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Frank C. Ogg, Jr.

THE JOHNS HOPKINS UNIVERSITY
CARLYLE BARTON LABORATORY
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FOREWORD

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This report has been reviewed and is approved.

Approved: /s/ EDWARD N. MUNZER
Chief, Electronic Warfare Laboratory
Directorate of Intelligence & Electronic Warfare

Approved: ROBERT J. QUINN, JR., Col, USAF
Director of Intelligence & Electronic Warfare

ABSTRACT

The detection of stochastic processes in noise is considered, under the assumption that neither the signal nor the noise need be Gaussian. The detector structure is found in terms of the semi-invariants of the signal and noise processes. The general detector structure is extremely complicated, but a threshold form may be obtained. For symmetric processes with zero mean and independent sampling, the energy detector is obtained. Error probabilities are computed for the energy detector with non-Gaussian signal process and/or non-Gaussian noise. It is shown that large degradations in sensitivity occur if the noise is highly impulsive in character, but the non-Gaussian character of the signal process is found to have very little effect on the detector sensitivity.

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I. INTRODUCTION

The purpose of this study is to examine the detection problem for stochastic waveforms in noise, under the assumption that neither the waveform nor the noise need be Gaussian.

The detection of signals having some randomized parameters is well known, and the detection of Gaussian processes has been studied extensively[1,2]. The background noise is almost always assumed to be Gaussian, although a small amount of work has been done on detection in other types of noise[3,4]. The detection of signals which consist of known waveforms with some random parameters (for example, random amplitude or phase) is important in radar and in some types of communication systems which use a small number of fixed waveforms to transmit digital data. The detection of Gaussian processes is important in radio astronomy and in radiometry.

The transmitted signals used in most communication systems do not fit into either model. A communication signal usually has an extremely complex statistical structure, and in many cases of practical importance the statistical structure is unknown and cannot be represented by a convenient mathematical model. Speech is a case in point.

The assumption of Gaussian background noise is valid in most applications, since the noise is usually dominated by receiver noise. In some situations of practical interest, this is not the case. Non-Gaussian background noise is found in underwater sound propagation, in VLF reception, and in reception in the presence of countermeasures or other man-made interference.

Under these circumstances it is natural to inquire whether a more complete statistical description of the signal and noise might be of value. We shall assume only that the signal and noise are additive and independent of each other, and that certain statistical quantities (moments or semi-invariants) exist. Due to well known analytical difficulties, we shall not usually be able to obtain closed-form solutions. Threshold detectors and series expansions are, however, possible and indicate the general nature of the optimum detectors.

II. STATISTICAL PRELIMINARIES

An arbitrary random process $y(t)$ may be described at a single time $t = t_1$ by its ensemble statistics of the first order. The first order distribution is

$$W_1(y_1, t_1) = \text{Prob}\left[y \le y_1 \text{ at } t = t_1\right] \quad , \tag{1}$$

which gives the distribution of values of $y(t)$ at time $t = t_1$, when a large ensemble of similar processes are measured simultaneously.

If $y(t)$ is measured at two times $t = t_1$ and $t = t_2$, the two first-order distributions are not usually independent if $t_1 - t_2$ is sufficiently small. The second-order distribution is their joint distribution

$$W_2(y_1, t_1; y_2, t_2) = \text{Prob}\left[y \le y_1 \text{ at } t = t_1 \text{ and } y \le y_2 \text{ at } t = t_2\right]. \tag{2}$$

The second-order distribution is related to the first-order distribution by

$$W_2(y_1, t_1; y_2 t_2) = W_1(y_1, t_1) \cdot W_2(y_2, t_2 | y_1, t_1) \quad , \tag{3}$$

where the conditional distribution is

$$W_2(y_2, t_2 | y_1, t_1) = \text{Prob}\left[y \le y_2 \text{ at } t = t_2 \text{ given } y \le y_1 \text{ at } t = t_1\right]. \tag{4}$$

Similarly, the n-th order distribution $W_n(y_1, t_1; y_2, t_2; \ldots; y_n, t_n)$ describes the ensemble statistics of measurements taken at n times: $t_1, t_2, \ldots, t_n$. Conditional distributions are defined by obvious generalizations of (4).

The n-th order distribution is a distribution in n random variables and has the usual statistical properties of an n-variable distribution. The expected value of any function $G(y_1, y_2, \ldots, y_n)$ of the variables is

$$E(G) = \int \cdots \int G(y_1, \ldots, y_n)\, dW_n(y_1, t_1; \ldots ; y_n, t_n) . \tag{5}$$

The characteristic function is

$$\phi(u_1, u_2, \ldots, u_n) = E\left[\exp(iu_1 y_1 + iu_2 y_2 + \ldots + iu_n y_n)\right], \tag{6}$$

and the moments and semi-invariants may be defined in the usual way as the coefficients of the expansions of ϕ and log ϕ into power series.

The distribution W_n may be discrete, continuous, or of mixed type. It is continuous if the distribution function W_n is continuous and if the density function

$$w_n(y_1, t_1; \ldots ; y_n, t_n) = \frac{\partial^n W_n}{\partial_{y_1} \partial_{y_2} \ldots \partial_{y_n}} \tag{7}$$

exists and is continuous everywhere except possibly on a set of probability zero.

The random process defined by this ensemble of random waveforms $y(t)$ is stationary (in the strict sense) if the hierarchy of distributions W_n is independent of the time of measurement. The first-order distribution W_1 is entirely independent of the time of measurement, and the higher-order distributions depend only on the time differences $t_2 - t_1, t_3 - t_1, \ldots, t_n - t_1$.

In practice, measurements of statistical quantities cannot be made on an ensemble of random waveforms because only a single source of the random process in question is available. We must then measure the time statistics, that is, the statistics of a single member $y(t)$ of the ensemble. Let $t = t_o$ be an epoch, and consider measurements made at times $t = t_1, t_2, \ldots, t_n$. Let $y_k = y(t_o + t_k)$ and let $G(y_1, \ldots, y_n)$ be an arbitrary function of the measured quantities. Then the time average of G is

$$\langle G(y_1, y_2, \ldots, y_n) \rangle = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} G(y_1, \ldots, y_n)\, dt_o \quad . \quad (8)$$

This average exists if G is a bounded function. It can easily be shown that the time average is independent of the epoch t_o and depends only on the time differences $t_2 - t_1, t_3 - t_1, \ldots, t_n - t_1$. This average can therefore be expected to be equal to the ensemble average E(G) defined by (5) only for stationary processes. It is not sufficient for the process to be stationary, since, if it consists of two subprocesses with different statistics, the time average (8) will depend on which subprocess the waveform y(t) belongs to. If, however, the process is stationary and has no stationary subprocess whose probability is different from one or zero, then the process is ergodic and the time average (8) is equal with probability one to the ensemble average (5).

In this case, we may determine the statistical properties of the n-th order distribution $W_n(y_1, t_1; \ldots ; y_n, t_n)$ by measurement of the corresponding time averages of the single random waveform y(t) which is at our disposal. The average value of y(t) is

$$\langle y \rangle = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} y(t_o + t)\, dt_o \quad , \qquad (9)$$

and its mean-square intensity or average power is

$$\langle y^2 \rangle = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} \left[y(t_o + t) \right]^2 dt_o \quad . \qquad (10)$$

Similarly, moments of all orders are defined by

$$\langle y_1^{v_1} y_2^{v_2} \dots y_n^{v_n} \rangle = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} \left[y(t_o + t_1) \right]^{v_1}$$

$$\cdot \left[y(t_o + t_2) \right]^{v_2} \dots \left[y(t_o + t_n) \right]^{v_n} dt_o \quad . \tag{11}$$

These time averages are equal with probability one to the corresponding moments $E(y)$, $E(y^2)$, $E(y_1^{v_1} y_2^{v_2} \dots y_n^{v_n})$ of the distribution W_n.

The average $\langle y_1 y_2 \rangle$ is the autocorrelation function of $y(t)$. The various higher-order moments are similarly equal to higher-order correlation functions of $y(t)$. Knowledge of these higher order correlation functions is equivalent to knowledge of some of the higher-order moments of the ensemble distributions.

We need not assume in the following that the processes considered are ergodic or even stationary, since we shall usually assume that we know the n-th order ensemble distribution. Our results will be obtained in terms of the moments of the ensemble distribution, and these moments are equal to the corresponding higher-order correlation functions when the process is ergodic. Higher-order moments and higher-order correlation functions are merely different descriptions of the same statistical quantities.

III. THE STATIONARY POISSON PROCESS

A very general class of random processes can be constructed from sums of the form

$$y(t) = \sum_{m=o}^{\infty} a_m\, u(t-t_m) \quad , \tag{12}$$

where u(t) is a known waveform, and a_m and t_m are randomly distributed amplitudes and epochs. If we assume that the epochs t_m are uniformly and independently distributed in time, then the probability that exactly N impulses u(t) will appear in an observation interval $\left[t_o,\ t_o + T\right]$ is given by the Poisson distribution

$$P_T(N) = \frac{(\bar{n}T)^N}{N!} \exp\left[-\bar{n}T\right] \quad , \tag{13}$$

where $\bar{n}$ is the average number of impulses per unit time.

The n-th order distribution of y(t) is

$$W_n(y_1, t_1; \ldots ; y_n, t_n) = \sum_{N=o}^{\infty} P_T(N)\, W_n(y_1, t_1; \ldots ; y_n, t_n | N), \tag{14}$$

where $W_n(y_1, t_1; \ldots ; y_n, t_n | N)$ is the n-th order distribution assuming that exactly N impulses are present. The characteristic function of this conditional distribution is

$$\phi(u_1, \ldots, u_n | N) = E\left[\exp\left(\sum_{\ell=1}^{n} iu_\ell\, y_N(t_\ell)\right)\right] \quad , \tag{15}$$

where $y_N(t)$ is a waveform of the form (12) with exactly N impulses with the observation interval. If the amplitudes a_m have the distribution w(a), then

$$\phi(u_1, \ldots, u_n | N) = \left[\int_{-\infty}^{+\infty} da\, w(a) \cdot \frac{1}{T} \int_{t_o}^{t_o+T} \exp\left[ia \sum_{\ell=1}^{n} u_\ell\, u(t_\ell - t) dt\right]\right]^N . \tag{16}$$

Then the n-th order distribution (14) has the characteristic function

$$\phi(u_1,\ldots,u_n) = \sum_{N=o}^{\infty} \frac{(\bar{n}T)^N}{N!} \exp\left[-\bar{u}T\right] \cdot$$

$$\left[\frac{1}{T}\int_{-\infty}^{+\infty} w(a) \int_{t_o}^{t_o+T} \exp\{ia \sum_{\ell=1}^{n} u_\ell u(t_\ell-t)\}\right]^N \qquad (17)$$

$$= \exp\left[\bar{n}\int_{-\infty}^{+\infty} w(a)\int_{t_o}^{t_o+T} \{\exp(ia\sum_{\ell=1}^{n} u_\ell u(t_\ell-t))-1\}dt\right].$$

As $T \rightarrow \infty$, the limits of integration become $(-\infty, +\infty)$.

The semi-invariants of this process are the most convenient means of description. Expanding $\log \phi(u_1,\ldots,u_n)$ in a power series, we find that the semi-invariants of order $\ell_1 + \ell_2 + \ldots + \ell_n = L$ are

$$\lambda^{(L)}_{\ell_1\ell_2\ldots\ell_n} = \bar{n}\int_{-\infty}^{+\infty} a^L w(a)\,da \int_{-\infty}^{+\infty} \left[u(t_1-t)\right]^{\ell_1}\ldots\left[u(t_n-t)\right]^{\ell_n} dt. \qquad (18)$$

The first-order semi-invariant (mean value) of the process is

$$\lambda^{(1)} = \bar{n}\,(\bar{a}) \int_{-\infty}^{+\infty} u(t)\,dt \qquad , \qquad (19)$$

and the second-order semi-invariants are

$$\lambda^{(2)}_{jk} = \lambda^{(2)}(t_j-t_k) = \bar{n}(\overline{a^2}) \int_{-\infty}^{+\infty} u(t)u(t+t_j-t_k)dt, \qquad (20)$$

with similar expressions for the higher-order semi-invariants.

The limiting case as $\bar{n} \rightarrow \infty$ of the Poisson process is the Gaussian process. All semi-invariants (18) become infinite as $\bar{n} \rightarrow \infty$, but we may normalize the process so that its mean is zero and its second-order semi-invariants are independent of $\overline{n}$ by using the normalized variable $y^*(t) = y(t) - \lambda_1/\sqrt{\overline{n}}$. The semi-invariants of the normalized process are

$$\lambda^{*(L)}_{\ell_1 \ell_2 \ldots \ell_n} = \frac{1}{(\overline{n})^{L/2-1}} \overline{a^L} \int_{-\infty}^{+\infty} \left[u(t_1-t)\right]^{\ell_1} \ldots \left[u(t_n-t)\right]^{\ell_n} dt \qquad (21)$$

for $L \geq 2$, with $\lambda^{*(1)} = 0$. As $\overline{n} \rightarrow \infty$, all semi-invariants of order $L \geq 3$ vanish, and we obtain a Gaussian process with second-order semi-invariants

$$\lambda^{(2)}_{jk} = \overline{a^2} \int_{-\infty}^{+\infty} u(t)\, u(t+t_j-t_k)\, dt \quad . \qquad (22)$$

The Poisson processes can be used to construct a random process having given higher-order correlation functions, if a deterministic waveform u(t) can be found with the required higher-order correlation functions.

The Poisson process is a representation of the output of a linear device which is driven by a series of impulses with random amplitude and time of arrival, since the waveform u(t) may be taken to be the impulse response of the linear device. It may be used to represent shot noise in receivers, atmospheric static in VLF reception, or swept-frequency jamming in a narrow-band receiver.

IV. DETECTOR STRUCTURE WITH INDEPENDENT SAMPLES

The likelihood ratio for a known signal S in noise whose distribution is F(V) is

$$\Lambda(V;S) = \frac{F(V-S)}{F(V)} \tag{23}$$

is we assume that the noise is additive. If a class of possible signals with prior distribution $\sigma(S)$ is to be detected, we use the averaged likelihood ratio

$$\Lambda(V) = \int_{\Omega} \sigma(S)\,\Lambda(V;S)\,dS = \frac{\int_{\Omega}\sigma(S)F(V-S)dS}{F(V)}, \tag{24}$$

where Ω is the class of possible signals. We wish to examine the characteristics of this likelihood ratio for arbitrary signal statistics $\sigma(S)$ and noise statistics F(V). The optimum detector is a threshold on the likelihood ratio. Our purpose is to determine its structure and then to find error probabilities.

In order to reduce the distributions $\sigma(S)$ and F(V) to finite-order distributions, we consider only sample values of the time-functions S and V at times $t = t_1, t_2, \ldots, t_n$ and call these sample values $S_1, S_2, \ldots, S_n$ and $V_1, V_2, \ldots, V_n$. The distributions $\sigma(S)$ and F(V) are then n-th order distributions of the kind discussed in Section II.

We now further assume that the samples taken at different times are independent in both signal and noise. This assumption will be removed in Section V. With independent sampling, the likelihood ratio is

$$\Lambda(V_1,\ldots,V_n) = \prod_{j=1}^{n} \Lambda(V_j) = \prod_{j=1}^{n} \frac{\int_{-\infty}^{+\infty} \sigma(S_j)F(V_j-S_j)dS_j}{F(V_j)} \tag{25}$$

and we need only consider one factor of the product. The detector structure is most conveniently obtained in terms of the logarithm

$$\lambda(V_1, \ldots, V_n) = \sum_{j=1}^{n} \lambda(V_j) = \log \Lambda(V_1, \ldots, V_n)$$

$$= \sum_{j=1}^{n} \{\log \int_{-\infty}^{+\infty} \sigma(S_j)\, F(V_j - S_j)\, dS_j - \log F(V_j)\} \quad . \tag{26}$$

The expression (26) may be transformed by introducing the characteristic functions of the first-order distributions of signal and noise, which are

$$\phi_{S_j}(u) = \int_{-\infty}^{+\infty} \exp\left[iuS_j\right] \sigma(S_j)\, dS_j \tag{27}$$

and

$$\phi_{V_j}(u) = \int_{-\infty}^{+\infty} \exp\left[iuV_j\right] F(V_j)\, dV_j \quad . \tag{28}$$

It is easily shown that the characteristic function of the distribution $F(V_j - S_j)$ is $\exp\left[iuS_j\right] \phi_{V_j}(u)$ and that the characteristic function of the distribution

$$\int_{-\infty}^{+\infty} \sigma(S_j)\, F(V_j - S_j)\, dS_j \quad , \tag{29}$$

which has the form of a convolution, is $\phi_{S_j}(u)\, \phi_{V_j}(u)$. Since these quantities are characteristic functions, that is, Fourier transforms of the distributions, we may write the distributions as inverse Fourier transforms in the form

$$\int_{-\infty}^{+\infty} \sigma(S_j) F(V_j - S_j)\, dS_j = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \exp\left[-iuV_j\right] \phi_{S_j}(u) \phi_{V_j}(u)\, du \tag{30}$$

and

$$F(V_j) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \exp\left[-iuV_j\right] \phi_{V_j}(u)\, du \quad . \tag{31}$$

Then each term in the summation (26) may be written

$$\begin{aligned} \lambda(V_j) &= \log \int_{-\infty}^{+\infty} \sigma(S_j)\, F(V_j - S_j)\, dS_j - \log F(V_j) \\ &= \log \int_{-\infty}^{+\infty} \exp\left[-iuV_j\right] \phi_{S_j}(u)\, \phi_{V_j}(u)\, du \\ &\quad - \log \int_{-\infty}^{+\infty} \exp\left[-iuV_j\right] \phi_{V_j}(u)\, du \quad . \end{aligned} \tag{32}$$

This transformed expression for the likelihood ratio has two advantages over the form (26). It is easy to obtain power series expansions in the observed data V_j, since we need only expand the exponential factor in a power series and integrate term-by-term. The observed data V_j here appear only in the exponential factor. The second advantage is the possibility of expanding the characteristic functions in terms of moments or semi-invariants of the signal and noise distributions and thus obtaining a description of the detector in terms of these moments or semi-invariants.

Expanding the integrals in (32) into power series in V, we have

$$\int_{-\infty}^{+\infty} \exp\left[-iuV_j\right] \phi_{S_i}(u)\, \phi_{V_j}(u)\, du = \sum_{k=0}^{\infty} a_k V_j^k \tag{33}$$

and

$$\int_{-\infty}^{+\infty} \exp\left[-iuV_j\right] \phi V_j(u)\, du = \sum_{k=0}^{\infty} b_k V_j^k \quad , \tag{34}$$

where

$$a_k = \frac{(-i)^k}{k!} \int_{-\infty}^{+\infty} u^k \phi_{S_j}(u)\, \phi_{V_j}(u)\, du \tag{35}$$

$$b_k = \frac{(-i)^k}{k!} \int_{-\infty}^{+\infty} u^k \phi_{V_j}(u)\, du \quad .$$

The coefficients a_k and b_k are "moments" of the characteristic functions. It is easy to show that these coefficients may be expressed in the form

$$b_k = \frac{(-i)^k}{2\pi k!} \frac{d^n F(0)}{dV^n} \tag{36}$$

with a similar expression for a_k in terms of the distribution (29). These coefficients could have been obtained by expanding (26) directly.

The logarithms may be expanded in power series in V_j, with the result

$$\lambda(V_j) = \sum_{k=0}^{\infty} d_k V_j^k \quad . \tag{37}$$

The first few coefficients are

$$\begin{aligned} d_o &= \log a_o - \log b_o \\ d_1 &= \frac{a_1}{a_o} - \frac{b_1}{b_o} \\ d_2 &= \frac{2a_o a_2 - a_1^2}{2a_o^2} - \frac{2b_o b_2 - b_1^2}{2b_o^2} \quad . \end{aligned} \tag{38}$$

If the random processes are assumed to be stationary, then the coefficients a_k and b_k will be the same for each component of the sampled signal, and the detector structure is

$$\lambda(V) = d_o n + d_1(V_1+V_2+\ldots+V_n) + d_2(V_1^2+V_2^2+\ldots+V_n^2) + d_3(V_1^3+V_2^3+\ldots+V_n^3) + \ldots \quad (39)$$

We must now evaluate the coefficients d_k from (35) and (38), which does not seem to be possible in closed form in any very useful way. We may, however, obtain series expansions in terms of the semi-invariants of the processes. These semi-invariants are defined by

$$\log \phi_{S_j}(u) = \sum_{m=1}^{\infty} \frac{\lambda_m (iu)^m}{m!} \quad (40)$$

$$\log \phi_{V_j}(u) = \sum_{m=1}^{\infty} \frac{K_m (iu)^m}{m!} \quad .$$

The semi-invariants of the distribution (29) are the sums $\lambda_m + K_m$ of the semi-invariants for signal and noise whenever the noise is additive. For a Gaussian distribution, only the first two semi-invariants are non-zero.

The coefficients are then found to be

$$a_k = \frac{(-i)^k}{k!} \int_{-\infty}^{+\infty} u^k \exp\left[\sum_{m=1}^{\infty} \frac{(\lambda_m+K_m)(iu)^m}{m!} \right] du \quad (41)$$

But we may write

$$\exp\left[\sum_{m=3}^{\infty} \frac{(K_m+\lambda_m)(iu)^m}{m!} \right] = 1 + \frac{(K_3+\lambda_3)(iu)^3}{3!} + \frac{(K_4+\lambda_4)(iu)^4}{4!} + \frac{(\lambda_5+K_5)(iu)^5}{5!} + \frac{\left[\lambda_6+K_6+10(\lambda_3+K_3)^2\right](iu)^6}{6!} \quad (42)$$

$$+ \ldots$$

and then (41) becomes

$$a_k = \frac{(-i)^k}{k!} \int_{-\infty}^{+\infty} u^k \exp\left[(\lambda_1+K_1)(iu) - \frac{(\lambda_2+K_2)}{2} u^2\right] du \quad (43)$$

$$+\frac{(-i)^k}{k!} \sum_{m=3}^{\infty} \frac{C_m(i)^m}{m!} \int_{-\infty}^{+\infty} u^{k+m} \exp\left[(\lambda_1+K_1)(iu)-\frac{(\lambda_2+K_2)}{2} u^2\right] du$$

where C_m is the m-th coefficient in the series (42). The integrals in (43) can be readily evaluated in terms of λ_1+K_1 and λ_2+K_2 from the known moments of the Gaussian distribution. The coefficients b_k are obtained from (43) by putting $\lambda_m = 0$ for all m.

Although the evaluation can easily be carried out generally, we shall be interested primarily in the detection of processes with zero mean in noise of zero mean. If we set $\lambda_1 = K_1 = 0$, a considerable simplification results and we obtain:

$$a_o = \sqrt{\frac{2\pi}{\lambda_2+K_2}}\left[1+\frac{3(\lambda_4+K_4)}{4!}\left(\frac{1}{\lambda_2+K_2}\right)^2 - \frac{15(\lambda_6+K_6+10(\lambda_3+K_3)^3)}{6!}\left(\frac{1}{\lambda_2+K_2}\right)^3+\ldots\right]$$

$$a_1 = \sqrt{\frac{2\pi}{\lambda_2+K_2}}\left[-\frac{(\lambda_3+K_3)}{3!}\left(\frac{1}{\lambda_2+K_2}\right)^2 + \frac{15(\lambda_5+K_5)}{5!}\left(\frac{1}{\lambda_2+K_2}\right)^3 + \ldots\right]$$

$$a_2 = \sqrt{\frac{2\pi}{\lambda_2+K_2}}\left[-\frac{1}{2}\left(\frac{1}{\lambda_2+K_2}\right) + \frac{15(\lambda_6+K_6+10(\lambda_3+K_3)^2)}{6!}\left(\frac{1}{\lambda_2+K_2}\right)^3+\ldots\right]$$

Similar expressions for b_k are obtained by putting $\lambda_m = 0$. From these, by (38), the leading terms of d_1 and d_2 are found to be:

$$d_1 = -\frac{1}{3!}\left[\frac{K_3+\lambda_3}{(K_2+\lambda_2)^2} - \frac{K_3}{K_2^2}\right] + \frac{15}{5!}\left[\frac{K_5+\lambda_5}{(K_2+\lambda_2)^3} - \frac{K_5}{K_2^3}\right] + \ldots$$

$$d_2 = -\frac{1}{2!}\left[\frac{1}{\lambda_2+K_2} - \frac{1}{K_2}\right]$$

$$+\frac{15}{6!}\left[\frac{\lambda_6+K_6+10(\lambda_3+K_3)^2+3(\lambda_4+K_4)}{(\lambda_2+K_2)^3} - \frac{K_6+10K_3^2+3K_4}{K_2^3}\right] + \ldots$$

For Gaussian distributions of both signal and noise the exact result is given by the first term of d_2, with all other coefficients zero.

The detector structure may now be found in a threshold form. If V is small enough, only the first few terms of the power series (39) need be considered. Generally, a detector which is optimized for small input signals will be useful for large signals as well. We may therefore consider a threshold detector of the form

$$\lambda_T(V) = d_1(V_1 + \ldots + V_n) + d_2(V_1^2 + \ldots + V_n^2) \quad . \tag{44}$$

It is difficult to determine the validity of this approximation, other than to state that it is certainly valid for sufficiently small input signals. It is also difficult to determine the coefficients d_1 and d_2, since the series for them do not seem to converge very rapidly.

The threshold form (44) of the optimum detector indicates the general character of the detector structure. It is found that the optimum threshold detector is of approximately the same form as the optimum detector for a Gaussian process. The weighting coefficients are functions of all the moments (or semi-invariants) of both the signal process and the noise process. The first-order term is not zero unless both signal and noise processes are symmetric, zero means not being sufficient. The exact optimum detector is a function of all powers of the input data.

The principal conclusion of this section is that the energy detector, which is optimum for Gaussian processes in Gaussian noise, is an optimum threshold detector for signal and noise processes which are both symmetric with zero means. In this case the higher-order statistics of the processes are irrelevant to the detector structure, if one is satisfied with a threshold detector, although the higher-order statistics have a significant effect on the error probabilities. If the processes are unsymmetric or have non-zero means, then the linear term may not be neglected and the detector structure is more complicated. The general threshold detector is dependent on all orders of signal and noise statistics, since d_1 and d_2 will appear implicitly and not as an inessential factor.

V. DETECTOR STRUCTURE WITH DEPENDENT SAMPLES

Extension of the results of Section IV to dependent samples can be carried out by the same methods. The likelihood ratio (24) may be expressed in terms of the n-th order characteristic functions

$$\phi_V(u_1, \ldots, u_n) = \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \exp\left[iu_1V_1 + \ldots + iu_nV_n\right] \tag{45}$$

$$\cdot\ F(V_1, \ldots, V_n)\, dV_1 \ldots dV_n$$

and

$$\phi_S(u_1, \ldots, u_n) = \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \exp\left[iu_1S_1 + \ldots + iu_nS_n\right] \tag{46}$$

$$\cdot\ \sigma(S_1, \ldots, S_n)\, dS_1 \ldots dS_n \quad .$$

The characteristic function of the distribution

$$\int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \sigma(S_1, \ldots, S_n)\, F(V_1 - S_1, \ldots, V_n - S_n) dS_1 \ldots dS_n \tag{47}$$

is

$$\phi_V(u_1, \ldots, u_n)\, \phi_S(u_1, \ldots, u_n) \quad .$$

The distributions $F(V_1, \ldots, V_n)$ and (47) may be written as inverse Fourier transforms, in the forms

$$F(V_1, \ldots, V_n) = \left(\frac{1}{2\pi}\right)^n \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \exp\left[-iu_1V_1 + \ldots - iu_nV_n\right]$$

$$\cdot\ \phi_V(u_1, \ldots, u_n)\, du_1 \ldots du_n \tag{48}$$

and

$$(47) = \left(\frac{1}{2\pi}\right)^n \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \exp\left[-iu_1V_1 + \ldots -iu_nV_n\right]$$

$$\cdot\ \phi_S(u_1, \ldots, u_n)\ \phi_V(n_1, \ldots, u_n)\ du_1 \ldots du_n \ . \qquad (49)$$

Series expansions are obtained as before by expanding the exponentials, giving for (48) the series

$$F(V_1, \ldots, V_n) = \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \cdots \sum_{j_n=0}^{\infty} b^{(J)}_{j_1 j_2 \ldots j_n} V_1^{j_1} V_2^{j_2} \ldots V_n^{j_n} \qquad (50)$$

where $J = j_1+j_2+\ldots+j_n$ is the degree of the term in $V_1, \ldots V_n$. The coefficients are

$$b^{(J)}_{j_1 j_2 \ldots j_n} = \left(\frac{1}{2\pi}\right)^n \frac{1}{j_1! j_2! \ldots j_n!} \int_{-\infty}^{+\infty} (iu_1)^{j_1} (iu_2)^{j_2} \ldots (iu_n)^{j_n}$$

$$\cdot\ \phi_V(u_1, u_2, \ldots, u_n)\ du_1\ du_2 \ldots du_n \ . \qquad (51)$$

Similar expressions are obtained for the distribution (47).

If the process is stationary, then all coefficients of order $J = 1$ are equal, and all coefficients of order $J = 2$ depend only on the time differences of the times of measurement of the corresponding samples V_j and V_k.

The series (50) must now be transformed into a series for the logarithm of the distribution $F(V_1, V_2, \ldots, V_n)$. The first few terms of this series are readily obtained, and the detector is found to have the form

$$\lambda(V_1, \ldots V_n) = d_o + d_1 (V_1 + \ldots + V_n)$$

$$+ \sum_{j=1}^{n} \sum_{k=1}^{n} d_{jk} V_j V_k + \ldots \ . \qquad (52)$$

Explicit expressions for the coefficients in the series (52) may be obtained in terms of the coefficients of the series (50) and the analogous series for the distribution (47). It is found that

$$d_o = \log a^{(0)} - \log b^{(0)}$$

$$d_1 = \frac{a^{(1)}}{a^{(0)}} - \frac{b^{(1)}}{b^{(0)}}$$

$$d_{jk} = \frac{a^{(2)}_{jk}}{a^{(0)}} - \frac{b^{(2)}_{jk}}{b^{(0)}} \quad \text{if } j \neq k \tag{53}$$

$$d_{jj} = \frac{a^{(2)}_{jj}}{a^{(0)}} - \frac{\left[a^{(1)}\right]}{2\left[a^{(0)}\right]^2} - \frac{b^{(2)}_{jj}}{b^{(0)}} + \frac{\left[b^{(1)}\right]}{2\left[b^{(0)}\right]^2} \quad .$$

Expressions for these quantities may be found as before in terms of the semi-invariants of the signal and noise processes.

The detector structure for dependent samples is seen to be quite analogous to that for independent samples. Threshold detectors may be defined in the same manner.

VI. ERROR PROBABILITIES

The approximate computation of error probabilities is simple when a large number of independent samples are available or when the observation time is long compared to the reciprocal band-width of the noise and the signal process. In this case, the distribution of the detector output is approximately Gaussian, and we need only compute its first and second order moments.

To simplify the calculations, we shall compute error probabilities only for independent samples. Extension to dependent samples is obvious, and the results should be similar when the observation time is long. We shall also consider only the quadratic threshold detector with symmetrical signal and noise distributions, which has the form

$$\lambda_T(V) = V_1^2 + V_2^2 + \ldots + V_n^2 \quad . \tag{54}$$

In this case, the mean and variance of the detector output are easily calculated and well-known. The mean is

$$\alpha_1 = \sum_{k=1}^{n} \overline{V_k^2} = n\,\overline{V^2} \tag{55}$$

and the second moment is

$$\alpha_2 = \sum_{k=1}^{n} \overline{V_k^4} + \sum_{\substack{j=1 \\ j\neq k}}^{n} \sum_{k=1}^{n} \overline{V_j^2}\;\overline{V_k^2}$$

$$= n\,\overline{V^4} + n(n-1)\,(\overline{V^2})^2 \tag{56}$$

where $\overline{V^m}$ represents the m-th moment of any one sample of the waveform V. The variance is then

$$\sigma^2 = \alpha_2 - \alpha_1^2 = n\overline{V^4} - n(\overline{V^2})^2 \quad . \tag{57}$$

These quantities may be conveniently expressed in terms of the semi-invariants of the signal and noise processes. When both signal and noise are present, we have

$$\alpha_1 = n(K_2 + \lambda_2)$$
$$\sigma^2 = n(K_4 + \lambda_4) + 2n(K_2 + \lambda_2)^2 \quad . \qquad (58)$$

When only noise is present the mean and variance are given by the same expressions with $\lambda_4 = \lambda_2 = 0$.

The fourth-order moments (or semi-invariants) play an important role in the error probabilities. If two processes with the same power (equal λ_2) are to be detected, the probabilities of detection will differ significantly when the fourth-order statistics are different.[3]

probabilities for more general detector structures may be treated similarly. If a polynomial of order K is used to represent a threshold detector, then the input statistics of all orders up to 2K will be needed to determine the error probabilities. If the samples are dependent the cross-moments will appear, again up to order 2K.

Computations of error probabilities are given in Figures 1, 2, and 3. In each case, n = 100 independent samples have been used. The effects of the fourth-order statistics have been represented by the coefficient of skewness

$$\gamma_2 = \frac{K_4}{K_2^2} \text{ or } \frac{\lambda_4}{\lambda_2^2} \qquad (59)$$

of the noise or signal distributions. A Gaussian distribution has $\gamma_2 = 0$. Calculations have been made for $\gamma_2 = 0, 1, 5, 10,$ and 20. If the non-Gaussian distributions are considered to be Poisson processes (Section III), then increasing coefficients of skewness represent increasing impulsive character of the noise.

Each curve represents the probability of detection P_d of the signal process with a Neyman-Pearson detector whose false alarm probability is 10^{-2}. For convenience, the noise variance K_2 is taken to be unity. The probability of detection P_d is plotted against the signal variance λ_2, which when normalized may be considered to be the signal-to-noise power ratio.

Figure 1 shows the effects of non-Gaussian background noise. The signal process is Gaussian for all curves shown. The increasingly impulsive character of the noise, as γ_2 increases, degrades the probability of detection of the signal process. With highly impulsive noise a substantial increase in the signal power may be needed to obtain the same probability of detection.

Figure 2 shows the effects of non-Gaussian signals received in Gaussian background noise. Some degradation of the probability of detection occurs, but the degradation is much smaller than that observed with non-Gaussian background noise.

Figure 3 shows the probability of detection of a non-Gaussian process in non-Gaussian noise when both processes have the same coefficient of skewness. The degradation in P_d is only slightly larger than that observed when the noise alone is non-Gaussian.

VII. CONCLUSIONS

The optimum detector for a non-Gaussian stochastic signal in non-Gaussian noise can be obtained as a power series in the observed data. Explicit formulas for the coefficients of this power series in terms of the characteristic functions of the signal and noise distributions have been calculated. Series expansions may be found for the coefficients in terms of the semi-invariants of the signal and noise distributions, but the convergence of these formal expansions is questionable.

For weak signals, the case of principal interest, threshold detectors, which are optimum for sufficiently small input signals, may be obtained by truncating the power series at some convenient number of terms. If the power series is truncated after quadratic terms, a detector is obtained which is exactly optimum for Gaussian signal and noise distributions and which is optimum as a threshold detector for non-Gaussian signal and noise.

If the signal and noise distributions are both symmetric with zero mean, the linear terms are zero and the quadratic terms alone appear. If the distributions are not symmetric, the threshold detector is a combination of linear and quadratic terms. The coefficients of these terms usually involve all moments of both signal and noise distributions, but in some cases the coefficients may be removed as inessential factors.

The error probabilities for the quadratic detector are strongly influenced by the fourth-order statistics of the signal and noise processes, as is well-known[3,4]. Computations of sensitivity for various coefficients of skewness of the signal and noise distributions have been made for the quadratic detector with independent samples. A Neyman-Pearson detector with a false-alarm probability of 10^{-2} has been used. A considerable loss of sensitivity is observed with non-Gaussian background noise. With a skewness coefficient of 20, as much as 5 db more signal power may be needed to obtain the probability of detection found with Gaussian noise background. Arbitrarily large losses of sensitivity may be obtained with very large skewness coefficients, which correspond to highly impulsive noise.

The detection of a non-Gaussian process on the same basis shows that losses of sensitivity are relatively small. The probabilities of detection are only slightly less than those obtained with a Gaussian signal of the same power.

When a non-Gaussian signal is detected in non-Gaussian noise, the probability of detection is degraded by about the same amount as if only the noise were non-Gaussian. It appears that the higher-order statistics of the signal process are relatively unimportant, while those of the noise process are quite significant.

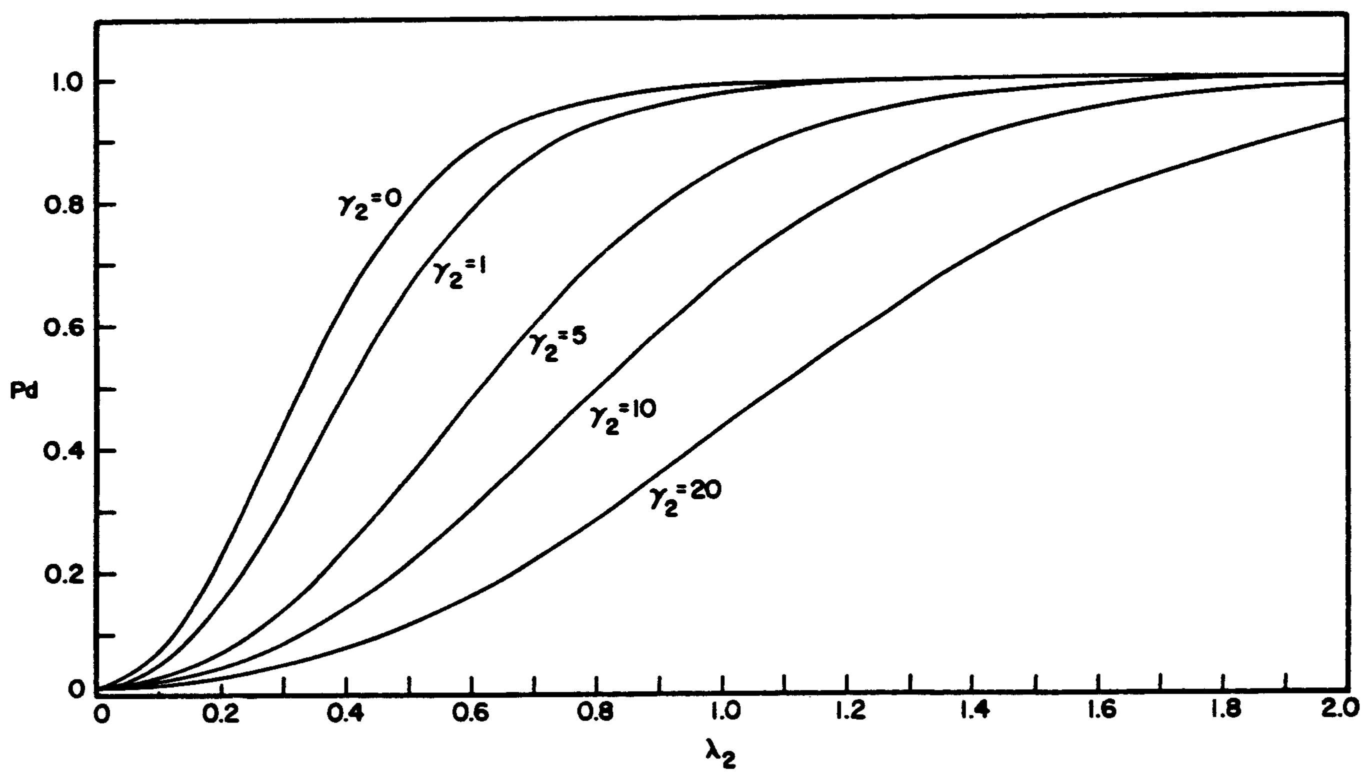

FIGURE 1 PROBABILITY OF DETECTION: GAUSSIAN SIGNAL IN NON-GAUSSIAN NOISE.

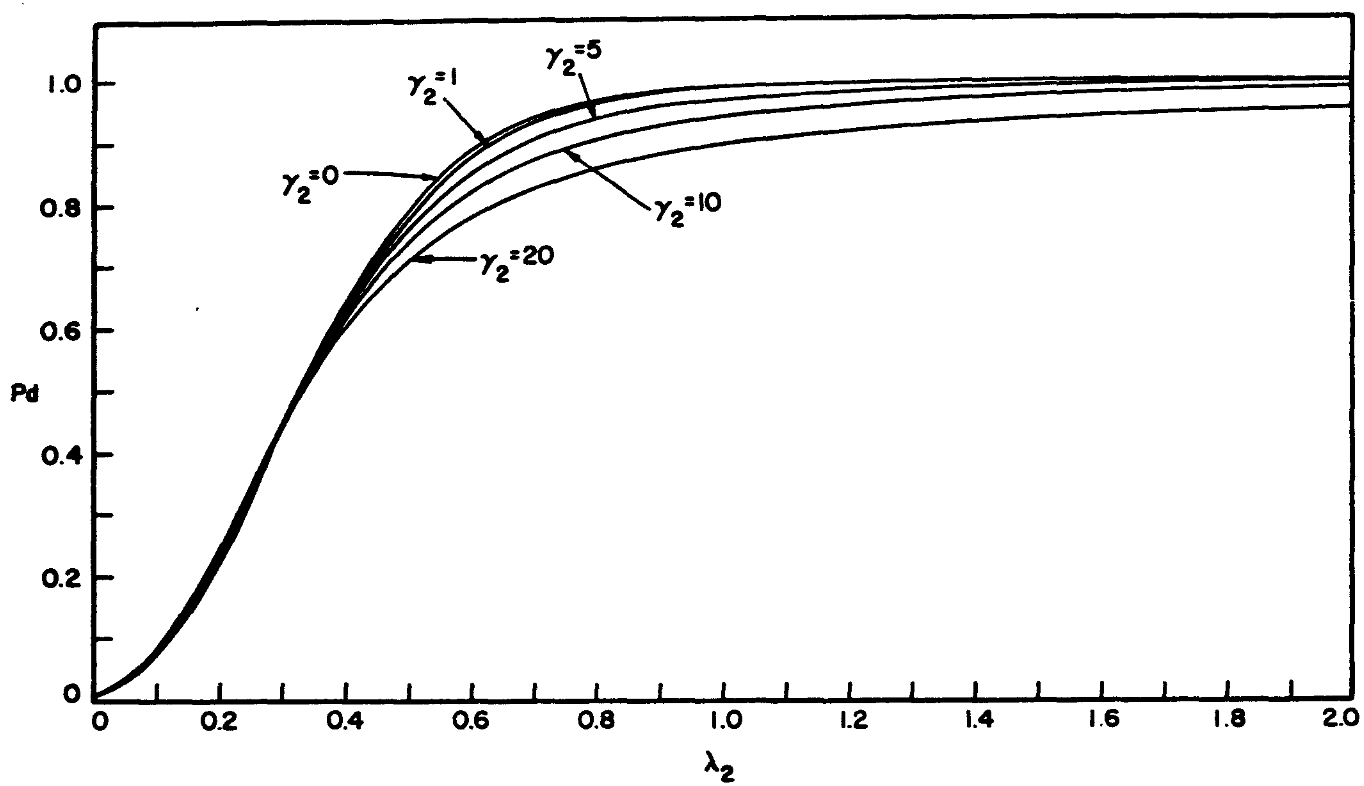

FIGURE 2 PROBABILITY OF DETECTION: NON-GAUSSIAN SIGNAL IN GAUSSIAN NOISE.

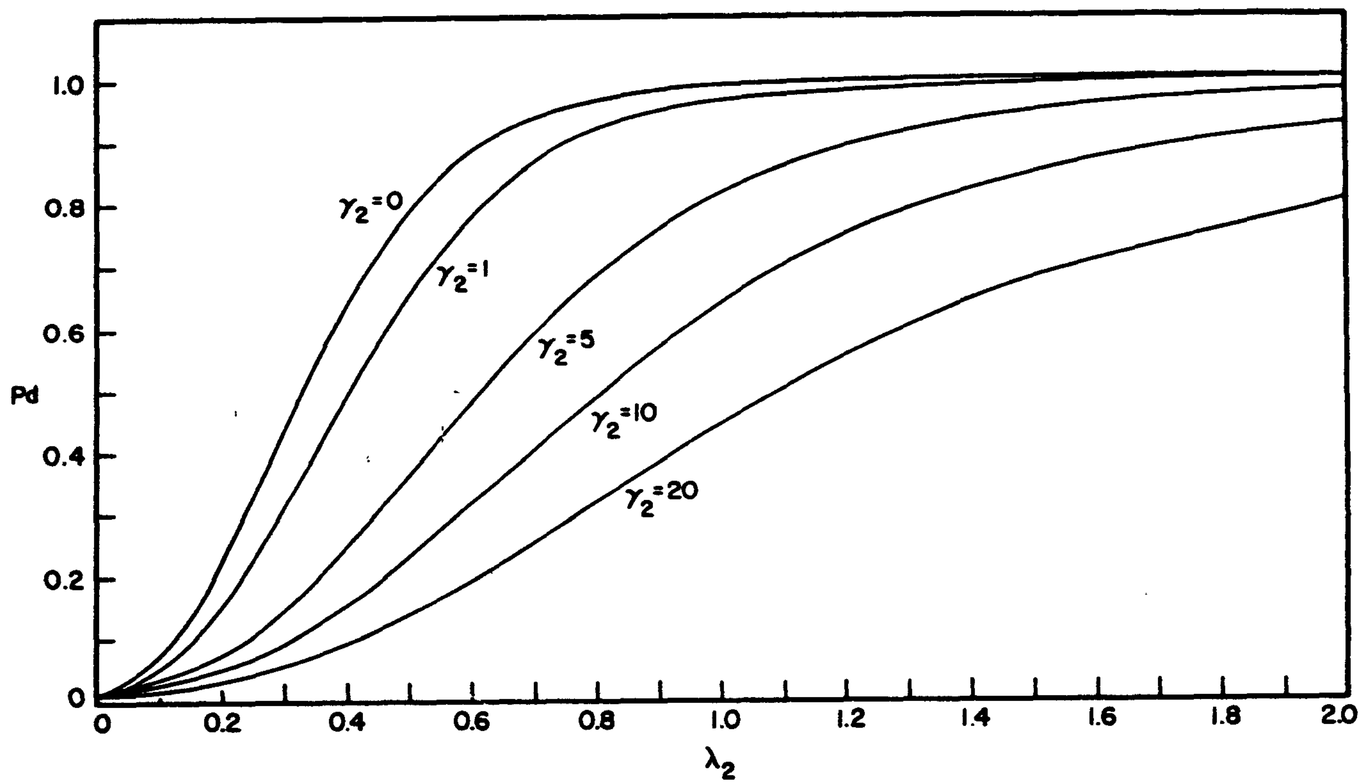

FIGURE 3 PROBABILITY OF DETECTION: NON-GAUSSIAN SIGNAL IN NON-GAUSSIAN NOISE.

REFERENCES

1. D. Middleton, "An Introduction to Statistical Communication Theory," McGraw-Hill Book Co., New York, 1960.

2. C. W. Helstrom, "Statistical Theory of Signal Detection," Pergamon Press, New York, 1960.

3. J. A. Magness, "Spectral Response of a Quadratic Device to Non-Gaussian Noise," J. Appl. Phys., Vol. 25, p. 1357, 1954.

4. D. Middleton, "Acoustic Signal Detection by Simple Correlators in the Presence of Non-Gaussian Noise. I. Signal-to-Noise Ratios and Canonical Forms," Jour. Acoust. Soc. Am., Vol. 34, p. 1598, October 1962.