}{\partial r} + \frac{\bar{u}}{r} - \epsilon_\ell \right) \tag{3.53}$$

and from Hooke's law

$$\sigma_r = \frac{E}{1 - \nu^2} \left(\frac{\partial u}{\partial r} + \frac{\nu u}{r} \right) \tag{3.54}$$

Now, from (3.52b) $\sigma_r = \bar{\sigma}_r$ at $r = k$. Then, from Eqs. (3.54), (3.55) and (3.52a) one obtains at $r = k$

$$\frac{\partial \bar{u}}{\partial r} - \frac{\partial u}{\partial r} = - \frac{\sigma_r}{2E} (1 - \nu + \nu^2) - \frac{\bar{u}}{k} \left(\frac{1}{2} - \nu \right) + \epsilon_\ell \tag{3.55}$$

This shows that there is jump in $\partial u/\partial r$ at $r = k$. Equations similar to (3.54) and (3.55) can be written for σ_θ. Then

$$\sigma_\theta - \bar{\sigma}_\theta = - \frac{uE}{k} \frac{1 - 2\nu}{1 - \nu^2} + \frac{E}{1 + \nu} \left(\frac{\nu}{1 - \nu} \frac{\partial u}{\partial r} - \frac{\partial \bar{u}}{\partial r} + \frac{E \epsilon_\ell}{1 + \nu} \right) \tag{3.56}$$

From this equation and (3.55) one can see that there is a jump in σ_θ.

(vii) Problem After the Disk is Completely Locked

As the angular velocity ω is increased the disk becomes completely locked when (see Eq. 3.51)

$$\rho \omega^2 R^2 = \frac{8E\epsilon_\ell}{1 - 2\nu}$$

After the disk is completely locked, i.e. for

$$\rho\omega^2 R^2 > \frac{8E\epsilon_f}{1-2\nu}$$

the stresses, strains and displacements in the disk are given by Eqs. (3.45)-(3.47). Now, the boundary conditions are

$$r = 0 \qquad \bar{u} = 0$$

$$r = R \qquad \bar{\sigma}_r = 0$$

Then, the constants C_1 and C_2 can be evaluated. In this case they are

$$C_1 = \frac{1}{3}\left[\frac{7\rho\omega^2 R^2(1+\nu)}{16E} + \epsilon_f\right]$$

$$C_2 = 0$$

The expressions for the stresses can then be written as

$$\bar{\sigma}_r = \frac{7\rho\omega^2}{16}(R^2 - r^2)$$

$$\bar{\sigma}_\theta = \frac{\rho\omega^2}{16}(7R^2 - 5r^2)$$

Similarly, the corresponding strains and displacements can be obtained. The variation of σ_r and σ_θ are shown in the Fig. 3.8.

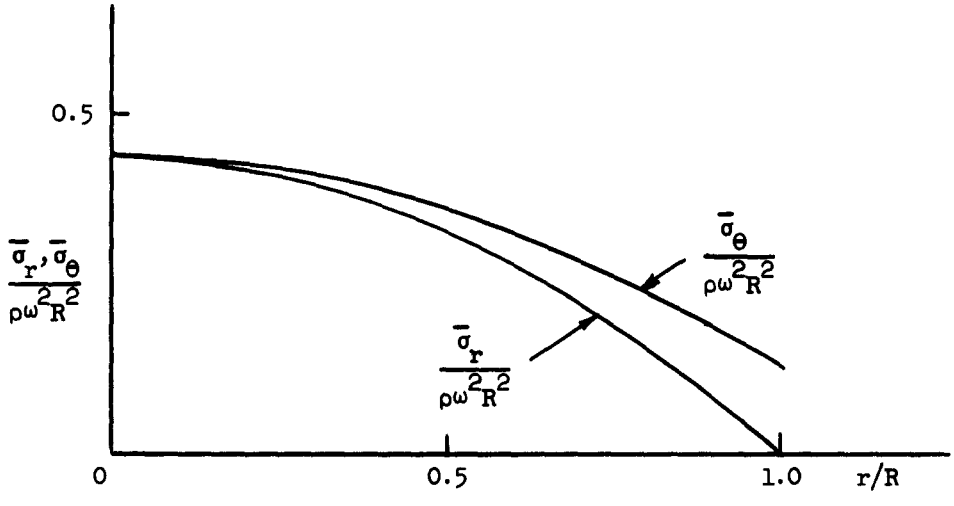

0.5
$\frac{\bar{\sigma}_r, \bar{\sigma}_\theta}{\rho\omega^2R^2}$
$\frac{\bar{\sigma}_\theta}{\rho\omega^2R^2}$
$\frac{\bar{\sigma}_r}{\rho\omega^2R^2}$
0
0.5
1.0
r/R

Fig. 3.8.

IV. PROBLEMS OF EQUILIBRIUM – DISTORTIONAL LOCKING

IV.1 Stresses Around a Small Spherical Cavity in a Body Subjected to Uniform External Pressure

We shall consider a body of distortional locking material subjected to uniform external pressure (see Fig. 4.1) with a spherical cavity of radius a. The radius a is assumed to be very small compared with the dimensions of the body. In the absence of the spherical cavity the stress distribution in the body corresponds to one of uniform hydrostatic stress throughout the body. The spherical cavity modifies this stress distribution. However the effect of the cavity is not felt appreciably farther away from the cavity, i.e., at distances from the center of the cavity which are large multiples of a. The modification of the uniform hydrostatic stress distribution in the body due to the spherical cavity will now be investigated.

(i) Elastic Solution

Let us consider a large sphere of radius b concentric with the sphere of radius a. The radius b is assumed to be very large compared with a. The modification of the stress distribution due to the presence of the cavity of radius a is very small at the external surface of the large sphere of radius r = b. Then it can be assumed that at r = b $\gg$ a the stress distribution corresponds to one of uniform hydrostatic stress.

As a final step in the solution, the derived formulae are modified by letting the radius b go to ∞. These simplified formulae are useful in application to bodies whose dimensions are very large compared with the radius of the spherical cavity a.

In the elastic hollow sphere of inner radius a and outer radius b, the stresses, strains and displacements should correspond to the following boundary values

$$\begin{aligned} r = a \qquad & \sigma_r = 0 \\ r = b \qquad & \sigma_r = -p \end{aligned} \tag{4.1}$$

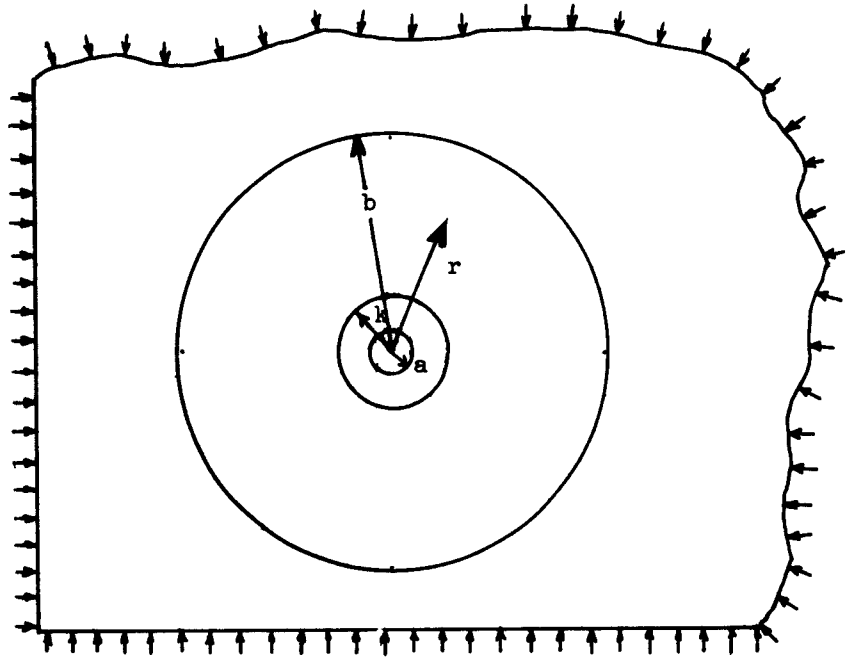


Fig. 4.1.

The geometry and the boundary values are spherically symmetric. The stresses σ_r, σ_θ, the strains ϵ_r, ϵ_θ and the displacement u should then satisfy the equilibrium equation (σ_θ, ϵ_θ, are the normal stress and normal strain in any direction in the plane normal to the radius vector $\vec{r}$)

$$\frac{d\sigma_r}{dr} + 2\frac{\sigma_r - \sigma_\theta}{r} = 0 \tag{4.2}$$

The stress-strain relations

$$\begin{aligned} \sigma_r + 2\sigma_\theta &= 3K(\epsilon_r + 2\epsilon_\theta) \qquad (a) \\ \sigma_r - \sigma_\theta &= 2G(\epsilon_r - \epsilon_\theta) \qquad (b) \end{aligned} \tag{4.3}$$

and the kinematic relations.

$$\epsilon_r = \frac{du}{dr}, \qquad \epsilon_\theta = \frac{u}{r} \tag{4.4}$$

By substituting (4.3) and (4.4) in the equilibrium equation (4.2) one obtains the following differential equation for u:

$$\frac{d^2u}{dr^2} + \frac{2}{r}\frac{du}{dr} - \frac{2u}{r^2} = 0 \tag{4.5}$$

Integration of Eq. (4.5) yields

$$u = C_1 r + \frac{C_2}{r^2} \tag{4.6}$$

The corresponding stresses and strains can then be calculated from Eqs. (4.4) and (4.5). Thus,

$$\begin{aligned} \epsilon_r &= C_1 - \frac{2C_2}{r^3} \\ \epsilon_\theta &= C_1 + \frac{C_2}{r^3} \end{aligned} \tag{4.7a}$$

$$\sigma_r = 3KC_1 - 4G\frac{C_2}{r^3}$$

$$\sigma_\theta = 3KC_1 + 2G\frac{C_2}{r^3} \tag{4.7b}$$

From the boundary conditions (4.1) and the stresses (4.7b), one obtains

$$C_2 = -\frac{p}{4G}\frac{a^3b^3}{b^3 - a^3}$$

$$C_1 = -\frac{p}{3K}\frac{b^3}{b^3 - a^3} \tag{4.8}$$

If one considers the limit as $b \to \infty$, one obtains

$$C_2 = -\frac{pa^3}{4G}, \qquad C_1 = -\frac{p}{3K} \tag{4.9}$$

Then, one can write the following formulae for u, ϵ_r, ϵ_θ, σ_r and σ_θ in the body which deforms like a completely elastic body:

$$\left.\begin{aligned}
u &= -\frac{p}{3K}r - \frac{pa^3}{4G}\frac{1}{r^2} && \text{(a)}\\
\epsilon_r &= -\frac{p}{3K} + \frac{pa^3}{2G}\frac{1}{r^3} && \text{(b)}\\
\epsilon_\theta &= -\frac{p}{3K} - \frac{pa^3}{4G}\frac{1}{r^3} && \text{(c)}\\
\sigma_r &= -p + \frac{pa^3}{r^3} && \text{(d)}\\
\sigma_\theta &= -p - \frac{pa^3}{2r^3} && \text{(e)}
\end{aligned}\right\} \tag{4.10}$$

However, the body is made of distortionally locking material. Therefore, locking takes place when the characteristic locking function

$$\phi = \phi(J_2') = 0$$

In this case the locking condition is assumed to be of the following form

$$(\epsilon_r - \epsilon_\theta)^2 + (\epsilon_\theta - \epsilon_\phi)^2 + (\epsilon_\phi - \epsilon_r)^2 - 2\epsilon_\ell^2 = 0 \qquad (4.11)$$

Because $\epsilon_\theta = \epsilon_\phi$, this equation simplifies to

$$\epsilon_r - \epsilon_\theta = \pm \epsilon_\ell \qquad (4.12)$$

From Eqs. (4.10) and (4.12) one can see that the body starts locking at

$$r = a$$

when

$$p = \frac{4G\epsilon_\ell}{3} \qquad (4.13)$$

In the next sub-section the stresses in the body after incipient locking will be investigated.

(ii) Problem after Incipient Locking

If the pressure is increased from the value given by Eq. (4.13) the locked region in the body increases. Because of spherical symmetry, one can assume that the region $a < r < k$ (Fig. 4.1) is locked while the region $r > k$ is still elastic. k is the radius defining the elastic-locking interface when the external pressure $p > 4G\epsilon_\ell/3$. Then, the stresses, strains and displacements in the locked region should satisfy Eqs. (4.14) through (4.17), where the barred letters denote the quantities in the locked region.

(a) Equilibrium Equation

$$\frac{d\bar{\sigma}_r}{dr} + 2\,\frac{\bar{\sigma}_r - \bar{\sigma}_\theta}{r} = 0 \qquad (4.14)$$

(b) Dilatational Elasticity

$$\bar{\sigma}_r + 2\bar{\sigma}_\theta = 3K(\bar{\epsilon}_r + 2\bar{\epsilon}_\theta) \tag{4.15}$$

(c) Ideal Locking Condition

$$\bar{\epsilon}_r - \bar{\epsilon}_\theta = \epsilon_\ell \tag{4.16}$$

(d) Kinematics

$$\left.\begin{aligned} \bar{\epsilon}_\theta &= \frac{\bar{u}}{r} \\ \bar{\epsilon}_r &= \frac{d\bar{u}}{dr} \end{aligned}\right\} \tag{4.17}$$

The expressions for stresses, strains and displacements in the elastic region are still given by Eqs. (4.6) and (4.7). Boundary conditions for the problem are

$$\left.\begin{aligned} r = a \qquad & \bar{\sigma}_r = 0 && \text{(a)} \\ r = k \qquad & \bar{\sigma}_r = \sigma_r && \text{(b)} \\ & \bar{u} = u && \text{(c)} \\ & \epsilon_r - \epsilon_\theta = \epsilon_\ell && \text{(d)} \\ r = b \qquad & \sigma_r = -p && \text{(e)} \end{aligned}\right\} \tag{4.18}$$

(iii) Stresses and Strains in the Locked Region

From Eqs. (4.16) and (4.17) one has

$$\frac{d\bar{u}}{dr} - \frac{\bar{u}}{r} - \epsilon_\ell = 0 \tag{4.19}$$

The solution of this equation is

$$\bar{u} = C_3 r + \epsilon_\ell r \log r \tag{4.20}$$

Then,

$$\left.\begin{aligned} \bar{\epsilon}_r &= C_3 + \epsilon_\ell(1 + \log r) \\ \bar{\epsilon}_\theta &= C_3 + \epsilon_\ell \log r \end{aligned}\right\} \qquad (4.21)$$

From Eqs. (4.15) and (4.21) one can write

$$\bar{\sigma}_\theta = \frac{1}{2} \left[3K(3C_3 + 3\epsilon_\ell \log r + \epsilon_\ell) - \bar{\sigma}_r\right] \qquad (4.22)$$

By substituting this expression in the equilibrium equation (4.14) one obtains

$$\frac{d\bar{\sigma}_r}{dr} + \frac{3\bar{\sigma}_r}{r} - \frac{3K}{r} (3C_3 + 3\epsilon_\ell \log r + \epsilon_\ell) = 0 \qquad (4.23)$$

which, when solved, yields the following expression for $\bar{\sigma}_r$

$$\left.\begin{aligned} \bar{\sigma}_r &= \frac{3C_4}{r^3} + 3KC_3 + 3K\epsilon_\ell \log r \\ &\text{whence} \\ \bar{\sigma}_\theta &= -\frac{3}{2}\frac{C_4}{r^3} + 3KC_3 + 3K\epsilon_\ell \log r + 3K\frac{\epsilon_\ell}{2} \end{aligned}\right\} \qquad (4.24)$$

By substituting Eqs. (4.20), (4.24), (4.6), (4.7a-e), in the boundary conditions (4.18b-e), one obtains

$$\left.\begin{aligned} C_1 &= -\frac{p}{3K} - \frac{4G}{3K}\frac{k^3}{b^3}\frac{\epsilon_\ell}{3} \\ C_2 &= -\frac{k^3\epsilon_\ell}{3} \\ C_3 &= -\frac{p}{3K} - \frac{4G}{3K}\frac{k^3}{b^3}\frac{\epsilon_\ell}{3} - \frac{\epsilon_\ell}{3} - \epsilon_\ell \log k \\ C_4 &= +\frac{1}{9}(4G + 3K)k^3\epsilon_\ell \end{aligned}\right\} \qquad (4.25)$$

These constants have been expressed in terms of k and p. The relationship between k and p can be obtained from Eq. (4.18a). Thus

$$(4G + 3K)\frac{k^3}{a^3}\frac{\epsilon_\ell}{3} - p - K\epsilon_\ell + 3K\epsilon_\ell \log\frac{a}{k} - 4G\frac{k^3}{b^3}\frac{\epsilon_\ell}{3} = 0 \tag{4.26}$$

Taking the limit as $b \to \infty$, i.e., as $a/b, (k/b) \to 0$, Eqs. (4.25) and (4.26) will become

$$\left.\begin{aligned} C_1 &= -\frac{p}{3K} \\ C_2 &= -\frac{k^3\epsilon_\ell}{3} \\ C_3 &= -\frac{p}{3K} - \epsilon_\ell \log k - \frac{\epsilon_\ell}{3} \\ C_4 &= +(4G + 3K)\frac{k^3\epsilon_\ell}{9} \\ (4G + 3K)\frac{k^3}{a^3}\frac{\epsilon_\ell}{3} &+ 3K\epsilon_\ell \log\frac{a}{k} = p + K\epsilon_\ell \end{aligned}\right\} \tag{4.27}$$

Now, the expressions for the stresses are

$$\left.\begin{aligned} \bar{\sigma}_r &= +(4G + 3K)\frac{k^3}{r^3}\frac{\epsilon_\ell}{3} - p + 3K\epsilon_\ell \log\frac{r}{k} - K\epsilon_\ell \\ \bar{\sigma}_\theta &= -\frac{1}{2}(4G + 3K)\frac{k^3}{r^3}\frac{\epsilon_\ell}{3} - p + 3K\epsilon_\ell \log\frac{r}{k} + \frac{K\epsilon_\ell}{2} \end{aligned}\right\} \tag{4.28}$$

Variations of $\bar{\sigma}_r$, $\bar{\sigma}_\theta$ with the radius r (for various values of k) are plotted on Fig. 4.2. In Fig. 4.3, variations k/a with $p/K\epsilon_\ell$ are plotted for different values of G/K and b/a.

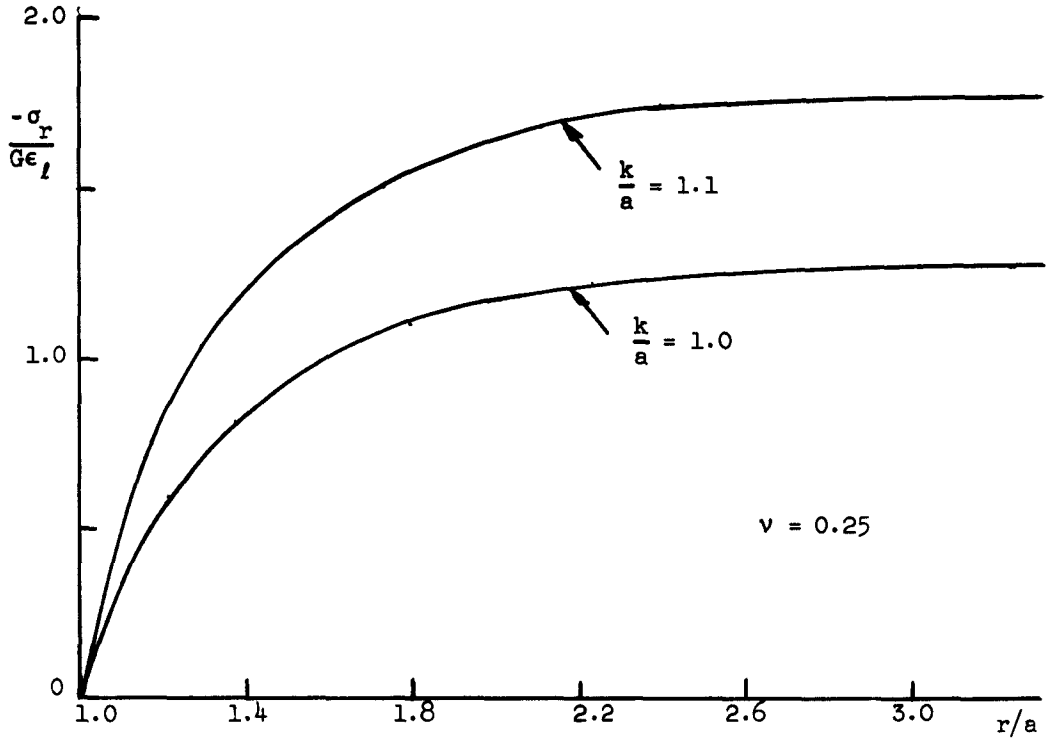


Fig. 4.2a.

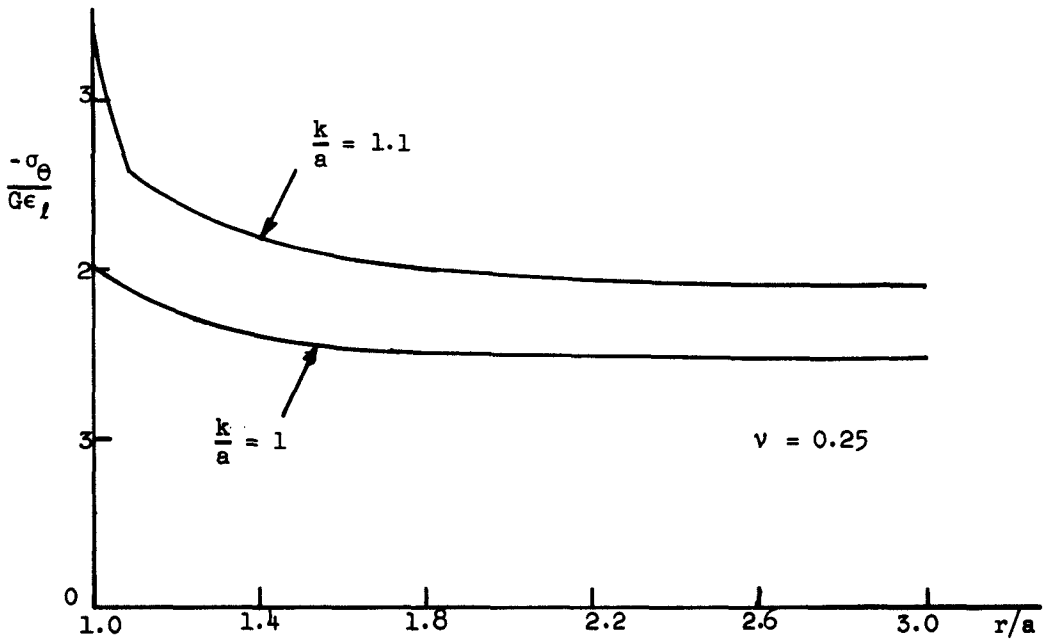


Fig. 4.2b.

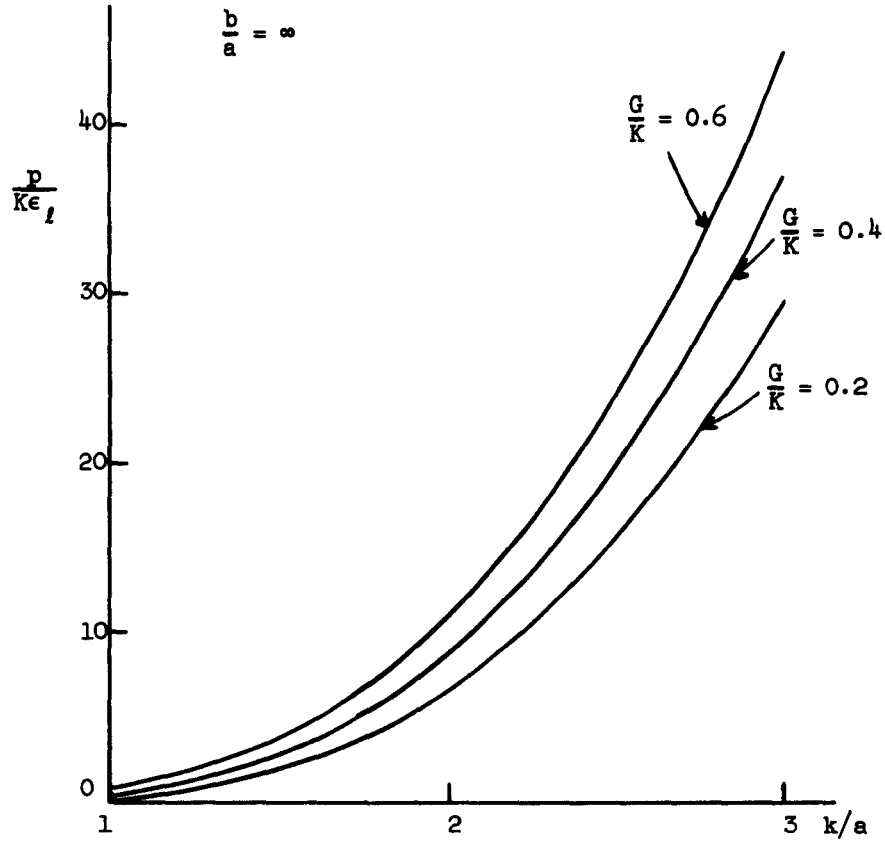


Fig. 4.3a

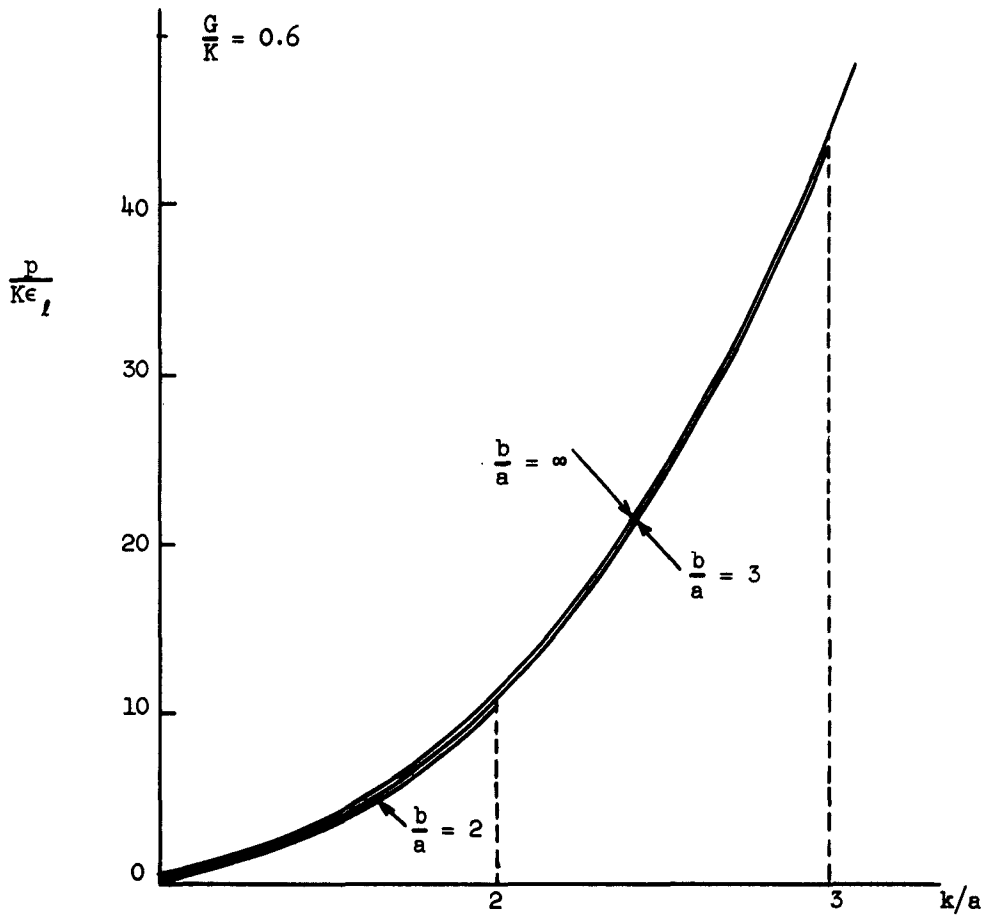


Fig. 4.3b.

The expressions (4.28) show that there is a stress concentration at $r = a$. The stress σ_θ at $r = a$ is given by

$$\sigma_\theta = -\frac{3}{2}\left(p + 3K\epsilon_\ell \log \frac{k}{a}\right) \tag{4.29}$$

When $k \to a$, the stress concentration factor reduces to the elastic value of $3/2$.

(iv) Apparent Bulk Modulus

One can measure the reduction in volume ΔV of a solid sphere subjected to uniform external pressure p, by measuring the reduction in the outer radius of the sphere, i.e.,

$$\Delta V = -4\pi b^2 (u)_{r=b} \tag{4.30}$$

where $2b$ is the diameter of the sphere. This equation is consistent with the assumption of small displacements.

If the sphere were made of elastic material, ΔV can be calculated from Eq. (4.10a), with $a = 0$, i.e.,

$$\Delta V = +4\pi b^3 \frac{p}{3K}$$

Then one can obtain the bulk modulus as

$$K = \frac{p}{\Delta V/V}$$

On the other hand, if the sphere has a small cavity of radius a, and one measures the reduction in volume by measuring the reduction in the outer radius only and calculates the bulk modulus from the preceding equation one obtains

$$\bar{K} = \frac{(b^3 - a^3)/b^3}{\frac{1}{K} + \frac{3a^3}{4Gb^3}} \tag{4.31a}$$

This value of the bulk modulus $\bar{K}$ is defined as the apparent bulk modulus. $\bar{K}$ is lower than the true value of the bulk modulus K. This means that the presence of a hole softens the elastic sphere.

Let us now consider a sphere made of distortionally locking material and having a spherical cavity of radius a. Then from Eqs. (4.20) and (4.30a) the value of the apparent bulk modulus $\bar{K}_\ell$ is

$$\bar{K}_\ell = \frac{1}{\frac{1}{\bar{K}} + \frac{\epsilon_\ell}{p}\frac{k^3}{b^3}\left(1 + \frac{4G}{3K}\right)} \tag{4.31b}$$

The value of $\bar{K}_\ell/K$ has been plotted in Fig. 4.4 for values of b/a = 2,3, and $\nu = 0.25$. One can see that locking slightly hardens the material. For $b/a = \infty$, $k/b = 0$ for finite p and $\bar{K}_\ell = K$.

Now, let us consider a solid body of volume $\bar{V}$ as shown in Fig. 4.5 with several small cavities. Let the volume enclosed by each cavity be v, the number of cavities n and the diameter of the cavity 2a. If the cavities are evenly distributed as shown in the figure, one can imagine the volume $\bar{V}$ to be made of n fictitious spheres of diameter $2b = 2\sqrt[3]{3\bar{V}/4n}$. Then the approximate value of the reduction in volume $\Delta\bar{V}$ computed by measuring only the changes in the outer dimensions of the body is

$$\Delta\bar{V} = p\bar{V}\left(\frac{1}{K} + \frac{3a^3}{4Gb^3}\right)\frac{b^3}{b^3 - a^3}$$

if the body is elastic and

$$\Delta\bar{V}_\ell = p\bar{V}\left[\frac{1}{\bar{K}} + \frac{\epsilon_\ell}{p}\frac{k^3}{b^3}\left(1 + \frac{4G}{3K}\right)\right]$$

if the body is locked. Then one can calculate the apparent bulk modulus $\bar{K}$. The variation of $\bar{K}$ with p for n = 12, 4, 1 and $V/v = 324$ are shown in Fig. 4.6.

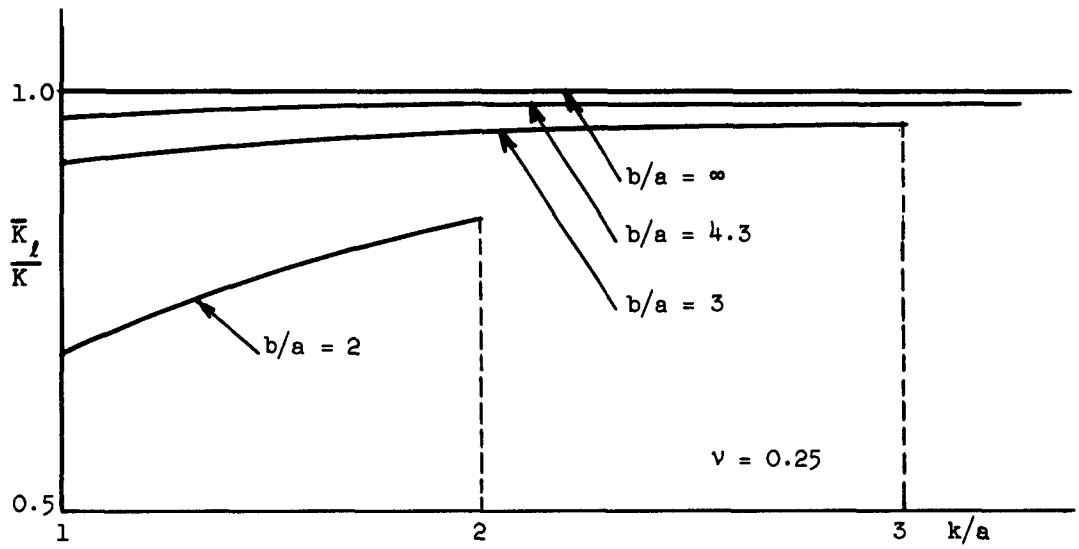


Fig. 4.4.

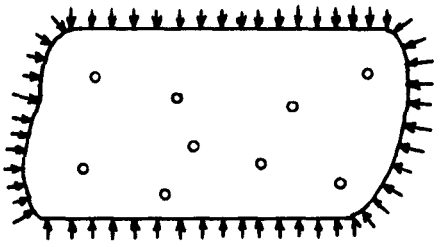


Fig. 4.5.

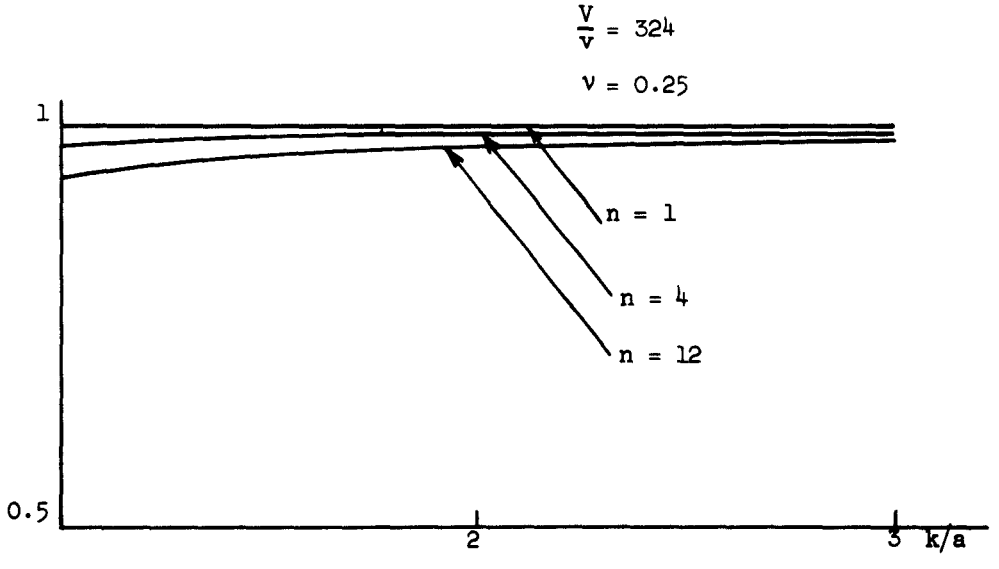


Fig. 4.6.

IV.2. Stresses Around a Small Spherical Inclusion in a Body Subjected to Uniform External Pressure

In this section we shall consider a distortionally locking body containing a small spherical rigid inclusion of radius a. The radius of the spherical inclusion is assumed to be very small compared with the other dimensions of the body. If the body is subjected to a uniform external pressure, the stress distribution will be different from the uniform hydrostatic stress. This stress distribution will now be studied.

(i) Elastic Solution

One can follow a procedure very similar to that of section IV.1. Then, we shall again consider a large sphere of radius b concentric with the sphere of radius a. The radius b is assumed to be very large compared with a. Then the difference between the stress distribution in the body and the uniform hydrostatic stress distribution will be very small at the outer radius r = b.

Then, one can assume that the stress distribution at $r = b \gg a$ is essentially a uniform hydrostatic stress. After deriving the formulae, the expressions are simplified by considering the limit as b tends to ∞.

When the applied external pressure p is very small, the hollow sphere of inner radius a and outer radius b deforms as an elastic body. Then the general expressions for stresses, strains and displacements are given by Eqs.(4.6) and (4.7). In this case the boundary conditions can be expressed by the following equations

$$\left.\begin{array}{ll} r = a & u = 0 \\ r = b & \sigma_r = -p \end{array}\right\} \qquad b \to \infty \qquad (4.32)$$

By using these boundary conditions, one can evaluate the constants C_1 and C_2 in Eqs. (4.6) and (4.7). They are

$$C_1 = -\frac{p}{3K + 4G\frac{a^3}{b^3}}$$

$$C_2 = \frac{p}{\frac{3K}{a^3} + \frac{4G}{b^3}} \qquad (4.33)$$

Taking the limit as $b \to \infty$, one has

$$C_1 = -\frac{p}{3K}, \quad C_2 = \frac{pa^3}{3K} \qquad (4.34)$$

The expressions for the stresses and strains are then given by the following equations:

$$\epsilon_r = -\frac{p}{3K} - 2\frac{p}{3K}\frac{a^3}{r^3}$$

$$\epsilon_\theta = -\frac{p}{3K} + \frac{p}{3K}\frac{a^3}{r^3} \qquad (4.35)$$

$$\sigma_r = -p - \frac{4G}{3K}p\frac{a^3}{r^3}$$

$$\sigma_\theta = -p + \frac{2G}{3K}p\frac{a^3}{r^3} \qquad (4.36)$$

The locking condition for the material is again assumed to be of the form of Eq. (4.12). Then, the body deforms as an elastic body if every where

$$\epsilon_\theta - \epsilon_r = \frac{p}{K}\frac{a^3}{r^3} < \epsilon_\ell \qquad (4.37)$$

and the body begins to lock at $r = a$ when

$$\frac{p}{K} = \epsilon_{\ell} \tag{4.38}$$

(ii) Problem After Incipient Locking

If

$$p \geq K\epsilon_{\ell} \tag{4.39}$$

the body contains a region which is distortionally locked. Because of spherical symmetry one can assume that the region $a < r < k$ is locked while the region $r > k$ is still elastic. The expressions for the stresses, strains and displacements in the elastic region are given by Eqs. (4.6) and (4.7). However, the stresses, strains and displacements in the locked region must satisfy the equilibrium equation (4.14), dilatational elasticity (4.15), kinematics (4.17) and the following ideal locking condition.

$$\bar{\epsilon}_{\theta} - \bar{\epsilon}_{r} = \bar{\epsilon}_{\ell} \tag{4.40}$$

The boundary conditions for the problem are

$$\left.\begin{array}{ll} r = a & \bar{u} = 0 \\ r = k & \bar{u} = u \\ & \bar{\sigma}_{r} = \sigma_{r} \\ & \epsilon_{\theta} - \epsilon_{r} = \epsilon_{\ell} \\ r = b & \sigma_{r} = -p \\ (b \to \infty) & \end{array}\right\} \tag{4.41}$$

(iii) Stresses and Strains in the Locked Region

From Eqs. (4.40) and (4.17), one has

$$\frac{d\bar{u}}{dr} - \frac{\bar{u}}{r} + \epsilon_{\ell} = 0 \tag{4.42}$$

The solution of this equation yields the following result:

$$\left.\begin{aligned} \bar{u} &= C_3 r - \epsilon_\ell r \log r \\ \bar{\epsilon}_r &= C_3 - \epsilon_\ell (1 + \log r) \\ \bar{\epsilon}_\theta &= C_3 - \epsilon_\ell \log r \end{aligned}\right\} \qquad (4.43)$$

Then, the dilation elasticity Eq. (4.3a) yields the following relationship between $\bar{\sigma}_\theta$ and $\bar{\sigma}_r$:

$$\bar{\sigma}_\theta = \frac{1}{2}\left[3K(3C_3 - 3\epsilon_\ell \log r - \epsilon_\ell) - \bar{\sigma}_r\right] \qquad (4.44)$$

This equation together with the equilibrium equation (4.2) then results in a differential equation for $\bar{\sigma}_r$, i.e.,

$$\frac{d\bar{\sigma}_r}{dr} + \frac{3\bar{\sigma}_r}{r} - \frac{3K}{r}(3C_3 - 3\epsilon_\ell \log r - \epsilon_\ell) = 0 \qquad (4.45)$$

The solution of this equation is

$$\left.\begin{aligned} \bar{\sigma}_r &= \frac{3C_4}{r^3} + 3KC_3 - 3K\epsilon_\ell \log r \\ \text{and} \\ \bar{\sigma}_\theta &= -\frac{3}{2}\frac{C_4}{r^3} + 3KC_3 - 3K\epsilon_\ell \log r - \frac{3K\epsilon_\ell}{2} \end{aligned}\right\} \qquad (4.46)$$

Eqs. (4.6), (4.7), (4.43), (4.46) and the boundary conditions (4.41) yield the following values for the constants C_1, C_2, C_3 and C_4.

$$C_1 = -\frac{p}{3K} + \frac{4G}{3K}\frac{k^3}{b^3}\frac{\epsilon_\ell}{3}$$

$$C_2 = \frac{k^3}{3}\epsilon_\ell$$

$$C_3 = \epsilon_\ell \log a$$

$$C_4 = \frac{k^3}{3}\left(-3K\,\epsilon_\ell \log \frac{a}{k} - p + 4G\,\frac{k^3}{b^3}\,\frac{\epsilon_\ell}{3} - 4G\,\frac{\epsilon_\ell}{3}\right) = -\,k^3[3K+4G]\,\frac{\epsilon_\ell}{9}$$

The relationship k and p is expressed in the form of the following equation:

$$-\frac{1}{3}\frac{p}{K} + \frac{4G}{3K}\frac{k^3}{b^3}\frac{\epsilon_\ell}{3} + \frac{\epsilon_\ell}{3} = \epsilon_\ell \log \frac{a}{k} \tag{4.48}$$

The stresses are then

$$\left.\begin{aligned}
\bar{\sigma}_r &= -(3K + 4G)\,\frac{\epsilon_\ell k^3}{3r^3} + 3K\epsilon_\ell \log \frac{a}{r} && a \le r \le k \\
\bar{\sigma}_\theta &= (3K + 4G)\,\frac{\epsilon_\ell k^3}{6r^3} + 3K\epsilon_\ell \log \frac{a}{r} && a \le r \le k \\
\sigma_r &= -p + \frac{4G}{3}\,k^3\epsilon_\ell\left(\frac{1}{b^3} - \frac{1}{r^3}\right) && k \le r \le b \\
\sigma_\theta &= -p + 2G\,\frac{k^3}{3}\,\epsilon_\ell\left(\frac{2}{b^3} + \frac{1}{r^3}\right) && k \le r \le b
\end{aligned}\right\} \tag{4.49}$$

If one considers the limiting values as $b \to \infty$, the constants C_1, C_2, C_3 and C_4 become

$$C_1 = -\frac{p}{3K}$$

$$C_2 = \frac{k^3}{3}\,\epsilon_\ell$$

$$C_3 = \epsilon_\ell \log a$$

$$C_4 = \frac{k^3}{3}\left(3K\epsilon_\ell \log\frac{k}{a} - p - 4G\frac{\epsilon_\ell}{3}\right)$$

Then, the relationship between k and p can be expressed in the form of the following equation

$$-\frac{1}{3}\frac{p}{K} + \frac{\epsilon_\ell}{3} = \epsilon_\ell \log\frac{a}{k}$$

The expressions for the stresses are then

$$\left.\begin{aligned}
\bar{\sigma}_r &= -\frac{\epsilon_\ell k^3}{3r^3}(3K + 4G) + 3K\epsilon_\ell \log\frac{a}{r} && a \le r \le k \\
\bar{\sigma}_\theta &= \frac{\epsilon_\ell k^3}{6r^3}(3K + 4G) + 3K\epsilon_\ell \log\frac{a}{r} - 3K\frac{\epsilon_\ell}{2} && a \le r \le k \\
\sigma_r &= -p - \frac{4G}{3}\frac{k^3}{r^3}\epsilon_\ell && r > k \\
\sigma_\theta &= -p + \frac{2G}{3}\frac{k^3}{r^3}\epsilon_\ell && r > k
\end{aligned}\right\} \quad (4.50)$$

The variation of σ_r and σ_θ for various values of k/a are shown in Fig. 4.7. b/a has been assumed to be ∞. In Fig. 4.8 variations of $p/K\epsilon_\ell$ with k/a are plotted for different values of G/K and b/a. The preceding expressions for the stresses yield the following values for the stresses at the surface of the rigid inclusion.

$$\bar{\sigma}_r\Big|_{r=a} = \frac{k^3}{a^3}\left(-3K\epsilon_\ell \log\frac{a}{k} - p - 4G\frac{\epsilon_\ell}{3}\right) = -\frac{\epsilon_\ell k^3}{3a^3}(3K + 4G)$$

$$\bar{\sigma}_\theta\Big|_{r=a} = \frac{k^3}{2a^3}\left(-3K\epsilon_\ell \log\frac{a}{k} - p - 4G\frac{\epsilon_\ell}{3}\right) + K\frac{\epsilon_\ell}{2} = \frac{\epsilon_\ell k^3}{6a^3}(3K+4G) - \frac{3K\epsilon_\ell}{2}$$

If the body were completely elastic the stresses at the surface of the inclusion would be

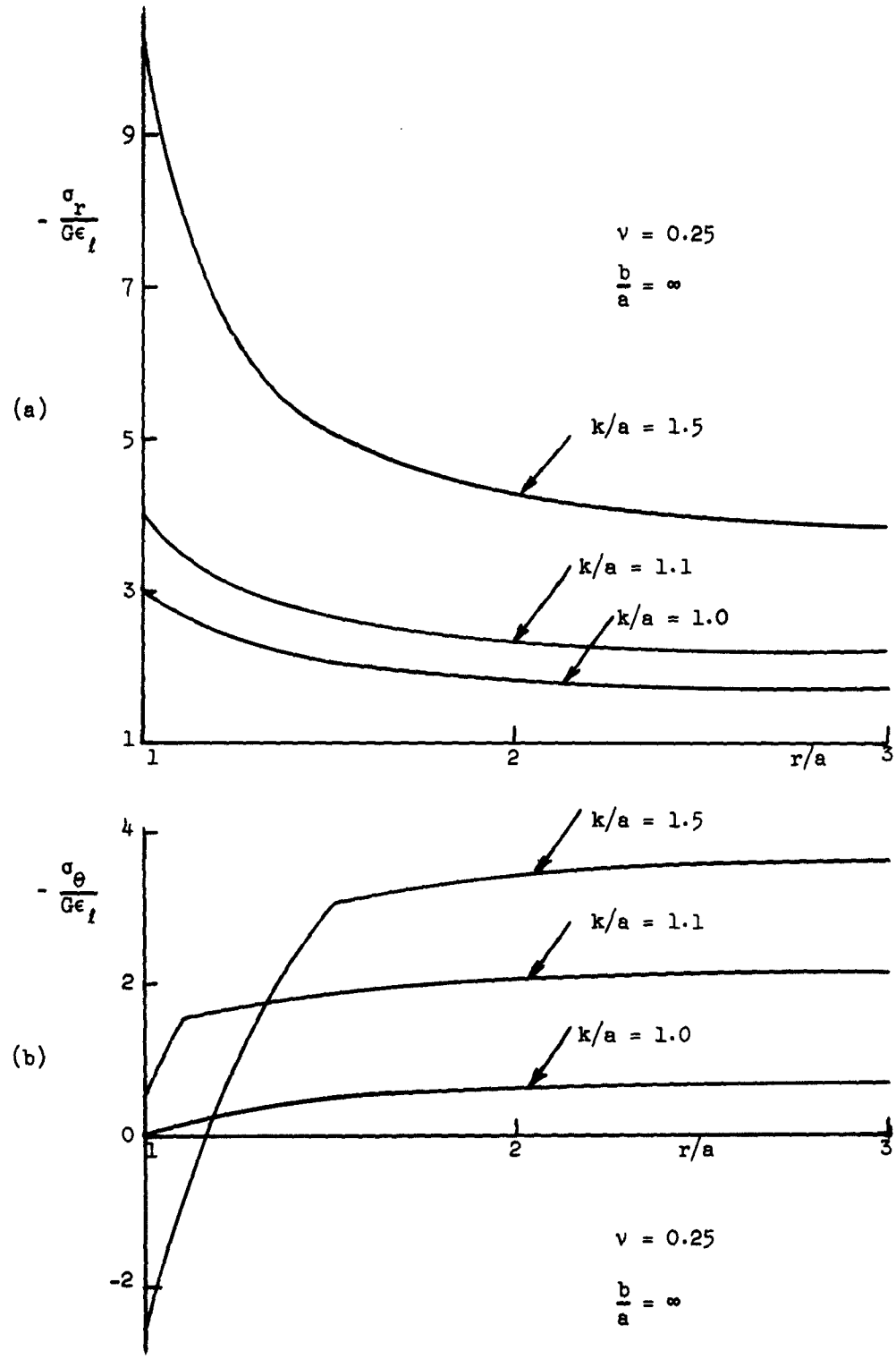

(a)
$-\frac{\sigma_r}{G\epsilon_l}$
9
7
5
3
1
1
2
3
r/a
ν = 0.25
b/a = ∞
k/a = 1.5
k/a = 1.1
k/a = 1.0
(b)
$-\frac{\sigma_\theta}{G\epsilon_l}$
4
2
0
-2
1
2
3
r/a
k/a = 1.5
k/a = 1.1
k/a = 1.0
ν = 0.25
b/a = ∞

Fig. 4.7.

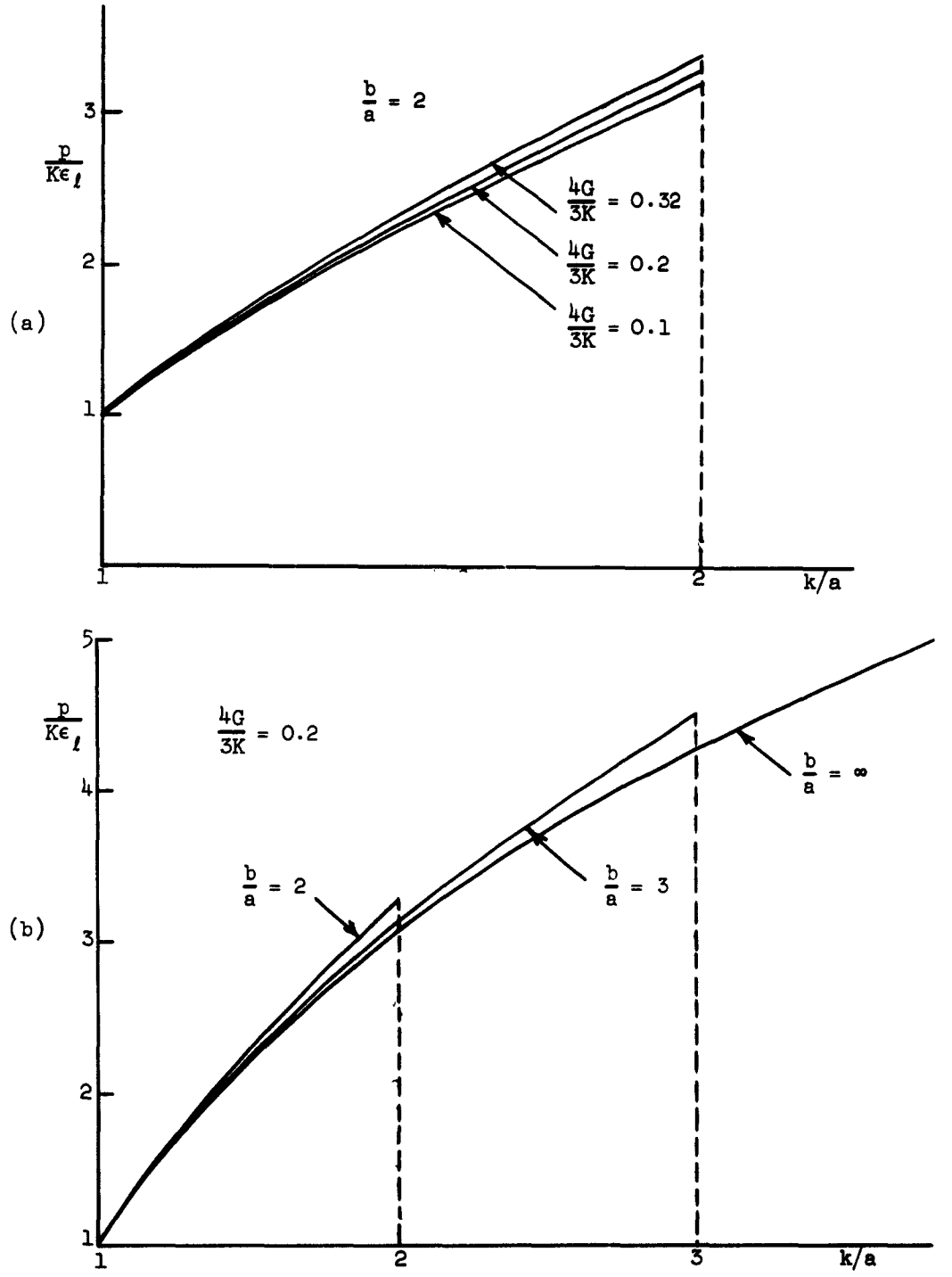

(a)
$\frac{p}{K\epsilon_l}$
$\frac{b}{a} = 2$
$\frac{4G}{3K} = 0.32$
$\frac{4G}{3K} = 0.2$
$\frac{4G}{3K} = 0.1$
k/a
(b)
$\frac{4G}{3K} = 0.2$
$\frac{b}{a} = \infty$
$\frac{b}{a} = 2$
$\frac{b}{a} = 3$
k/a

Fig. 4.8.

$$\sigma_r = -p\left(1 + \frac{4G}{3K}\right)$$

$$\sigma_\theta = -p\left(1 - \frac{2G}{3K}\right)$$

(iv) Apparent Bulk Modulus

By following a procedure similar to that of section IV.1, an apparent bulk modulus can be calculated in this case:

$$\bar{K} = \frac{K + 4Ga^3/b^3}{1 - \frac{a^3}{b^3}} \qquad (4.51a)$$

if the material is elastic, and

$$\bar{K}_l = \frac{1}{\frac{1}{K} - \frac{\epsilon_l}{p}\frac{k^3}{b^3}\left(1 + \frac{4G}{3k}\right)} \qquad (4.51b)$$

if the material is locked. The variation $\bar{K}_l/K$ for different values of b/a are shown in Fig. 4.9a. In Fig. 4.9b, variation of $\bar{K}_l/K$ with k/a for $b/a = 3, \infty$ are plotted for the case of a spherical cavity and a spherical inclusion. The different effects of the cavity and the inclusion can be seen in the figure.

IV.3. Stresses Around a Circular Cylindrical Hole in a Body, in Plane Strain Condition, Subjected to Uniform Pressure Along the Edges of Every Cross-Section

We shall now consider a body (Fig. 4.10) whose length in the z-direction is large compared with the dimensions in x and y directions. The body is subjected to uniform pressure p applied at the edges of every cross-section perpendicular to the z-direction. The sections normal to the z-direction are assumed to be restrained from deformation in that direction. The stresses in such a body with a circular cylindrical hole of radius a will be investigated. "a" is very small compared with the dimensions of the body.

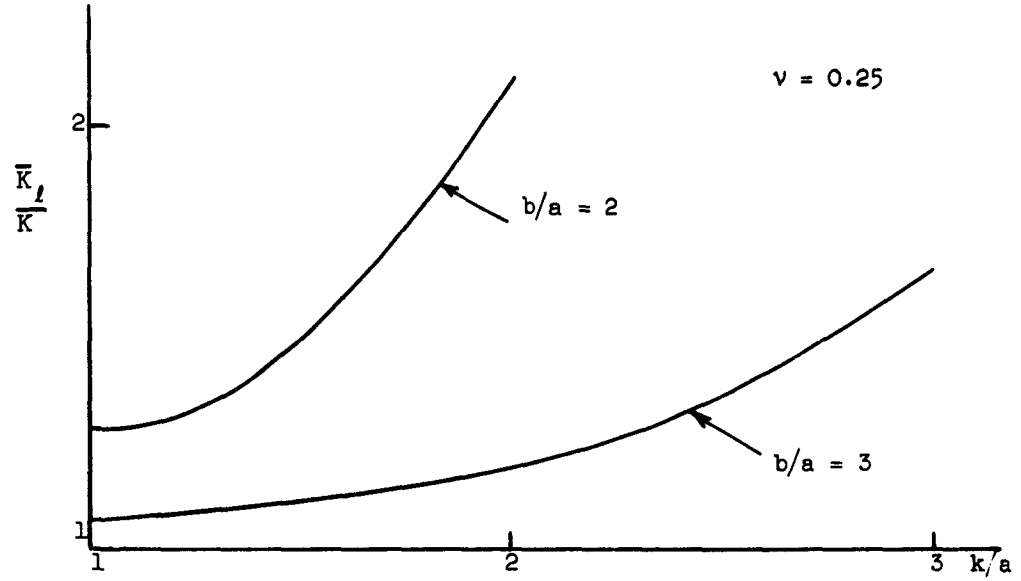


Fig. 4.9a.

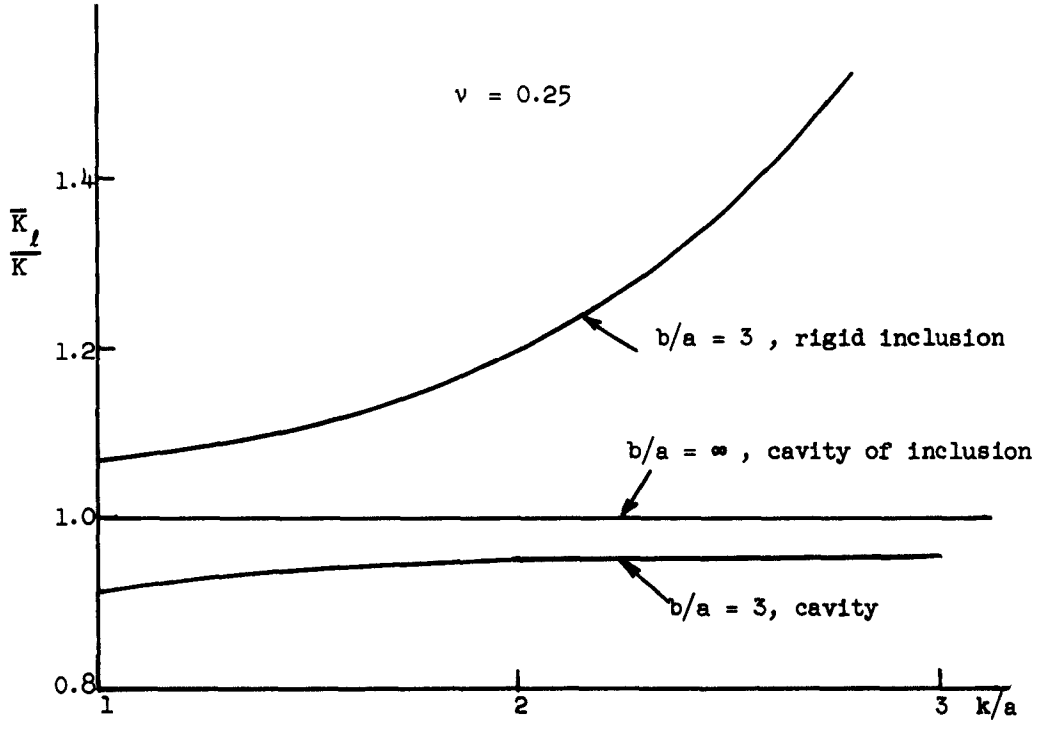


Fig. 4.9b.

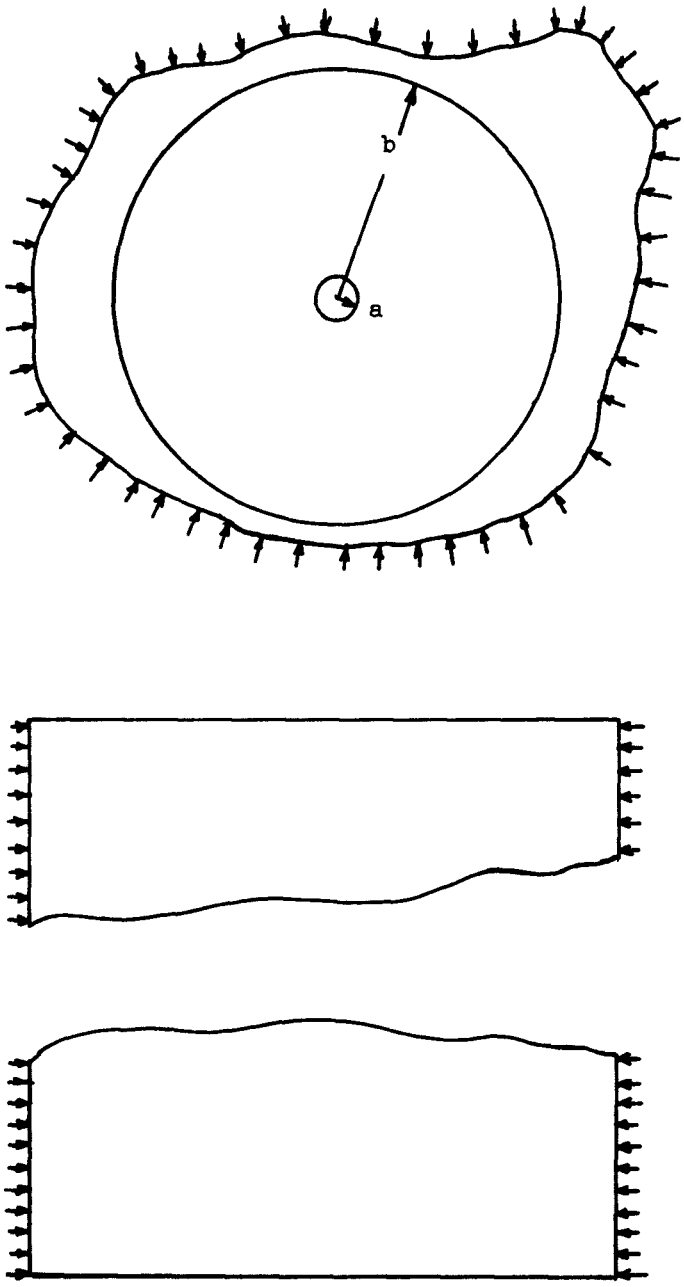


Fig. 4.10.

(i) Elastic Solution

When the applied pressure is small enough, the body deforms like a completely elastic body. We shall anlayze the stresses under such conditions by considering a circular cylinder of radius b concentric with the cylinder of radius a. If $b \gg a$ the stresses at b are essentially a uniform pressure p. Then one can assume (cylindrical polar coordinates have been used for analytical convenience) at

$$\left.\begin{aligned} r &= b \qquad \sigma_r = -p \\ \text{and at} & \\ r &= a \qquad \sigma_r = 0 \end{aligned}\right\} \tag{4.52}$$

If we now assume an axisymmetric stress distribution in the region $a \leq r \leq b$ we have the general expression for stresses[3]*

$$\left.\begin{aligned} \sigma_r &= \frac{C_1}{r^2} + C_2 \\ \sigma_\theta &= -\frac{C_1}{r^2} + C_2 \end{aligned}\right\} \tag{4.53}$$

By using the boundary conditions (4.52) we can determine the constants C_1 and C_2. Then the stresses are

$$\left.\begin{aligned} \sigma_r &= -\frac{b^2 p}{b^2 - a^2}\left(1 - \frac{a^2}{r^2}\right) \\ \sigma_\theta &= -\frac{b^2 p}{b^2 - a^2}\left(1 + \frac{a^2}{r^2}\right) \end{aligned}\right\} \tag{4.54}$$

The corresponding strains are

$$\epsilon_r = p\,\frac{1+\nu}{E}\,\frac{a^2 b^2}{b^2 - a^2}\left(\frac{1}{r^2} - \frac{1-2\nu}{b^2}\right) \tag{4.55a}$$

*Ref. 3, p.59.

$$\epsilon_\theta = p \frac{1+\nu}{E} \frac{a^2 b^2}{b^2 - a^2} \left(- \frac{1}{r^2} - \frac{(1-2\nu)}{b^2} \right) \tag{4.55b}$$

$$\epsilon_z = 0 \tag{4.55c}$$

The body is made of distortionally locking material. In this case the condition of distortional locking is assumed to be of the form

$$\left| \epsilon_i - \epsilon_j \right| = \epsilon_\ell \tag{4.56}$$

where ϵ_i and ϵ_j are to be chosen out of three principal strains such that $|\epsilon_i - \epsilon_j|$ is a maximum. From Eq. (4.56) one can see that locking takes place at $r = a$ when

$$\epsilon_r - \epsilon_\theta = \frac{2(1+\nu)}{E} \frac{pb^2}{b^2 - a^2} = \epsilon_\ell \tag{4.57}$$

(ii) Problem After Incipient Locking

If the pressure

$$p > \frac{E\epsilon_\ell (b^2 - a^2)}{2(1+\nu) b^2} \tag{4.58}$$

the body contains a region which is locked and a region which is elastic. Because of axisymmetry we can expect that the region $a \leq r \leq k$ is locked while the region $r \geq k$ is elastic. The expresssions for the stresses and strains in the elastic region are still given by Eqs. (4.53) and (4.56) while the stresses in the locked region should satisfy:

(a) The Equilibrium Equation

$$\frac{d\bar{\sigma}_r}{dr} + \frac{\bar{\sigma}_r - \bar{\sigma}_\theta}{r} = 0 \tag{4.59}$$

(b) Dilatation Elasticity

$$\bar{\epsilon}_r + \bar{\epsilon}_\theta = \frac{1}{3K} (\bar{\sigma}_r + \bar{\sigma}_\theta + \bar{\sigma}_z) \tag{4.60}$$

(c) The Ideal Locking Condition

$$\bar{\epsilon}_r - \bar{\epsilon}_\theta = \epsilon_\ell \tag{4.61}$$

(d) Kinematics

$$\bar{\epsilon}_r = \frac{d\bar{u}}{dr} \qquad \bar{\epsilon}_\theta = \frac{\bar{u}}{r} \tag{4.62}$$

(d) The Incremental Stress-Strain Relationship which will be derived in the next sub-section.

The boundary conditions for the problem can be expressed by the following equations

$$\left.\begin{aligned} r = a : \quad & \bar{\sigma}_r = 0 \\ r = k : \quad & \bar{\sigma}_r = \sigma_r \\ & \bar{u} = u \\ & \epsilon_r - \epsilon_\theta = \epsilon_\ell \\ r = b : \quad & \sigma_r = -p \end{aligned}\right\} \tag{4.63}$$

(iii) Incremental Type of Stress-Strain Relationship

In the present problem the locking condition is given by

$$\phi = \epsilon_\theta - \epsilon_r - \epsilon_\ell = 0$$

As discussed in Chapter II, the stress components $\bar{\sigma}_r$, $\bar{\sigma}_\theta$ and $\bar{\sigma}_z$ can be represented as

$$\left.\begin{aligned} \bar{\sigma}_r &= \bar{\sigma}_{r\ell} + \bar{\sigma}_{re} \\ \bar{\sigma}_\theta &= \bar{\sigma}_{\theta\ell} + \bar{\sigma}_{\theta e} \\ \bar{\sigma}_z &= \bar{\sigma}_{z\ell} + \bar{\sigma}_{ze} \end{aligned}\right\} \tag{4.64}$$

where $\bar{\sigma}_{r\ell}$, $\bar{\sigma}_{\theta\ell}$ and $\bar{\sigma}_{z\ell}$ are the locked parts of the stress components, and $\bar{\sigma}_{re}$, $\bar{\sigma}_{\theta e}$, $\bar{\sigma}_{ze}$ are the elastic parts of the stress components. From the Eq. (2.27a) we have

$$\left.\begin{aligned} \delta\bar{\sigma}_{\theta\ell} &= \delta\bar{\lambda}\frac{\partial\phi}{\partial\epsilon_\theta} = \delta\bar{\lambda} \\ \delta\bar{\sigma}_{r\ell} &= \delta\bar{\lambda}\frac{\partial\phi}{\partial\epsilon_r} = -\delta\bar{\lambda} \\ \delta\bar{\sigma}_{z\ell} &= 0 \end{aligned}\right\} \qquad (4.65)$$

The relationship between the elastic components of stress and strain are given by Hooke's law, i.e.,

$$\left.\begin{aligned} \bar{\sigma}_{re} - \bar{\sigma}_{\theta e} &= 2G(\bar{\epsilon}_r - \bar{\epsilon}_\theta) \\ \bar{\sigma}_{ze} - \bar{\sigma}_{re} &= -2G\bar{\epsilon}_r \\ \bar{\sigma}_{ze} - \bar{\sigma}_{re} + \bar{\sigma}_{\theta e} &= 3K(\bar{\epsilon}_r + \bar{\epsilon}_\theta) \end{aligned}\right\} \qquad (4.66)$$

From this equation and the locking condition (4.61) we have

$$\delta(\bar{\sigma}_\theta - \bar{\sigma}_r) = \delta(\bar{\sigma}_{\theta e} + \bar{\sigma}_{\theta\ell} - \bar{\sigma}_{re} - \bar{\sigma}_{r\ell}) = 2\delta\bar{\lambda} \qquad (4.67a)$$

and

$$\delta\bar{\sigma}_z = \delta\bar{\sigma}_{ze} + \delta\sigma_{z\ell} = \delta\sigma_{ze} = \nu(\delta\sigma_{re} + \delta\sigma_{\theta e}) = \nu(\delta\bar{\sigma}_r + \delta\bar{\sigma}_\theta) \qquad (4.67b)$$

Equation (4.67b) can be integrated to yield

$$\bar{\sigma}_z = \nu(\bar{\sigma}_r + \bar{\sigma}_\theta) \qquad (4.68)$$

Another useful formula can be obtained by using (4.68) and the dilatational elasticity Eq. (4.60)

$$\bar{\sigma}_r + \bar{\sigma}_\theta = \frac{3K}{1+\nu}(\bar{\epsilon}_r + \bar{\epsilon}_\theta) \qquad (4.69)$$

(iv) Solution in the Locked Region

From Eqs. (4.61) and (4.62) we get the following differential equation in $\bar{u}$.

$$\frac{d\bar{u}}{dr} - \frac{\bar{u}}{r} - \epsilon_\ell = 0 \tag{4.70}$$

which when solved yields the following result.

$$\left.\begin{aligned} \bar{u} &= C_3 r + \epsilon_\ell r \log r \\ \bar{\epsilon}_r &= C_3 + \epsilon_\ell (1 + \log r) \\ \bar{\epsilon}_\theta &= C_3 + \epsilon_\ell \log r \end{aligned}\right\} \tag{4.71}$$

Then from Eqs. (4.69) and (4.71) we have

$$\bar{\sigma}_\theta = \frac{3K}{1+\nu} \quad (2C_3 + 2\epsilon_\ell \log r + \epsilon_\ell) - \bar{\sigma}_r \tag{4.72}$$

The equilibrium equation (4.59) then reduces to a differential equation in $\bar{\sigma}_r$.

$$\frac{d\bar{\sigma}_r}{dr} + \frac{2\bar{\sigma}_r}{r} = \frac{3K}{(1+\nu)} \quad (2C_3 + 2\epsilon_\ell \log r + \epsilon_\ell) \frac{1}{r} \tag{4.73}$$

Hence,

$$\bar{\sigma}_r = \frac{3K}{1+\nu} C_3 + \frac{3K}{1+\nu} \epsilon_\ell \log r + \frac{C_4}{r^2} \tag{4.74}$$

(v) Matching the Boundary Conditions

The solution in the elastic region is given by Eqs. (4.53) and (4.56). By using the boundary conditions (4.63) we find

$$C_1 = \frac{E}{2(1+\nu)} \quad k^2 \epsilon_\ell \tag{4.76a}$$

$$C_2 = -\frac{E}{2(1+\nu)} \frac{k^2}{b^2} \epsilon_\ell - p \tag{4.76b}$$

$$C_3 = -\epsilon_\ell \log k - \frac{\epsilon_\ell}{2} - \frac{1-2\nu}{2} \frac{k^2}{b^2} \epsilon_\ell - \frac{p(1+\nu)(1-2\nu)}{E} = -\epsilon_\ell \log a - \epsilon_\ell (1-\nu) \frac{k^2}{a^2} \tag{4.76c}$$

$$C_4 = \frac{E\epsilon_\ell(1-\nu)}{(1+\nu)(1-2\nu)}\ k^2 \tag{4.76d}$$

and the relationship between k and p is given by the following equation

$$p = \frac{E\epsilon_\ell}{(1+\nu)(1-2\nu)}\left[(1-\nu)\frac{k^2}{a^2} + \log\frac{a}{k} - \frac{1}{2} - \frac{1-2\nu}{2}\frac{k^2}{b^2}\right] \tag{4.77}$$

Now if we consider the limit as $b \to \infty$ the expressions for C_1, C_2, C_3 and C_4 become

$$\left.\begin{aligned} C_1 &= \frac{E}{2(1+\nu)}\,k^2\epsilon_\ell \\ C_2 &= -p \\ C_3 &= -\epsilon_\ell \log k - \frac{\epsilon_\ell}{2} - p\,\frac{(1+\nu)(1-2\nu)}{E} \\ C_4 &= \frac{E\epsilon_\ell(1-\nu)}{(1+\nu)(1-2\nu)}\ k^2 \end{aligned}\right\} \tag{4.78}$$

Then the stresses are given by the following equations:

$$\sigma_r = \frac{E\epsilon_\ell}{2(1+\nu)}\ \frac{k^2}{r^2} - p$$

$$\sigma_\theta = -\frac{E\epsilon_\ell}{2(1+\nu)}\,\frac{k^2}{r^2} - p$$

$$\bar{\sigma}_r = \frac{E\epsilon_\ell}{(1+\nu)(1-2\nu)}\ \left[(1-\nu)\frac{k^2}{r^2} + \log\frac{r}{k} - \frac{1}{2}\right] - p$$

$$\bar{\sigma}_\theta = -\frac{E\epsilon_\ell}{(1+\nu)(1-2\nu)}\ \left[(1-\nu)\frac{k^2}{r^2} - \log\frac{r}{k} - \frac{1}{2}\right] - p$$

where k follows from

$$p = \frac{E\epsilon_\ell}{(1+\nu)(1-2\nu)} \left((1-\nu)\frac{k^2}{a^2} + \log\frac{a}{k} - \frac{1}{2} \right)$$

The stress σ_θ at $r = a$ is

$$\bar{\sigma}_\theta = -2p + \frac{K\epsilon_\ell}{(1+\nu)} \log\frac{k}{b} \tag{4.79}$$

This is higher than the value of σ_θ derived on the assumption of elastic behavior of the material. In Fig. 4.11, the variation of σ_r and σ_θ with r is shown for different values of k. b/a has been assumed to be ∞. In Fig. 4.12a, $p/E\epsilon_\ell$ has been plotted against k/a for different values of ν. The effect of different b/a is shown in Fig. 4.12b. In concluding this section, one can observe that the problem of computing stresses around a circular cylindrical rigid conclusion in a body, in plane strain conditions subjected to uniform pressure along the edges of every cross-section can be worked out in a similar way.

IV.4. An Example to Illustrate the Limitations on the Displacement Boundary Condition

It has been proved in section I-13 that if the displacements are prescribed on the surface of the body so that they increase from zero value proportionally, the prescribed displacement can be increased cnly up to that value which makes the body completely locked. This will now be illustrated by a simple example.

We consider a hollow sphere of inner radius a and outer radius b with prescribed displacement $u = n\alpha$ at $r = a$ and $u = n\beta$ at $r = b$. α is assumed to be greater than β. The prescribed displacements are assumed to increase from their zero value proportionally, i.e., n increases from its value zero to its present value. For small values of n the sphere deforms like a completely elastic body. Further, this problem has spherical symmetry. Then the stresses, strains and displacements are given by Eqs. (4.7) and (4.6). By using the boundary conditions the constants C_1 and C_2 can be evaluated. Thus,

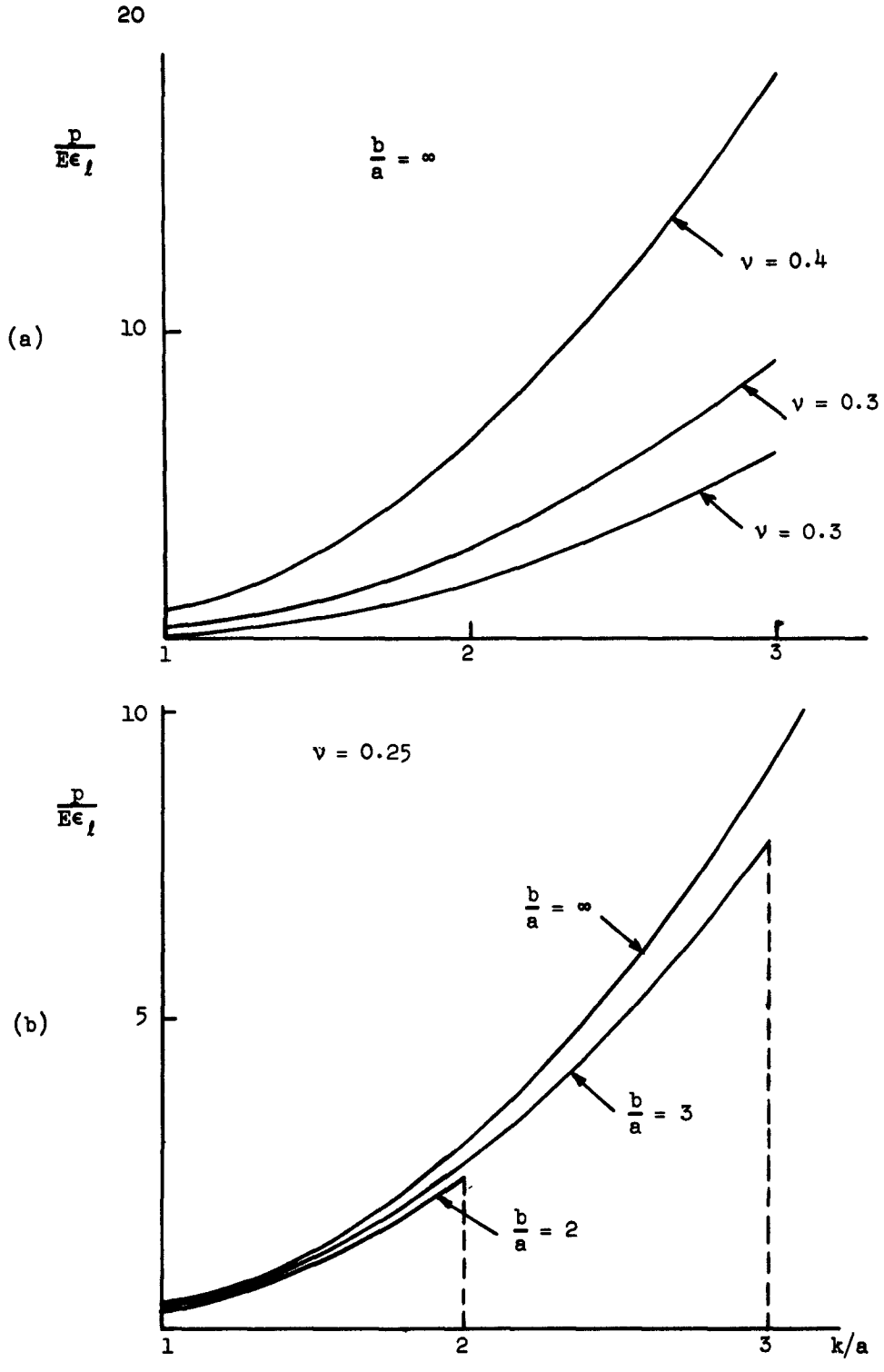

20
$\frac{p}{E\epsilon_l}$
$\frac{b}{a} = \infty$
$\nu = 0.4$
(a)
10
$\nu = 0.3$
$\nu = 0.3$
1
2
3
10
$\nu = 0.25$
$\frac{p}{E\epsilon_l}$
$\frac{b}{a} = \infty$
(b)
5
$\frac{b}{a} = 3$
$\frac{b}{a} = 2$
1
2
3
k/a

Fig. 4.8.

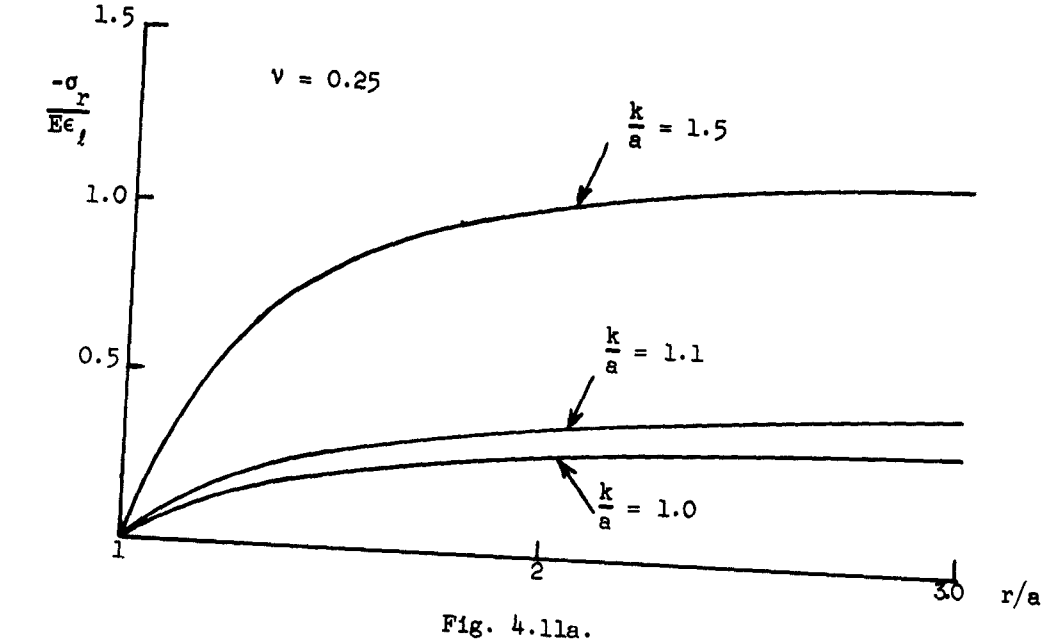


Fig. 4.11a.

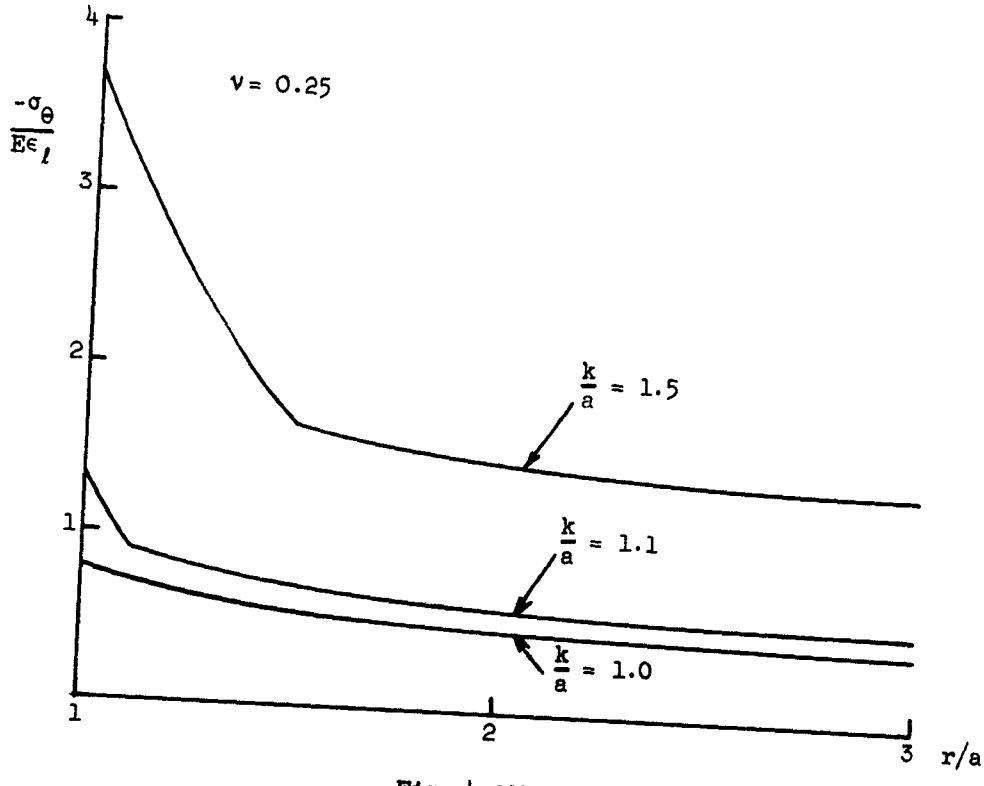


Fig. 4.11b.

$$\left.\begin{aligned} C_2 &= na^2b^2\,\frac{\alpha b - \beta a}{b^3 - a^3} \\ C_1 &= \frac{n\alpha}{a} - \frac{nb^2}{a}\,\frac{\alpha b - \beta a}{b^3 - a^3} \end{aligned}\right\} \qquad (4.80)$$

The stresses, strains and displacements in the body can then be calculated using these values of C_1 and C_2. The sphere is capable of locking distortionally. The locking condition is assumed to be of the form

$$(\epsilon_\theta - \epsilon_r)^2 = \epsilon_\ell^2 \qquad (4.81)$$

From Eqs. (4.7a) and (4.87) $\epsilon_\theta - \epsilon_r$ can be computed. Its value is

$$\epsilon_\theta - \epsilon_r = 3na^3b^2\,\frac{\alpha\frac{b}{a} - \beta}{b^3 - a^3}\,\frac{1}{r^2}, \quad \alpha > \beta \qquad (4.82)$$

This has the maximum at $r = a$. Then the sphere begins to look at $r = a$ when

$$n = \frac{\epsilon_\ell}{3b^2}\,\frac{b^3 - a^3}{\alpha\frac{b}{a} - \beta} \qquad (4.83)$$

(i) Problem After Incipient Locking

As n is further increased from its value given by (4.83) the sphere will contain a region which is locked and a region which is elastic. From spherical symmetry we can assume that the region $a < r < k$ is locked and the region $k < r < b$ is still elastic.

In the region $a < r < k$ the general expressions for stresses, strains and displacements are given by Eqs. (4.20), (4.21), and (4.24) with $+\epsilon_\ell$ replaced by $-\epsilon_\ell$, i.e.,

$$\left.\begin{aligned}
\bar{u} &= C_3 r - \epsilon_\ell r \log r \\
\bar{\epsilon}_r &= C_3 - \epsilon_\ell (1 + \log r) \\
\bar{\epsilon}_\theta &= C_3 - \epsilon_\ell \log r \\
\bar{\sigma}_r &= \frac{3C_4}{r^3} + 3KC_3 - 3K\epsilon_\ell \log r
\end{aligned}\right\} \qquad (4.84)$$

The general expressions for stresses in elastic region are again given by Eqs. (4.6). Now, the boundary conditions can be expressed by the following equations

$$\left.\begin{aligned}
r = a \qquad & \bar{u} = n\alpha \\
r = b \qquad & u = n\beta \\
r = k \qquad & u = \bar{u} \\
& \sigma_r = \bar{\sigma}_r \\
& \epsilon_r - \epsilon_\theta = -\epsilon_\ell
\end{aligned}\right\} \qquad (4.85)$$

Then

$$\left.\begin{aligned}
C_1 &= \frac{n\beta}{b} - \frac{\epsilon_\ell}{3}\frac{k^3}{b^3} \\
C_2 &= +\epsilon_\ell \frac{k^3}{3} \\
C_3 &= \frac{n\alpha}{a} + \epsilon_\ell \log a \\
C_4 &= \frac{k^3 \epsilon_\ell}{9} \quad (4G - 3K)
\end{aligned}\right\} \qquad (4.86)$$

The relationship between n and k is given by

$$\frac{n\beta}{b} - \frac{\epsilon_\ell}{3}\frac{k^3}{b^3} - \left(\frac{n\alpha}{a} + \epsilon_\ell \log a\right) = -\frac{\epsilon_\ell}{3} - \epsilon_\ell \log k \qquad (4.87)$$

The sphere will be completely locked when $k = b$, i.e., when

$$n = b \frac{\epsilon_\ell \log \frac{b}{a}}{\alpha \frac{b}{a} - \beta} \tag{4.88}$$

(ii) Problem After the Sphere is Completely Locked

Let us now attempt to study the sphere in the completely locked state. In this state the expressions for stresses, strains and displacement throughout the sphere are given by Eqs. (4.84). The boundary conditions are

$$r = a: \qquad \bar{u} = n\alpha \tag{4.89a}$$

$$r = b: \qquad \bar{u} = n\beta \tag{4.89b}$$

From (4.89a)

$$C_3 = \epsilon_\ell \log a + \frac{n\alpha}{a} \tag{4.90}$$

From (4.89b)

$$C_3 = \epsilon_\ell \log b + \frac{n\beta}{b} \tag{4.91}$$

Contradicting conditions (4.90) and (4.91) for b require that

$$\epsilon_\ell \log \frac{a}{b} = n \left(\frac{\beta}{b} - \frac{\alpha}{a} \right)$$

This can be reconciled for any n only if $a = b$. Therefore, the proportional increase of surface displacement is not possible for

$$n > \frac{\epsilon_\ell b \log \frac{b}{a}}{\alpha \frac{b}{a} - \beta}$$

V. WAVE PROPAGATION IN LOCKING MEDIA

V.1 One-Dimensional Wave Propagation

(i) Stress-Strain Relationship in the Uniaxial Stress System and Volumetric Locking

In this section we shall study the wave propagation under a uniaxial stress system. As a first step toward the study, the stress-strain relationship in a volumetrically locked region applicable in this case will be derived.

The locking condition for the material is assumed to be

$$\epsilon_x + \epsilon_y + \epsilon_z = \epsilon_\ell \tag{5.1}$$

After locking, the deviatoric parts of stress and strain tensors are still related by Hooke's law. In the one-dimensional stress system the stress tensor consists of only one component σ_x. The strain tensor contains three components of normal strain ϵ_x, ϵ_y, ϵ_z. If the deviatoric parts of strain and stress tensors are obtained and Hooke's law is used one obtains

$$\left.\begin{aligned} 2\bar{\sigma}_x &= 2G(2\bar{\epsilon}_x - \bar{\epsilon}_y - \bar{\epsilon}_z) \\ -\bar{\sigma}_x &= 2G(-\bar{\epsilon}_x + 2\bar{\epsilon}_y - \bar{\epsilon}_z) \\ -\bar{\sigma}_x &= 2G(-\bar{\epsilon}_x - \bar{\epsilon}_y + 2\bar{\epsilon}_z) \end{aligned}\right\} \tag{5.2}$$

Only two of these three relations are independent. By simplifying these equations one obtains

$$\bar{\epsilon}_y = \bar{\epsilon}_z \quad , \quad \bar{\sigma}_x = 2G(\bar{\epsilon}_x - \bar{\epsilon}_y) \tag{5.3}$$

Eqs.(5.3) together with the locking condition yield the following relationships

$$\bar{\sigma}_x = 3G\bar{\epsilon}_x - G\epsilon_\ell \tag{5.4a}$$

$$\bar{\epsilon}_y = \frac{\epsilon_\ell}{3} - \frac{\bar{\sigma}_x}{6G} \tag{5.4b}$$

These equations express the relationship between σ_x, ϵ_x and ϵ_y in the locked region.

The locking condition (5.1) can be expressed in terms of stresses also. If the material is elastic

$$\epsilon_x + \epsilon_y + \epsilon_z = \frac{1 - 2\nu}{E} (\sigma_x + \sigma_y + \sigma_z)$$

In this case $\sigma_y = \sigma_z = 0$. Then Eq.(5.1) becomes

$$\sigma_x = \frac{E\epsilon_\ell}{1 - 2\nu} \tag{5.5}$$

Then one can state the following inequalities. The material is elastic if

$$\sigma_x \leq \frac{E\epsilon_\ell}{1 - 2\nu}$$

and locked if

$$\sigma_x \geq \frac{E\epsilon_\ell}{1 - 2\nu}$$

The stress-strain diagram is as shown in Fig.5.1

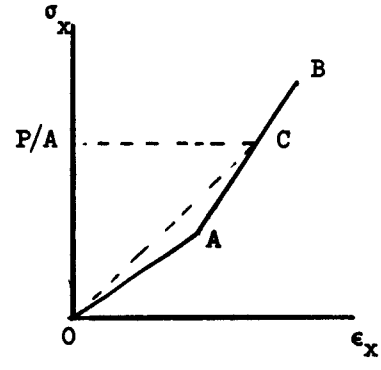


Fig. 5.1.

The straight lines OA and AB are defined by the following equations

$$OA: \quad \epsilon_x = \frac{\sigma_x}{E}$$

$$AB: \quad \epsilon_x = \frac{\sigma_x}{3G} + \frac{\epsilon_\ell}{3}$$

(ii) Equation of Motion

Let us consider a bar of uniform cross section made of elastic-locking material. It is assumed that the average stresses on any cross section can be approximated by a one-dimensional stress system. Then, one can apply Newton's law of motion to an element of bar of length dx as shown in Fig.5.2 . The displacement in the x-direction is represented by u and the velocity by $v = \partial u/\partial t$.

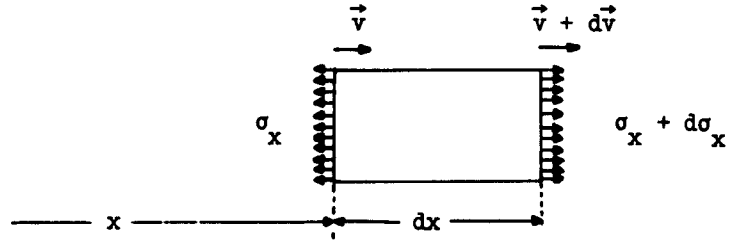


Fig. 5.2.

In further discussion of one-dimensional wave propagation the usual notation of barred letters to denote the stresses, strains and displacements in the locked region will not be applied because we are working most of the time only with the stresses in the locked region. The equation of motion can then be written as

$$\frac{\partial \sigma_x}{\partial x} = \rho \frac{\partial v}{\partial t} = \rho \frac{\partial^2 u}{\partial t^2}$$

In general the stress-strain law can be expressed as

$$\sigma = f(\epsilon) \tag{5.6}$$

If one defines the normal force N such that

$$N = A\sigma_x$$

and the mass per unit length μ as

$$\mu = A\rho$$

the equation of motion and the stress-strain relationship can be re-written in the following way

$$N' = \mu\ddot{u} \quad , \quad N = Af(u') \tag{5.7}$$

where $(\,)'$ represents partial differentiation with respect to r and $(\dot{})$ represents partial differentiation with respect to t. Elimination of N from Eq.(5.7) yields

$$A\frac{df}{du'}u'' = \mu\ddot{u} \tag{5.8}$$

By introducing the symbol

$$E(u') = \frac{df}{du'}$$

equation (5.8) becomes

$$u'' = \frac{\mu}{AE(u')}\ddot{u} \tag{5.9}$$

From this equation one can infer that there is a wave of velocity c such that

$$c^2 = \frac{AE(u')}{\mu} \tag{5.9a}$$

moving in the locked region.

Alternately, one can derive the equation of motion by considering the discontinuities at the wave front.

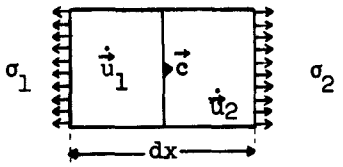


Fig. 5.3.

By the impulse momentum theorem one can write (See Fig. 5.3)

$$dtA(\sigma_{x1}-\sigma_{x2}) = \rho A(\dot{u}_2 - \dot{u}_1)dx$$

i.e.,

$$\sigma_{x2} - \sigma_{x1} = -\rho c(\dot{u}_2 - \dot{u}_1) \tag{5.10}$$

or

$$N_1 - N_2 = \mu c(\dot{u}_2 - \dot{u}_1) \tag{5.10a}$$

where

$$c = \frac{dx}{dt}$$

By introducing the jump symbol

$$\Delta( ) = ( )_2 - ( )_1$$

one can rewrite equation (5.10a) in the following way

$$\mu c \Delta\dot{u} = -\Delta N \tag{5.11}$$

From the condition of continuity of displacements along the wave front, one can write

$$u_2 - u_1 = \Delta u = 0$$

By differentiating this equation along the wave front in the x-direction one obtains

$$\frac{D}{Dx}(\Delta u) = \left(\frac{\partial}{\partial x} + \frac{1}{c}\frac{\partial}{\partial t}\right)\Delta u = \Delta u' + \frac{1}{c}\Delta \dot{u} = 0 \tag{5.12}$$

Eqs. (5.11) and (5.12) can be simplified to obtain

$$\mu c^2 \Delta u' = \Delta N \tag{5.13}$$

If one makes the assumption that the discontinuities $\Delta \dot{u}$, ΔN, $\Delta u'$ carried by the wave are infinitesimal quantities, one can write

$$\Delta N = A\Delta f(u') = AE(u')\Delta u'$$

Then,

$$\mu c^2 \Delta u' = AE(u')\Delta u'$$

i.e.,

$$c = \sqrt{\frac{EA}{\mu}} \tag{5.14}$$

The result represented by Eq.(5.14) is the same as that obtained by the differential equation. However, by the study at wave front we have learned that the results (5.14) and (5.9a) are true only for infinitesimal discontinuities. Then the next problem will be to study whether there is a possibility of the discontinuity being finite. As a first step toward the study the variation of discontinuity during the propagation of the wave will be studied in the next section.

(iii) Variation of Discontinuity During the Propagation of the Wave

By differentiating Eq.(5.11) along the line of propagation of the wave in the x-direction, one

$$\mu c \frac{D}{Dx}(\Delta\dot{u}) + \mu c'\Delta\dot{u} + \frac{\mu\dot{c}}{c}\Delta\dot{u} = -\Delta N' - \frac{1}{c}\Delta\dot{N} \tag{5.15}$$

From Eq. (5.7) one can write that

$$\Delta N' = \mu\Delta\ddot{u} \tag{5.16a}$$

$$\Delta\dot{N} = \Delta(EA\dot{u}') \tag{5.16b}$$

Eq.(5.16b) can then be written as

$$\Delta\dot{N} = A\frac{dE}{du'}\dot{u}'\Delta u' + \mu c^2\Delta\dot{u}' \tag{5.17}$$

This equation is for infinitesimal waves only. Further, from Eq.(5.9a) one has

$$2cc' = \frac{A}{\mu}\frac{dE}{du'}u''$$

$$2c\dot{c} = \frac{A}{\mu}\frac{dE}{du'}\dot{u}' \tag{5.18}$$

By substituting (5.16a), (5.17), and (5.18) in Eq.(5.15), the following equation is obtained:

$$2\mu c\frac{D}{Dx}(\Delta\dot{u}) + \frac{A}{2c}\frac{dE}{du'}(u'' + \frac{1}{c}\dot{u}')\Delta\dot{u} + \frac{A}{c}\frac{dE}{du'}\dot{u}'\Delta u' = 0$$

Then, by using Eq.(5.12) this equation can be simplified to the following form

$$\frac{D}{Dx}(\Delta\dot{u}) + \frac{A}{4\mu c^2}\frac{dE}{du'}(u'' - \frac{1}{c}\dot{u}')\Delta\dot{u} = 0 \tag{5.19}$$

This is a differential equation in $\Delta\dot{u}$. It will be used in the next sub-section to study the effect of variation of discontinuity as the wave propagates into the bar. It should be noted that this equation is valid only for infinitesimal waves.

(iv) Case of Unloading Wave

Let us consider a bar prestressed to $N = P$. Let a stress-discontinuity of the magnitude

$$N(0,t) = - PI(t)$$

be sent through the bar from the end $x = 0$. In this equation $I(t)$ represents the unit step function, i.e., the force P is suddenly released at the end $x = 0$. Further, it is assumed that the discontinuity $- P$ is made up of elementary waves of magnitude $- \Delta P$ which propagate successively through the bar.

At first $u(x,t)$, $\dot{u}(x,t)$ are equal to zero throughout the bar. Then, the second term in Eq.(5.19) is zero. Therefore, $\Delta\dot{u}$ is constant throughout the bar; i.e., as the first elementary wave propagates through the bar, the wave creates a state of constant velocity and constant stress behind the wave. Then, one can conclude that u'' and $\dot{u}'$ are zero behind the wave. Thus, for the second elementary wave of magnitude $-\Delta P$, the second term of the Eq.(5.19) is again zero. The Eq.(5.19) again yields the same result, i.e., $\Delta\dot{u}$ = constant throughout the bar. That is, the propagation of the second elementary wave through the bar also creates a state of constant stress and constant velocity behind the wave.

Thus, one can see that all elementary waves spread through the bar, similar to the linearly elastic waves. However, they propagate with different speeds as given by Eq.(5.9a). It means that after the nth elementary wave has passed through an arbitrary point x_1 of the bar, the stress at the point is $P - n\Delta P$. The speed of propagation of the nth elementary wave is

$$c_n = \sqrt{\frac{E_n A}{\mu}} \tag{5.20}$$

where E_n is the slope of the stress-strain curve evaluated at

$$\sigma_x = \sigma_{xn} = \frac{n\Delta P}{A}$$

i.e.,

$$E_n = \left(\frac{d\sigma_x}{dt}\right)_{\sigma_x = \sigma_{xn}}$$

From Fig.5.1

$$E_n = 3G \quad \text{if} \quad \sigma_x > \frac{E\epsilon_l}{1-2\nu}$$

$$= E \quad \text{if} \quad \sigma_x < \frac{E\epsilon_l}{1-2\nu}$$

Also, $3G > E$. Then, one can see that the unloading wave falls apart into two waves with speeds $\sqrt{3G/\rho}$ and $\sqrt{E/\rho}$.

Further, according to Eq.(5.11) the elementary waves add a contribution

$$\Delta\dot{u} = -\frac{\Delta N}{\mu c}$$

to the velocity $\dot{u}$. Therefore, the particle velocity at a point allowing an axial force $P - n\Delta P$ is

$$\dot{u}(x,t) = -\frac{1}{\mu}\int_0^{n\Delta P}\frac{dN}{c(N)} = -\frac{A}{\mu}\int_0^{\frac{n\Delta P}{A}}\frac{d\sigma_x}{c(\sigma_x)} \qquad (5.21)$$

From Eq.(5.21) it is evident that $u(x,t)$ is a function of the stress σ_x only. Thus it does not depend on the stress distribution along the length of the bar.

If one considers the upper limit of the integral in Eq.(5.21) as

$$n\Delta P = P$$

one obtains the velocity at the end of the bar immediately after the application of the discontinuity.

(v) Case of the Loading Wave

Let us now consider the unloaded bar $x > 0$. Let a stress discontinuity of intensity

$$N = PI(t)$$

be sent through the bar from the end $x = 0$. Again, one can consider the stress discontinuity to be made of elementary waves of intensity ΔP . The first of these waves travels with a speed $c = c(0)$ while the final elementary wave travels with a speed

$$c = c(P/A)$$

(see Fig.5.1a), which is greater than that of the first wave, i.e., the last wave catches up with the first at a certain point. Similarly, all the elementary waves introduced at a later time can catch up with those introduced when $n\Delta P < E\epsilon_{\ell}/1-2\nu$. However, when the two waves catch up with each other they cannot pass one another as two waves approaching from opposite directions, in an elastic bar. In this case the wave that overtakes passes from a domain of higher stress to a lower stress. Hence it becomes the slowest wave. Therefore, one can see that the waves do not separate after they catch up with each other. They move together with a common speed, i.e., the discontinuity at the wave front starts accumulating and becomes finite. One can no longer apply the formulae derived on the assumption of an infinitesimal discontinuity. In order to find the speed of propagation of the wave carrying a finite discontinuity, one must use Eq.(5.13) which is applicable for finite discontinuities, i.e.,

$$c^2 = \frac{N_1 - N_2}{\mu(u_1' - u_2')} = \frac{1}{\rho}\frac{\sigma_1 - \sigma_2}{\epsilon_1 - \epsilon_2} \tag{5.22}$$

In this equation and in further analysis, the subscript x of σ_x and ϵ_x has been dropped.

In concluding, one can see that the case of loading produces a phenomenon which is quite different from that of unloading. Stress and velocity discontinuities in the case of loading do not fall apart as in the case of unloading. They move together as a finite wave (called the shock wave). The speed of this wave at any point is given by Eq.(5.22).

(vi) Impossibility of an Unloading Shock

The relationships (5.13) and (5.22) are valid whether the load increases or decreases. The concept of loading shock has been explained on the basis of these equations. Then, one may ask the question whether an unloading shock exists? If not, what are the reasons preventing the existence of an unloading shock? This can be explained by energy considerations.

Let us consider the loading wave. When the stress changes from σ_1 to σ_2 and the velocity from $\dot{u}_1$ to $\dot{u}_2$, the change in kinetic energy is

$$U_1 = \frac{1}{2}\mu dx\left(\dot{u}_2^2 - \dot{u}_1^2\right)$$

The increase in elastic energy during the same process is

$$U_2 = dx \int_{N_1}^{N_2} N(u')du'$$

Net work done during the process is

$$U_3 = -(N_1\dot{u}_1 - N_2\dot{u}_2)dt$$

Then one can define the energy loss U as

U = increase in kinetic energy + increase in elastic energy - work done

i.e.,

$$Udx = \frac{1}{2}\mu dx\left(\dot{u}_2^2 - \dot{u}_1^2\right) + dx\int_{N_1}^{N_2} N(u')du' + \frac{1}{c}(N_1\dot{u}_1 - N_2\dot{u}_2)dx \qquad (5.23)$$

The $\sigma - \epsilon$ curve is of the shape shown in Fig.5.4

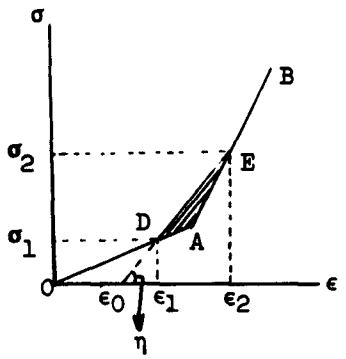


Fig. 5.4.

When the curvature is concave upwards the integral can be estimated in the following way:

$$\int_{N_1}^{N_2} N(u')du' = A \int_{\sigma_1}^{\sigma_2} \sigma d\epsilon = \frac{A}{2}(\sigma_1 + \sigma_2)(\epsilon_2 - \epsilon_1) - C \qquad (5.24)$$

where C is the shaded area in the figure. Further, C is a positive quantity.

Also, from Eq.(5.22) one can write

$$\mu c^2 = \eta A$$

where

$$\eta = \frac{\sigma_1 - \sigma_2}{\epsilon_1 - \epsilon_2}$$

i.e.,

$$\epsilon_2 - \epsilon_1 = \frac{1}{\eta}(\sigma_2 - \sigma_1)$$

Therefore,

$$\int_{\sigma_1}^{\sigma_2} \sigma d\epsilon = \frac{A}{2\eta}\left(\sigma_2^2 - \sigma_1^2\right) - C$$

Further, from the figure one can write

$$\sigma_1 = \eta(\epsilon_1 - \epsilon_0)$$

$$\sigma_2 = \eta(\epsilon_2 - \epsilon_0)$$

Then, the energy loss U is

$$U = \frac{\eta A}{2c^2}\left(\dot{u}_2^2 - \dot{u}_1^2\right) + \frac{A}{2\eta}\left(\sigma_2^2 - \sigma_1^2\right) + \frac{A}{c}\left(\sigma_2\dot{u}_2 - \sigma_1\dot{u}_1\right) - C$$

$$= \frac{\eta A}{2}\left[-\left(\frac{\sigma_1}{\eta} + \frac{\dot{u}_1}{c}\right)^2 + \left(\frac{\sigma_2}{\eta} + \frac{\dot{u}_2}{c}\right)^2\right] - C$$

$$= \frac{\eta A}{2}\left[-\left(u' + \frac{\dot{u}_1}{c} - \epsilon_0\right)^2 + \left(+u_2' + \frac{\dot{u}_2}{c} - \epsilon_0\right)^2\right] - C \qquad (5.25)$$

The terms in the square brackets cancel each other because of continuity of displacements. Then

$$U = -C \qquad (5.26)$$

This means that there is more energy than needed which makes the process possible.

For an unloading wave the signs of all energy terms are reversed, including that of the contribution C derived from the nonlinearity of the stress-strain relationship. Instead of a loss we would then have mechanical energy produced from nothing. This shows that upon unloading the wave front disperses; a shock is not possible.

(vii) Interaction of two Waves Approaching from Opposite Directions

Let us consider two waves A and B approaching from opposite directions. The wave A is carrying a stress discontinuity σ_1 and velocity discontinuity v_1. The wave B is carrying a stress

discontinuity σ_2 and the velocity discontinuity v_2. Further, it is assumed that

$$\sigma_1 \leq \sigma_2 \leq \frac{E}{1-2\nu}\epsilon_\ell \tag{5.27}$$

but

$$\sigma_1 + \sigma_2 > \frac{E}{1-2\nu}\epsilon_\ell$$

Then the waves A and B are elastic waves moving with the elastic wave speed $c = \sqrt{E/\rho}$. Further, it is assumed that the region through which neither of the waves has passed is stress free. Then from Eq.(5.10) one can write the following relationships between stress and velocity discontinuities

$$\left.\begin{aligned}\sigma_1 &= \rho c v_1 \\ \sigma_2 &= \rho c v_2\end{aligned}\right\} \tag{5.28}$$

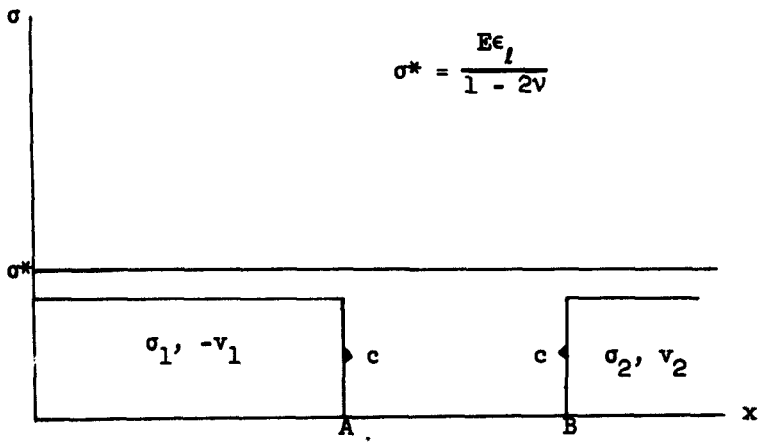


Fig. 5.5.

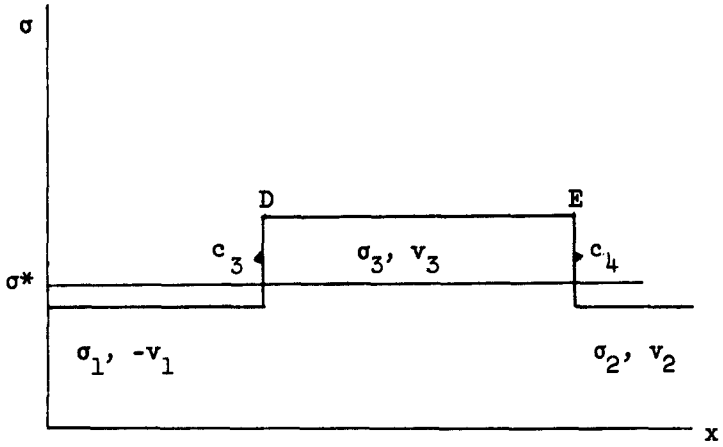


Fig. 5.6.

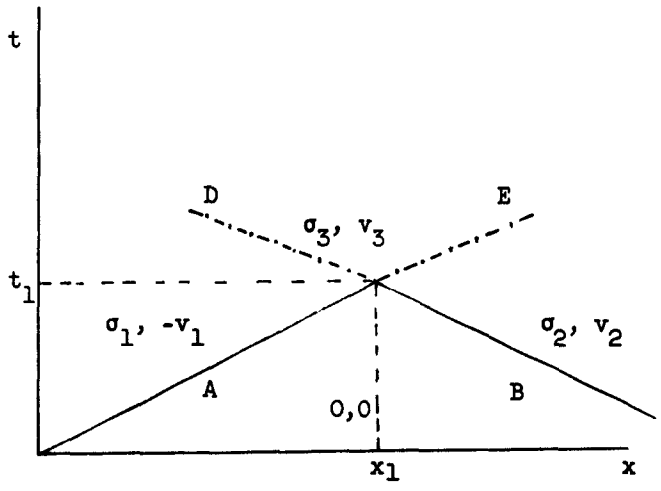


Fig. 5.7.

The $x-t$ diagram for the propagation of the wave has been indicated in Fig.5.7. At a certain point $x = x_1$, $t = t_1$ the two waves meet. Then two different waves D and E moving with speeds c_3 and c_4 emerge from the point as indicated in Figs.5.6 and 5.7. The stress and velocity discontinuities carried by these waves can be calculated in the following way.

By applying Eq.(5.10) to the waves D and E one obtains

$$\left.\begin{aligned} \sigma_3 - \sigma_1 &= -\rho c_3 (v_1 + v_3) \\ \sigma_3 - \sigma_2 &= \rho c_4 (v_2 - v_3) \end{aligned}\right\} \qquad (5.29a)$$

By using Eqs.(5.28), these equations can be written as

$$\left.\begin{aligned} \sigma_3 - \rho c v_1 &= -\rho c_3 (v_1 + v_3) \\ \sigma_3 - \rho c v_2 &= \rho c_4 (v_2 - v_3) \end{aligned}\right\} \qquad (5.29b)$$

Further, from Eq.(5.22)

$$c_3^2 = \frac{1}{\rho} \frac{\sigma_1 - \sigma_3}{\epsilon_1 - \epsilon_3}$$

$$c_4^2 = \frac{1}{\rho} \frac{\sigma_2 - \sigma_3}{\epsilon_2 - \epsilon_3}$$

But from Eq.(5.4) and Hooke's law

$$\epsilon_1 = \frac{\sigma_1}{E}$$

$$\epsilon_2 = \frac{\sigma_2}{E}$$

$$\epsilon_3 = \frac{\sigma_3}{3G} + \frac{\epsilon_\ell}{3}$$

Then

$$\left.\begin{aligned} c_3^2 &= \frac{1}{\rho}\,\frac{\sigma_1 - \sigma_3}{\frac{\sigma_1}{E} - \frac{\sigma_3}{3G} - \frac{\epsilon_l}{3}} \\ c_4^2 &= \frac{1}{\rho}\,\frac{\sigma_2 - \sigma_3}{\frac{\sigma_2}{E} - \frac{\sigma_3}{3G} - \frac{\epsilon_l}{3}} \end{aligned}\right\} \qquad (5.30)$$

Equations (5.29) and (5.30) are four equations for four unknowns, σ_3, v_3, c_3 and c_4.

In the special case when $\sigma_1 = \sigma_2$ and $v_1 = v_2$, the right-hand sides of Eq. (5.30) become equal, indicating that $c_3 = -c_4$. The minus sign has been chosen because the waves D and E are moving in opposite directions. Equations (5.29) can then be written in the following way:

$$\sigma_3 - \rho c v_1 = -\rho c_3 (v_1 + v_3)$$

$$\sigma_3 - \rho c v_1 = \rho c_4 (v_1 - v_3)$$

whence

$$\left.\begin{aligned} v_3 &= 0 \\ -c_3 = +c_4 &= \frac{\sigma_3}{\rho v_1} - c \end{aligned}\right\} \qquad (5.30a)$$

Then, from Eqs. (5.30) and (5.29a)

$$c_3^2 = \frac{c_3 (v_1 + v_3)}{\frac{\sigma_1}{E} - \frac{\sigma_3}{3G} - \frac{\epsilon_l}{3}}$$

By using Eqs. (5.30a) this equation becomes

$$c_3 = \frac{1}{\frac{\sigma_1}{Ev_1} - \frac{\rho}{3G}(c - c_3) - \frac{\epsilon_\ell}{3v_1}} \tag{5.30b}$$

However, from Eq.(5.28)

$$c = \frac{\sigma_1}{\rho v_1}$$

Then, Eq.(5.30b) becomes

$$c_3^2 + c_3\left(\frac{\bar{c}^2}{c^2}c - c - \frac{\epsilon_\ell \bar{c}^2}{3v_1}\right) - \bar{c}^2 = 0$$

where

$$\bar{c} = \sqrt{\frac{3G}{\rho}}$$

is the wave velocity inside the locked region and $c = \sqrt{E/\rho}$ is the elastic wave velocity. If c_3 is determined by solving the quadratic equation, σ_3 can be obtained from Eq.(5.30a). However, the quadratic equation has two roots. We should investigate to see which of the two roots is the correct answer.

c_3 can be expressed as

$$c_3 = \frac{\alpha}{2} \pm \sqrt{\frac{\alpha^2}{4} + \bar{c}^2} \tag{5.30c}$$

where

$$\alpha = c + \frac{\epsilon_\ell \bar{c}^2}{3v_1} - \frac{\bar{c}^2}{c^2}c$$

Now

$$\frac{\bar{c}^2}{c^2} = \frac{3G}{\rho} \cdot \frac{\rho}{E} = \frac{3}{2(1+\nu)}$$

Also, we have

$$\sigma_1 \leq \frac{E\epsilon_\ell}{1-2\nu}$$

i.e.,

$$\frac{\sigma_1}{3\rho v_1} \leq \frac{E\epsilon_\ell}{3\rho v_1(1-2\nu)}$$

i.e.,

$$\frac{\epsilon_\ell}{3v_1} \geq \frac{1-2\nu}{3c}$$

or

$$\frac{\epsilon_\ell c^{-2}}{3v_1} \geq \frac{1-2\nu}{3} \cdot \frac{3}{2(1+\nu)}\, c$$

Then the minimum value of α is

$$\alpha_{min} = c\left(1 + \frac{1-2\nu}{2(1+\nu)} - \frac{3}{2(1+\nu)}\right) = 0$$

Therefore

$$\alpha \geq 0 .$$

The equal sign is applicable where $\sigma_1 = E\epsilon_\ell/1-2\nu$ i.e., in incipient locking. Therefore, the plus sign of the radical in Eq.(5.30c) gives a value of c_3 higher than $\bar{c}$. This is not possible because $\bar{c}$ corresponds to the slope of the straight line AB in Fig.5.1 or Fig.5.4, while c_3 is the slope of one of the secants such as DE or OC. Then the correct c_3 corresponds to the minus sign of the radical in Eq.(5.30c).

If in particular

$$\sigma_1 = \frac{E\epsilon_\ell}{1-2\nu}$$

c_3 becomes equal to $\sqrt{3G/\rho}$ because $\alpha = 0$. Then

$$\sigma_3 = v_1(c\rho + \sqrt{3G\rho}) .$$

(vii) Interaction of Two Waves Moving in the Same Direction

Now, let us consider two waves A and B as shown in Fig.5.8a and moving in the same direction. The wave B propagates into the undisturbed, stress free bar. It carries a stress discontinuity

$$\sigma_1 = \frac{E\epsilon_\ell}{1-2\nu}$$

and a velocity discontinuity v_1.

The wave A is assumed to carry a stress discontinuity of the magnitude $\sigma_2 - \sigma_1$ and to move with a velocity c_1. The value of σ_2 is assumed to be greater than σ_1. Then one can write the following relationships between stresses and velocities (see Eq.(5.10))

$$\sigma_1 = \rho c v_1$$

$$\sigma_2 - \sigma_1 = \rho c_1(v_2 - v_1)$$

where c is the elastic wave speed and c_1 can be calculated from the formula (5.22).

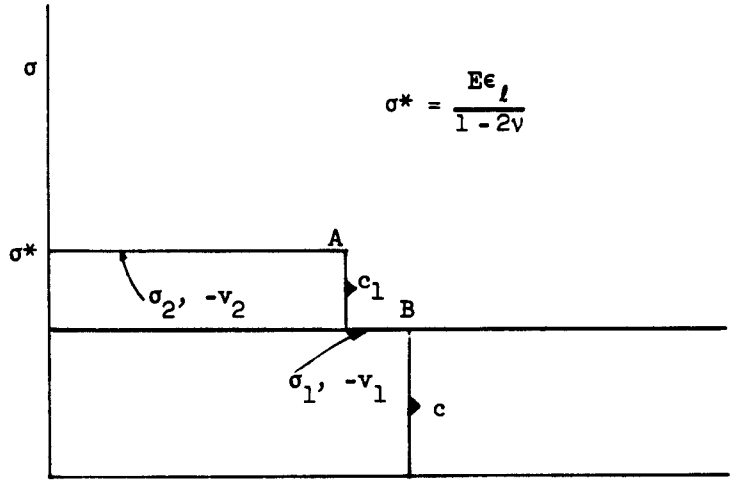


Fig. 5.8a.

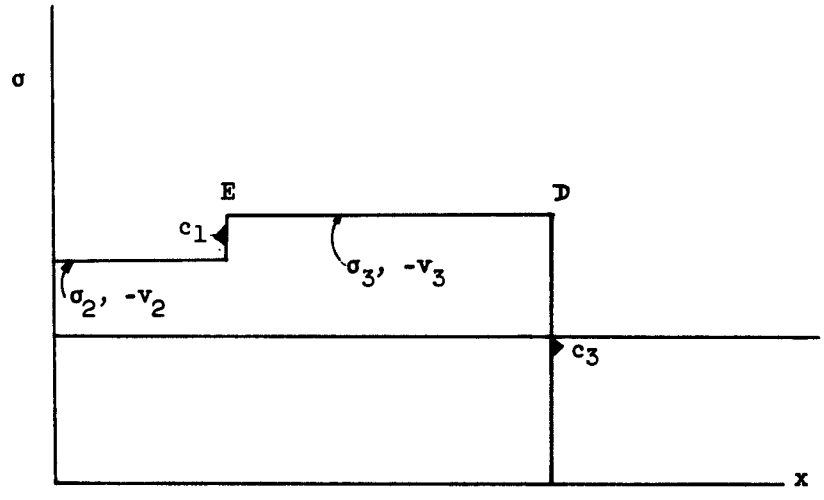


Fig. 5.8b.

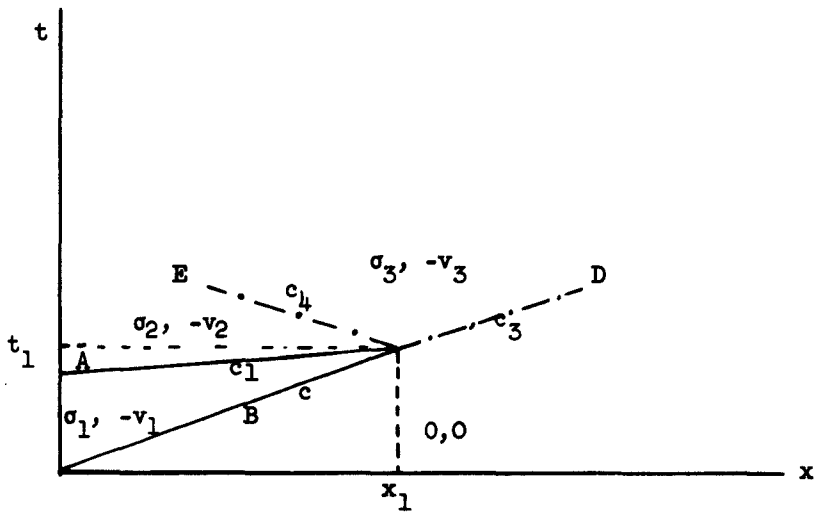


Fig. 5.8c.

i.e.,

$$c_1 = \sqrt{\frac{3G}{\rho}} = \bar{c}$$

$$c = \sqrt{\frac{E}{\rho}}$$

The x,t diagram is as shown in Fig.5.8c. The wave A meets the wave B at a certain point $x = x_1$, $t = t_1$. Then two waves D and E emerge from the point. The stress discontinuities $\sigma_3 - \sigma_2$ and σ_3 carried by these waves will now be calculated.

By applying Eq.(5.10) at the wave fronts D and E, one can write

$$\left.\begin{aligned} \sigma_3 &= \rho c_3 v_3 \\ \sigma_3 - \sigma_2 &= \rho c_4 (v_2 - v_3) \end{aligned}\right\} \qquad (5.31)$$

Further, from the equation (5.22)

$$c_3^2 = \frac{1}{\rho} \frac{\sigma_3}{\frac{\sigma_3}{3G} + \frac{\epsilon_\ell}{3}}$$

$$c_4^2 = \frac{1}{\rho} \frac{\sigma_2 - \sigma_3}{(\sigma_2 - \sigma_3)\frac{1}{3G}} = \frac{3G}{\rho} = c_1^2 = c^2 \qquad (5.32)$$

Eqs.(5.31) and (5.32) can be rewritten as

$$\sigma_3 = \rho c_3 v_3 \qquad (5.33a)$$

$$\sigma_3 - \sigma_2 = (v_2 - v_3)\sqrt{3G\rho} \qquad (5.33b)$$

$$c_3^2 = \frac{1}{\rho} \frac{\sigma_3 3G}{\sigma_3 + G\epsilon_\ell} \qquad (5.33c)$$

Eqs.(5.33) are three equations for three unknowns σ_3, v_3 and c_3, which can then be determined. Then we know the stress discontinuity, the velocity discontinuity and the speed of the wave after intersection.

From Eqs.(5.33a) and (5.33c)

$$c_3^2 = \frac{\sigma_3^2}{\rho^2 v_3^2} = \frac{1}{\rho} \frac{\sigma_3 3G}{\sigma_3 + G\epsilon_\ell}$$

i.e.,

$$\sigma_3 = \frac{3G\rho v_3^2}{\sigma_3 + G\epsilon_\ell}$$

From Eq.(5.33b) and this equation

$$\sigma_3^2 + \sigma_3 G\epsilon_\ell = (\sigma_3 - \sigma_2 - \sqrt{3G\rho}\, v_2)^2$$

i.e.,

$$\sigma_3 = \frac{\sigma_2^2 + 3G\rho v_2^2 + 2v_2\sigma_2\sqrt{3G\rho}}{G\epsilon_\ell + 2\sigma_2 + 2v_2\sqrt{3G\rho}}$$

The right-hand side of this equation contains known quantities. Hence, σ_3 can be calculated. Then c_3 and v_3 can be obtained from Eqs.(5.33c) and (5.33a).

v.2 Spherical Wave Propagation under Volumetric Locking

(i) Equations of Motion and the Speed of the Wave

In this section the dynamic response problem of an infinite, homogeneous, isotropic and volumetrically locking medium will be studied under spherically symmetric conditions. One can write the following equations governing the problem. In these equations $\bar{\sigma}_r$, $\bar{\epsilon}_r$, $\bar{u}$ are the normal stress, normal strain and displacement in the radial direction r. $\bar{\sigma}_\theta$, $\bar{\epsilon}_\theta$ are the normal stress and normal strain in any direction in the plane perpendicular to r-direction.

The following equations govern the motion:

a) Newton's Law

$$\frac{\partial \bar{\sigma}_r}{\partial r} + \frac{2\bar{\sigma}_r - \bar{\sigma}_\theta}{r} = \rho \frac{\partial^2 \bar{u}}{\partial t^2} \tag{5.34}$$

b) The Stress-Strain Law: The material is assumed to be capable of non-ideal locking. The results for ideal locking will be derived as a limiting case. From Eq.(2.36) and distortional elasticity

$$\bar{\sigma}_r + 2\bar{\sigma}_\theta = 3K\epsilon_\ell + \lambda(\bar{\epsilon}_r + 2\bar{\epsilon}_\theta - \epsilon_\ell) \quad , \quad \bar{\epsilon}_r + 2\bar{\epsilon}_\theta \geq \epsilon_\ell$$

$$\bar{\sigma}_r - \bar{\sigma}_\theta = 2G(\bar{\epsilon}_r - \bar{\epsilon}_\theta) \tag{5.35a}$$

These equations can be solved for $\bar{\sigma}_r$ and $\bar{\sigma}_\theta$

$$\bar{\sigma}_r = \epsilon_\ell\left(k - \frac{\lambda}{3}\right) + \frac{\lambda + 4G}{3}\,\bar{\epsilon}_r + \frac{2\lambda - 4G}{3}\,\bar{\epsilon}_\theta$$

$$\bar{\sigma}_\theta = \epsilon_\ell\left(k - \frac{\lambda}{3}\right) + \frac{\lambda - 2G}{3}\,\bar{\epsilon}_r + \frac{2\lambda + 2G}{3}\,\bar{\epsilon}_\theta \tag{5.35b}$$

c) Kinematics

$$\epsilon_r = \frac{\partial \bar{u}}{\partial r}, \qquad \epsilon_\theta = \frac{\bar{u}}{r} \tag{5.36}$$

From Eqs.(5.34) to (5.36) the following differential equation can be derived

$$\frac{\partial^2 \bar{u}}{\partial r^2} + \frac{2}{r}\frac{\partial \bar{u}}{\partial r} - 2\,\frac{\bar{u}}{r^2} = \frac{3\rho}{\lambda + 4G}\,\frac{\partial^2 \bar{u}}{\partial t^2} \tag{5.37}$$

If one introduces a potential ψ such that

$$u = \frac{\partial \psi}{\partial r}$$

Eq.(5.37) can be rewritten in the following way

$$c_1^2\,\frac{\partial}{\partial r}\left[\frac{1}{r^2}\,\frac{\partial}{\partial r}\left(r^2\,\frac{\partial \psi}{\partial r}\right)\right] = \frac{\partial}{\partial r}\,\frac{\partial^2 \psi}{\partial t^2}$$

i.e.,

$$c_1^2\,\nabla^2 \psi = \frac{\partial^2 \psi}{\partial t^2} \tag{5.38}$$

where

$$c_1^2 = \frac{\lambda + 4G}{3\rho}$$

and

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r}\right)$$

is the Laplacean operator under spherically symmetric conditions.

Now, one can see that Eq.(5.38) represents a wave equation with the wave speed

$$c_1 = \sqrt{\frac{\lambda + 4G}{3\rho}} \tag{5.39}$$

One must remember that the speed of the wave is given by this equation only if the wave is propagating in a medium which is already locked.

If the material is an ideally locking material, λ becomes infinity. Then the speed of the wave becomes infinity.

Alternately, one can derive the equations governing the problem by considering the discontinuities at the wave front. By applying the impulse-momentum theorem one can arrive at the following relationship (Fig.5.9):

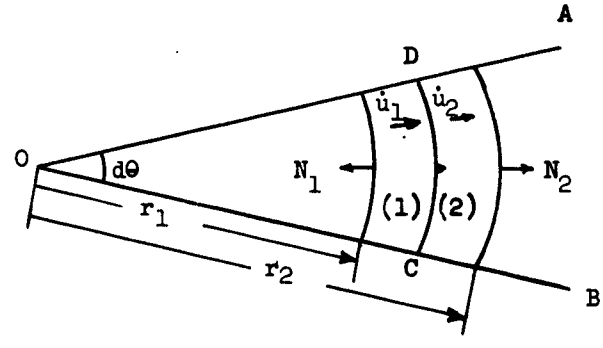


Fig. 5.9.

(In this figure AOB is a cone with vertex at the origin, and vertex angle being equal to $d\theta$. The wave CD sweeps through the radial distance $r_2 - r_1 = dr$ in time dt .)

$$(N_2 - N_1)dt = \rho\pi \frac{r_1^2 + r_2^2}{2} d\theta^2 dr(\dot{u}_1 - \dot{u}_2) \tag{5.40}$$

where N_1 and N_2 are the resultant forces and u_1 and u_2 are the radial velocities on sides (1) and (2) of the element shown in Fig.5.9. Further,

$$N_1 = \sigma_{r1}\pi r_1^2 d\theta^2$$

$$N_2 = \sigma_{r2}\pi r_2^2 d\theta^2$$

Then Eq.(5.40) can be rewritten as

$$(\sigma_{r2} - \sigma_{r1})\, r_1^2 d\theta^2 dt + \sigma_{r2}\pi(2r_1 dr d\theta^2 + dr^2 d\theta^2)dt$$
$$= (\dot{u}_1 - \dot{u}_2)\rho\pi r_1^2 d\theta^2 dr(\dot{u}_1 - \dot{u}_2)$$
$$+ \rho\pi d\theta^2 dr^2 (r_1 + \tfrac{dr}{2})(\dot{u}_1 - \dot{u}_2)$$

By dividing this equation by $\pi r_1^2 d\theta^2 dt$ and by considering the limit as dr and $d\theta$ tend to zero, this equation becomes

$$\sigma_{r2} - \sigma_{r1} = \rho c(\dot{u}_1 - \dot{u}_2) \tag{5.41}$$

where

$$c = \frac{dr}{dt}$$

Further, the displacements are continuous at the wave front. Then

$$u_1 - u_2 = 0$$

By differentiating this equation along the wave front in r-direction one can write

$$(u_1' - u_2') + \frac{1}{c}(\dot{u}_1 - \dot{u}_2) = 0$$

where $(\,)'$ denotes partial differentiation with respect to r and $(\dot{})$ denotes partial differentiation with respect to t. Then, from Eq.(5.41) one has

$$c^2 = \frac{1}{\rho}\,\frac{\sigma_{r2} - \sigma_{r1}}{\epsilon_{r2} - \epsilon_{r1}} \tag{5.42}$$

if both the sides (1) and (2) are in the locked region and the locking is nonideal. One can derive the following value of the wave speed from Eq.(5.42), the stress-strain relationship (5.35) and the displacement continuity relationship, namely:

$$c = \sqrt{\frac{\lambda + 4G}{3\rho}}$$

This is the same value as that obtained in Eq.(5.39). Further, as in the case of one-dimensional wave propagation, when region (2) is elastic and region (1) is locked, it can be shown that the loading waves catch up with each other and move as a shock wave with finite amplitude, while the unloading waves fall apart. Then that wave which brings the material to a state of incipient unloading propagates with the speed just given, and it is followed by an elastic wave of speed $\sqrt{\lambda e + 2G}/\rho$.

(ii) Infinite Medium with Pressure Suddenly Applied at the Edge of a Spherical Cavity

In this section we shall consider an infinite medium (made of a volumetrically locking material) with a spherical cavity of radius a as indicated in Fig. 5.10. The stresses, strains and displacements in the medium due to a pressure p suddenly applied at the cavity surface will be studied. Further, it is assumed that the pressure p is maintained at the cavity surface for all times, i.e.,

$$\bar{\sigma}_r\Big|_{r=a} = -p \quad \text{for} \quad t > 0 .$$

When the pressure p is suddenly applied at the cavity surface a wave starts spreading into the medium. At a certain time t this wave front will be at a certain radius $r = k(t)$. Then, the region $a \leq r \leq k(t)$ will be a volumetrically locked region if

(a) the pressure p is of sufficient magnitude to cause volumetric locking,

(b) the strains in the region $0 \leq r \leq k(t)$ are nowhere of a magnitude which would cause unlocking.

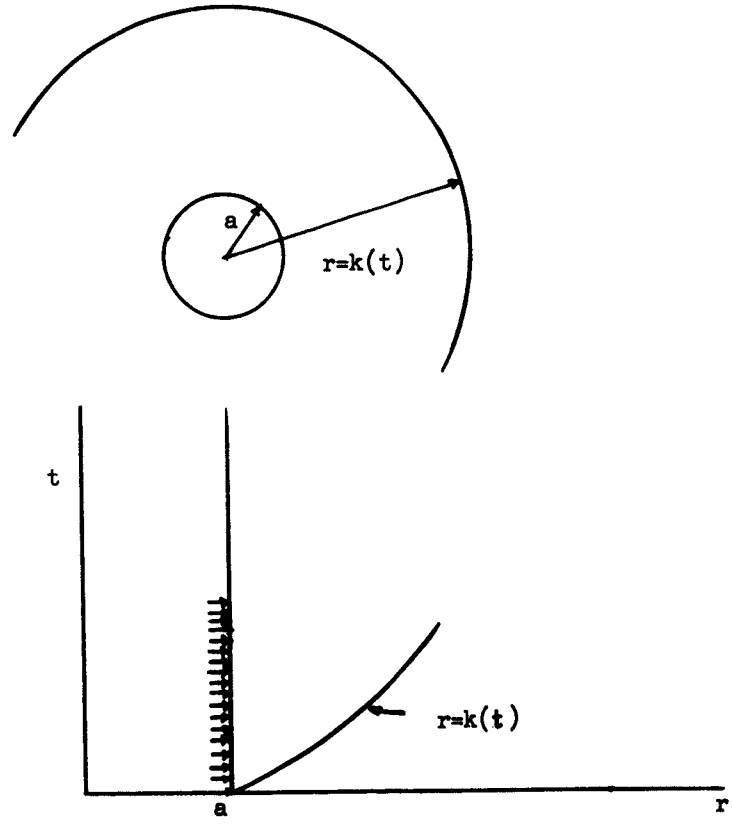


Fig. 5.10.

The preceding requirements will be checked later.

The stresses, strains and displacements in the region $0 \leq r \leq k(t)$ are governed by the following equations; (equation of motion, distortional elasticity, locking condition and kinematics)

$$\frac{\partial \bar{\sigma}_r}{\partial r} + 2\,\frac{\bar{\sigma}_r - \bar{\sigma}_\theta}{r} = \rho\,\frac{\partial \bar{u}}{\partial t^2} \tag{5.43}$$

$$\bar{\sigma}_r - \bar{\sigma}_\theta = 2G(\bar{\epsilon}_r - \bar{\epsilon}_\theta) \tag{5.44}$$

$$\bar{\epsilon}_r + 2\bar{\epsilon}_\theta = -\epsilon_\ell \tag{5.45}$$

$$\begin{aligned} \bar{\epsilon}_r &= \frac{\partial \bar{u}}{\partial r} \\ \bar{\epsilon}_\theta &= \frac{\bar{u}}{r} \end{aligned} \tag{5.46}$$

The locking is assumed to be ideal. Further, it is assumed to take place in compression only. Then the locking condition (5.45) holds throughout the locked region. Eqs.(5.45) and (5.46) then yield the following differential equation for u.

$$\frac{\partial \bar{u}}{\partial r} + \frac{2\bar{u}}{r} + \epsilon_\ell = 0$$

with the general solution

$$\bar{u} = \frac{f(t)}{r^2} - \frac{\epsilon_\ell}{3}\,r \tag{5.47}$$

where $f(t)$ is an arbitrary function of time. Then,

$$\bar{\epsilon}_r = -\frac{2f(t)}{r^3} - \frac{\epsilon_\ell}{3} \tag{5.48a}$$

$$\bar{\epsilon}_\theta = \frac{f(t)}{r^3} - \frac{\epsilon_\ell}{3} \tag{5.48b}$$

Then, from distortional elasticity one obtains the following relationship

$$\bar{\sigma}_r - \bar{\sigma}_\theta = -6G \frac{f(t)}{r^3} \tag{5.49}$$

The equilibrium equation and this equation then yield the following differential equation for σ_r

$$\frac{\partial \bar{\sigma}_r}{\partial r} = 12G \frac{f(t)}{r^4} + \rho \frac{\ddot{f}(t)}{r^2}$$

A simple integration yields the following expression for the stress:

$$\bar{\sigma}_r = -4G \frac{f(t)}{r^3} - \rho \frac{\ddot{f}(t)}{r} + g(t) \tag{5.49a}$$

Eqs.(5.47), (5.48) and (5.49) constitute the general solution in the region $0 \leq r \leq k(t)$. The solution is determined if the functions $f(t)$, $g(t)$ and $k(t)$ are determined. Therefore, the investigation in the next section will be concerned with determining these functions.

(iii) Functions $f(t)$, $g(t)$ and $k(t)$

For $t > 0$ the boundary condition at the cavity surface $r = a$ is

$$\bar{\sigma}_r = -p \tag{5.50}$$

At the wave front the displacements are continuous. Further, it is assumed that, for the time $t < 0$, the infinite medium is at rest and is stress free. The pressure p is applied at time $t = 0$. Then the region ahead of the wave front $r = k(t)$ is at rest and is stress free, i.e., at $r = k(t)$

$$\bar{u} = 0 \tag{5.51}$$

and hence

$$\bar{\epsilon}_\theta = 0$$

Further, the region $0 \leq r \leq k(t)$ is in a state of ideal locking and, therefore at $r = k(t)$

$$\bar{\epsilon}_r + 2\bar{\epsilon}_\theta = -\epsilon_\ell$$

whence

$$\bar{\epsilon}_r = -\epsilon_\ell \tag{5.52}$$

Then, from Eq. (5.42)

$$c^2 = \left(\frac{dk}{dr}\right)^2 = \frac{1}{\rho} \frac{(\bar{\sigma}_r)_{r=k(t)}}{-\epsilon_\ell}$$

i.e.,

$$(\bar{\sigma}_r)_{r=k(t)} = -\rho\, \epsilon_\ell \left(\frac{dk}{dt}\right)^2 \tag{5.53}$$

The conditions (5.50), (5.51) and (5.53) will be used to determine the functions $f(t)$, $g(t)$, and $k(t)$. From Eqs. (5.50) and (5.49), one has

$$g(t) = \frac{4Gf(t)}{a^3} + \rho \frac{\ddot{f}(t)}{a} - p \tag{5.53a}$$

Then

$$\bar{\sigma}_r = 4Gf(t)\left(\frac{1}{a^3} - \frac{1}{r^3}\right) + \rho\ddot{f}\left(\frac{1}{a} - \frac{1}{r}\right) - p \tag{5.54}$$

From Eqs. (5.51) and (5.47) one obtains

$$f = \frac{1}{3}\epsilon_\ell k^3 \tag{5.56a}$$

Then,

$$\bar{u} = \frac{\epsilon_\ell}{3}\left(\frac{k^3(t)}{r^2} - r\right) \tag{5.56b}$$

$$\bar{\epsilon}_r = -\frac{\epsilon_\ell}{3} - \frac{2}{3}\epsilon_\ell \frac{k^3(t)}{r^3} \tag{5.56c}$$

$$\bar{\epsilon}_\theta = -\frac{\epsilon_\ell}{3} + \frac{1}{3}\epsilon_\ell \frac{k^3(t)}{r^3} \tag{5.56d}$$

$$\bar{\sigma}_r = \frac{4G}{3}\epsilon_\ell k^3(t)\left(\frac{1}{a^3} - \frac{1}{r^3}\right) + \rho\epsilon_\ell\left(\frac{1}{a} - \frac{1}{r}\right)(k^2\ddot{k} + 2k\dot{k}^2) - p \tag{5.56e}$$

Then from Eqs. (5.53) and (5.56e) the following differential equation for $k(t)$ can be obtained:

$$k\left(\frac{k}{a} - 1\right)\ddot{k} + \left(2\frac{k}{a} - 1\right)\dot{k}^2 + \frac{4G}{3\rho}\left(\frac{k^3}{a^3} - 1\right) - \frac{p}{\rho\epsilon_\ell} = 0 \tag{5.57}$$

If the function $k(t)$ is obtained by integrating this equation, the functions $f(t)$ and $g(t)$ can be determined from Eqs. (5.56a) and (5.53a). The initial conditions for the differential equation (5.57) are

$$k(0) = a$$

$$-\rho\epsilon_\ell(\dot{k}^2)_{t=0} = [\sigma_r]_{t=0,r=a} = -p \tag{5.58}$$

Before proceeding to integrate the differential equation, the condition under which the region $0 \leq r \leq k(t)$ could be a locked region will be investigated in the next sub-section.

(iv) <u>The Conditions Under Which the Locked Region $0 < r < k(t)$ Exists</u>

The region $0 \leq r \leq k(t)$ can be a locked region if the stresses satisfy the inequality

$$-(\bar{\sigma}_r + 2\bar{\sigma}_\theta) \geq 3K\epsilon_\ell \tag{5.59}$$

where $3K$ is the bulk modulus

$$K = \frac{E}{3(1 - 2\nu)}$$

From the distortional elasticity σ_θ is related to σ_r (see Eq. 5.49). Then

$$\bar{\sigma}_\theta = \bar{\sigma}_r - 2G(\epsilon_r - \epsilon_\theta) = \sigma_r + \frac{6Gf(t)}{r^3}$$

Then the inequality (5.59) can be written in the following way (see Eq. 5.54)

$$3p - 3\rho\ddot{f}\left(\frac{1}{a} - \frac{1}{r}\right) - 12G\frac{f}{a^3} \geq 3K\epsilon_\ell \tag{5.60}$$

In order that the locked region $a \leq r \leq k(t)$ does exist, the inequality must hold at least for $r = k = a$, $t = 0$, i.e.,

$$p \geq \frac{3K + 4G}{3}\epsilon_\ell \tag{5.60a}$$

for $\nu = 0.25$, $3K = 5G$. Then this inequality becomes $p/G\epsilon_\ell \geq 3$. Also, from the inequality (5.60), one can see that the material will unlock if, at any place,

$$3p - 3\rho\ddot{f}\left(\frac{1}{a} - \frac{1}{r}\right) - \frac{12Gf}{a^3} = K\epsilon_\ell \tag{5.60b}$$

This will occur first where r is maximum, i.e., at $r = k(t)$.

Further, from Eq. (5.42) and the incipient locking conditions, the following value for dk/dt at the instant of unlocking can be derived

$$\frac{dk}{dt} = \sqrt{\frac{\sigma_{r2} - \sigma_{r1}}{\epsilon_{r2} - \epsilon_{r1}}}\,\frac{1}{\rho}$$

Material ahead of the wave is at rest and stressfree. Then $\sigma_{r2} = \epsilon_{r2} = 0$. Therefore

$$\frac{dk}{dt} = \sqrt{\frac{\bar{\sigma}_{r1}}{\rho\bar{\epsilon}_{r1}}}$$

At the instant of unlocking σ_{r1} and ϵ_{r1} are related by the elastic law. Then

$$\frac{dk}{dt} = \sqrt{\frac{1}{\rho} \frac{\lambda_e(\epsilon_{r1} + 2\epsilon_{\theta 1}) + 2G\epsilon_{r1}}{\epsilon_{r1}}}$$

where λ_e is Lamé's constant. Also at $r = k(t)$

$$\epsilon_{\theta 1} = \frac{u_1}{r} = 0$$

Then

$$\frac{dk}{dt} = \sqrt{\frac{\lambda_e + 2G}{\rho}}$$

which is the speed of the elastic irrotational wave.

(v) Solution of the Differential Equation (5.57)

Equation (5.57) is an ordinary, nonlinear differential equation for $k(t)$. It may be solved by numerical integration. However, before any numerical procedure can be started, it is necessary to clarify the behavior of the solution at $k = a$. This will be done first, and then Milne's method will be used for numerical integration. Also, to prepare the equation for numerical work it is non-dimensionalized by introducing the following new variables

$$\frac{k}{a} = q \qquad t = \frac{3\rho a^2}{4G} T$$

In this notation, Eq. (5.57) reads as follows.

$$q(q - 1) \frac{d^2q}{dT^2} + (2q - 1) \left(\frac{dq}{dT}\right)^2 + (q^3 - 1) - \alpha^2 = 0 \qquad (5.61)$$

where

$$\alpha^2 = \frac{3}{4} \frac{p}{G\epsilon_l}$$

The initial conditions for this differential equation are

$$T = 0 \qquad q = \frac{k}{a} = 1 \tag{5.62}$$

and (see Eqs. 5.53, 5.58, and 5.62)

$$T = 0 \qquad \frac{dq}{dT} = \sqrt{\frac{3\rho a^2}{4G\epsilon_\ell}}\,\frac{1}{a}\,\frac{dk}{dt} = \alpha \tag{5.63}$$

The power series solution is assumed in the following form.

$$q = 1 + \alpha T + b_2 T^2 + b_3 T^3 + b_4 T^4 + b_5 T^5 \tag{5.64}$$

Because this power series will be used to evaluate the values of q and its derivatives for $T < 0.003$, it is reasonable to expect that the first few terms of the series should be sufficient to evaluate d^2q/dt^2 to the accuracy of three decimal places. Therefore, the coefficients of T^6 and higher powers are neglected in the series.

Then by keeping only terms up to the fifth and lower powers of T, one can evaluate the individual terms of the differential equation (5.61). By equating like powers of T the coefficients $b_2 \cdots b_5$ can be calculated.

For a particular value of $p/G\epsilon_\ell = 4$ the coefficients are as follows (From Eq. 5.60a, for $\nu = 0.25$, one can see that the value of $p/G\epsilon_\ell$ creates a locked region.)

$$b_1 = 3$$

$$b_2 = -\frac{3}{2}$$

$$b_3 = \sqrt{3}$$

$$b_4 = -3.6$$

$$b_5 = 8.09$$

The expression for q can then be written in the following way

$$q = 1 + 3T - \frac{3}{2}T^2 + \sqrt{3}\,T^3 - 3.6T^4 + 8.09\,T^5$$

This power series has been used to start the numerical integration. The typical numerical solutions for $p/G\epsilon_l = 4$ is shown in the following table. Unlocking takes place at $q = 1.142$ and $T = 0.088$. The stresses, strains and displacements have been plotted in Fig. 5.12. Fig. 5.11 shows the position of wave front and wave velocity.

T	$\dot{q}$	$\ddot{q}$	q
0.0	-3.0	1.73206	1.0
0.01	-2.899	1.70257	1.01717
0.02	-2.809	1.67404	1.03405
0.03	-2.714	1.64649	1.05066
0.04	-2.650	1.61962	1.06700
0.05	-2.574	1.59353	1.08305
0.06	-2.505	1.56813	1.09887
0.07	-2.441	1.54341	1.11441
0.08	-2.381	1.51930	1.12974
			unlocks*
0.09	-2.328	1.49577	1.14480

*Unlocks at $q = 1.142$ $\dot{q} = 1.5$

In the next sub-section the problem after unlocking will be studied.

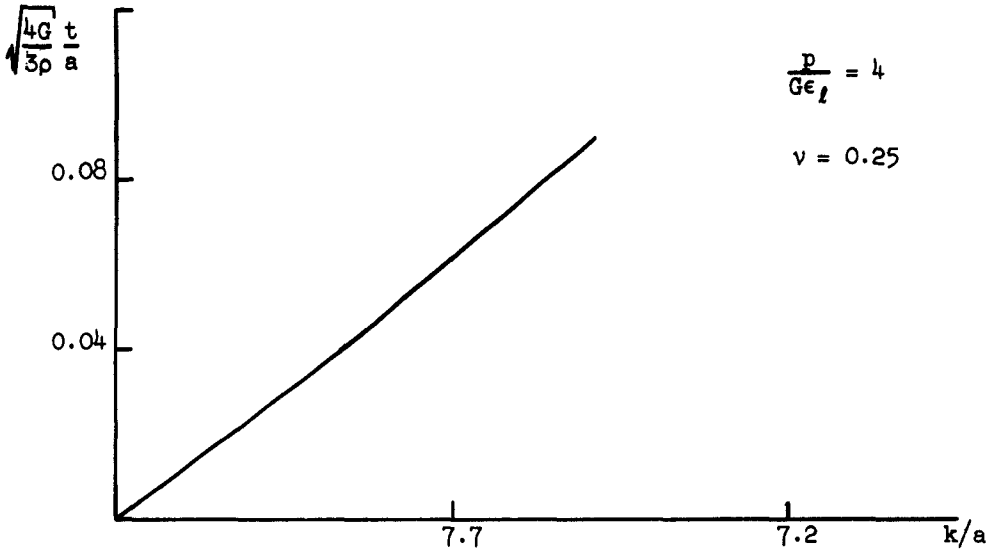


Fig. 5.11a. Position of Wave Front.

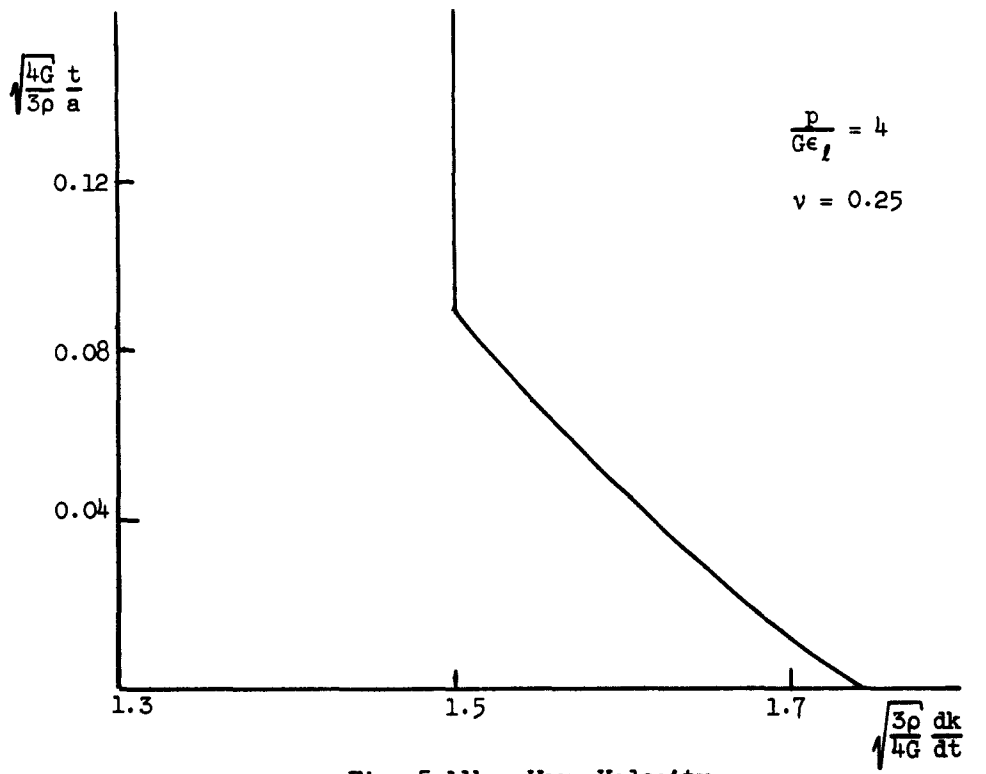


Fig. 5.11b. Wave Velocity.

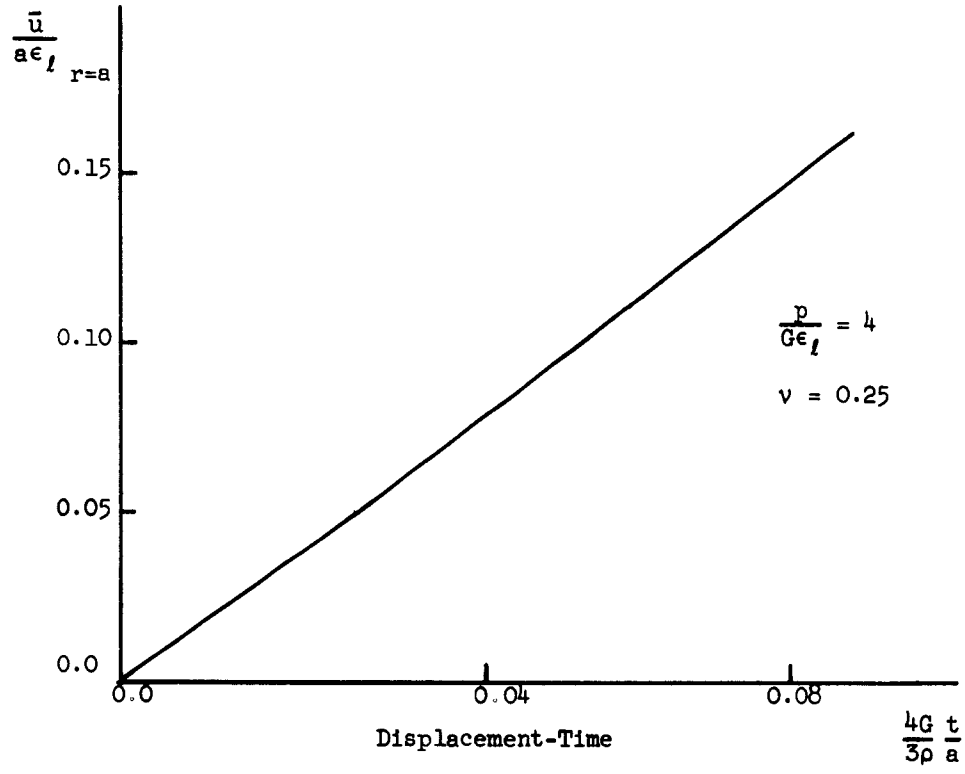

$\frac{\bar{u}}{a\epsilon_\ell}_{r=a}$
0.15
0.10
0.05
0.0
0.0
0.04
0.08
$\frac{p}{G\epsilon_\ell} = 4$
$\nu = 0.25$
Displacement-Time
$\frac{4G}{3\rho}\frac{t}{a}$

Fig. 5.12a.

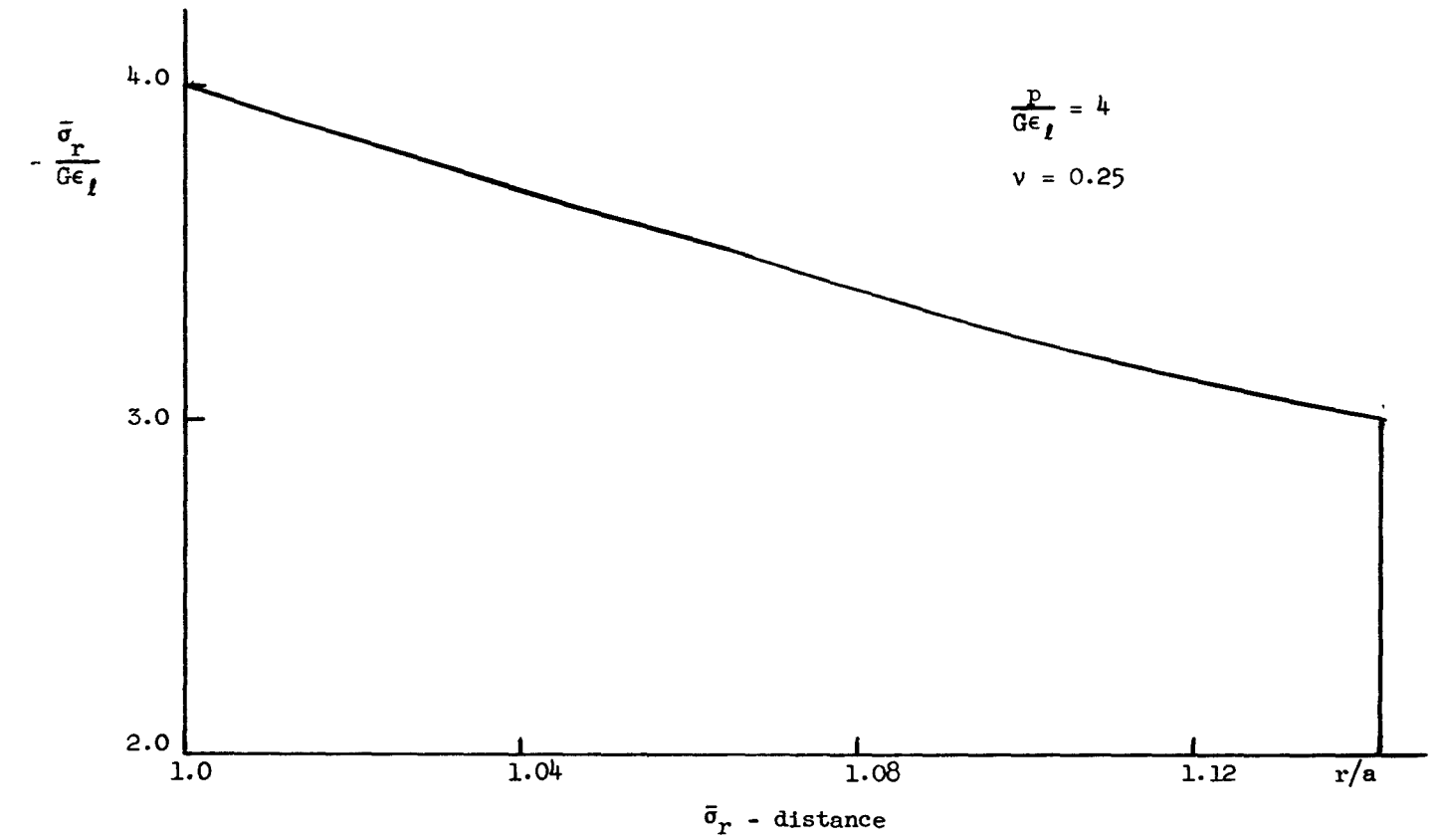

4.0
3.0
2.0
$\frac{\bar{\sigma}_r}{G\epsilon_l}$
$\frac{p}{G\epsilon_l} = 4$
$\nu = 0.25$
1.0
1.04
1.08
1.12
r/a
$\bar{\sigma}_r$ - distance

Fig. 5.12b.

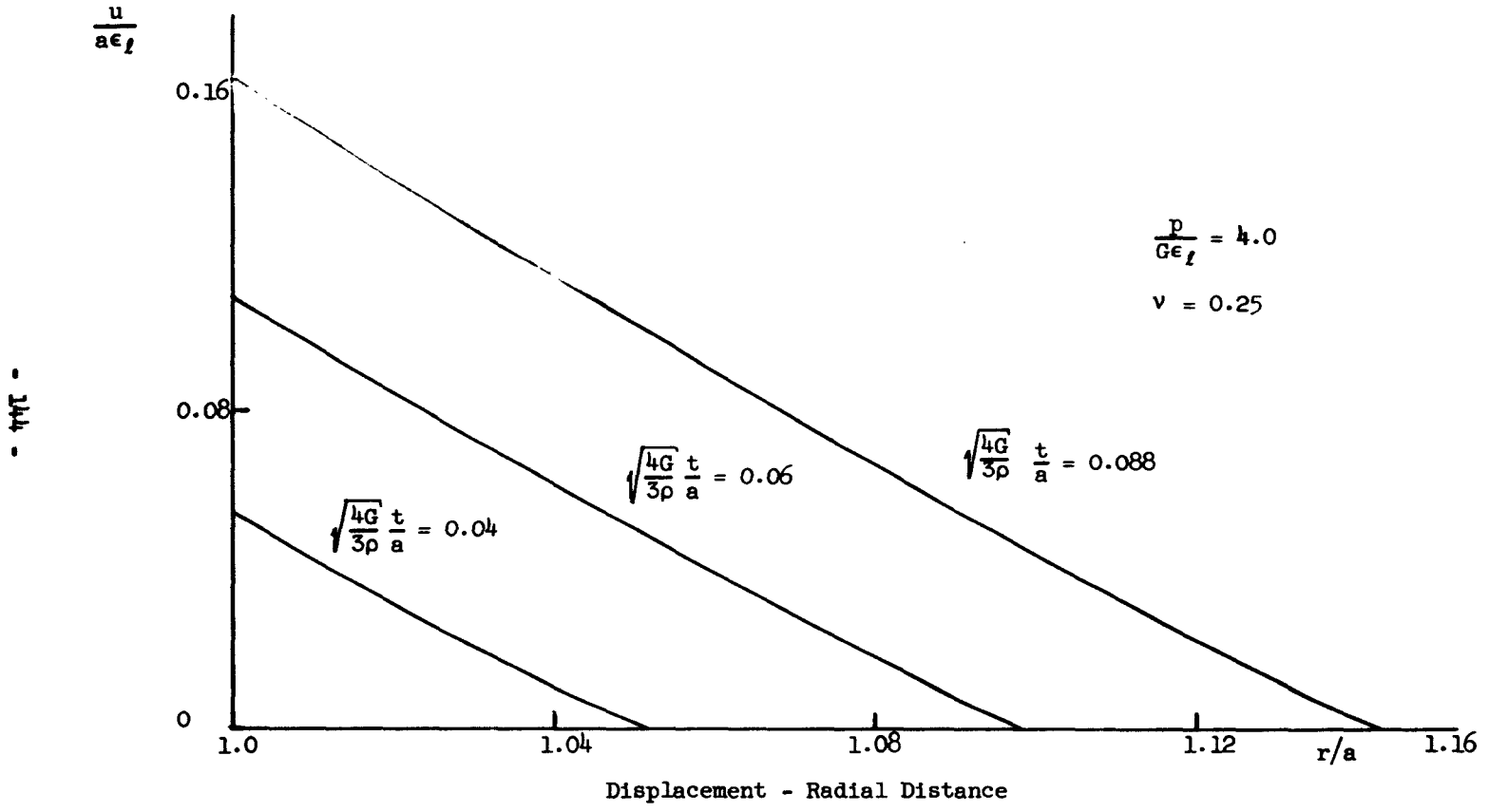

u / aε_ℓ
0.16
0.08
0
1.0
1.04
1.08
1.12
1.16
r/a
p / Gε_ℓ = 4.0
ν = 0.25
√(4G/3ρ) t/a = 0.088
√(4G/3ρ) t/a = 0.06
√(4G/3ρ) t/a = 0.04
Displacement - Radial Distance

Fig. 5.12c.

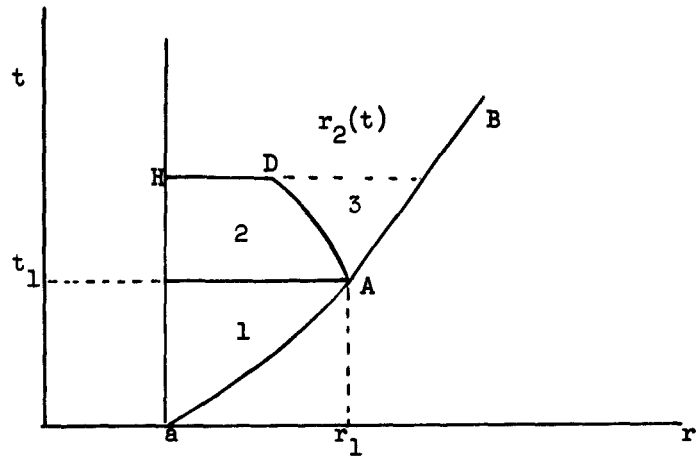


Fig. 5.13.

vith Problem After Unlocking

It has been shown (see p. 140) that unlocking begins at the wave front at point A (Fig. 5.13) where $k(t_1) = r_1$. When the wave has reached this point, its velocity has the value c of elastic waves, as everywhere on the wave front $u = 0$, and the stress at A is $\sigma_r = \rho c^2 \epsilon_r$. From here on the wave continues as an elastic wave going outward, as shown in Fig. 5.12.

The region behind the elastic wave may contain a locked region (2) and an elastic region (3), as indicated in the figure. The wave front AB is a straight line, its slope corresponds to the elastic wave speed c. The problem is now to obtain the stresses, strains and displacements in the elastic and locked regions such that they match on the interface AD which is still unknown. Further, the solution must satisfy the appropriate boundary and initial conditions.

The general solution in the locked region is still given by Eqs. (5.47), (5.48) and (5.49). The general solution in the elastic region will be derived in the next section.

(vii) General Solution in the Elastic Region

The dynamic response in the elastic region is governed by the following equations:

(a) Equation of Motion (see Eq.(5.34)

$$\frac{\partial \sigma_r}{\partial r} + 2\,\frac{\sigma_r - \sigma_\theta}{r} = \rho\,\frac{\partial^2 u}{\partial t^2}$$

(b) Stress-Strain Relationship

$$\sigma_r = \lambda_e(\epsilon_r + 2\epsilon_\theta) + 2G\epsilon_r$$

$$\sigma_\theta = \lambda_e(\epsilon_r + 2\epsilon_\theta) + 2G\epsilon_\theta$$

where λ_e is Lamé's constant and $2G$ is the shear modulus of the material.

(c) Kinematics

$$\epsilon_r = \frac{\partial u}{\partial r}\ ,\quad \epsilon_\theta = \frac{u}{r}$$

From these equations the following differential equation in u can be derived

$$\frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} - \frac{2}{r^2}u = \frac{\rho}{\lambda_e + 2G}\frac{\partial^2 u}{\partial t^2}$$

As explained in p.128, this equation can be rewritten in the following form by introducing the potential function ψ such that $u = \partial\psi/\partial r$:

$$c^2\,\frac{\partial}{\partial r}\left[\frac{1}{r^2}\,\frac{\partial}{\partial r}\left(r^2\,\frac{\partial \psi}{\partial r}\right)\right] = \frac{\partial}{\partial r}\,\frac{\partial^2 \psi}{\partial t^2}$$

where

$$c = \sqrt{\frac{\lambda_e + 2G}{\rho}}$$

Integration of this equation yields

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\ \frac{\partial \psi}{\partial r}\right) = \frac{1}{c^2}\frac{\partial^2 \psi}{\partial t^2} + C_1$$

where C_1 is a constant. This equation can also be written as

$$\left(\frac{\partial^2}{\partial r^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)(r\psi) = C_1 r \tag{5.66}$$

The general solution of this equation is

$$r\psi = f_1(r+ct) + f_2(r-ct) + \frac{C_1 r^3}{6} \tag{5.67}$$

Once ψ is known, the displacement u and hence the stresses and strains can be calculated.

We are seeking solution in a region which extends to infinity. Thus, there are no incoming waves. Further the elastic waves start at time $t = t_1$, and at radius $r = r_1$. Therefore, the potential function ψ in the elastic region can be written in the following form:

$$\psi = \frac{1}{r} f_3\left(t - t_1 - \frac{r - r_1}{c}\right) + \frac{C_1 r^2}{6} \tag{5.68}$$

In the further analysis the abbreviation $\bar{t} = t - t_1$ will be used. Then

$$u = -\frac{1}{rc} f_3'\left(\bar{t} - \frac{r - r_1}{c}\right) - \frac{1}{r^2} f_3\left(\bar{t} - \frac{r - r_1}{c}\right) + \frac{C_1 r}{3} \tag{5.69}$$

where $(\ )'$ denotes differentiation with respect to the argument of the function.

(viii) Boundary and Interface Conditions

The general solutions in the locked and elastic regions are given by Eqs.(5.47) to (5.49) and (5.69). They contain arbitrary functions $f(\bar{t})$, $g(\bar{t})$ and f_3. These functions and the unknown radius

$r = r_2(\bar{t})$ of the interface can be determined from the following conditions.

(a) At the wave front the displacement is zero; i.e., at $r = r_1 + c\bar{t}$:

$$u = 0 \tag{5.70}$$

(b) At the surface of the cavity, the radial stress is $-p$; i.e., at $r = a$:

$$\bar{\sigma}_r = -p \tag{5.71}$$

(c) At the interface the radial stress, the particle velocity and the displacement are continuous; i.e., at $r = r_2(\bar{t})$:

$$u = \bar{u} \tag{5.72a}$$

$$\sigma_r = \bar{\sigma}_r \tag{5.72b}$$

$$\dot{u} = \dot{\bar{u}} \tag{5.72c}$$

Eqs.(5.72b) and (5.72c) are not independent of each other because the continuity of the radial stress implies the continuity of the particle velocity across the interface.

(d) Further, the stresses in the locked region are in a state of incipient locking at the interface; i.e., at $r = r_2(\bar{t})$

$$\bar{\sigma}_r + 2\bar{\sigma}_\theta = -3K\epsilon_\ell \tag{5.73}$$

From this equation and the condition that the strains in the locked region satisfy the locking condition (5.45), we have at $r = r_2(\bar{t})$:

$$\bar{\sigma}_r + 2\bar{\sigma}_\theta = 3K(\bar{\epsilon}_r + 2\bar{\epsilon}_\theta) = -3K\epsilon_\ell \tag{5.74}$$

Also, from distortional elasticity in the locked region at $r = r_2(\bar{t})$

$$\bar{\sigma}_r - \bar{\sigma}_\theta = 2G(\bar{\epsilon}_r - \bar{\epsilon}_\theta) \tag{5.75}$$

From the fact that the elastic stresses and strains satisfy Hooke's law we have at $r = r_2(\bar{t})$

$$\sigma_r + 2\sigma_\theta = 3K(\epsilon_r + 2\epsilon_\theta) \tag{5.76a}$$

$$\sigma_r - \sigma_\theta = 2G(\epsilon_r - \epsilon_\theta) \tag{5.76b}$$

From Eqs.(5.75) and (5.76b) at $r = r_2(\bar{t})$

$$(\bar{\sigma}_r - \sigma_r) - (\bar{\sigma}_\theta - \sigma_\theta) = 2G\left[(\bar{\epsilon}_r - \epsilon_r) - (\bar{\epsilon}_\theta - \epsilon_\theta)\right]$$

From Eqs.(5.72a) and (5.72b) this equation becomes at $r = r_2(\bar{t})$

$$\bar{\sigma}_\theta - \sigma_\theta = -2G(\bar{\epsilon}_r - \epsilon_r) \tag{5.77}$$

Similarly from Eqs.(5.74) and (5.76a) at $r = r_2(\bar{t})$

$$\bar{\sigma}_\theta - \sigma_\theta = \frac{3K}{2}(\bar{\epsilon}_r - \epsilon_r) \tag{5.78}$$

From (5.77) and (5.78) one obtains at $r = r_2(\bar{t})$

$$\bar{\epsilon}_r - \epsilon_r = 0$$

and then $\epsilon_\theta = \bar{\epsilon}_\theta$ because of continuity of the radial displacement u at the interface. Then at $r = r_2(\bar{t})$

$$\epsilon_r + 2\epsilon_\theta = \bar{\epsilon}_r + 2\bar{\epsilon}_\theta = -\epsilon_\ell \tag{5.79}$$

This equation can be used in place of one of the equations (5.72a,b,c) or (5.73).

From Eq.(5.69) one can write the following expressions for strains and velocity in the elastic region:

$$\epsilon_r = \frac{\partial u}{\partial r} = \frac{1}{rc^2} f_3''\left(\tilde{t} - \frac{r - r_1}{c}\right) + \frac{2}{r^2 c} f_3'\left(\tilde{t} - \frac{r - r_1}{c}\right) + \frac{2}{r^3} f_3\left(\tilde{t} - \frac{r - r_1}{c}\right) + \frac{C_1}{3} \qquad (5.80a)$$

$$\epsilon_\theta = \frac{u}{r} = -\frac{1}{r^2 c} f_3'\left(\tilde{t} - \frac{r - r_1}{c}\right) - \frac{1}{r^3} f_3\left(\tilde{t} - \frac{r - r_1}{c}\right) + \frac{C_1}{3} \qquad (5.80b)$$

$$\dot{u} = -\frac{1}{rc} f_3''\left(\tilde{t} - \frac{r - r_1}{c}\right) - \frac{1}{r^2} f_3'\left(\tilde{t} - \frac{r - r_1}{c}\right) \qquad (5.80c)$$

Then, from Eqs. (5.79), (5.80a,b)

$$f_3''\left(\tilde{t} - \frac{r_2 - r_1}{c}\right) = -\epsilon_\ell r_2 c^2 - C_1 r_2 c^2 \qquad (5.81)$$

This is a useful relation.

Now let us consider the remaining boundary conditions. From Eqs. (5.69) and (5.70)

$$-\frac{1}{c(r_1 + c\tilde{t})} f_3'(0) - \frac{f_3(0)}{(r_1 + c\tilde{t})^2} + \frac{C_1}{3}(r_1 + ct) = 0$$

i.e.,

$$\begin{aligned} f_3'(0) &= 0 \\ f_3(0) &= 0 \\ C_1 &= 0 \end{aligned} \qquad (5.81a)$$

Then Eq. (5.81) becomes

$$f_3''\left(\bar{t} - \frac{r_2 - r_1}{c}\right) = -\epsilon_\ell c r_2^2 \tag{5.82}$$

From (5.72a), (5.69), and (5.47) one has

$$\frac{f(\bar{t})}{r_2^2(t)} - \frac{1}{3}\epsilon_\ell r_2(\bar{t}) = -\frac{1}{r_2 c} f_3'\left(\bar{t} - \frac{r_2 - r_1}{c}\right) + \frac{1}{r_2^2} f_3\left(\bar{t} - \frac{r_2 - r_1}{c}\right) \tag{5.83}$$

Similarly from (5.72c), (5.80c), (5.47) and (5.81a)

$$\frac{\dot{f}(\bar{t})}{r_2^2} = -\frac{1}{r_2 c} f_3''\left(\bar{t} - \frac{r_2 - r_1}{c}\right) - \frac{1}{r_2^2} f_3'\left(\bar{t} - \frac{r_2 - r_1}{c}\right) \tag{5.84}$$

From (5.71) and (5.49a)

$$g(\bar{t}) = 4G\frac{f(\bar{t})}{a^3} + \rho\frac{\ddot{f}(\bar{t})}{a} - p \tag{5.85}$$

Then, as before,

$$\bar{\sigma}_r = 4Gf(\bar{t})\left(\frac{1}{a^3} - \frac{1}{r^3}\right) + \rho\ddot{f}(t)\left(\frac{1}{a} - \frac{1}{r}\right) - p \tag{5.86}$$

This equation, distortional elasticity (5.44), Eq.(5.48) and the boundary condition (5.73) result in the following equation (more simply from (5.60b)):

$$4G\frac{f(\bar{t})}{a^3} + \rho\ddot{f}\left(\frac{1}{a} - \frac{1}{r_2}\right) - p = -K\epsilon_\ell \tag{5.87}$$

Eqs.(5.83), (5.84) and (5.87) are three equations for three unknowns f_1, f_2 and r_2. However, the use of the equation (5.81) simplifies the analysis.

From Eqs.(5.82) and (5.84) one obtains

$$f_3'\left(\bar{t} - \frac{r_2(\bar{t}) - r_1}{c}\right) = \epsilon_\ell c r_2^2 - \dot{f}(\bar{t}) \tag{5.88}$$

Now from this equation and Eq.(5.83) one has the following expression for f_3

$$f_3 = \frac{r_2}{c}\dot{f}(\bar{t}) - \frac{2}{3}\epsilon_\ell r_2^3 \tag{5.89}$$

Eq.(5.88) is a differential equation in $\bar{t}$ which is the independent variable. If both sides of this equation are differentiated with respect to $\bar{t}$, one obtains the following equation

$$\left[f_3''\left(\bar{t} - \frac{r_2(t) - r_1}{c}\right)\right]\left[1 - \frac{1}{c}\frac{dr_2}{d\bar{t}}\right] = 2\epsilon_\ell c r_2 \frac{dr_2}{d\bar{t}} - \ddot{f}(\bar{t}) \tag{5.90}$$

But, from Eq.(5.82)

$$f_3''\left(\bar{t} - \frac{r_2(\bar{t}) - r_1}{c}\right) = -\epsilon_\ell c^2 r_2$$

Then, from Eq.(5.90) and this equation, one can write the following expression for $\ddot{f}(\bar{t})$:

$$\ddot{f}(\bar{t}) = \epsilon_\ell c^2 r_2\left(\frac{1}{c}\frac{dr_2}{d\bar{t}} + 1\right) \tag{5.91}$$

If this expression for $\ddot{f}(t)$ is substituted in Eq.(5.87) one obtains

$$4G\frac{f(\bar{t})}{a^3} + \rho\epsilon_\ell c^2 r_2\left(\frac{1}{c}\frac{dr_2}{d\bar{t}} + 1\right)\left(\frac{1}{a} - \frac{1}{r_2}\right) = p - K\epsilon_\ell$$

i.e.,

$$f(\bar{t}) = \frac{a^3}{4G}(p - K\epsilon_\ell) - \rho\epsilon_\ell c^2\left(\frac{1}{c}\frac{dr_2}{d\bar{t}} + 1\right)\left(\frac{r_2}{a} - 1\right)\frac{a^3}{4G} \tag{5.92}$$

Then the following expressions for $\dot{f}$ and $\ddot{f}$ can be obtained by differentiating this equation with respect to $\bar{t}$

$$\dot{f}(\bar{t}) = -\rho\epsilon_{\ell}c^2\frac{a^3}{4G}\left[\left(\frac{dr_2}{d\bar{t}}\right)^2\frac{1}{ac}+\frac{r_2}{ac}\frac{d^2r_2}{d\bar{t}^2}-\frac{1}{c}\frac{d^2r_2}{d\bar{t}^2}+\frac{1}{a}\frac{dr_2}{dt}\right] \tag{5.93a}$$

$$\ddot{f}(t) = -\frac{\rho\epsilon_{\ell}c^2a^3}{4G}\left[\frac{2}{ac}\frac{dr_2}{d\bar{t}}\frac{d^2r_2}{d\bar{t}^2}+\frac{1}{ac}\frac{dr_2}{d\bar{t}}\frac{d^2r_2}{d\bar{t}^2}+\frac{r_2}{ac}\frac{d^3r_2}{d\bar{t}^3}-\frac{1}{c}\frac{d^3r_2}{d\bar{t}^3}+\frac{1}{a}\frac{d^2r_2}{d\bar{t}^2}\right] \tag{5.93b}$$

By equating the two different expressions, (5.91) and (5.93b), for $\ddot{f}(t)$ the following differential equation for $r_2(\bar{t})$ can be obtained

$$\frac{d^3r_2}{d\bar{t}^3}\frac{1}{c}\left(\frac{r_2}{a}-1\right)+\frac{d^2r_2}{d\bar{t}^2}\left(\frac{3}{ac}\frac{dr_2}{d\bar{t}}+\frac{1}{a}\right)+\frac{dr_2}{d\bar{t}}\left(\frac{4Gr_2}{a^3\rho c}\right)+\left(\frac{4Gr_2}{a^3\rho}\right)=0 \tag{5.94}$$

If this equation is solved for $r_2(\bar{t})$, $f(\bar{t})$ can be obtained from the Eq.(5.92). Then, $g(\bar{t})$ and f_3 can be obtained from Eqs.(5.85) and (5.89). The displacements, stresses, and strains can then be determined in elastic and locked regions.

(ix) Solution of the Differential Equation (5.94)

The differential equation (5.94) can be non-dimensionalized by introducing the following variables

$$R_2 = \frac{r_2}{a}$$

$$t - t_1 = \bar{t} = \sqrt{\frac{3\rho a^2}{4G}}\,\bar{T}$$

i.e.,

$$\bar{T} = 0 \quad , \quad t = t_1 \quad \text{or} \quad T = T_1$$

Then the equation can be written in the following form if the value of the Poisson's ratio is assumed to be equal to 0.25, i.e., $3K = 5G$ and (see Eq.(5.66a))

$$c\sqrt{\frac{3\rho}{4G}} = \sqrt{\frac{3K+4G}{3\rho}\,\frac{3\rho}{4G}} = \frac{3}{2}$$

$$\frac{d^3R_2}{d\bar{T}^3}(R_2 - 1) + \frac{d^2R_2}{d\bar{T}^2}\left(3\frac{dR_2}{d\bar{T}} + \frac{3}{2}\right) + 3R_2\frac{dR_2}{d\bar{T}} + \frac{9}{2}R_2 = 0 \qquad (5.95)$$

The initial conditions for the differential equation can be obtained from the conditions at the time of initial unlocking at the radius $r = r_1$, i.e., $r = r_1$ and $t = t_1$ or $\bar{t} = 0$, in the $x-t$ diagram (Fig.5.13).

For example, one can consider the case when $p/G\epsilon_\ell = 4$. Then, from Fig.5.11a

$$R_2(0) = 1.142$$

Now, the values of $dR(0)/d\bar{T}$ and $d^2R(0)/d\bar{T}^2$ can be obtained from Eqs.(5.92) and (5.93a) in the non-dimensionalized form, i.e.,

$$\frac{q^3}{3} = \left(\frac{p}{4G\epsilon_\ell} - \frac{5}{12}\right) - \frac{3}{4}\left(\frac{2}{3}R_2\frac{dR_2}{d\bar{T}} + R_2 - \frac{2}{3}\frac{dR_2}{d\bar{T}} - 1\right)$$

$$q^2\frac{dq}{d\bar{T}} = -\frac{3}{4}\left[\frac{2}{3}\left(\frac{dR_2}{d\bar{T}}\right)^2 + \frac{dR_2}{d\bar{T}} + \frac{2}{3}(R_2 - 1)\frac{d^2R_2}{d\bar{T}^2}\right]$$

Again, the values of $dR_2/d\bar{T}$ and $d^2R_2/d\bar{T}^2$ in the case when $p/G\epsilon_\ell = 4$ are

$$\bar{T} = 0 \ , \qquad \frac{dR_2}{d\bar{T}} = -0.2676 \ , \qquad \frac{d^2R_2}{d\bar{T}^2} = -25.229$$

Then, Eq.(5.86) can be integrated numerically. Milne's method is used for numerical integration. The typical numerical solution has been illustrated in the following table for the value of $p/G\epsilon_\ell = 4$:

$\bar{T}$	R_2	$\frac{dR_2}{d\bar{T}}$	$\frac{d^2R_2}{d\bar{T}^2}$	$\frac{d^3R_2}{d\bar{T}^3}$
0	1.142	-0.2676	-25.229	94.133
0.005	1.1404	-0.3930	-24.92	29.98
0.01	1.1381	-0.5173	-24.93	-33.66
0.015	1.1352	-0.6428	-25.26	-101.63
0.020	1.1317	-0.7705	-25.96	-178.76
0.025	1.1275	-0.9031	-27.07	-272.58
0.030	1.1227	-1.0421	-28.73	-393.36
0.035	1.1171	-1.1915	-31.08	-559.25
0.040	1.1108	-1.3546	-34.45	-801.84
→ * 0.045	1.1035	-1.5384	-39.31	-1180.07

* unlocking along GH (Fig.5.14)

The variations of r_2 , the stresses, strains and displacements in the region are illustrated in the Figs.5.14 and 5.15. It can be seen that unlocking takes place throughout the region GH (Fig.5.14) at time $\bar{T} = 0.044$, or $T = 1.32$. The solution for the problem must be worked separately for time $T \geq 1.32$.

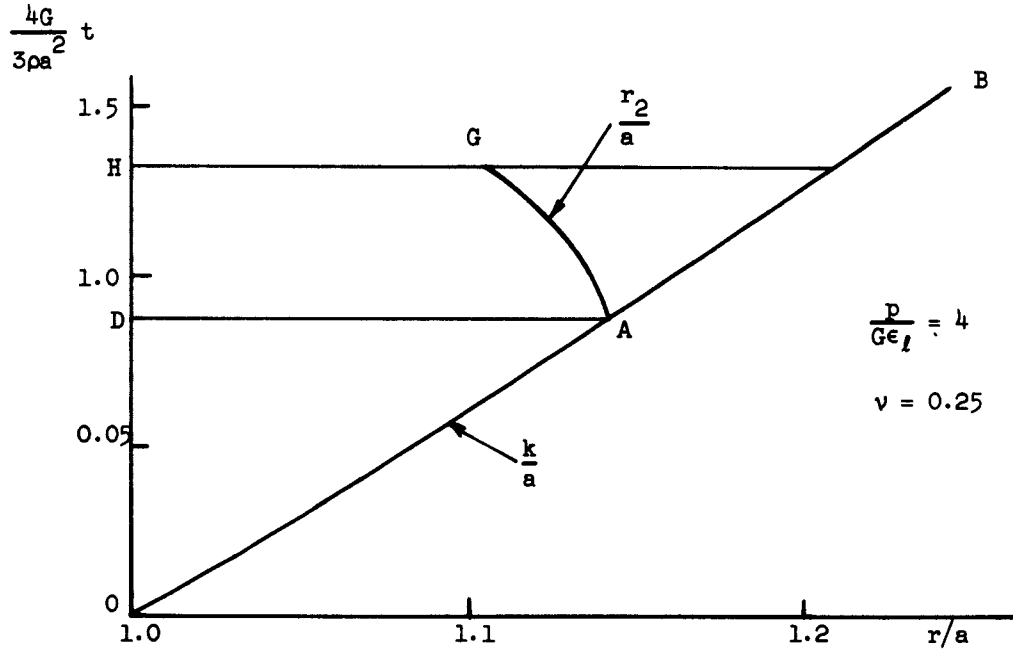


Fig. 5.14.

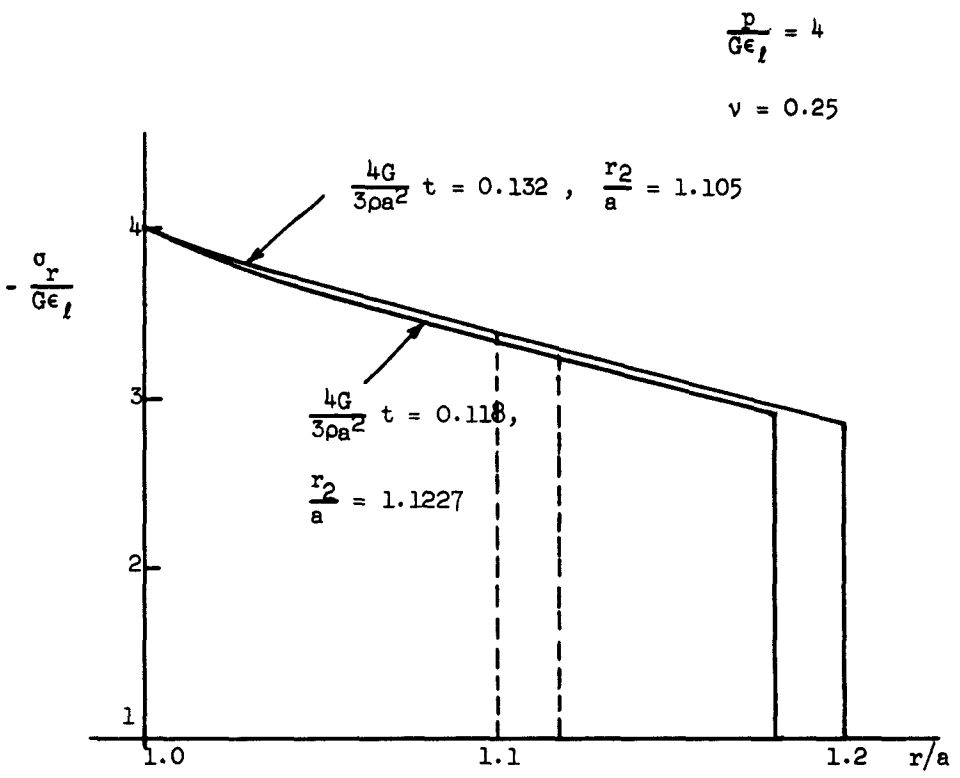

$\frac{p}{G\epsilon_l} = 4$
$\nu = 0.25$
$\frac{4G}{3\rho a^2} t = 0.132$, $\frac{r_2}{a} = 1.105$
$-\frac{\sigma_r}{G\epsilon_l}$
$\frac{4G}{3\rho a^2} t = 0.118$,
$\frac{r_2}{a} = 1.1227$
4
3
2
1
1.0
1.1
1.2
r/a

Fig. 5.15a

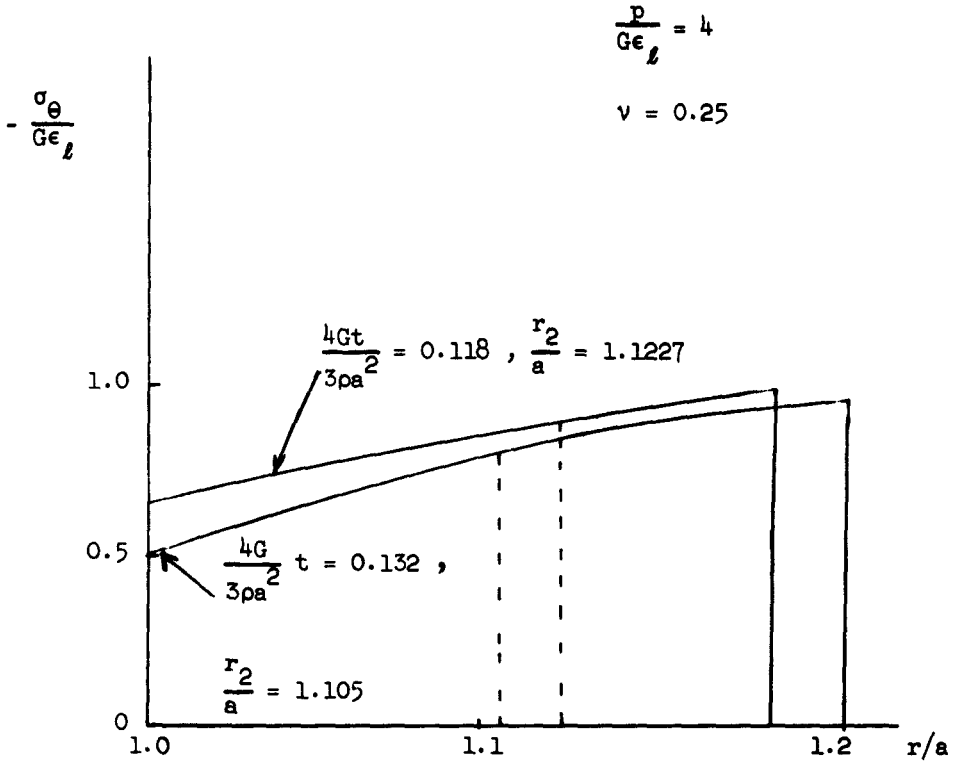

$\frac{p}{G\epsilon_\ell} = 4$
$\nu = 0.25$
$-\frac{\sigma_\theta}{G\epsilon_\ell}$
$\frac{4Gt}{3\rho a^2} = 0.118$, $\frac{r_2}{a} = 1.1227$
$\frac{4G}{3\rho a^2}\, t = 0.132$,
$\frac{r_2}{a} = 1.105$
1.0
0.5
0
1.0
1.1
1.2
r/a

Fig. 5.15b.

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