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# THE APPLICATION OF OCCULTATIONS TO GEODESY

S. W. Henriksen



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# THE APPLICATION OF OCCULTATIONS TO GEODESY

## PART 1: GENERAL THEORY

### Introduction

The sun and the moon have apparent daily motions across the sky, rising in the east and setting in the west. These motions are apparent in that they are presumably not motions of the sun and moon proper, but are the result of the earth's rotation. Hence these motions are shared by all the rest of the heavenly bodies -- the stars and the planets. That the sun and moon have other motions which are not related to the earth's rotation is shown by the different positions of these two from day to day with respect to the stars. This is especially obvious in the case of the moon, which can be seen to shift its position eastward with respect to the stars by a distance equal to its own diameter every hour. Since the moon has an angular diameter of about half a degree, in one day of 24 hours the moon will have moved 12° eastward with respect to the background stars, and hence will be observed to rise the time equivalent of 12°, or 48 minutes, later every day. A complete circuit of the star field takes about 27.3 days.

In its eastward motion among the stars, it is to be expected that the moon will pass in front of and hide such luminous bodies as may be along its path. Except where the body is our own sun, in which case an eclipse is said to occur, the passage of the moon across the line of sight between a star or planet and an observer is called an occultation of the star or planet by the moon. An eclipse of one sort or another seldom occurs more than twice a year; an occultation, on the other hand, can be observed almost every night with a pair of binoculars.

Both eclipses and occultations have been observed in order to find the geographic location of the observer (references 12, 13, 14). That such use can be made of eclipses and occultations stems basically from three circumstances:

(1) The moon is so close to the earth that a shift in the observer's position produces an appreciable change in the apparent position of the moon with respect to the surrounding stars (or the sun). At the sub-lunar point, a shift of 1000 meters in the observer's position produces a shift of about one-half second of arc in the position of the moon.

(2) The directions of the brighter stars occulted by the moon are well known, so that a bright star occultation gives an accurate space direction for that portion of the edge of the moon (at the instant of occultation) at which the occultation occurs.

(3) The position of the moon with respect to the earth is known from the exhaustive studies of E. W. Brown (reference 1) and others, as a result of many years of precise observations at observatories. At the instant of occultation there is established in space a fixed line on which the observer is known to lie. Since the occultation takes place at the edge of the moon and not at the center, to fix the line would require not only the coordinates of the center of the moon, but also the direction and distance from the center of the moon to that point on the edge at which the occultation was observed to take place (Figure 1).

In practice, the direction (from the moon's center to its edge) at which the occultation occurs cannot be measured accurately, and no attempt is made to measure it, so that there is actually a set of parallel lines determined, all pointing toward the same star and on any one of which the observer could lie. The intersection of this set of lines with the spheroid (taken to be the earth's surface) is an arc, a short segment of which is called a "line of position"(LOP). The direction from the center is always known well enough that on a map the LOP is practically a straight line. Two occultations give two lines of position, and the observation site lies at the intersection of these two lines.

Another deviation of practice from theory occurs because the lunar profile is not a perfect circle, but is highly irregular; that point on the edge of the moon at which the occultation occurs is at a different distance from the lunar center for each occultation and this distance is not accurately known. Hence, every occultation occurs at a different distance from the optical center of the moon; every additional occultation introduces another and different radius of the moon, and the ensuing set of equations always contains more unknowns than there are knowns. To remove this difficulty each occultation can be observed by two different observers; if the occultation occurs at the same point on the limb for both, one of these observations may be used to determine the radius of the moon for that occultation. When this is done, the situation is like that in the preceding paragraph where only the position angle (direction from the moon's center) in addition to the observer's position, was unknown, and the problem is again solvable.

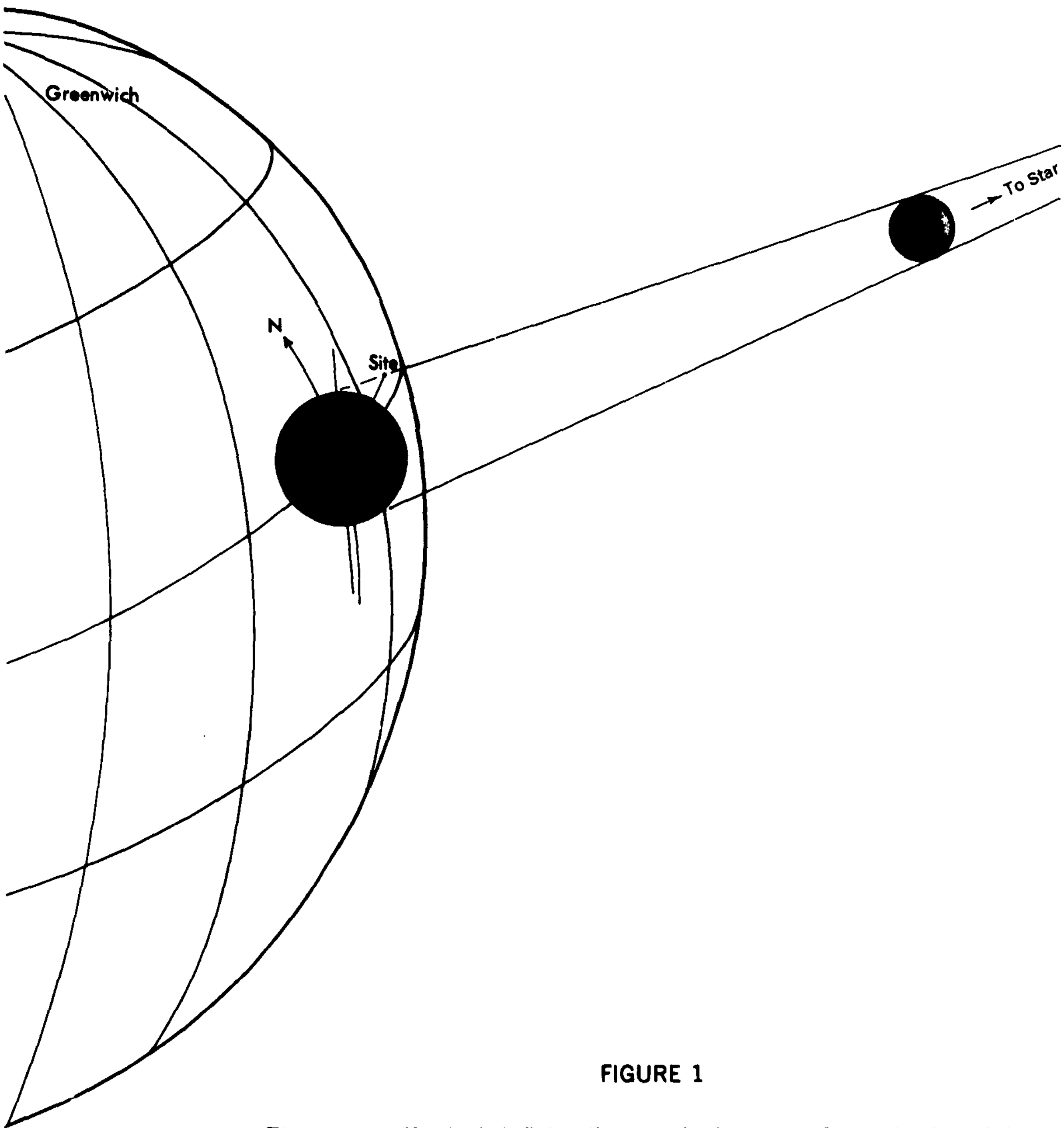



FIGURE 1

The star at effectively infinite distance, is the center for projection of the profile of an irregularly-shaped body, the Moon, onto an irregularly-shaped surface, the Earth. The Moon's surface is approximated by a sphere, its profile by a circle and the Earth's surface by an ellipsoid of revolution.

## I. Mathematical Formulation of the Problem

Mathematically, the simplest approach to the problem would be to go directly to the solution by determining the intersection of the spheroid surface with the set of lines tangent to the moon's edge and extending in the direction of the occulted star. Such an approach would, however, be very difficult to handle, since it would entail the solution of a complex system of simultaneous fourth-degree equations. Hence a less direct route to the solution was adopted long ago by astronomers and has been retained ever since. This indirect route permits most of the computations to be carried out as if the occultation occurred in a plane, and then brings the results from the plane up to the surface of the spheroid as a last step.

As was mentioned in the Introduction, the two major unknowns, other than the geodetic ones, occurring in the occultation problem are the position angle to the occultation and the distance from the center of the moon to the point on the edge at which the occultation occurs. After the basic occultation equations have been set up, as in section B below, two distinct ways of removing these two unknowns are possible. One, which is called the Single-Site Method, requires that occultations be observed from one site over as great a range of position angles as possible. When the geodetic unknowns and as many as possible of the systematic astronomical unknowns are solved for by the method of least squares, and the variations in the radius of the moon are ignored, it is expected that variations in the moon's radius will act in the solution as if they were randomly distributed with respect to the various observations, and hence will not appear in the values found for the geodetic unknowns. This method is discussed only briefly in section C below, because although it is of considerable interest in its own right, as well as closely related in theory to the Equal-Limb-Line Method, it has already been treated at length in a previous paper by Pamelia A. Henriksen (reference 2). It is included here largely for the sake of completeness and to throw additional light on the less obvious problems associated with occultation survey by the Equal-Limb-Line Method.

The Equal-Limb-Line Method, as stated in the Introduction, removes the uncertainty in the radius of the moon by using a second observer to determine the radius for the same occultation as that being viewed by the first observer. Since no reliance is placed on statistics to produce results, the accuracy of this method is inherently greater than that of the Single-Site Method. Section D below covers the Equal-Limb-Line Method in detail.

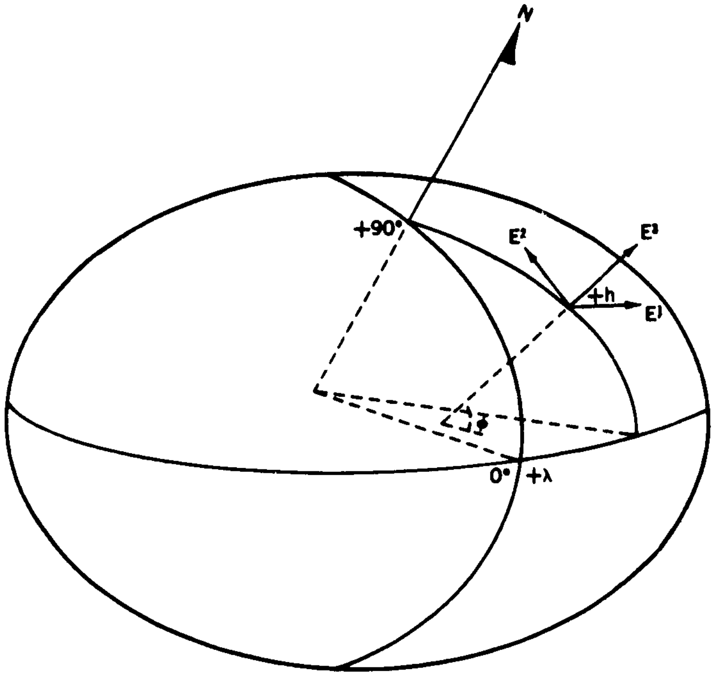



FIGURE 2

Geodetic and topocentric coordinates of the observer ($\lambda, \phi$, h, and $E^i$ systems)

Before anything is done toward setting up the equations describing the occultation phenomenon, however, a clear statement must be made concerning the frames of reference in which the equations will be used. This is not a simple subject, since the data which come from the observations, from the ephemeris of the moon, and from the star catalogs are referred to many distinct frames of reference which are related to each other in a complex fashion. Also, because the computation procedure is indirect, a chain of intermediate reference systems is introduced which must be distinguished from the primary reference systems. Section A below will discuss simply the frames of reference used in the theory.

A. Frames of Reference.

1. The observer's coordinates (Figure 2) are given first with respect to some standard spheroid as:

$$\lambda, \phi, h.$$

The spheroid is assumed to be defined (Bomford, reference 3) so that its minor axis is parallel to the earth's axis of rotation; a major axis is assigned the value $\lambda = 0$, and is parallel to the meridian of Greenwich, and the perpendicular at the datum point $\lambda_0$, $\phi_0$ is defined to coincide with the vertical (the direction of gravity) there at a depth $h_0$ below the surface of the earth. Usually, $h$ will be given initially not with respect to the reference spheroid but with respect to mean sea level -- essentially, the co-geoid. The height of the co-geoid above the spheroid, $\Delta h$, must be added to the given $h_s$ to get the height, $h$, above the spheroid.

Most computations are carried out using the International spheroid ( $a = 6,378,388$ meters, $1/f = 297$) as the reference surface. When, as often happens, the survey data on the occultation site are with reference to some other reference surface, a transformation between the two spheroids is necessary. The procedure used is given in Appendix D.

2. A rectangular coordinate system with origin at the center of the spheroid is used in preference to the geographic system for computations (Figure 3); this system is defined by the equations:

$$\begin{aligned} X^1 &= (N+h) \cos\lambda \cos\phi \\ X^2 &= (N+h) \sin\lambda \cos\phi \qquad (1) \\ X^3 &= [N(1-e^2) + h] \sin\phi. \end{aligned}$$

N is the radius of curvature in the prime vertical and can be computed from

$$N = a \left[1-e^2 \sin^2 \phi\right]^{-\frac{1}{2}}. \tag{2}$$

a is the semi-major axis of the spheroid.

3. A topocentric rectangular coordinate system with origin near or at the observer is used to relate the occultation data directly to the geodetic coordinates of the observer. This system (Figure 2), designated the $E^i$-system, is related to the $X^i$-system by the equations:

$$E^i = R_\phi \, R_\lambda \, (X^i - X^i_o) \tag{3}$$

$$R = \begin{bmatrix} -\sin\lambda & +\cos\lambda & 0 \\ -\cos\lambda & -\sin\lambda & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{4}$$

$$R_\phi = \begin{bmatrix} 1 & 0 & 0 \\ 0 & +\sin\phi & -\cos\phi \\ 0 & +\cos\phi & +\sin\phi \end{bmatrix} \tag{5}$$

The $X^i_o$ are the $X^i$-system coordinates of the topocentric system origin.

4. Most of the computations are carried out in an intermediate reference system whose origin is also at the center of the spheroid (see below), but whose orientation is with reference to the moon's and star's coordinates rather than with reference to earth-fixed coordinates (references 11, 12, 13). This system will be denoted by lower-case letters ($x^1$, $x^2$, $x^3$). The $x^3$-axis is parallel to the line joining the star and the center of the moon and is positive in the direction of the star; the $x^2$-axis lies in the plane of $x^3$ and the spheroid's minor-axis and is positive to the North; the $x^1$-axis is perpendicular to $x^3$ and is positive towards the East (Figure 4). The x-system is therefore related to the $X^i$-system by the equations:

$$\left[x^i\right] = \left[R_{\delta_*}\right] \left[R_{\mu_*}\right] \left[X^i\right]. \tag{6}$$

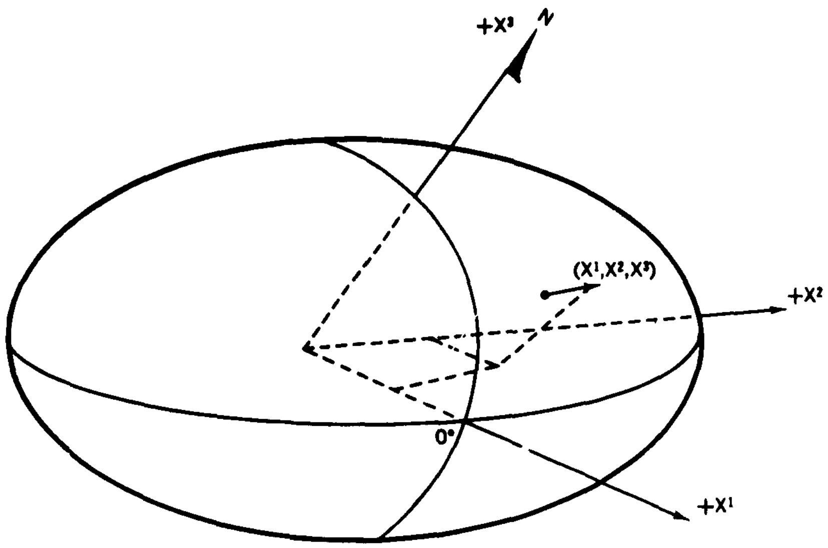



FIGURE 3

Rectangular coordinates, Earth-fixed (the $X^i$-system)

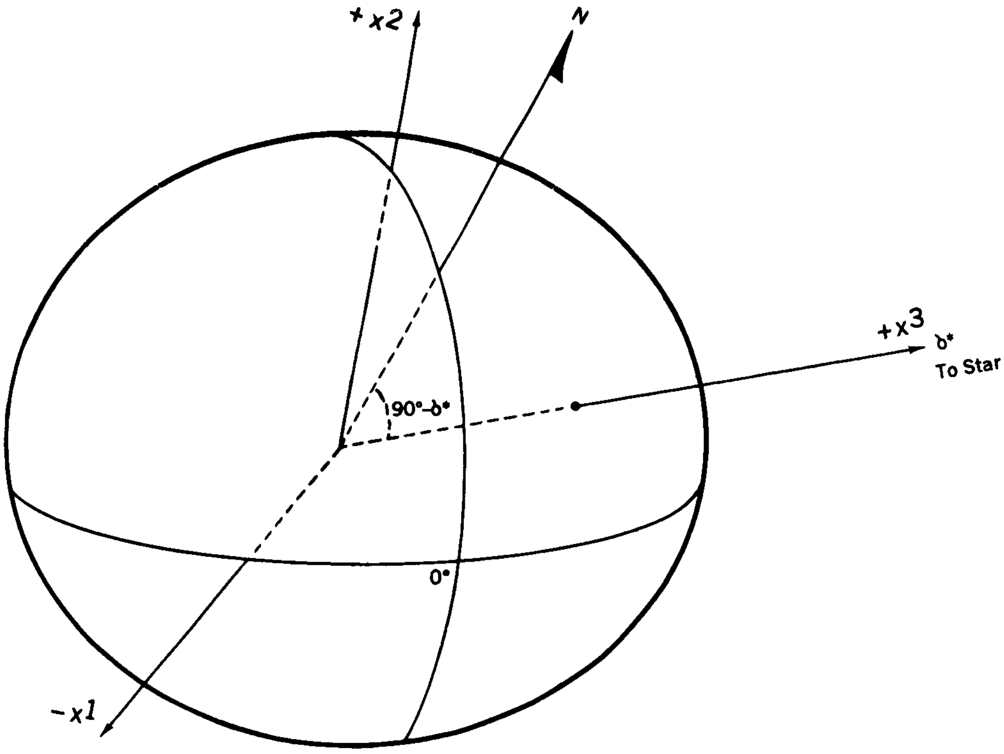



FIGURE 4

Rectangular coordinates (the $x^i$-system)

Here

$$R_{\mu*} = \begin{bmatrix} +\sin\mu_* & +\cos\mu_* & 0 \\ -\cos\mu_* & +\sin\mu_* & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{7}$$

$$R_{\delta*} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & +\sin\delta_* & +\cos\delta_* \\ 0 & -\cos\delta_* & +\sin\delta_* \end{bmatrix} \tag{8}$$

$\mu_*$ is the Greenwich hour angle of the star.

$$\mu_* \equiv GST - \alpha_* \tag{9}$$

$$GST = GST_{ho} + (\text{time of observation})$$

If the $[\bar{x}]$-system is referred to some space-fixed system instead of to an earth-fixed system, the relation between the two can be expressed as a rotation:

$$[R_\theta] = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Here the $[X]$ system has rotated through the angle $\theta$ since zero time when it made the angle $\theta_o$ with the space-fixed systems, and $\alpha_*$, $\delta_*$ are the right ascension and declination of the star (GST is the Greenwich hour angle of the star at the instant considered).

In observing an occultation, the optical axis of the telescope is kept pointed at the star, and the time at which the star disappears (or reappears if an emersion is being observed) is recorded. The optical axis is therefore constantly parallel to to the $x^3$-axis, and no distinction is made between events occurring at different points along this axis. As far as the observer is concerned, his world of events lies in the $x^1\ x^2$ plane. Hence it is customary to further simplify the mathematics by doing most of the computations in the $x^1\ x^2$ plane, which is called the "fundamental plane" after Bessel, who first introduced it. Coordinates in this plane are often denoted by $\xi^1$, $\xi^2$, or $\xi, \eta$.

These four systems are basic to the mathematics of occultations. There are a few others, principally those associated with the motion of the moon and with the lunar surface, but except for the above four, none plays any great role in the reduction of the observation data. Considerable care must be taken, however, in the use of these four systems. In the first place, the observations give data which are or can be referred to systems with origin at the center of the reference spheroid. The equations of the moon's motion, on the other hand, are set up in a geocentric system -- i.e., a system with origin at the earth's center of gravity; at the present time, the position of the earth's center of gravity with respect to any reference spheroid is known to better than ± 100 meters. Hence a certain amount of error is introduced; a more detailed study of this point is made in Part II. In the second place, if more than one observing site is used, and if the sites are on different datums, there will be a separate $x^i$-, $X^i$-, $E^i$-system for each site. The systems for the separate sites will be designated by subscripts.

B. Basic Equations

1. General:

There are four physical bodies which enter into the occultation equations; these are the star, the moon, the earth, and the observer. They are all tied together by one parameter -- the time. The star's position is given by the vector:

$$\vec{r}_* = (\alpha_*, \delta_*, r_*),$$

where $\alpha_*$, $\delta_*$ are the star's apparent place coordinates, but since the distance $\vec{r}_*$ is of no interest to occultations, the unit vector $\vec{r}_*(\alpha_*, \delta_*)$, which gives the direction of the star, can be used instead. $r_*$ is gotten in the usual manner from a catalog of star directions; the occulted stars lie within 30° of the celestial equator. In the above vectors, $\alpha_*$, $\delta_*$ and $r_*$ are all functions of the time $t$, but since the period of time over which a particular occultation can be usefully observed is less than three hours (usually less than 1.5 hours), $\vec{r}_*(t)$ for such stars is practically constant.

The moon's position is given by the vector:

$$\vec{r}_m(\alpha_m, \delta_m, r_m),$$

where $\alpha_m$, $\delta_m$ and $r_m$ are also time dependent. The elements of the vector are most easily gotten from one of the published lunar ephemerides (reference 6). They, of course, change appreciably within a period of three hours, and could be written explicitly as

$$r_m\left[\alpha_m(t), \ \delta_m(t), \ r_m(t)\right]$$

or, since ephemerides give the horizontal equatorial parallax instead of the distance, as

$$\vec{r}_m \left[\alpha_m(t),\ \delta_m(t),\ \pi_m(t)\right].$$

In the ephemerides the elements given refer to a point which is called the center of mass of the moon, and not to the center of the visible disk. Hence account must be taken, in correlating observations with the ephemeris predictions, of the displacement of the optical center from the ephemeris center. The equation of a spheroid in a space-fixed system is

$$\vec{r} \cdot S \cdot \vec{r} = 1 \tag{10}$$

where $\underline{S}$ is a dyadic with elements which are functions of time.

Since the occultation occurs at the <u>edge</u> of the moon, as viewed from the earth, the equation of the line on which the observer and star lie is:

$$\varkappa \vec{R} = \vec{r}_m + \vec{\rho}_m + \vec{r}_* \tag{11}$$

where $\rho_m$ is the vector from the center of the moon to the point on the surface at which the occultation occurs, and $\varkappa$ is a variable. The vector $\vec{\rho}_m$ is a complicated function of the time since the moon's motion is compounded from many independent motions such as librations, etc.

$$\vec{\rho}_m = \rho_m \left[\ell_m(t),\ b_m(t),\ \rho_m(t)\right]. \tag{12}$$

Here $\ell_m(t)$, $b_m(t)$, $\rho_m(t)$ are the selenographic coordinates of that point on the moon at which the line observer-to-star $\varkappa R$ is tangent when the occultation occurs. Our knowledge of the shape of the surface is insufficient to permit prediction of $\ell_m$, $b_m$, and $\rho_m$, and as mentioned before, certain assumptions or procedures must be adopted to get around this difficulty. Taking it for granted that this has been done, the solution is gotten by solving simultaneously the equations:

$$\varkappa \vec{R} = \vec{r}_m + \vec{\rho}_m + \vec{r}_* \tag{13}$$

$$\varkappa^2 1 = \vec{R} \cdot S \cdot \vec{R} \tag{14}$$

Because of the indeterminacy in $\vec{\rho}_m$, the first equation will be not that of a single line, but will define a set of generating elements of a cylinder; the common solution of these two equations will give a curve instead of a point. If n occultations are observed, the vectors in these equations become 3 x n matrices, and we have

$$\mathcal{X}[R] = [r_m] + [\rho_m] + [r_*] \tag{15}$$

$$2\,1 = [R]^T\,[S][\vec{R}] \tag{16}$$

When $n > 2$, the solution obviously becomes indeterminate since the rank of the matrix is then less than the order, and recourse must be had to least squares.

2. The Fundamental Plane Equations:

The general approach given above has not been used for computations because of the difficulty of solving the equations with small mechanical computers. Instead, as much of the computation as possible is done on the fundamental plane, $\overline{x^1\,x^2}$. When the equations are formulated in $\overline{x^1}$, $\overline{x^2}$ coordinates, a tremendous simplification takes place. $r_*$ disappears formally, since $r_*$ is parallel to $x^3$; $\vec{\rho}_m$ projected onto $x^1\,x^2$ becomes identical with the observed radius of the moon, and $r_m$ projected onto $\overline{x^1\,x^2}$ shrinks approximately to the size of the earth's radius or less. The radius vector from the spheroid center to the observer is also projected onto $\overline{x^1\,x^2}$, and the combination of the two equations (15) and (16) on the fundamental plane is then very simple.

Let $\vec{\sigma}$ be the projection of $\vec{\rho}_m$ onto the plane $\overline{x^1\,x^2}$, and let $\xi^3$ be the vector along the $x^3$-axis. Then

$$\vec{\sigma} = \left\{\vec{\rho}_m - (\vec{\rho}_m\,\vec{\xi}^3)\,\xi^3\right\} \tag{17}$$

which maps $\vec{\rho}_m$ into $\vec{\sigma}$. So,

$$\begin{aligned}
&\vec{r}_m\ [\alpha_m,\ \delta_m,\ r_m] \rightarrow [\xi_m(t)]\\
&\vec{r}_*\ [\alpha_*,\ \delta_*,\ r_*] \rightarrow 0\\
&\vec{\rho}_m\ [\ell_m,\ b_m\ \rho_m] \rightarrow [\sigma]\\
&\vec{r}\ [\lambda,\ \phi,\ h] \rightarrow [\xi(t)].
\end{aligned} \tag{18}$$

The simultaneous solution of (14) and (18) is given by

$$[\sigma]^T[\sigma] = \Big[[\xi] - [\xi_m]\Big]^T\Big[[\xi] - [\xi_m]\Big]. \qquad (19)$$

This is a second-degree equation. To make it easier to work with, it can be linearized by taking the differential of both sides, to give the first-degree equation in $[\Delta\xi]$ and $[\Delta\xi_m]$

$$\sigma\Delta\sigma = \Big[[\xi] - [\xi_m]\Big]^T\Big[[\Delta\xi] - [\Delta\xi_m]\Big]. \qquad (20)$$

From here on, the treatment is either by the <u>single-site</u> method or by the <u>equal-limb-line</u> method. The former will be treated first.

### C. <u>Single-Site Method of Survey</u>

The equation (20) above is so simple in form that it gives a misleading idea of the number of important quantities which actually influence the results. In the equal-limb-line method this is not too important, since most of these influencing factors are removed by the control observations; in the single-site method it is very important to identify every influencing factor and to evaluate its size in order to be able to decide whether the factor can be ignored or must be solved for along with the geodetic unknowns. To emphasize the presence of these factors, equation (20) is rewritten as:

$$\Delta\sigma = \frac{1}{\sigma}\left\{\Big[[\xi^i] - [\xi^i_m]\Big]^T\left[\frac{\partial\xi^i}{\partial u^k}\right] - \Big[[\xi^i] - [\xi^i_m]\Big]^T\left[\frac{\partial\xi^i_m}{\partial u^k}\right]\right\}\left[\Delta u^k\right] \qquad (21)$$

where

$$\left[\xi^i\right] = \left[\xi^i(u^k)\right], \quad \left[\xi^i_m\right] = \left[\xi^i_m(u^k)\right]$$

The $[u^k]$ are the factors whose influence on the results is to be determined. A large part of the evaluation is of a quantitative nature and will be covered in Part II, <u>Error Analysis</u>. At this point, only a general discussion of the particular factors involved is needed.

The most obvious factors are the geodetic corrections:

$$\Delta E^1 \equiv \Delta E, \text{ the correction to the easting}$$
$$\Delta E^2 \equiv \Delta N, \text{ the correction to the northing} \qquad (22)$$
$$\Delta E^3 \equiv \Delta H, \text{ the correction to the height above the spheroid,}$$

which enter into the variables $[\Delta\xi]$. The $[\Delta\xi_m]$ give, of course, the factors:

$$\begin{array}{c} \Delta\alpha_m \\ \Delta\delta_m \\ \Delta r_m \end{array} \qquad (23)$$

defining the corrections to the position of the center of the moon, and

$$\begin{array}{c} \Delta\ell_m \\ \Delta b_m \\ \Delta\rho_m \end{array} \qquad (24)$$

which define corrections involved in locating the occulting feature.

Implicit in the equation, however, and not literally present are a number of other quantities. First, there are the quantities

$$\begin{array}{c} \Delta\alpha_* \\ \Delta\delta_* \end{array} \qquad (25)$$

which were used to define the space-fixed directions of the $x^i$-system. Next, there are the quantities

$$\Delta x^i_{mo} \qquad (26)$$

which enter into the analysis because the equation was set up as if everything were referred to the observer's reference spheroid and datum at P, whereas, in fact, the position $[X_m]$ of the moon is taken from the Brown's Tables of the Motion of the Moon (reference 4)( or from an ephemeris based on these tables, reference 6), and these tables refer $[X_m]$ to a system which is at least approximately geocentric -- that is, which has its origin at the center of gravity of the earth. The vector $[\Delta X^i_{mo}]$ is therefore the displacement between the spheroidocentric system and the putative geocentric system origins.

Another quanity which must be considered is $\Delta T$, the error in the recorded time of occultation, which enters not only into $[\xi_m]$ but also, through $[R_\theta]$, into $[\xi]$. The time error is of two kinds: instrumental and astronomical. The former arises from errors in the mechanical equipment used in recording and reading the time associated with the occultation, while the latter arises principally from the difference between the time scale in which the observations are carried out (U.T.) and the "time" scale associated with the $[X_m]$ given in Brown's Tables (E.T.). This difference in scales arises, as is well known, from the irregular rotation of the earth (which rotation measures U.T.) as compared with the revolution of the moon about the earth (which measures E.T.), and from the failure of Brown's tables to represent the motion of the moon. The former effect is apparently very much greater than the latter.

In addition, there is a quantity $[\Delta \rho_m]$ analogous to the $[\Delta X^i_{mo}]$ which represents the displacement of the visible or geocentric center of the moon, which is what is observed, from the center of mass of the moon, which is given as $[X_m]$ in Brown's tables. Very little is known regarding the actual value of $[\Delta \rho_m]$; only the $\Delta b_m$ component (i.e., the latitude component) is known well enough to be given a value in computations.

The distance $\sigma$ itself contains several different types of error which, however, cannot be solved for but must either be eliminated by a least-squares adjustment, or allowed for in the reduction by use of the best available information. The most important of these is the quantity $h_m$, which is the height of the occulting feature above the lunar mean datum. $h_m$ is different for each occultation, hence cannot be found from the analysis; it can be estimated to within 200 meters from Hayn's charts (reference 5) or from the charts of Dr. Watts of the U.S. Naval Observatory (reference 6). Only the latter charts can be considered to be accurate to ± 200 meters, although Hayn's charts are contoured at 0".2 or 300-meter intervals.

When the list of factors is further investigated, it is found that the $[\Delta\alpha_*, \Delta\delta_*]$, $[\Delta\alpha_m, \Delta\delta_m, \Delta r_m]$ can be broken down still farther. The $[\Delta\alpha_*, \Delta\delta_*]$ will, like $h_m$, vary from occultation to occultation. In addition, because of the nature of occultations, they cannot be separated from the gross values of $[\Delta\alpha_m, \Delta\delta_m]$. Hence in the solution they will be lumped with $[\Delta\alpha_m, \Delta\delta_m, \Delta r_m]$. The corrections $[\Delta\alpha_m, \Delta\delta_m, \Delta r_m]$ to the moon's position are themselves influenced by the errors in the parameters of the moon's motion. These parameters may be variously chosen; the paper (reference 2) by Pamelia A. Henriksen selects the following correction factors:

$$
\begin{aligned}
\Delta r &\equiv \text{correction to the longitude of perigee} \\
\Delta\Omega &\equiv \text{correction to the longitude of the node} \\
\Delta e &\equiv \text{correction to the eccentricity of the moon's orbit} \\
\Delta i &\equiv \text{correction to the angle between lunar and stellar axes} \\
\Delta\xi &\equiv \text{correction to the obliquity of the ecliptic} \\
\Delta\varphi &\equiv \text{correction to the position of the equinox.}
\end{aligned}
\tag{27}
$$

The $\Delta T$ could also be included in the group of factors involved in the moon's motion.

Taking all the above correction factors as a set $[\Delta u^k]$, the equation (21) can then be used to determine the $[\Delta u^k]$ if sufficient values of $\Delta\sigma$ are available. If there are J observations ($j = 1$ to $J$, $J > K$), the observation equations are

$$
\left[\Delta\sigma_j\right] = \left[\frac{1}{\sigma_j}\right]\left\{\left[[\xi_j] - [\xi_{mj}]\right]^T \left[\frac{\partial\xi^i}{\partial u^k}\right]_j - \left[[\xi_j] - [\xi_{mj}]\right]^T \left[\frac{\partial\xi_m^i}{\partial u^k}\right]_j\right\}\left[\Delta u^k\right]
\tag{28}
$$

$$
\text{or } \left[\Delta\sigma_j\right] = \left[[\xi_j] - [\xi_{mj}]\right]^T \left[U_j\right] \cdot \left[\Delta u^k\right].
\tag{29}
$$

In order for the least squares adjustment to give reasonable values for $[\Delta u^k]$, there must be sufficient observations to provide an approximately Gaussian distribution for the $\Delta r_*$ and the $h_m$; i.e., $J \gg K$.

Then

$$\left[\Delta u^k\right] = \left\{\left[\left[\xi_j - \xi_{mj}\right]^T \left[U\right]_j\right]^T \left[\left[\xi_j - \xi_{mj}\right]^T \left[U\right]\right]\right\}^{-1} \left[\left[\xi_j - \xi_{mj}\right]^T \left[U\right]_j\right]^T \left[\Delta\sigma_j\right] \tag{30}$$

or, in abbreviated form,

$$\left[\Delta u^k\right] = \left[W\right] \left[\Delta\sigma_j\right]. \tag{31.1}$$

$$\left[X_1\right]_{Final} = \left[X\right] + \left[\Delta X_1\right].$$

The $[X]$ are related to the $[E]$ and the $[\Delta X]$ to the $[\Delta E]$ by equations 3, 4, 5 and the partials of these equations. However, the

$$\frac{\partial\left[R_\phi\right]}{\partial\left[X\right]} \quad \text{and} \quad \frac{\partial\left[R_\lambda\right]}{\partial\left[X\right]}$$

are zero, so that

$$\left[\Delta E\right] = \left[R_\phi\right]\left[R_\lambda\right]\left[\Delta X\right] \tag{31.2}$$

The problem cannot be considered solved by this equation since it is the position $[X]$ of the observer with respect to a geocentric system of axes which has been found, and this is of no value by itself. What is wanted is the relation of $[X_1]_{Final}$ to the $[X]$ of some standard datum and spheroid as, for example, North American Datum and the International spheroid, or British Datum and the Airy spheroid. In order to get this relation, a similar series of observations would have to be carried out independently on the standard reference datum and spheroid.

Let $[X_{g2}]$ be the geocentric coordinates of a point in a reference system different from that of $[X_1]$, and let $[X_{s2}]$ be the coordinates of this same point referred to the reference datum and spheroid of that system. Then

$$\left[X_{g2}\right] = \left[X_{s2}\right] + \left[\Delta X_{s2}\right], \tag{32}$$

and the $[\Delta X_s]$ are found as before from the equation

$$\left[\Delta u^k\right]_n = \left[W_n\right]\left[\Delta\sigma_{nj}\right], \tag{33}$$

and equation (31.2).

Since

$$\Delta u^k_1 = \Delta u^k_n$$

when two or more datums are involved it is best to solve all the observation equations simultaneously, except for the geodetic variables.

### D. Equal-Limb-Line Method of Survey

Return now to the equation (20) of section B,

$$\sigma \Delta \sigma = \left[[\xi] - [\xi_m]\right]^T \left[[\Delta\xi] - [\Delta\xi_m]\right] \tag{34}$$

The $\Delta\sigma$ can be separated into two parts:

$$\Delta\sigma \equiv \Delta\sigma \qquad \Delta\sigma \tag{35}$$

$\Delta\sigma_g$ is that part of the error in $\Delta\sigma$ which arises from errors in the geodetic coordinates of the observer, while $\Delta\sigma_m$ is that part of the error in $\Delta\sigma$ which arises from errors in the observed lunar radius (and hence includes error in $\vec{r}_m$, $\vec{\rho}_m$, and $\vec{r}_*$). If the occultation is observed at two different sites, $P_1$ and $P_2$,

$$\begin{aligned} \sigma_1 \Delta \sigma_1 &= \left[[\xi]_1 - [\xi_m]_1\right]^T \left[[\Delta\xi]_1 - [\Delta\xi_m]_1\right] \\ \sigma_2 \quad \sigma_2 &= \left[[\xi]_2 - [\xi_m]_2\right]^T \left[[\Delta\xi]_2 - [\Delta\xi_m]_2\right] \end{aligned} \tag{36}$$

First, point $P_2$ will be so chosen that

$$\Delta[\xi]_2 = 0. \tag{37}$$

This assumption will be more fully discussed in Part II, in the section on geodetic errors. For the present, it will be justified by the rather obvious statement that any datum and spheroid can be chosen as the basic datum and spheroid to which all positions are to be referred, in which case, points already geodetically connected to that datum can be considered as having no geodetic error. There will actually be some error associated with $\vec{r}_{20}$ and hence with $[\xi]_2$, and we could write:

$$\vec{r}_2 = \vec{r}_{02} \pm \vec{S}_2, \tag{37}$$

but at the moment it is enough to state that the datum and spheroid on which $P_2$ is located are the datum and spheroid taken as known. Then $P_2$ can usually be located anywhere within a fairly large area without violating the assumption, and:

$$\sigma_1 \Delta \sigma_1 = \left[[\xi]_1 - [\xi_m]_1\right]^T \left[[\Delta\xi]_1 - [\Delta\xi_m]_1\right] \qquad (39.1)$$

$$\sigma_2 \Delta \sigma_2 = \left[[\xi]_2 - [\xi_m]_2\right]^T \left[-[\Delta\xi_m]_2\right] \qquad (39.2)$$

Note that $[\xi]_1$ and $[\xi]_2$ and hence $[\Delta\xi_1]$ and $[\Delta\xi_2]$ refer to the observing sites' coordinates, while it is the corrections $[\Delta\xi]_{1d}$ and $[\Delta\xi]_{2d}$ to the datum coordinates which are sought. The relationship between $[\Delta\xi]$ and $[\Delta\xi]_d$ is not easily expressed in the $E^i$-system, where the rigidity of the geodetic net requires that

$$[\Delta E] = [\Delta E]_d \qquad (40)$$

Secondly, $P_2$ is chosen in such a manner that, at the time $t_2$,

$$\sigma_{m2} = \sigma_{m1} \qquad (41)$$

and therefore

$$\delta_{m2} = \sigma_{m1}.$$

This is in practice an extremely restrictive condition. Lack of accurate knowledge of the shape of the moon's surface makes it imperative that the occultations as observed from $P_2$ and $P_1$ have $(\ell_{m2}, b_{m2})$ as close as possible to $(\ell_{m1}, b_{m1})$. $\ell_m$ and $b_m$ are the seleonographic longitude and latitude of the point on the moon at which the occultation is seen to occur. When the moon's librations in latitude and longitude are resolved into components with axes parallel to $x^3$ and $x^2$, it is seen that only that part whose axis is parallel to $x^3$ can be accounted for. This component is a rotation in position angle $\psi$. Using a theorem discovered by O'Keefe,

$$\frac{dx}{dt} = \frac{2\pi}{T_m} \sin i_f \qquad (42)$$

where $T_m$ is the moon's sidereal period; $i_f$ is the inclination of the moon's axis of rotation to the fundamental plane; and $\psi$ is the position angle (reference 7). Use of this equation helps determine the coordinates of $P_2$, since

$$\sin\psi = \frac{\xi^2 - \xi^2_m}{\sigma}\ ; \quad \cos\psi = \frac{\xi^1 - \xi^1_m}{\sigma} \qquad (43)$$

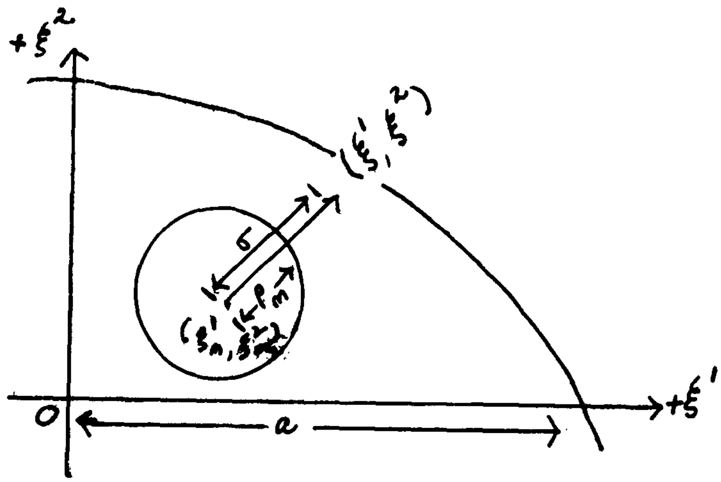
Once the coordinates $[\xi]_1$ are fixed, $[\xi]_2$ becomes a function of the time:

$$\left[\xi\right]_2 = \begin{bmatrix} \sin \psi_1 + \int_{t_1}^{t_2} \frac{2\pi}{T_m} \sin i_f \\ \cos \psi_1 + \int_{t_1}^{t_2} \frac{2\pi}{T_m} \sin i_f \end{bmatrix} + \left[\xi_m\right] \qquad (44)$$

This (second) condition therefore limits the choice of points available to $P_2$ to a line which is the locus of the projection of $\vec{r}_{m1}(\ell_{m1}, b_{m1})$ onto the fundamental plane.

Once $P_2$ has been chosen to satisfy the second condition, the final equation is derived by subtracting equation (39.2) from equation (39.1). Then

$$\begin{aligned} \sigma_1 \cdot \sigma_2 = & \left[ \frac{[\xi]_1 - [\xi_m]_1}{\sigma_1} \right]^T \left[ [\Delta\xi]_1 \right] \\ & - \left[ \frac{[\xi]_1 - [\xi_m]_1}{\sigma_1} \right]^T \left[ \Delta[\xi_m]_1 \right] \\ & + \left[ \frac{[\xi] \quad [\xi_m]_2}{\sigma_2} \right]^T \left[ \Delta[\xi_m] \right]_2 \end{aligned} \qquad (45)$$



But at $t_1$ and $t_2$, beaause of the choice of the coordinates of $P_2$

$$\sigma_1(t_1) = \sigma_2(t_2)$$
$$\Delta\sigma_{m1}(t_1) = \Delta\sigma_{m2}(t_2) \tag{46}$$

Hence as can also be seen immediately from the diagram,

$$\left[\frac{[\xi]_1 - [\xi_m]_1}{\sigma_1}\right]^T \left[\Delta[\xi_m]_1\right] = \left[\frac{[\xi_m]_2 - [\xi_m]_2}{\sigma_2}\right]^T \left[\Delta[\xi_m]\right] \tag{47}$$

and therefore

$$\Delta\sigma_1 - \Delta\sigma_2 = \Delta\sigma_{g1}$$

$$\Delta\sigma_{g1} = \left[\frac{[\xi]_1 - [\xi_m]_1}{\sigma_1}\right]^T \left[\Delta\xi\right]_1 \tag{48}$$

This is the basic equation for the equal-limb-line method of survey giving the geodetic corrections $[\xi]_1$ as a function of the difference $[\Delta\sigma_g]$ between the observed radii of the moon at sites $P_1$ and $P_2$.

The $[\Delta\xi]$ must, of course, be related to the $[\Delta E_d]$ which are the corrections to the datum of $P_1$ on the surface of the earth.

The relations are provided by the equations

$$\begin{aligned} [\xi] &= [R\,\delta_*]\,[R\,\mu_*]\,[X] + [R\,\delta_*]\,[R\,\mu_*]\,[R_\lambda]^{-1}\,[R_\phi]^{-1}\,[E] \\ [\xi_m] &= [R\,\delta_*]\,[R\,\mu_*]\,[X_m] + [R\,\delta_*]\,[R\,\mu_*]\,[R_\lambda]^{-1}\,[R_\phi]^{-1}\,[E_m] \end{aligned} \tag{49}$$

Here $[X]$ and $[X_m]$ refer to the usual astronomic coordinate system with center at the center of spheroid, $X^3$-axis along the polar axis, and $X^1$ parallel to the meridian of Greenwich. $[E]$ and $[E_m]$ are topocentric systems with origin at the observer's local reference point, $E^3$, along the normal at that point, and $[E^1]$ tangent to a parallel of latitude. The $[R]$'s are the obvious rotation matrices which rotate from the $[X]$ and $[E]$ systems to the fundamental plane system as indicated by subscripts.

Writing

$$[R] = [R\,\delta_*]\,[R\,\mu_*]$$
$$[P] = [R\,\delta_*]\,[R\,\mu_*]\,[R\lambda]^{-1}\;[R\varphi]^{-1} \qquad (50)$$

and differentiating,

$$[\Delta\xi] = [P]\,[\Delta E] + [R]\,[\Delta X]$$
$$+\left[\frac{\partial P}{\partial u^n}\right][E]\,[\Delta u^n] + \left[\frac{\partial R}{\partial u^n}\right][X]\,[\Delta u^n] \qquad (51)$$

where $[u^n] = [\delta, \mu, \lambda, \varphi, h]^T$ for n = 1, 2, 3, 4, 5, in that order.

The quantity $\Delta u^2$ can be separated into two parts:

$$\Delta u^2 = \Delta u^{21} + \Delta u^{22}$$
$$\Delta u_* = \Delta\varphi - \Delta\alpha_* \qquad (52)$$

where $\Theta$ is the G.S.T., $\Delta\Theta$ is certainly less than $0^s.01$ or 5 meters, and can be neglected in the computations. It would, in any case, be random in character and would therefore be among the quantities appearing in the r. m. s. error after the least squares adjustment. The errors $\Delta\lambda$, $\Delta\varphi$, and $\Delta h$ are survey corrections; these could presumably be found if enough observations were available. For most sites, however, the total number of equal-limb-lines cannot be expected to be more than six or seven in number, and it would not be desirable to include $\Delta\lambda$ and $\Delta\varphi$ among the unknowns to be found. Since the two quantities are certainly not greater than $\Delta\Theta$, these also will be ignored in the solution. This leaves the unknowns $\Delta\delta_*$ and $\Delta\alpha_*$ and here again an analysis of the expected magnitudes shows that the errors contributed by each will not exceed a few meters. The total error caused by dropping $\left[\Delta u^n\right]$ altogether -- i.e., assuming

$$\left[\Delta u^n\right] = \left[0\right] \tag{53}$$

is less than 10 meters, which can be tolerated. Equation (51) therefore becomes

$$\left[\Delta \xi\right] = \left[P\right] \left[\Delta E\right] + \left[R\right] \left[\Delta X\right] \tag{54}$$

all quantities being evaluated at point $P_1$ (and therefore at time $t_1$). The quantities $\left[\Delta E\right]$ and $\left[\Delta X\right]$ both represent corrections to the position of $P_1$; the latter, however, is the error in an arbitrarily selected quantity, the assigned coordinates of the origin of the system at $P_1$, and hence

$$\left[\Delta X\right] \equiv 0. \tag{55.1}$$

The final relationship is therefore

$$\left[\Delta \xi\right] = \left[P\right] \left[\Delta E\right] \tag{55.2}$$

and the elements of $\left[P\right]$ are the direction cosines

$$\cos(\gamma^i E^j).$$

When this is substituted back into the basic equation (48), there results:

$$\Delta \sigma_{g1} = \left[\frac{\left[\xi\right]_1 - \left[\xi_m\right]_1}{\sigma_1}\right]^T \left[P\right]\left[\Delta E\right] \tag{56}$$

or

$$\Delta \sigma_{g1} \equiv \left[A\right]^T \left[\Delta E\right]_1 \tag{57}$$

where $\left[A\right]$ is a vector.

If j observations are made from $P_1$ and $P_2$, the coordinate corrections to $E_1$ are given by the equation:

$$\left[\Delta E_1\right] = \left\{ \left[A_j\right]^T \left[A_j\right] \right\}^{-1} \left[A_j\right]^T \left[\Delta \sigma_j\right]. \tag{58}$$

The variances $\left[S^i_m\right]$ associated with the $\left[\Delta E_1\right]$ are given by:

$$\left[A^{1j}\right] \left[A_{j1}\right] \left[S^i_m\right] = S\left[\delta^{\ell}_m\right] \tag{59}$$

where the S is the variance of the observation error distribution. These $\left[S^i_m\right]$ are referred to the $E^i$-system by rotation of the axes; the off-diagonal variances can be reduced to zero, since the $\left[S^i_m\right]$ matrix is symmetrical. Hence, in the two-dimensional case, it is possible to find a matrix $V^{\ell}_k$ such that

$$V^{\ell}_k = S^{\ell}_k \; V^{\ell}_k. \tag{60}$$

The rotation Ⓗ is given by

$$\left[V^{\ell}_k\right] = \left[R\right]^T \left[S^i_m\right] \left[R\right] \tag{61.1}$$

where

$$\left[R\right] = \begin{bmatrix} \cos H & \sin H \\ -\sin H & \cos H \end{bmatrix} \tag{61.2}$$

or, when the variance $\left[\Delta E^3\right]$ is explicitly included,

$$R = \begin{bmatrix} \cos H & \sin H & 0 \\ -\sin H & \cos H & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{61.3}$$

These equations define the semi-major axis, $V^1_1$, and the semi-minor axis, $V^2_2$, of the ellipse, as well as the angle Ⓗ between the semi-major axis and the $E^1$-axis.

These equations are easily generalized to the n-dimensional case, which is covered briefly in Appendix E.

Because of the few observations available from any one site, it is highly desirable to restrict the solutions at first to those unknowns whose errors are greatest. The component is hence usually set arbitrarily equal to zero:

$$\Delta E^3 = 0,$$

and the solution limited to the determination of $\Delta E^1$ and $\Delta E^2$, or, what is the same thing for the purpose of computation, $\Delta \lambda \cos$ and $\Delta \phi$. In this connection it is worth pointing out again that the occultation data can provide no information about the $x^3$-component of events. In occultations involving point $P_1$ and $P_2$, which are 2000 km or more apart, the many practical restrictions on the observation conditions -- per cent illumination of the moon, time after evening twilight or before morning twilight, etc. -- often combine to force the observations at $P_1$ to be made either near the horizon or near the zenith. In the first case the $E^1$-component will be poorly determined by the solution, and in the second case the $E^3$-component would be poorly determined.

Up to now it has been assumed that the corrections are being applied directly to the site coordinates. For a proper solution the corrections should be applied to the datum coordinates, since it is assumed in the derivation of the equations that the site coordinate errors result from the datum coordinate errors (survey errors being neglected, as explained earlier). Where the observation sites are only a hundred kilometers apart, no significant error is introduced by neglecting the difference between $[\Delta E^i]_{observer}$ and $[\Delta E^i]_{datum}$. At distances much greater than this, such a simplifying assumption cannot be made, and a transformation from $[E^i]_{observer}$ to $[E^i]_{datum}$ is necessary.

This transformation can be made by using the formulas of Hristow (reference 8) connecting the change in position of a point $P_1$ to the change of position of a point $P_2$. These formulas can be put into the form:

$$\begin{bmatrix} d\lambda \cos\phi \\ d\varphi \\ dA \\ dS \\ da \\ df \end{bmatrix}_{obs.} = \begin{bmatrix} T \end{bmatrix} \begin{bmatrix} d\lambda \cos\phi \\ d\varphi \\ dA \\ dS \\ da \\ df \end{bmatrix}_{datum} \tag{61.4}$$

where $\begin{bmatrix} T \end{bmatrix}$ is a function of the separation $\Delta\phi$, $\Delta\lambda$ between $P_2$ and $P_1$. For all practical purposes, df and da, the changes in the flattening and in the semi-major axis of the spheroid, can be set equal to zero when only a small number of observations are available. When many observations are available, covering a large area of the world, a solution including df and da could be significant. This is not the case at present.

Since $P_2$ is tied to $P_1$ by geodetic survey, it may be assumed that ds and $DA_{datum}$ are not to be changed by the translation. The resulting equation is therefore:

$$\begin{bmatrix} \Delta E^1 \\ \Delta E^2 \\ \Delta E^3 \end{bmatrix}_{observer} = \begin{bmatrix} T \end{bmatrix} \begin{bmatrix} \Delta E^1 \\ \Delta E^2 \\ \Delta E^3 \end{bmatrix}_{datum} \tag{61.5}$$

where $\begin{bmatrix} T^1 \end{bmatrix}$ is the matrix $\begin{bmatrix} T \end{bmatrix}$ suitably modified (see Appendix F). This transformation can now be introduced into (57) to give:

$$\Delta \sigma_{g1} = \begin{bmatrix} A \end{bmatrix}^T \begin{bmatrix} T^1 \end{bmatrix} \begin{bmatrix} \Delta E \end{bmatrix}_{datum\ 1} \tag{61.6}$$

which becomes

$$\left[\Delta E\right]_{datum} = \left\{\left[T^1\right]^T\left[A\right]\left[A\right]^T\left[T^1\right]\right\}^{-1}\left\{\left[T^1\right]^T\left[A\right]\right\}\left[\Delta\delta\right] \qquad (61.7)$$

analogously to (58).

The above equations /(57) - (61.7)/ have been formulated for the case where only two points, $P_1$ and $P_2$ were involved, and where $P_1$ was the "unknown" point and $P_2$ the control point. In an occultation survey program covering a large area, the distinction between "unknown" and "control" points will become largely artificial, with a particular site $P_k$ being referred sometimes to an "unknown", sometimes to a "control" datum. A network of occultation sites will therefore be set up in the area being surveyed, and the equation (61.7) must be modified accordingly. In the light of the discussion in Section C of the Single-Site Method Survey, it is obvious that an adjustment of an equal-limb-line network in which no one point is held fixed (i.e., for which $\left[\Delta\xi\right] = 0$, will not have any meaning.) Some point $P_k$ in the network must be assumed known, so that for it

$$\left[\Delta\xi_k\right] = 0 \qquad (62)$$

The occultation equations will then be /referring back to equation (57)/:

$$\begin{aligned} \Delta\sigma_j^k &= A_{ji}^{kn}\,\Delta E_n^i \\ \Delta E_k^i &\equiv 0 \end{aligned} \qquad (63)$$

and the solution will be given by:

$$\Delta E_n^i = \left\{\left[A_{ji}^{kn}\right]^T\left[A_{ji}^{kn}\right]\right\}^{-1}\left[A_{ji}^{kn}\right]^T\left[\Delta\sigma_j^k\right] \qquad (64)$$

or the suitable variation according to (61.7). The variances are then found as before, by an obvious extension of the previous equations.

Suppose D distinct datums are involved, and let these datums be numbered $d = 1, 2, 3, \ldots\ldots D$. Each datum is to be fixed on the basis of observations from s sites, referred to that datum, so that

$$s_d = 1_d, 2_d, 3_d, \ldots\ldots sd$$

Furthermore, each site will have observed $q_{sd}$ occultations

$$q_{sd} = 1_{sd}, 2_{sd}, 3_{sd}, \ldots\ldots Q_{sd}.$$

To simplify the notation, this can be written:

$$\begin{aligned} d &= 1, 2, 3, \ldots. D \\ sd &= 1d, 2d, 3d, \ldots. sd \\ qsd &= 1sd, 2sd, 3sd, \ldots. Qsd. \end{aligned}$$

Then

$$\left[\Delta \sigma_{qsde}\right] = \left[A^{i}_{qsd}\right]\left[\Delta E^{i}_{sd}\right] + \left[B^{i}_{qsd}\right]\left[\Delta E^{i}_{te}\right] \qquad (64.1)$$

where

$$t, e \neq s, d.$$

From (61.5) and (61.7) and the above equation (64.1), it then follows that the final set of equations can be written as:

$$\left[\Delta\sigma_{qsde}\right] = \left[A^{i}_{qsd}\right]\left[T^{i}_{sd}\right]\left[\Delta E^{i}_{d}\right] + \left[B^{i}_{qsd}\right]\left[T^{i}_{te}\right]\left[\Delta E^{i}_{e}\right] \tag{64.2}$$

$$= \left[A^{i}_{qsd}\right]\left[\Delta E^{i}_{d}\right] + \left[H^{i}_{qsd}\right]\left[\Delta E^{i}_{e}\right]$$

or, enlarging the vectors and matrices and inverting,

$$\Delta E^{i}_{j} = \left\{\left[K\right]^{T}\left[K\right]\right\}^{-1}\left[K\right]^{T}\left[\Delta\sigma\right] \tag{64.3}$$

where the $\left[\Delta E\right]$ vector now contains all the datum corrections.

When more than two sites lie on the same equal-limb-line, and only two sites are involved, it is obvious that the total addition to the geodetic information is not equal to the number of permutations available, but is actually a good deal less. These cases can be taken care of by weighting the observations. For example, the most common case, where two observations are made from one end of a line and one observation from the other, could lead to three equations involving.

$$\Delta\sigma_{12},\ \Delta\sigma_{23},\ \Delta\sigma_{31}$$

where $P_2$ and $P_3$ are on one datum and $P_1$ on another. The quantity $\Delta\sigma_{23}$ is of no importance to the adjustment scheme; $\Delta\sigma_{12}$ and $\Delta\sigma_{31}$ are of importance in relating the two datums and the single observation there and give two equations. The point $P_1$ is, however, common to both equations. Whereas, if the equations were being solved by the method of individual occultations (Part II), there would be three equations corresponding to the three values $\Delta\sigma_{g1}$, $\Delta\sigma_{g2}$, $\Delta\sigma_{g3}$, as opposed to the four equations if there were four independent observations. Hence each of the two $\Delta\sigma_{1j}$ equations is to be weighted by 3/4 for a total of 3/2. A table for relative weighting of observation equations referring to the same equal-limb-line can be drawn up in the following manner to take care of the most common cases.

RELATIVE WEIGHTING

| No. of Sites on Line | | No. of Equations | Weight per Equation | Total Weight |
|---|---|---|---|---|
| Datum 1 | Datum 2 | | | |
| 1 | 1 | 1 | 1 | 1 |
| 1 | 2 | 2 | 0.75 | 1.5 |
| 1 | 3 | 3 | 0.5 | 1.5 |
| 2 | 2 | 3 | 0.75 | 2.25 |

E. <u>Other Forms of the Occultation Equations</u>

The fundamental relation between $[\xi]$ and $[\xi_m]$ can also be written as

$$[\sigma_j] = [\xi_j][\xi_{mj}]^T[\sin \chi_j]$$

$$[\Delta\sigma_j] = [\Delta\xi_j - \Delta\xi_{mj}]^T[\sin\chi_j] \qquad (65)$$

$$+ [\xi_j - \xi_{mj}]^T[\cos\chi_j \quad \Delta\chi_j]$$

(since $x_2 = \frac{\pi}{2} - \chi$).

Going through the same procedure as before, it is easy to show that the final equations are, formally,

$$[\Delta\sigma_j] = [\Delta s_j] + [\Delta s_j]^T[\Delta\chi_j] \qquad (66)$$

where $[\Delta\sigma_j]$ has the usual meaning. $[\Delta S_j]$ is the $\overline{x^1 x^2}$-plane component of the correction to $[E]$, and $[S_j]^T[\Delta\chi]$ is the $x^1 x^2$-plane component perpendicular thereto. The equations can be considered as giving the correction components $[\Delta E]$ or $[\Delta\xi]$ in still another system, the $s^1$-system, where the axes are designated as

$$\begin{cases} s^1 = s \\ s^2 = \gamma \\ s^3 = \xi^3 \end{cases} \qquad (67)$$

The s-axis makes an angle $\chi^1$ with the $\xi^1$-axis, and the x-axis makes an angle $\chi$ with the $\xi$-axis. Then

$$\begin{aligned}\left[\Delta r\right] &= \left[R\right]\left[E\right] + \left[r\right]\left[E\right] \\ &\equiv \left[Q\right]\left[E\right]\end{aligned} \tag{68}$$

The rotation matrices $[R]$, $[r]$ and $[Q]$ then create vectors $\vec{R}$, $\vec{r}$, and $\vec{Q}$, whose sum is $\overrightarrow{\Delta\sigma}$, which can itself be considered as a component of a 3-vector $[\Delta\sigma]$ whose components $\sigma^2$ and $\sigma^3$ are identically zero. The rest of the solution can be carried out just as before and is no different mathematically from the previous methods. The interest lies in the fact that $[\Delta\sigma_j]$ can be transformed into a correction in the $E^1$ $E^2$-plane,

$$\begin{aligned}\left[R_\xi\right]\left[\Delta\sigma\right] &= \left[R_\xi\right]\left[\Delta S\right] + \left[R_\xi\right]\left[S\right]^T\left[\Delta\chi\right] \\ \Delta S_j &= A_j \quad \Delta r \quad + \quad B_j \quad \Delta P\end{aligned} \tag{69}$$

where $\Delta r$ and $\Delta P$ are the radial and tangential components, respectively, of the corrections. If only one of these is solved and the other is ignored, then the question is whether to set $\Delta r$ or $\Delta P$ equal tozzero for a particular occultation. If $\Delta P = 0$, equation (69) defines lines

$$1 = (E^1_j - A_j \;\Delta r \;\sin\; \Theta_j)\;\sin\; \Theta_j \tag{70}$$

and the corrected position of the point $P_1$ is obtained by solving these equations for the intersection of the straight lines. If, on the other hand, $\Delta r = 0$, equation (69) defines circles of radius $B_j\,\Delta P$, and the corrected $E^1$ and $E^2$ lie at the common intersection of these circles.

### F. Relative Position Method

The equations and methods described heretofore are satisfactory if a set of related geodetic positions referred to one particular datum is being built up. This is especially true when the set is global, so that the positions can all be referred to an "absolute" coordinate system (in this case that system to which the lunar orbit is referred). Where no such unified set is envisioned or possible, however, the previous equations are open to the objection that they contain implicitly the assumption that an "absolute" coordinate system is involved when it is postulated that

$$\Delta X^i_2 = 0$$

for one of a pair of occultation sites, the implication is present that the lunar orbit is also fully defined for that site. This is not so, and the resulting error will not be removed unless a global set of positions is involved.

To make the equations useable for relative position determination, there is no change required in the equations other than to drop the condition

$$\Delta X_2^i = 0,$$

and to substitute for it some relation which does not involve knowledge of the absolute coordinate system. The required relation is arrived at through the obvious, and simply proven, considerations that the vectors $\Delta X_1^i$ and $\Delta X_2^i$ must, if they represent relative shifts, be equal in magnitude but opposite in signs; the unknown relation between the lunar orbit coordinate system and the local geodetic systems is the same at both places. The resulting equation is therefore:

$$\left[a_{ik}^2\right] + \left[a_{ik}^1\right]\left[\Delta X^i\right] = \sigma_{21k},$$

which is solved for the $\left[\Delta X^i\right]$ by the same method as before.

There is no reason why this equation cannot also be used in setting up a global, connected set of positions. From the relative nature of the $\Delta X_j^i$, it is obvious, of course, that the end product will be "floating" in the sense that two (if $\Delta X_j^3$ is involved) or three parameters must be arbitrarily specified unless the connection is closed.

The preceding discussion has covered principally the data analysis aspect of occultations. For the prediction of occultations, the same equations would, of course, apply. However, additional work must be done to select the stars, compute the "equal-limb-line positions, and so on. The procedures have been put into a form suitable for mechanical calculation by the Department of Geodesy, Army Map Service; these procedures will be published in a forthcoming report entitled "Mechanical Procedures for Occultation Prediction."

## PART II. ERROR ANALYSIS

### Introduction

The equations which govern the use of occultations for surveying are incomplete as they are given in Part I of this report. since they do not state the range of errors which can affect the results. These errors in the results arise from errors in the constants and original data, and their sizes are decided by both the sizes of the input errors and the equations themselves. An analysis of the errors therefore concerns itself first with the sources of errors in the input, and secondly with the growth of these errors as they pass through the occultation equations.

As is seen from the equations in Part I, the sources of error are:

1. Star positions
2. Moon positions
3. Geodetic coordinates of the observer
4. Timing of the occultation.

Errors in star position and moon position combine with the (unknown) lunar radius at the point of occultation to produce an apparent lunar radius which is greater than or less than the standard assumed radius of the moon. Errors in geodetic position, by moving the observers off the "equal-limb-line" positions, affect the results by amounts $\Delta g$ which are approximately,

$$\Delta g = \frac{\partial hm}{\partial P} \Delta \phi$$

where $\frac{\partial h}{\partial P}$ is the average slope in the vicinity of the occultation feature on the moon, and $\Delta \phi$ is the error in latitude. Errors in timing are transformed into errors in position correction which are approximately equal to the time error multiplied by the average velocity of the moon's projection onto the observer's horizon plane.

## II. Error Analysis

### A. Error Sources

#### 1. Star Position Uncertainties:

The occultation survey method, using two independent observers, is not sensitive to reasonably sized errors in the star coordinates. Such errors, practically inseparable from errors in the moon's position, cancel out almost completely in the first-order approximation to the exact occultation equations. Sample computations used in the prediction star, positions differing by 1", have given paths differing by only 1 or 2 meters over a 1000-kilometer length. However, it is obviously desirable to reduce all errors to a minimum consistent with computational ability so as to produce the number of unknowns that must be accounted for.

The source of the star positions used is of major importance here. Since only stars occulted by the moon need be considered, attention can be restricted to the zodiacal zone. Furthermore, it is desirable to have as homogeneous a set of stars as possible. The requirements of (a) high precision, (b) zodiacal zone, (c) homogeneity, and (d) adequate coverage to 9th magnitude, effectively limit the choice of sets to (1) the Robertson Zodiacal Catalog (ref. 17) and (2) the Yale Zone Catalogs between +30° and -30° (ref. 18) The FK4 contains too few stars to be useful as does the N30; the Boss General Catalog is inaccurate (one second of arc or more r.m.s. error). Hence, all stars used have been from either the Robertson or Yale Zone catalogs, with corrections being made where necessary for the systematic differences between the two. The necessary corrections are made by using the conversion tables given in the Yale Zone and Robertson catalogs, going in each case from the system in question to the Boss General Catalog and thence to the appropriate second system.

Table I. Star Position Error

| Catalog | Absolute Value of p. e. |
|---|---|
| Yale Zone (1) | < 0"5 |
| Zodiacal (2)(3) | < 0"5 |

(1) Evaluated from Yale Zone Catalogs for Zones between +30° and -30° declination. See Table II.

(2) Approximate evaluation from 50 random samples in catalog.

(3) Systematic differences between the Zodiacal Catalog and the General Catalog run from about +0"45 to -0"22 in right ascension and from +0"26 to +0"07 in declination.

TABLE II. Probable Errors in Direction
Yale Zone Catalogs

| Zone | $\alpha$ | $\delta$ | $d\alpha/dt$ | $d\delta/dt$ | 1960 Total |
|---|---|---|---|---|---|
| -30 to -27 | ± 0.105 | ± 0.105 | ± 0.011 | ± 0.010 | ± 0.41 |
| -22 to -27 | ± 0.105 | ± 0.105 | ± 0.011 | ± 0.010 | ± 0.41 |
| -20 to -22 | ± 0.105 | ± 0.105 | ± 0;011 | ± 0.009 | ± 0.41 |
| -18 to -20 | ± 0.120 | ± 0.105 | ± 0.011 | ± 0.009 | ± 0.43 |
| -14 to -18 | ± 0.120 | ± 0.105 | ± 0.012 | ± 0.009 | ± 0.43 |
| -10 to -14 | ± 0.120 | ± 0.105 | ± 0.011 | ± 0.011 | ± 0.46 |
| - 6 to -10 | ± 0.105 | ± 0.105 | ± 0.008 | ± 0.008 | ± 0.32 |
| - 2 to - 6 | ± 0.105 | ± 0.105 | ± 0.008 | ± 0.008 | ± 0.32 |
| + 1 to - 2 | ± 0.115 | ± 0.115 | ± 0.006 | ± 0.007 | ± 0.24 |
| + 5 to + 1 | ± 0.115 | ± 0.115 | ± 0.007 | ± 0.007 | ± 0.27 |
| +10 to + 5 | ± 0.115 | ± 0.115 | ± 0.008 | ± 0.007 | ± 0.27 |
| +15 to +10 | ± 0.115 | ± 0.115 | ± 0.008 | ± 0.006 | ± 0.26 |
| +20 to +15 | ± 0.115 | ± 0.115 | ± 0.005 | ± 0.006 | ± 0.23 |
| +25 to +20 | ± 0.095 | ± 0.095 | ± 0.005 | ± 0.005 | ± 0.26 |
| +30 to +25 | ± 0.095 | ± 0.095 | ± 0.007 | ± 0.005 | ± 0.31 |

From an analysis of the errors introduced by each of the two catalogs, as summarized in Tables I and II, it appears that there is no significant difference between the errors of the two catalogs (proper motions being taken into account). Except for systematic differences between the two catalogs, there is no hindrance to the simultaneous use of both catalogs in the reduction of occultations. The different backgrounds of the two catalogs (the Robertson Catalog is "fundamental" and relates to an earth-fixed system, while the Yale Zone Catalog is differential and hence more "absolute" in nature) have no significance as far as the use of the catalogs for occultations is concerned.

In summary, then, it can be assumed that the probable errors in right ascension and declination of the directions of the stars used are less than $0''.5$, and hence that the standard deviations are less than $0''.75$. The corresponding error introduced into the computed geodetic position is less than 2 meters.

2. Moon Position Uncertainties

The phrase "center of the moon" is inadequate for defining a point since the definition used is operational and actually varies with the method being studied. There are three kinds of lunar centers which are of interest:

a. The center of mass, or, more precisely, the center of gravity.

b. The optical center.

c. The operational center.

The center of gravity of the moon is at present defined by Brown's equations for the motion of the moon (ref. 4) as modified in the Improved Lunar Ephemeris (ref. 19). It is a set of coordinates (latitude, longitude, and parallax) to which the center of the moon is said to belong, and which are computable as a function of time alone. These coordinates, established by the equations of motion, do not correspond to any point that can be located by observation, because (a) the theory is based on values of basic parameters, such as the size and shape of the earth, whose errors are not known; and (b) the theory refers its predictions to the gravitational center of the earth approximately, while actual observations do not have access to this center.

The right ascension and declination of the moon are given in the Improved Lunar Ephemeris (ref. 19), and in national ephemerides of 1960 and following, to:

| | |
|---|---|
| 0".015 | right ascension |
| 0".01 | declination. |

These were obtained, however, by interpolation in the computed longitude and latitude of the moon as given by Brown's equations, and are really limited in precision only by the abilities of the computing machine used to evaluate the equations, and could be carried out to any number of figures. The constants entering into the computation are good to $10^{-5}$ seconds of arc, and this together with neglect of terms, machine rounding-off error, etc., limits computed positional precision to about ± 0".001. As is discussed further under Time Uncertainties, the precision is here also the accuracy. The reason for this is that at the present time all discrepancies between the computed coordinates and coordinates obtained from observations are assumed to be caused by irregularities in the observations, and primarily in the real time. As far as occultations and most other observational techniques are concerned, the positional uncertainty in the gravitational center of the moon can be taken as:

(a) From published ephemerides $< 0''.02$

(b) From special ephemerides $< 0''.002$

(c) From specific computations $\sim 0''$

b. The only center for the moon which is "directly" observable is the optical center. Even the gravitational center of the moon is not free of optical influences, since the constants entering into the equations of motion were determined from optical observations. It is true that the connection is tenuous, since a very large number of optical observations enter into the constant determinations, so that random variations could be expected to be averaged out. Systematic effects do appear and must be accounted for; the initial work by Brown (ref. 1, 4) gives a value of -0".5 for the difference in latitude between the optical center and the gravitational center. This value accounts explicitly for all residuals in the latitude equation not otherwise accounted for, and hence is not entirely satisfactory. Furthermore, there is an asymmetry between the eastern and western portions of the lunar limb, as is shown by existing profile measurements. This asymmetry does not show up in the equations of motion.

A more direct way of finding the optical center is to fit circular or elliptical arcs to lunar profiles and to resect from these. Or, what is the same thing, to fit a circle or ellipse to profile measurements to give best fit in the least squares sense. The latest and probably best measurement of the relation between center of figure and center of gravity is that of Watts and Scott (ref. 23), which gives - 0".60 ± 0".08 for the difference in latitude.

The value given in the American Ephemeris and Nautical Almanac to be used for the calculation of eclipses is - 0".5.

c. The definition of the moon's center is that in which the observations (occultations, eclipses, etc.) themselves provide the center. In this way, centers peculiar to each method arise. For example, a careful statistical analysis in 1952 by Mr. G. Reuning of some 2000 visually observed occultations gave an optical center minus ephemeris center value of $\Delta\delta$ = 0".4 (ref. 24). Similarly, eclipse reductions provide temporary centers pertaining to the eclipse.

TABLE III. Lunar Position Uncertainties

| Uncertainty in: | | Absolute | Relative |
|---|---|---|---|
| Difference between Optical and Gravitational Centers | | 0".1 | ~ 0" |
| Orbit | (2) | < 0".02 | ~ 0" |
| Right Ascension | | < 0".015 | ~ 0" |
| Declination | | < 0".01 | ~ 0" |
| Radius | (3) | 0 | 0 |
| Height | (4) | 0".12 | << 0".09 |

To compare a theoretical position of the moon, as defined by the Improved Lunar Ephemeris, with the actual position of the moon as defined by, say, transit or moon camera observations, is not very easy, since instrumental errors must be also accounted for. A simpler and almost as valid method of comparison is to compute the difference between the Improved Lunar Ephemeris position and a position based on the same equations but with more "modern" values for some of the fundamental constants.

If we take the following values as possible values,

| | |
|---|---|
| a | 6,378,150 meters |
| f | 1/298.3 |
| $g_E$ | 978.036 cm/sec$^2$ |
| M/E | 1/81.32 |

for the semi-major axis of the earth-fitting spheroid, the flattening, the mean gravitational acceleration at the equator, and the ratio of lunar to terrene mass, then the moon's horizontal equatorial parallax as given by Brown's equations requires correction by a factor

$$1 - 6.72 \times 10^{-5}.$$

while the lunar ephemeris computed by the various observations from these equations requires a correcting factor:

$$1 - 2.05 \times 10^{-5}.$$

Furthermore, all terms with the coefficient $\alpha_1$, which contain M/E and a, must be multiplied by the correction factor

$$1 - 1.805 \times 10^{-3}.$$

A list of those terms which are sufficiently changed to affect the position by 0".001 is given in Table IV. Further changes, in addition to those in which the dynamical parallax occurs directly, are required by a change from Brown's value of 1/294 for the flattening to 1/298.3. These changes are of the size

$$1 - 2.92517 \times 10^{-2}.$$

Terms affected to the extent of 0".001 are listed in Table V.

TABLE IV. List of Terms Affected by Changes in Major and Minor Axes Dimensions of Ellipsoid

| Number | Value | Change | Affected |
|---|---|---|---|
| 21 | - 125.154 | + 0.250 | Longitude |
| 55 | - 8.466 | + 0.017 | Longitude |
| 56 | + 18.609 | - 0.037 | Longitude |
| 57 | + 3.215 | - 0.006 | Longitude |
| 61 | + 18.023 | - 0.036 | Longitude |
| 62 | + 0.560 | - 0.001 | Longitude |
| 118 | - 0.986 | + 0.001 | Longitude |
| 119 | + 1.750 | - 0.003 | Longitude |
| 120 | + 1.225 | - 0.002 | Longitude |
| 123 | + 1.267 | - 0.002 | Longitude |
| 129 | - 1.089 | + 0.002 | Longitude |
| 138 | + 0.584 | - 0.001 | Longitude |
| 627 | - 0.978 | + 0.002 | Parallax |

TABLE V. List of Terms Affected by Change of Ratio of Major to Minor Axis of Ellipsoid

| Number | Change | Term Affected |
|---|---|---|
| **1. Arguments** | | |
| 1375 | - 0.212 | L |
| 1376 | - 0.008 | L |
| 1401 | + 0.061 | $\varpi$ |
| 1407 | - 2.807 | ☊ |
| 1408 | - 0.456 | ☊ |
| 1409 | - 0.054 | ☊ |
| **2. Periodic Terms** | | |
| 3 | - 0.007 | Longitude |
| 7 | - 0.025 | Longitude |
| 8 | + 0.019 | Longitude |
| 25 | + 0.002 | Longitude |
| 26 | + 0.001 | Longitude |
| 33 | + 0.001 | Longitude |
| 51 | - 0.009 | Longitude |
| 314 | - 0.007 | Latitude |
| 326 | + 0.025 | Latitude |
| 328 | + 0.019 | Latitude |
| 339 | - 0.002 | Latitude |

## 4. The Radius of the Moon

As stated previously, the principal sources of information on lunar heights are the charts of Hayn, Weimer, and Watts. Using these charts, it becomes a matter of indifference what the mean surface is, as long as the proper value is used with each chart. From the method of construction of Watts' profile, the major error is introduced in the process of finding the mean surface. The various errors can be summarized as:

| | |
|---|---|
| actual surface to mean surface | ± 0."07 |
| mean surface to center (optical) | ± 0.2 |
| optical center to mass center | ± 0.1 |
| Total | ± 0."23 |

Although the computations for equal-limb-line occultations presume that the occulting feature is located precisely, the actual situation is quite different. A major uncertainty arises through the lack of ability to account for (a) the uncertainty in profile caused by uncertainty in ground position, and (b) the uncertainty in profile on the moon's edge because of libration in longitude between occultations at the two or more ground stations.

The difference between the assumed position of a ground point and its position referred to the lunar orbit reference system will, in all cases except where initial survey was grossly inadequate, be less than 3 km., measured along the reference ellipsoid. The greatest uncertainty will occur when this distance is along the lunar edge and when the slope of the profile in that region is maximal. The radial uncertainty would then be

$$\Delta r_{max} = \pm 1500 \text{ meters}$$

with respect to the radius at the other ground sites. Since station locations are changed from occultation to occultation to diminish the difference between assumed "true" positions, after one or two occultations the radial uncertainty would be reduced to correspond approximately to the uncertainty in the occultation method itself. Furthermore, occultations are selected which occur at maria or similarly flat areas in which the maximum slope is closer to ± 5°. A realistic estimate of the radial uncertainty maximum arising from station position uncertainty would then be

$$\Delta r_{max} = 30 \text{ meters.}$$

Since the maximum time between occultations on an equal-limb survey is less than 2 hours, the radial uncertainty is the sum of the uncertainties from the longitudinal libration position at the different occultation times In 2 hours the longitudinal libration is less than 9', and the maximum radial uncertainty attributable to introduction of a new feature by longitudinal libration would be between ± 15 meters certainty, and most of the time, therefore, much less.

5. <u>Observer's Position Uncertainties</u>

In theory, the latitude, longitude, and height of an observer are clearly defined quantities. In the simplest case, a spheroid (ellipsoid with two equal axes) of size and shape approximately that of the earth is selected. A point on this spheroid is chosen to represent a given latitude and longitude on the earth, the position usually being chosen so that its geodetic coordinates coincide with or are close to the actually measured coordinates of the physical point on the earth. The spheroid is then oriented (in theory) until its minor axis is parallel to the earth's mean axis of rotation and its major axis, labelled X', points parallel to the plane of the Greenwich meridian. Again, in theory, the spheroid is translated bodily until the chosen point on it lies on the vertical through the selected point on the earth, and it is then further translated, always without rotation, until the separation of the two points is equal to some pre-selected distance. The point on the earth's surface is referred to as the <u>datum point</u>, and its latitude, longitude, and height above or below the spheroid, together with the size and shape of the spheroid, are sufficient to define the geodetic system. Subsequent points located on the earth's surface, with respect to the datum point, are transferred to the corresponding points on the spheroid.

In the case of a large area such as North America, the process of transferring points from the earth's surface to the spheroid surface can introduce large errors. These errors arise because, at great distances from the datum point, the triangulation errors which increase with distance become larger. Furthermore, the distance between the surface of the earth, on which the measurements are made, and the surface of the spheroid, to which positions are referred, becomes less well known with increasing distance from the datum point, at which the spheroid-earth difference is defined.

Where there is no continuous chain of survey from the datum point to a given point on the earth's surface, another source of error enters. This is the unknown amount of translation needed to bring the spheroid proper to the first system into coincidence (at the origin) with that proper to the second point.

The majority of occultations observed so far by the equal-limb-line method have been such that one of the two observation points lies within 10 km of the datum point. However, a substantial number have the observation points at considerable distances from the datum point, and hence this distance must be considered.

Where extensive triangulation networks exist, the survey is of high-order accuracy, and an error of 1 cm/km in the distance between points on the network itself can be assumed. Points not on the network are located to within ± 1 meter horizontally, with respect to the nearest points in the network. For a point 1000 km from the datum point, the error will therefore be approximately ± 10 meters, horizontally, the local error being insignificant. This error is usually that in distance along a sea-level surface. Because of the not-very-well-known separation between the sea-level surface and the spheroid, an additional amount must be added to show this. There are no universally valid data on the error thus introduced; there are, however, excellent approximations available, based on assumptions regarding the size and shape of the sea-level surface, which make a value of ± meters in 1000 meters a reasonable upper bound.

Although even a very long triangulation chain represents a high order of accuracy as far as distance is concerned, the accuracy of the coordinates at the end of the chain would be very low unless independent means were available for controlling the direction of the chain. This control, generally afforded by other chains orthogonal to the first, and by periodic determinations of astronomic azimuth along the chain, can be assumed for the purposes of the analysis to be present. It may be noted that, in almost all cases where occultations have been observed, the observation sites have been well within 1000 km of the datum point and usually within 100 km of the datum point.

These errors or uncertainties, together with a number of others of much less importance arising from errors in the geoid above spheroid and other values, are all considered as lying within one datum system. Another large error arises from the unknown relationship between positions measured on spheroids which are located in different, unrelated datum systems. Because of the definition of the spheroid and datum or because of the mathematical procedures used in computation of coordinates, all spheroids are similarly oriented. The errors therefore arise from the errors in measurement of the distance between spheroid centers. There is, of course, no value known for the error in this distance in any case of interest; only the precision is known.

Furthermore, since occultation surveys depend on a knowledge of the lunar orbit to some extent, the error in the distance between the spheroid center and gravity center is of importance. A value of ± 200 meters is adopted here for the quantity. This value is probably greater than necessary, but not unreasonably so. By the very nature of occultation surveying, this particular error appears entirely in the unknown site coordinates, which have to be corrected in any case. The effect of the error, together with the errors within the datum, is to cause a displacement of the unknown site along the lunar limb and perpendicular thereto, which means that the unknown site is going to obtain a final correction whose error is proportional to the amount by which the actual feature observed on the moon differed from that predicted by the equal-limb-line calculations. This was discussed earlier in the section on lunar-position uncertainties.

6. Time Uncertainties.

The principal sources of time uncertainties are listed in Table VI, together with estimates of the uncertainty magnitudes. The uncertainties arise from:

a. The nature of time itself and its definition.

b. The determination of time and its propagation to the receiver.

c. The reception, amplification, and recording of time.

d. The propagation of the light signal.

e. Reception, amplification, and recording.

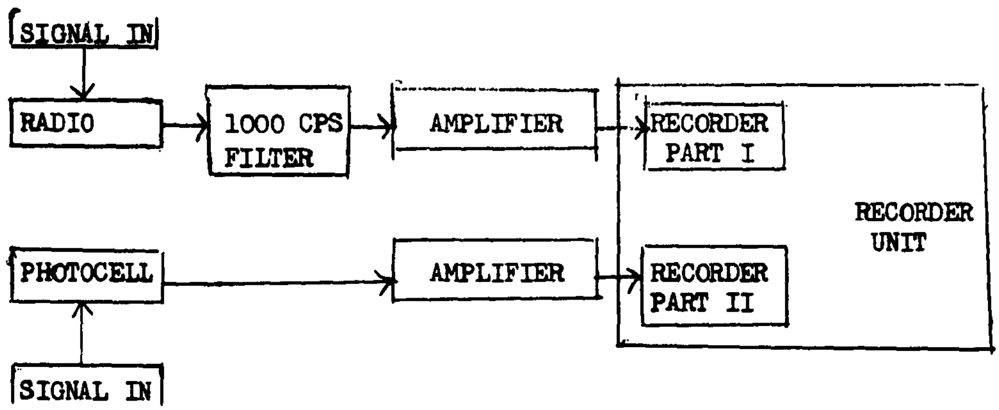
f. Measurement of the record.

Total time uncertainty is found to be less than $0^s.01$. In spite of the many still unanswered questions in a check of the above list, this value of $0^s.01$ is considered reliable and probably close to an upper bound. The reason is that the largest part of this uncertainty arises from the measurement resolution limit, which was taken as 0.5 mm upper bound, and which is accessible to verification, and from the radio propagation time uncertainty of $0^s.002$, which also can be evaluated, although with slightly less confidence. The unknowns in the time problem are at least an order of magnitude smaller than this.

TABLE VI. Time Uncertainty Estimates

| | | Absolute Delay (μs) | Relative Uncertainty (μ |
|---|---|---|---|
| 1. Delay in Electronic Circuitry | (1) | | |
| a. Photomultiplier | (2) | 0.1 | 0.004 |
| b. Cable, photomultiplier to amplifier | | 7000 | 2 |
| c. Amplifier | | | |
| d. Cable, amplifier-recorder | | < 500 | < 10 |
| e. Recorder | | | |
| f. Radio recevier | | < 500 | < 10 |
| g. Filter, 1000-cps band pass | | | |
| 2. Delay in Mechanical Components | (1) | | |
| a. Recorder pen lengths | | | < 1000 |
| b. Galvanometer inertia | (3) | 10,000 | < 1000 |
| c. Chart speed | | < 1000 | < 1000 |
| 3. Optical Uncertainties | | | |
| a. Light travel time | | 1,000,000 | < 0.1 |
| b. Diffraction edge position | (4) | 700 | < 100 |
| 4. Radio Propagation Uncertainties | | | |
| Radio propagation time | | 30,000 | < 2000 |
| 5. Direct Time Errors | | | |
| a. Emission time correction | (5) | 50 | < 1 |
| b. Ephemeris Time correction | (6) | 10,000 | < 1 |
| c. Polar motion: rotation of earth | (5) | 500 | < 5600 |
| 6. Measurement Errors | | | |
| a. Resolution (0.5 mm) | | 4000 | 1 |
| b. Cross-chart correction (0.1 mm) | | 800 | 1100 |
| TOTAL | | | < 8400 |

(1) Tests have been made to measure the time delays present in the equipment between signal (light or radio) reception and recording. The system paths considered are shown in the following diagram:




The amplifiers are of the same type; the recorder units (galvanometer and pens) are of the same type, housed together, and with common chart drive and chart. Figures 5 and 6 show the two test methods. Variants of these methods were also used, with individual components of the systems being tested as well.

(2) The transit time in nine- or ten-stage photomultiplier tube is around $10^{-7}$ to $10^{-8}$ seconds, but the variation in this is many orders of magnitude less. Engstrom (ref. 28) and Karolus (ref. 26), among others, give values for the transit times. There is some caution necessary in the evaluation of transit times. A single electron at the input to an eleven-stage tube will, according to the literature, produce a pulse of $10^{-8}$-second width at half amplitude. The signal transit time is the time of start of the pulse, but the measured time will depend upon the sensitivity, averaging properties, etc., of the measuring instrument, and could be the pulse half-width point.

(3) The galvanometer response time was evaluated separately and as part of the over-all system. Appendix A gives the derivation of the pen-response equations.

(4) The moon acts as a diffracting plane to produce a diffraction pattern. The effect of this on the shape of the occultation curve and on the time determination is discussed in Appendix B.

(5) The radio signals are, for obvious technical reasons, seldomly emitted at the instant which they are supposed to indicate. The actual times of emission are determined by monitoring of station broadcasts by time centers such as Greenwich Observatory; the necessary corrections are published periodically in time bulletins. The major source for such corrections has been the U. S. Naval Observatory Time Bulletins. The bulletin values must, of course,

TEST SET-UP FOR MEASURING
DELAY IN TIME-SIGNAL CIRCUIT

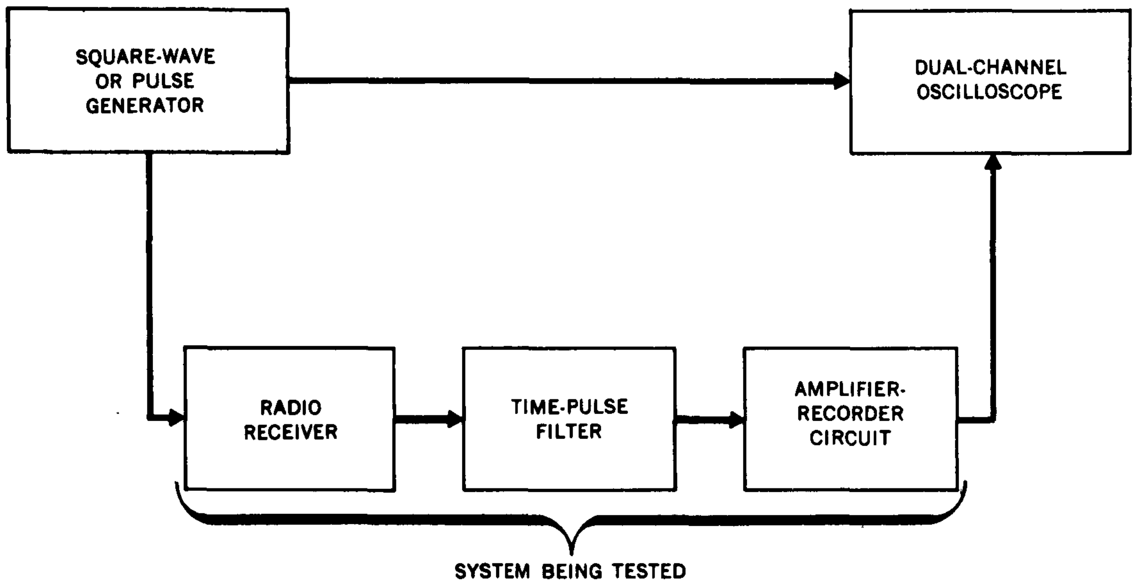



FIGURE 5

TEST SET-UP FOR MEASURING
DELAY IN STARLIGHT CIRCUIT

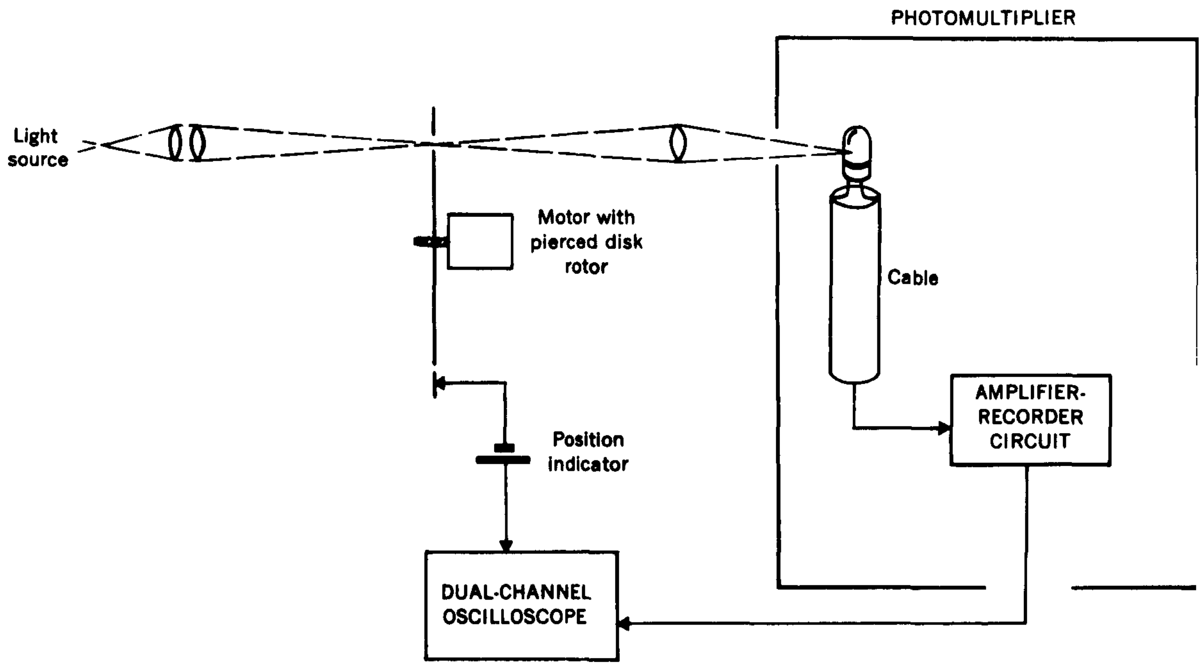



FIGURE 6

be corrected for time of transmission from the station to the observatory; since only the difference in time of reception by two equal-limb-line time observers is involved, this transit-time correction is of no importance where the same radio-station signals are used.

Further corrections must be made for the variation of latitude caused by the motion of the pole and for the seasonal variation in the rotation of the earth. The necessary correction constants are computed by the Bureau International de l'Heure (ref. 27); preliminary values are provided in the time bulletins.

(6) The observations of occultations are related to time through radio signals sent out by radio stations, or to locally generated time signals from a clock which is calibrated against the radio signals. The problem of accounting for the difference between the time recorded by the occultation equipment and the time - Universal Time - at which the time signal should have been emitted is complex, but the individual steps necessary to the solution are well understood. This is not the case with the relation between Ephemeris Time and Universal Time. Universal Time and its variants Universal Time-1 and Universal Time-2 lend themselves to precise operational definition; Ephemeris Time, on the other hand, has no physical existence, is defined theoretically in terms of an unrecoverable unit, and can be related to other time standards only through empirical methods. There is no theoretical relationship, for instance, between the Ephemeris Time second and the second defined by an atomic molecular oscillator.

The unit of Ephemeris Time (ref. 19) is the tropical year defined by the mean motion of the sun in longitude at January 0, 1900, $12^h$ E.T. The ephemeris second is defined as $(31,556,925.9747)^{-1}$ times the topical year at the above epoch. Since Ephemeris Time is designed explicitly to give positions of the major solar system bodies which agree with observations, the relation of Ephemeris Time to other kinds of time must be through observation of these bodies. The body easiest to observe is the moon, and therefore the relation between Ephemeris Time and Universal Time is gotten at present through the lunar motion. The principal techniques utilized are occultations, transit observations, and photography of the moon against a star background [the moon-camera method of Markowitz (ref. 32)] . Although from the standpoint of operational definition there is a different kind of Ephemeris Time for every member of the solar system, these fine distinctions are quite lost in the observational errors associated with measurement of:

$$\Delta T \equiv E.T. - U.T.$$

This is true also of the difference between E.T. as defined by Brown's theory and E.T. as defined by a numerically correct theory.

(7) The $UT_2$ time of emission of the signal is not the same as the time stated by the signal, but may differ from it by many milliseconds even if the radio station is operating properly. The U. S. Naval Observatory monitors the time signals from a large number of time-transmitting stations, as do Greenwich; Tokyo Astronomical Observatory, et al., and the times of receptions are published as $UT_2$ times. Since the monitoring station whose published reception times are used in the reduction is usually within a hundred miles of the transmitting station whose signals are used for timing the occultations, the transmission time error from this source is certainly less than 100 $\mu$s. Likewise, the monitoring errors themselves should be less than 100 $\mu$s, and, in fact, the times of reception are for most stations given to 0.1 ms.

(8) In most cases, the occultation observing site is within 4000 km of a time-transmitting station, and one-hop propagation may be assumed. Giving the F2 layer a height of 400 km, the maximum difference in transmission times are given by:

| | | |
|---|---|---|
| Straight-line: | 13.3 ms | 0.3 |
| One-hop: | 13.6 ms | 0.8 |
| Two-hop: | 14.4 ms | |

The maximum error is therefore less than 1 ms, and will in fact usually be less than 0.5 ms.

(9) There are no good theoretical values for the time delay involved in the movement of the signal from the antenna to the recorder input. The maximum delay occurs in the 1000 cps UTC band pass filter, and direct measurements of the time delay show that this is about 1.3 ms.

7. Aberration

Aberration, the apparent displacement in position of an object, caused by the finite velocity of light, is not of any great importance in the equal-limb-line method. The reason for this is that the observations are paired in such a manner as to cancel the effects of aberration on the results. First, since at a particular point on the earth, an occultation consists of the alinement of star, lunar feature, and observer, the star and lunar aberration effects are exactly equal. Second, the time difference between observations of an occultation from two different sites is so small that annual aberration is insignificant. Since the interval between the observations in an equal-limb-line occultation is almost always less than 2½ hours, the difference in the annual aberration at the two points in the orbit must be less than:

$$\Delta^2 \cos\beta = + k \sin(s-\lambda)\,\Delta\,(s-\lambda)$$

$$\Delta^2 \beta = - k \sin\beta \cos(s-\lambda)\,\Delta(s-\lambda).$$

Since $(s-\lambda)$, over a period of 3 hours, is less than $3 \times 10^{-3}$ the aberration differences are less than 0".06, and the maximum difference is therefore less than 0".1. This is insignificant when even a rough approximation to the aberration correction is made; an error of 10 km in estimating the amount of correction necessary would introduce less than 0".005 error.

The diurnal aberration, 0".32 cos $\varphi$ is somewhat more important. The changes in right ascension and declination are given by

$$\Delta\,\alpha = - 0''.32 \cos\varphi \cos H \sec\delta$$

$$\Delta\,\delta = + 0''.32 \cos\varphi \sin H \sin\delta$$

(Reference 11). Latitudes are generally restricted to ± 60°, the declination to ± 30°, the hour angles H can extend almost to ±90°. The diurnal variation limits are therefore

| | Max. | Min. |
|---|---|---|
| $\Delta\,\alpha$ = | - 0".37 | 0 |
| $\Delta\,\delta$ = | ± 0".16 | 0 |

From the equal-limb-line view, the difference between the values of diurnal variation at the two places is of primary importance. Here

| | Max. | Min. |
|---|---|---|
| $\Delta\delta$ = | 0 | 0 |
| $\Delta H$ = | 85° | 0 |
| $\Delta\varphi$ = | ±30° | 0 |

and the variation in aberration is as great as for the single-site type of observation. The error in the estimate of the aberration is compounded of the errors in

(1) radius of the earth ( $< 1:10^4$)

(2) velocity of light ( $< 1:10^5$)

(3) cosine of latitude of observer ( $< 1:10^4$).

It is therefore less than 0".0001 and can be ignored.

From what is stated at the beginning of this section, it is safe to neglect the errors arising from aberration effects in the lunar orbit. In the Improved Lunar Ephemeris, page x, it is mentioned that terms depending on the eccentricity of the lunar orbit are omitted (= periodic terms of 0".009 size). Rules for inclusion are given in Transactions of the International Astronomical Union, No. VII, page 175 (1950).

B. Error Effects

At this point it is well to summarize the conclusions of the section just preceding. That section gave, as reasonable values for the errors in the various physical quantities affecting an occultation, the following:

| | | |
|---|---|---|
| Star Position Errors | ---- | $< 0''.8$ |
| Moon Position Errors | ---- | $< 0''.3$ |
| Observer's Position Errors | | |
| known datum | ---- | $< 1''.0$ |
| unknown datum | ---- | $< 60''$ |
| Time Error | ---- | $< 0^s.01$ |
| Aberration Error | ---- | $< 0''.005$ |

The combined effects of these errors are given by the equation:

$$\begin{bmatrix} \Delta E^1 \\ \Delta E^2 \\ \Delta E^3 \end{bmatrix} = \begin{bmatrix} \frac{\partial E^1}{\partial \delta_m} & \frac{\partial E^1}{\partial \alpha_m} & \frac{\partial E^1}{\partial \delta_*} & \frac{\partial E^1}{\partial \alpha_*} & \frac{\partial E^1}{\partial E^1_o} & \frac{\partial E^1}{\partial E^2_o} & \frac{\partial E^1}{\partial E^3_o} & \frac{\partial E^1}{\partial t} \\ \frac{\partial E^2}{\partial \delta_m} & \frac{\partial E^2}{\partial \alpha_m} & \frac{\partial E^2}{\partial \delta_*} & \frac{\partial E^2}{\partial \alpha_*} & \frac{\partial E^2}{\partial E^1_o} & \frac{\partial E^2}{\partial E^2_o} & \frac{\partial E^2}{\partial E^3_o} & \frac{\partial E^2}{\partial t} \\ \frac{\partial E^3}{\partial \delta_m} & \frac{\partial E^3}{\partial \alpha_m} & \frac{\partial E^3}{\partial \delta_*} & \frac{\partial E^3}{\partial \alpha_*} & \frac{\partial E^3}{\partial E^1_o} & \frac{\partial E^3}{\partial E^2_o} & \frac{\partial E^3}{\partial E^3_o} & \frac{\partial E^3}{\partial t} \end{bmatrix} \begin{bmatrix} \Delta \delta_m \\ \Delta \alpha_m \\ \Delta \delta_* \\ \Delta \alpha_* \\ \Delta E^1_o \\ \Delta E^2_o \\ \Delta E^3_o \\ \Delta t \end{bmatrix}$$

where the matrix elements have the obvious interpretation. The aberration error is present in the equation, but combined with the star and moon errors.

Now the quantities $[\Delta E^i]$ enter into the theory through the equation

$$[\Delta\sigma^j] = [a^j_i][\Delta E^i].$$

The $\Delta\sigma^j$ are the "observed" differences in the lunar radii at the occultation equal-limb-line points for occultation j, while the $a^j_i$ are the corresponding coefficients. In order that $\Delta E^i$ may be determined, it is essential that three or more values of $\Delta\sigma^j$ be available. That is,

$$\Delta E^i = \Delta E^i(\Delta\sigma^1, \Delta\sigma^2, \Delta\sigma^3, \ldots.)$$

and the equation is

$$[\Delta E^i] = \left\{[a^j_i]^T [a^j_i]\right\}^{-1} [a^j_i]^T [\Delta\sigma^j]$$

For convenience, let us define

$$\begin{array}{lll} u^1 = \delta_m & \qquad & u^2 = \alpha_m \\ u^3 = \delta_* & & u^4 = \alpha_* \\ u^5 = E^1_o & & u^6 = E^2_o \\ u^8 = t & & u^7 = E^3_o \end{array}$$

Then the successive derivatives of the ( $\Delta E^i$), when strung together in a Taylor's series, give

$$\delta(\Delta E^i) = \sum \frac{\partial \Delta E^i}{\partial u^k} \Delta u^k$$

$$= \sum_{j,k} \frac{\partial(\Delta E^i)}{\partial(\Delta\sigma^j)} \frac{\partial(\Delta\sigma^j)}{\partial u^k} \Delta u^k$$

and finally

$$\left[\delta(\Delta E^{i})\right] = \left\{\left[a_i^j\right]^T\left[a_i^j\right]\right\}^{-1}\left[a_i^j\right]^T\left[\frac{\partial(\Delta\sigma^j)}{\partial u^k}\right]\left[\Delta u^k\right]$$

This equation is merely the mathematical way of saying that the errors present in the occultation results are to be found from the errors in the radius of the moon measured on the fundamental plane, magnified by inverse projection of these latter errors from the fundamental plane back onto the surface of the earth (back into the observer's local horizon).

There are several ways of continuing the solution of the error-analysis problem. One, which is the simplest when there is convenient access to an automatic computer, is to compute, for various combinations of values of the $u^k$'s, values of $\sigma$ for unit variations (for each combination of $u^k$'values) in each of the $u^k$ successively, i.e., compute

$$\sigma = \sigma(u^1, u^2, u^3, u^4, u^5, u^6, u^7)$$

$$\sigma + \Delta\delta_1 = \sigma(u^1 + \Delta u^1, u^2, u^3, u^4, u^5, u^6, u^7)$$

and so on, so that, approximately,

$$\frac{(\sigma + \Delta\sigma^k) - \delta}{\Delta u^k} \approx \frac{\partial\sigma}{\partial u^k}$$

(Because, as will be remembered from the definitions in Part I, $\Delta\delta$ is defined as

$$\Delta\sigma \equiv \sigma_2 - \sigma_1$$

where $\sigma_2$ and $\sigma_1$ are the radii of the moon as observed at points 2 and 1, respectively, the quantity

$$\frac{\partial\Delta\sigma}{\partial u^k} = \frac{\partial\sigma^3}{\partial u^k} - \frac{\partial\sigma^1}{\partial u^k} .$$

The linearity of the preceding equations hence allows the problem to be worked for $\sigma^2$ and $\sigma^1$ separately. This separation is assumed immediately above and following.)

The Coefficients having been evaluated, it remains only to compute, for each $\underline{k}$,

$$\left( \frac{\partial \sigma^2}{\partial u^k} - \frac{\partial \sigma^1}{\partial u^k} \right) \Delta u^k$$

and to sum over $\underline{k}$. The result will, of course, be applicable only to the situation as defined by the set of $u^k$'s chosen, but it would not be too difficult to (a) repeat the procedure for more sets of $u^k$'s; (b) select a representative $u^k$ in the first place; or (c) select the $u^k$ for maximum $\delta(\Delta\sigma)$.
A representative case is provided by an occultation of date

| | |
|---|---|
| $t_1$ | 1957 June $14^d$ $7^h$ $25^m$ $0^s.6774$ |
| $\lambda_1$ | 119° 03' 36".000 W. |
| $\phi_1$ | 34° 13' 00".000 N. |
| $h_1$ | 0 meters |

Star 28 896 (AMS catalog no.)

| | |
|---|---|
| $t_2$ | 1957 June $14^d$ $8^h$ $02^m$ $24^s.000$ |
| $\lambda_2$ | 102° 16' 37".679 W. |
| $\phi_2$ | 36° 12' 00.585 N. |
| $h_2$ | 0 meters |

The individual variations are given in Table VII. (The conversion between $\Delta E^1$ and $\Delta\lambda$, $\Delta\varphi$ is given closely enough by the relations

$$\Delta E^1 = r_{mean} \cos \phi \, \Delta\lambda$$

$$\Delta E^2 = r_{mean} \, \Delta\phi$$

although more precise values can be obtained from the usual formulae or from one of the well-known tables of functions on the International ellipsoid.)

Table VII. Parameter Variations

| | $\Delta\sigma_1$ | $\Delta\sigma_2$ |
|---|---|---|
| $\Delta\delta_*$ | - 205.9 | - 205.2 |
| $\Delta\alpha_*$ | - 1604.5 | - 1605.4 |
| $\Delta\delta_m$ | 212.8 | 200.0 |
| $\Delta\alpha_m$ | 1631.1 | 1631.3 |
| $\Delta\lambda_1$ | - 31.8 | - 31.0 |
| $\Delta\phi_1$ | - 6.4 | - 0.5 |
| $\Delta h_1$ | 7.3 | - 5.3 |
| $\Delta t_1$ | 664.2 | 676.0 |
| $\Delta\pi_m$ | 265.9 | - 1414.6 |

The rule for propagation of errors gives

$$(s\cdot d)^2 = \sum_k \left(\Delta \frac{\partial \xi_o}{\partial u^k} \quad \Delta u^k\right)^2$$

where

$$\left(\frac{\partial \xi}{\partial u^k}\right)_{\text{site 2}} - \left(\frac{\partial \xi}{\partial u^k}\right)_{\text{site 2}} \equiv \Delta \frac{\partial \xi}{\partial u^k}$$

Attention must be paid to the non-equal time errors, but when we are through, we get, using the values given at the start of this section,

$$(s\cdot d_{\Delta\sigma})_{\text{partial}} = 23.87295$$

To this r.m.s $\sigma$ we may add the r.m.s. error arising from the non-equal-limb-line observation circumstances. If it is assumed that the Watts limb profiles, although good absolutely to only 180 meters, are good relatively to much better - say 60 meters, then corrections can be made to $\sigma$ on this basis. That is, although height

$$\text{r.m.s.e. } (h_{B,\text{Watts}} - h_{B,\text{True}}) \approx 100 \text{ meters}$$

$$\text{r.m.s.e. } (h_{A,\text{Watts}} - h_{A,\text{True}}) \approx 180 \text{ meters}$$

$$\text{r.m.s.e } (h_{B,\text{Watts}} - h_{A,\text{Watts}} \approx 30 \text{ meters.}$$

Hence

$$\sigma_{2\,\text{Total}} = 23.87495^2 + 3600 \text{ meters}$$

$$= 570.0132375 + 3600 = 4170.0132375 \text{ meters}$$

Taking as before the worst case, we shall bring this value up to the surface of the earth for an observation at an altitude of 10° (the usual limit in selecting an occultation is 15°). Hence

$$\text{r.m.s.e. } \underline{d} = \sec^2 80° \times \sigma$$

$$= 33$$

$$= 33 \times 64.578$$

$$= 2131.074. \text{ meters}$$

A moment will be taken here for a short digression to discuss the roles of "numerical" error analysis and "algebraic" error analysis. Until fairly recently, the derivation of a parameter of a function from computed values of the function at different points in its domain, rather than by computation from an analytic expression for the parameter has been frowned upon by mathematicians as being "unmathematical" and less desirable. This attitude, unfortunately, has been adopted too often in the other sciences where mathematics is applied, and there is very little justification any more for the attitude. The validity of a method is determined largely by its correctness and by its ease of application. Until the last decade, the latter consideration made it feasible to spend a long time deriving a formula if the formula could shorten the computing time; now, in most cases, a computation can be carried out much more rapidly than can the derivation of a formula. Realization is gradually growing that algebra is only a tool to be used in getting to an end, and is not an end in itself, and that analytic formulations must be justified by the end for which they are used, not by the cultural environment in which they are used.

After this digression, we can go on to look at the analytic formulation of the error analysis. As was seen, the major problem remaining is that of finding, for each observation point

$$\frac{\partial \sigma}{\partial u^k}.$$

This is fortunately quite simple. The basic equation to be used is that given in Part 1:

$$\sigma^2 = \sum_i \left(\xi^i_{obs} - \xi^i_{1\ moon}\right)^2$$

so that

$$\Delta\sigma = \frac{1}{\sigma}\left[(\xi^1_0 - \xi^1_m)(\Delta\xi^1_0 - \Delta\xi^1_m) + (\xi^2_0 - \xi^2_m)(\Delta\xi^2_0 - \Delta\xi^2_m)\right]$$

The basic equations are

$$\begin{bmatrix} \xi^1_0 \\ \xi^2_0 \\ \xi^3_0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\delta_* & -\sin\delta_* \\ 0 & \sin\delta_* & \cos\delta_* \end{bmatrix} \begin{bmatrix} \cos h & 0 & \sin h \\ 0 & 1 & 0 \\ -\sin h & 0 & \cos h \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi^1 & \sin\phi^1 \\ 0 & -\sin\phi^1 & \cos\phi^1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \rho \end{bmatrix}$$

and correspondingly

$$\begin{bmatrix} \xi^1_m \\ \xi^2_m \\ \xi^3_m \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\delta_* & -\sin\delta_* \\ 0 & \sin\delta_* & \cos\delta_* \end{bmatrix} \begin{bmatrix} \cos(\alpha_m - \alpha_*) & 0 & +\sin(\alpha_m - \alpha_*) \\ 0 & 1 & 0 \\ -\sin(\alpha_m - \alpha_*) & 0 & \cos(\alpha_m - \alpha_*) \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\delta_m & \sin\delta_m \\ 0 & -\sin\delta_m & \cos\delta_m \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ r \end{bmatrix}$$

where

$$r = \frac{\rho}{\sin \pi_m}.$$

The Jacobian $\left[\frac{\partial \xi^i}{\partial u^k}\right]$ is simply derived from these formulae since if we denote the successive rotation matrices in the equations for $[\xi^i]$ by $R_1\ R_2\ R_3\ P$, we have

$$\frac{\partial [\xi^i]}{\partial \delta_*} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \left[\xi^i\right]$$

$$\frac{\partial [\xi^i]}{\partial \left\{ {h \atop \Delta\alpha} \right\}} = R_1 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} R_2\ R_3\ P$$

$$\frac{\partial [\xi^i]}{\partial \left\{ {\varphi' \atop \delta_m} \right\}} = R_1\ R_2 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} R_3 P$$

$$\frac{\partial [\xi^i_e]}{\partial \rho} = R_1 R_2 R_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\frac{\partial [\xi^i_m]}{\partial \pi_m} = R_1 R_2 R_3 \begin{bmatrix} 0 \\ 0 \\ -r \cot \pi_m \end{bmatrix}$$

Expansions of particular terms are given below:

$$\frac{\partial \xi_0^1}{\partial \delta_*} = 0 \qquad \frac{\partial \xi_0^1}{\partial \alpha_*} = \rho \cos \varphi' \cos h$$

$$\frac{\partial \xi_0^2}{\partial \delta_*} = -\rho \left[ \sin \varphi' \sin \delta_* + \cos \varphi' \cos \delta_* \cos h \right]$$

$$\frac{\partial \xi_0^2}{\partial \alpha_*} = \rho \cos \varphi' \sin \delta_* \sin h$$

$$\frac{\partial \xi_0^1}{\partial \varphi'} = -\rho \sin \varphi' \sin h$$

$$\frac{\partial \xi_0^1}{\partial \lambda'} = \rho \cos \varphi' \cos h$$

$$\frac{\partial \xi_0^2}{\partial \varphi'} = \rho \left[ \cos \varphi' \cos \delta_* + \sin \varphi' \sin \delta^* \cos h \right]$$

$$\frac{\partial \xi_0^2}{\partial \lambda} = \rho \cos \varphi' \sin \delta_* \sin h$$

$$\frac{\partial \xi_0^3}{\partial \delta_*} = \rho \left[ \cos \delta_* \sin \varphi' - \sin \delta_* \cos \varphi' \cos h \right]$$

$$\frac{\partial \xi_0^3}{\partial \alpha_*} = -\rho \cos \delta_* \cos \varphi' \sin h$$

$$\frac{\partial \xi_0^3}{\partial \varphi'} = \rho \left[ \sin \delta_* \cos \varphi' - \cos \delta_* \sin \varphi' \cos h \right.$$

$$\frac{\partial \xi_0^3}{\partial \lambda} = -\rho \cos \delta_* \cos \varphi' \sin h$$

$$\frac{\partial \xi_0^3}{\partial t} = -\rho \cos \delta_* \cos \varphi' \sin h$$

$$\frac{\partial \xi_0^1}{\partial t} = \rho \cos \varphi' \cos h$$

$$\frac{\partial \xi_0^2}{\partial t} = \rho \cos \varphi' \sin \delta_* \sin h$$

Similarly, the coefficients for the moon's position errors may be derived.

$$\frac{\partial \xi_m^1}{\partial \delta_*} = 0$$

$$\frac{\partial \xi_m^2}{\partial \delta_*} = \frac{-\rho}{\sin \pi_m} \left[ \sin \delta_* \sin \delta_m + \cos \delta_* \cos \delta_m \cos(\alpha_m - \alpha_*) \right]$$

$$\frac{\partial \xi_m^1}{\partial \delta_m} = \frac{-\rho}{\sin \pi_m} \left[ \sin \delta_m \sin(\alpha_m - \alpha_*) \right]$$

$$\frac{\partial \xi_m^2}{\partial \delta_m} = \frac{\rho}{\sin \pi} \left[ \cos \delta_* \cos \delta_m + \sin \delta_* \sin \delta_m \cos(\alpha_m - \alpha_*) \right]$$

$$\frac{\partial \xi_m^1}{\partial \alpha_m} = \frac{\rho}{\sin \pi_m} \left[ \cos \delta_m \cos(\alpha_m - \alpha_*) \right]$$

$$\frac{\partial \xi_m^2}{\partial \alpha_m} = \frac{\rho}{\sin \pi_m} \left[ \sin \delta_* \cos \delta_m \sin(\alpha_m - \alpha_*) \right]$$

$$\frac{\partial \xi_m^1}{\partial \alpha_*} = \frac{-\rho}{\sin \pi_m} \left[ \cos \delta_m \cos(\alpha_m - \alpha_*) \right]$$

$$\frac{\partial \xi_m^2}{\alpha_*} = \frac{-\rho}{\sin \pi_m} \left[ \sin \delta_* \cos \delta_m \sin(\alpha_m - \alpha_*) \right]$$

$$\frac{\partial \xi_m^1}{\partial \pi_m} = -\cot \pi_m \, \xi_m^1$$

$$\frac{\partial \xi_m^2}{\partial \pi_m} = -\cot \pi_m \, \xi_m^2$$

$$\frac{\partial \xi^3_m}{\partial \delta_*} = \frac{\rho}{\sin \pi_m}\left[\cos \delta_* \sin \delta_m - \sin \delta_* \cos \delta_m \cos(\alpha_m - \alpha_*)\right]$$

$$\frac{\partial \xi^3_m}{\partial \delta_m} = \frac{\rho}{\sin \pi_m}\left[\sin \delta_* \cos \delta_m - \cos \delta_* \sin \delta_m \cos(\alpha_m - \alpha_*)\right]$$

$$\frac{\partial \xi^3_m}{\partial \alpha_m} = \frac{-\rho}{\sin \pi_m}\left[\cos \delta_* \cos \delta_m \sin(\alpha_m - \alpha_*)\right]$$

$$\frac{\partial \xi^3_m}{\partial \alpha_*} = \frac{\rho}{\sin \pi_m}\left[\cos \delta_* \cos \delta_m \sin(\alpha_m - \alpha_*)\right]$$

$$\frac{\partial \xi^3_m}{\partial \pi_m} = -\cot \pi_m \, \xi^3_m$$

$$\frac{\partial \xi^3_m}{\partial t} = \frac{\rho}{\sin \pi_m}\left[\left\{\sin \delta_* \cos \delta_m - \cos \delta_* \cos \delta_m \cos(\alpha_m - \alpha_*)\right\} \frac{\partial \delta_m}{\partial t}\right.$$

$$\left. - \cos \delta_* \cos \delta_m \sin(\alpha_m - \alpha_*) \frac{\partial \alpha_m}{\partial t}\right] - \sec \pi_m \, \xi^3_m \frac{\partial \pi_m}{\partial t}$$

$$\frac{\partial \xi^1_m}{\partial t} = \frac{\rho}{\sin \pi_m}\left[-\sin \delta_m \sin(\alpha_m - \alpha_*) \frac{\partial \delta_m}{\partial t}\right.$$

$$\left. + \cos \delta_m \cos(\alpha_m - \alpha_*) \frac{\partial \alpha_m}{\partial t}\right] - \sec \pi_m \, \xi^1_m \frac{\partial \pi_m}{\partial t}$$

$$\frac{\partial \xi^2_m}{\partial t} = \frac{\rho}{\sin \pi_m}\left[\left(\cos \delta_* \cos \delta_m + \sin \delta_* \sin \delta_m \cos(\alpha_m - \alpha_*)\right) \frac{\partial \delta_m}{\partial t}\right.$$

$$\left. + \sin \delta_* \cos \delta_m \sin(\alpha_m - \alpha_*) \frac{\partial \alpha_m}{\partial t}\right] - \sec \pi_m \, \xi^2_m \frac{\partial \pi_m}{\partial t}$$

The $\delta_*$ and $\alpha_*$ are here assumed to be constant; as far as the three-hour or less interval, which is all that concerns occultations, such an assumption is certainly valid.

In addition to the above quantities, there must be taken account of the natural errors which creep directly into $\Delta\sigma$ through our lack of complete knowledge of the moon's profile shape. Since these errors are directly additive to $\Delta\sigma$, and, furthermore, have been discussed in preceding sections, there will be no further investigation here of the error, but it will be included after evaluation of the other errors. Note that this same addition must be made to the $\delta(\Delta\sigma)$ determined previously by computation.

Evaluation of the above expressions for typical values, ranges of values, or for values leading to a maximum for $\delta(\Delta\sigma)$ is not difficult, but is time consuming. The values of the derivates of the types

$$\frac{\partial\delta}{\partial t}, \quad \frac{\partial\alpha}{\partial t}, \quad \frac{\partial\pi}{\partial t}$$

are gotten from any lunar ephemeris. The variable $\varphi'$ in these equations is the geocentric latitude, not the geodetic latitude. The difference between the two latitudes is not significant for error evaluation. Some simplification can be achieved by making a number of approximations. First of all, $\alpha_*$ and $\delta_*$ do not change appreciably between occultations at points 1 and 2, and, in fact, it is customary in the analyses of the data to use for both points an $\alpha_*$ and $\delta_*$ computed for a time half-way between the times at 1 and 2. Hence we can use the same $\alpha_*$ and $\delta_*$ for 1 and 2. Secondly, $\alpha_m$ and $\delta_m$ change at rates of 0ˢ5 per second and 0ʺ18 per second. Over the three-hour maximum interval between paired observations, this involves only about 1°5 change in $\alpha_m$ and 0°55 change in $\delta_m$. Hence we can use the same values for $\alpha_m$ and $\delta_m$ at both ends, merely adding on a small correction for the change. Furthermore, if a reasonable limit is being sought, rather than an exact (but obviously not accurate) limit, then the coefficients of the $\alpha$ and $\delta$ variations can themselves be approximated. This is all the more possible since $\alpha_m$ and $\delta_m$ will not differ greatly from $\alpha_*$ and $\delta_*$ at occultation time. For example, the correction to

$$\frac{\partial\xi^2}{\partial\delta_*}$$

is then

$$\Delta\left(\frac{\partial\xi_m^2}{\partial\delta_*}\right) = \frac{-1}{\sin\pi_m}\Big[\sin\delta_*\cos\delta_m - \cos\delta_*\sin\delta_m\cos(\alpha_m-\alpha_*)\Big]\Delta\delta_m$$

$$- \cos\delta_*\cos\delta_m\sin(\alpha_m-\alpha_*)\,\Delta\alpha_m$$

$$\approx \frac{-1}{\sin\pi_m}\ \sin\delta_*\cos\delta_*\,(1-\cos(\alpha_m-\alpha_*)\Delta\delta_m$$

$$-\cos^2\delta_*\sin(\alpha_m-\alpha_*)\Delta\alpha_m.$$

$$\approx \; -\cos^2 \delta_* \, \Delta\alpha_m \Delta\alpha \left(-\frac{1}{\sin \pi_m}\right)$$

$$\frac{1}{\sin \pi_m} \approx \frac{1}{.0175}$$

Set $\delta_* \approx 30°$

$$\alpha_m - \alpha_* = \Delta\alpha_{m*} \approx 1° \approx .01745^r$$

$$\Delta\alpha \approx 1.^\circ 5 \qquad .0262^r$$

$$= -\,3/4 \times 4.5 \times 10^{-4} \left(\frac{-1}{\sin \pi_m}\right)$$

$$= -\,3.3 \times 10^{-4} \times \left(\frac{-1}{\sin \pi_m}\right)$$

$$= -\,3.3 \times 10^{-4} \times 6.378 \times 10^{6} \times 57 \times 10^{-6}$$

$\simeq$ 0.6 meters per second of arc.

This quantity is then to multiply the $\Delta \delta_*$, since the other parts of

$$\frac{\partial \xi^2}{\partial \delta_*}$$

will cancel each other. The rest of the differential corrections can be gone through in the same manner. The major differences enter through the portions which depend upon the observer's coordinates since the coefficients here cannot be equated, the one to the other. The observer's coordinates may differ by 30° to 60° or more in latitude; the local hour angle h may differ by 90° or more at the two points. It is to be expected that the major errors will creep in at this point. As can be seen from the equations, the maximum for any one $\frac{\partial \xi}{\partial u^k}$ will be about 30 meters per second of arc.

The following partials are based on expected maximum variations:

PARTIAL DERIVATIVES - $\frac{\partial \xi^i}{\partial u^k}$

| $u^k$ \ $\xi^i$ | $\xi_0^1$ | $\xi_0^2$ | $\xi_0^3$ | $\xi_m^1$ | $\xi_m^2$ | $\xi_m^3$ |
|---|---|---|---|---|---|---|
| $\delta_*$ | 0 | - 14.67 | 17.56 | 0 | - 1822.15 | 0.08 |
| $\alpha_*$ | 3.92 | 11.10 | - 19.23 | - 1577.94 | - 13.81 | 23.92 |
| $\varphi'$ | - 22.21 | 21.49 | 7.89 | - | - | - |
| $\lambda$ | 3.92 | 11.10 | - 19.23 | - | - | - |
| t | 3.92 | 11.10 | - 19.23 | 52.39 | 26.87 | - 1.81 |
| $\delta_m$ | - | - | - | - 15.95 | 1822.24 | 0.08 |
| $\alpha_m$ | - | - | - | 1577.94 | 13.81 | -23.92 |
| $\pi_m$ | - | - | - | - 1582.16 | - 4.52 | -104390.92 |

All values expressed in meters/second of arc.

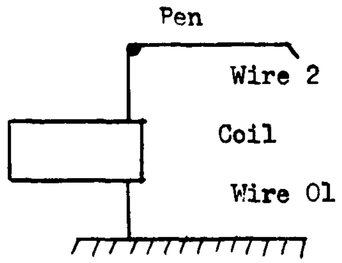
| $u^k$ | $\sin u^k$ | $\cos u^k$ | $\cot u^k$ | $\sec u^k$ |
|---|---|---|---|---|
| $\delta_*$ | .500 | .866 | | |
| $\varphi'$ | .707 | .707 | | |
| $\delta_m$ | .500 | .866 | | |
| $\alpha_m - \alpha_*$ | .0175 | .9999 | | |
| $\pi_m$ | .0175 | .9999 | 57.29 | 1.0002 |
| h | .985 | .174 | | |

$$\frac{\partial \delta_m}{\partial t} = \frac{660''}{3600^s} \qquad \rho = 6.378 \times 10^6$$

$$\frac{\partial \alpha_m}{\partial t} = \frac{1800''}{3600^s} \qquad \frac{\partial \pi_m}{\partial t} = \frac{30''}{3600^s}$$

## APPENDIX A: COMPENSATED GALVANOMETER EQUATIONS

The d'Arsonval galvanometer system can be shown schematically as




where $I_1$ and $I_2$ are the moments of inertia of the coil and pen, respectively, and $k_{01}$ and $k_{12}$ are the torsional stiffness of the anchoring wire and pen-coil wire, respectively. The flux density of the (permanent magnetic field is B, the effective length of the coil wire is L, and the current in the coil is i. If the back e.m.f. of the coil and drag are neglected,

$$i = E/R$$

where E is the signal voltage. To start with, E will be assumed to have the form $E=E_o 1_{(t)}$, so that $i = i_o 1_{(t)}$. The differential equations of motion are therefore (references 15, 16)

$$I_1 \frac{d^2 \theta_1}{dt^2} - k_{12} ( \theta_2 - \theta_1) + k_{01} \theta = -BLI \qquad (1)$$

$$I_2 \frac{d^2 \theta_2}{dt^2} + k_{12} ( \theta_2 - \theta_1) = 0$$

where $\theta_1$ and $\theta_2$ are deflections of the coil and of the pen, respectively from the rest, or "no current" position.

Rearranging,

$$I_1 \frac{d^2 \theta_1}{dt^2} + (k_{01} + k_{12})\,\theta_1 - k_{12}\,\theta_2 = -BLI \tag{2}$$

$$-k_{12}\,\theta_1 + I_2 \frac{d^2 \theta_2}{dt^2} + k_{12}\,\theta_2 = 0$$

The Laplace transform of these equations gives,

$$\begin{aligned} I_1 P^2 \phi_1 - I_1 P^2 \phi_{10} + (k_{01} + k_{12})\,\phi_1 - k_{12}\,\phi_2 &= -BLI \\ -k_{12}\,\phi_1 + I_2 P^2 \phi_2 - I_2 P^2 \phi_{20} + k_{12}\,\phi_2 &= 0 \end{aligned} \tag{3}$$

$$k = k_{01} + k_{12}$$

$$\begin{aligned} (I_1 P^2 + k)\,\phi_1 - k_{12}\,\phi_2 - I_1 P^2 \phi_{10} &= -BLI \\ -k_{12}\,\phi_1 + (I_2 P^2 + k_{12})\,\phi_2 - I_2 P^2 \phi_{20} &= 0 \end{aligned} \tag{4}$$

Assume also $\phi_{10} = 0 = \phi_{20}$

Then

$$\phi_2 = \frac{\begin{vmatrix} I_1 P^2 + k & -BLI \\ -k_{12} & 0 \end{vmatrix}}{\begin{vmatrix} I_1 P^2 + k & -k_{12} \\ -k_{12} & I_2 P^2 + k_{12} \end{vmatrix}} \tag{5}$$

or

$$\phi_2 = \frac{-k_{12}\,BLI}{I_1 I_2 P^4 + P^2(I_2 k + I_1 k_{12}) + (k - k_{12})k_{12}} \tag{6}$$

$$= \frac{\dfrac{-k_{12}\,BLI}{I_1 I_2}}{P^4 + P^2 \left(\dfrac{k}{I_1} + \dfrac{k_{12}}{I_2}\right) \dfrac{k_{12}}{I_2}} \tag{7}$$

$$\frac{\dfrac{-k_{12}\ BLI}{I_1\ I_2}}{P^4 + d_1\ P^2 + d_2} \tag{8}$$

The roots of the expression in the denominator are

$$\pm\ ir_1\ ,\ \pm\ ir_2,$$

so that the transform of $\phi_2$ is

$$\Theta_2 = \frac{-k_{12}\ BLI}{I_1\ \ I_2} \left(\frac{1}{r_1^2 - r_2^2}\right) \left(\frac{1 - \cos h\ r_1 t}{r_1^2} - \frac{1 - \cos h\ r_2 t}{r_2^2}\right) \tag{9}$$

If the back e.m.f. of the coil is taken into account, together with the drag of the pen across the paper, then several new terms are introduced.

We have

$$R_i + BL\ \frac{d\theta_1}{dt} = E \tag{10}$$

$$i = \frac{E}{R} \qquad \frac{BLr}{R}\ \frac{d\theta_1}{dt} \tag{11}$$

$$E = E_o 1(t) \tag{12}$$

$$I_1\ \ddot{\theta}_1 - k_{12}(\theta_2 - \theta_1) + k_{01}\theta_1 = \frac{-\ BLE}{R} \qquad \frac{B^2L^2r}{R}\ \dot{\theta}_1 \tag{13}$$

$$\left\{ \begin{array}{ll} I_1\ \ddot{\theta}_1 - \dfrac{B^2L^2r}{R}\ \dot{\theta}_1 + (k_{01} + k_{12})\ \theta_1 - k_{12}\theta_2 & = -\dfrac{BLE}{R} \\ I_2\ \ddot{\theta}_2 + n\ \dot{\theta}_2 + k_{12}\ (\theta_2 - \theta_1) & = 0 \end{array} \right. \tag{14}$$

$$
\begin{cases}
I_1(P^2\phi_1 - P^2\theta_{10} - P\,\dot{\theta}_{10}) - \dfrac{B^2L^2r}{R}(P\,\phi_1 - P\,\theta_{10}) \\[2ex]
\qquad + (k_{01} + k_{12})\,\phi_1 - k_{12}\,\phi_2 = -BLE_o/R \\[2ex]
I_2(P^2\phi_2 - P^2\theta_{20} - P\,\dot{\theta}_{20}) + n(P\,\phi_2 - P\,\theta_{20}) \\[2ex]
\qquad + k_{12}\,\phi_2 - k_{12}\phi_1 = 0
\end{cases}
\tag{15}
$$

Let $k = k_{01} + k_{12}$ (16)

$$\theta_{20} = \dot{\theta}_{20} = 0 = \dot{\theta}_{10} = \theta_{10} \tag{17}$$

So that

$$
\begin{cases}
\left[I_1P^2 + P\left(\dfrac{B^2L^2r}{R}\right) + k\right]\phi_1 - k_{12}\,\phi_2 = -\dfrac{BLE_o}{R} \\[2ex]
\left(I_2P^2 + nP + k_{12}\,\phi_2 - k_{12}\right)\phi_1 = 0
\end{cases}
\tag{18}
$$

These equations, solved for $\phi_2$, give

$$
\phi_2 = \frac{\begin{vmatrix} I_1P^2 - \dfrac{B^2L^2r}{R}\,P + k & -\dfrac{BLE_o}{R} \\ -k_{12} & 0 \end{vmatrix}}{\begin{vmatrix} I_1P^2 - \dfrac{B^2L^2r}{R}\,P + k & -k_{12} \\ -k_{12} & I_2P^2 + nP + k_{12} \end{vmatrix}}
\tag{19}
$$

which can be written

$$
\phi_2 = -\frac{k_{12}BLE_o}{R}\left[\frac{1}{(P-a_1)(P-a_2)(P-a_3)(P-a_4)}\right]
\tag{20}
$$

where the $a_j$ are the roots of the quartic in the denominator of (19) hence

$$\Theta_2 = + \frac{k_{12} BLE_o}{R} \sum_{\substack{j=1,2,3,4 \\ \text{cyclicly}}} \frac{(a_j - a_y + 1)\, e^{a_{j-1} t}}{\Pi(a_j - a_{j+1})} \quad (21)$$

Equation (21) gives the response of the galvanometer pen to a step function, and shows that the system is doubly resonant, with dampened oscillations because of the pen-drag and back-e.m.f. (Proper choice of the values of $I_1$, $I_2$, $k_{01}$ and $k_{12}$ removes the double resonance.) A somewhat more representative form for E would have been the ramp function.

## APPENDIX B: THE LIGHT CURVE

To an observer looking at an occultation with a telescope, the event appears to be instantaneous; in fact, it is even possible for physiological reasons for the observer to "see" the event before or after it actually occurs. A recording of the light intensity, however, shows that the event is far from instantaneous; a noise-free record would show a curve which is similar to that of a Fresnel diffraction curve for a straight edge. The slight (but important) differences are caused by:

1) the finite angular size of the source, which makes the influence of non-uniform light distribution over the star's surface felt;

2) the failure of the lunar surface to act like a straight edge.

These two differences permit, in some cases, the reconstruction of the stellar structure or the micro-analysis of the form of the lunar profile (References 20, 21). As far as the geodetic use is concerned, these details are of no significance and are neglected.

The diffraction pattern in its gross appearance can be constructed by evaluating the Fresnel integral

$$I = I_o\left[\left(\int_o^v \cos\frac{\pi v^2}{2}\,dv\right)^2 + \left(\int_o^v \sin\frac{\pi v^2}{2}\,dv\right)^2\right] \qquad (1)$$

The variable $\underline{v}$ is related to the distance along the diffraction pattern by the formula

$$v = \frac{y - y_p}{\sqrt{\frac{\lambda r_o}{2}}} \qquad (2)$$

Evaluating the integral for $\lambda = 550$ nm, $d = 380{,}000$ km, the intensity at the line observer at the instant of the occultation, (when observer, moon's edge, and star are collinear) is

$$I_{obs} = \frac{I_{average}}{4} \tag{3}$$

where $I_{average}$ is the average intensity before the immersion (or after the emersion). From the observer to the first maximum is approximately 12 meters.

These values must be modified to take account of the difference between the physical situation and the simplifying assumptions which were made above. These differences include:

a. the superposition of diffraction patterns from a range of wavelengths

b. the different response of the photomultiplier tube at different wavelengths

c. the varying brightness of the source at different wavelengths

d. the filtering action of the atmosphere and optics on the radiation from the source. At a particular distance $\underline{y}$ from the diffracting edge, the intensity as recorded would therefore be represented by the formula

$$I = K\int_{\lambda_1}^{\lambda_2} F(\lambda)\, G(\lambda)\, H(\lambda) \times \left[\left(\int_0^v \cos \frac{\pi v^2}{2}\, dv\right)^2 + \left(\int_0^v \sin \frac{\pi v^2}{2}\, dv\right)^2\right] d\lambda$$

where $F(\lambda)$ is the spectral response of the photomultiplier tube,

$G(\lambda)$ is the source relative brightness

$H(\lambda)$ is the atmospheric filtering function

and $K$ is a scaling factor.

The RCA 1P21 photomultiplier tube, widely used in astronomical photometry and used in the Army Map Service occultation surveys, has an S-4 spectral response. This scaled to a value of 1.00 at the 400 m maximum, the values are given at 50 m (50 nanometer) intervals in Table B-1. The source function can be represented by the Planck formula (reference 29)

$$I_2 = \frac{2C_1}{\lambda^5} (e^{C_2/\lambda T} - 1)^{-1}$$

where $C_1 = 4.992 \times 10^{-15}$ erg-cm

$C_2 = 1.438$ cm-deg

or, to a sufficient approximation, by a numerical table for the sun, a G 0 star. In Table B-2 is given a set of values combining $G(\lambda)$ and $H(\lambda)$, being the solar radiation at sea-level.

A simple picture of the effect of these various factors can be gotten by looking back at equation B (2). Taking the derivative of the function $y$, and keeping the position of the first maximum constant at $\bar{v} = 1.2$ (approximately),

$$\frac{dy}{y - y_p} = \frac{1}{\cdot} \frac{d\lambda}{\lambda}$$

The effective range of $\underline{d\lambda}$ is about $\pm$ 200 around the 550 nm point, so that the maximum at different wavelengths varies over about $\pm$ 2 meters from the 550 nm point. The close-by maximum and minimum will be shifted by about the same amount, so that the diffraction pattern may be expected to be considerably smoothed, in the recording, from what it would be in a monochromatic recording.

TABLE B-1
S-4 RESPONSE

| $\lambda$ (nm) | RESPONSE (erg/cm$^2$) |
|---|---|
| 0 ↑ | 0 |
| 300 | 0.0 |
| 350 | 0.90 |
| 400 | 1.00 |
| 450 | 0.90 |
| 500 | 0.70 |
| 550 | 0.40 |
| 600 | 0.10 |
| 650 | 0.03 |
| 700 | 0.00 |
| ↓ 00 | 0 |

TABLE B-2
SOLAR RADIATION AT 60° ZENITH DISTANCE AND AT SEA LEVEL

| $\lambda$ (cm) x $10^7$ | Q(erg sec$^{-1}$cm$^{-3}$) x $10^7$ | $Q_N$(Q400 = 1) |
|---|---|---|
| 300 | 0 | 0 |
| 350 | 0.188 | 0.40 |
| 400 | 0.470 | 1.006 |
| 450 | 1.006 | 2.14 |
| 500 | 1.215 | 2.58 |
| 550 | 1.190 | 2.53 |
| 600 | 1.167 | 2.48 |
| 650 | 1.173 | 2.49 |
| 700 | 1.108 | 2.35 |
| 750 | 0.867 | 1.84 |
| 800 | 0.857 | 1.82 |

## APPENDIX C: ANALYSIS OF EQUAL-LIMB-LINE OBSERVATIONS

The equations (13,14) as written enable one, apparently, to solve for the corrections $\overrightarrow{\Delta E_j}$ at several sites without the necessity of making any assumptions about the corrections at one of the sites. The reason for this apparent paradox is that the equations have hidden the lunar orbit errors within the geodetic corrections without explicitly saying so. This is shown by the following analyses.

Equation (19) can be written in vector form as the difference between two vectors

$$\vec{\sigma}_j \cdot \vec{\sigma}_j = (\vec{\xi}_j - \vec{\xi}_{jm})^2$$

$$|\vec{\sigma}_j| \equiv \sigma_j$$

$$\sigma_j = \frac{1}{\sigma_j} (\vec{\xi}_j - \vec{\xi}_{jm})^2$$

Let the true values of $\vec{\xi}_j$, $\vec{\xi}_{jm}$ be $\vec{X}_j$, $\vec{X}_{jm}$ so that

$$\vec{\xi}_j = \vec{X}_j + \overrightarrow{\Delta X}_j \;;$$

$$\vec{\xi}_{jm} = \vec{X}_{jm} + \overrightarrow{\Delta X}_{jm}$$

Then

$$\sigma_2 - \sigma_1 = \frac{(\vec{X}_2 + \overrightarrow{\Delta X}_2 - \vec{X}_{2m} - \overrightarrow{\Delta X}_{2m})^2}{\sigma_2} - \frac{(\vec{X}_1 + \overrightarrow{\Delta X}_1 - \vec{X}_{1m} - \overrightarrow{\Delta X}_{1m})^2}{\sigma_1}$$

$$= \frac{(\vec{X}_2 - \vec{X}_{2m})^2}{\sigma_2} - \frac{(\vec{X}_1 - \vec{X}_{1m})^2}{\sigma_1} + \frac{(\vec{X}_2 - \vec{X}_{2m}) \cdot \overrightarrow{\Delta X}_2}{\sigma_2}$$

$$- \frac{(\vec{X}_1 - \vec{X}_{1m}) \cdot \overrightarrow{\Delta X}_1}{\sigma_1} - \frac{(\vec{X}_2 - \vec{X}_{2m}) \cdot \overrightarrow{\Delta X}_{2m}}{\sigma_2} + \frac{(\vec{X}_1 - \vec{X}_{1m}) \cdot \overrightarrow{\Delta X}_{1m}}{\sigma_1}$$

$$+ \; 0\,(\Delta X_j)^2$$

Let $\sigma_2 - \sigma_1 = \Delta\sigma << \sigma_j$

Furthermore, by the conditions of equal-limb-line occultation observations, $(\vec{X}_2 - \vec{X}_{2m})^2 = (\vec{X}_1 - \vec{X}_{1m})^2$.

We can assume that, over the (at most) 2-hour interval between occultations $\overrightarrow{\Delta X}_{2m} = \overrightarrow{\Delta X}_{1m}$.

From these conditions it follows that

$$\Delta\sigma = - \underbrace{\frac{(X_1 - X_{1m})^2}{\sigma_1^2}\Delta\sigma}_{\text{I}} + \overbrace{\frac{(\vec{X}_2 - \vec{X}_{2m})}{\sigma_1} \cdot \Delta X_2 - \frac{(\vec{X}_1 - \vec{X}_{1m})}{\sigma} \cdot \overrightarrow{\Delta X}_1}^{\text{II}}$$

$$- \underbrace{\left[ \frac{(\vec{X}_2 - \vec{X}_{2m})}{\sigma_2} - \frac{(\vec{X}_1 - \vec{X}_{1m})}{\sigma_1} \right] \cdot \overrightarrow{\Delta X}_{1m}}_{\text{III}}$$

The equation actually used explicitly includes only term II without specifying the disposition of terms I and III. But term I is obviously the residual error $\Delta^2\sigma\rho$ in the lunar radius $\rho$ while the term III is approximately the component of the moon's position error in the direction of the tangent to the limb. Since term III varies from occultation to occultation, it cannot be solved for, and must be absorbed in the overall error. Fortunately, since the two terms in the pre-factor are very nearly equal, differing in direction only by the libration in latitude and in magnitude by the small error in the lunar radius, there is no great error incurred. The residual error $\Delta\sigma\rho$ is very small, as follows from breaking $\vec{\sigma}_j$ up into components

$$\vec{\sigma}_j = \overrightarrow{\Delta\sigma}_{jm} + \vec{\sigma}_{j\rho} + \overrightarrow{\Delta\sigma}_{jx}$$

where $\Delta\sigma_{jm}$ is the vectorial error in position of the moon's center,

$\sigma_{j\rho}$ is the lunar radius vector

$\Delta \sigma_x$ is the vectorial error in position of the observer.

Hence

$$\vec{\sigma_2} - \vec{\sigma_1} = (\vec{\sigma}_{2m} - \vec{\sigma}_{1m}) + (\vec{\sigma}_{2\rho} - \vec{\sigma}_{1\rho}) + (\vec{\sigma}_{2x} - \vec{\sigma}_{1x})$$

and

$$\overrightarrow{\Delta^2_{\sigma_\rho}} = \vec{\sigma}_{2\rho} - \vec{\sigma}_{1\rho}$$

## APPENDIX D: SPHEROID-TO-SPHEROID TRANSFORMATIONS

A point P with geodetic coordinates ( $\lambda_1$, $\phi_1$, $h_1$) is given, the coordinates being measured on and from a spheroid $\underline{s_1}$ whose semi-major axis is $\underline{a_1}$ and whose semi-minor axis is $\underline{b_1}$. It is required to find the geodetic coordinates of P, ( $\lambda_2$, $\phi_2$, $h_2$) measured on and from some other spheroid $s_2$ whose semi-major axis is $a_2$, whose semi-minor axis is $b_2$, and which is located in some known relation to $s_1$. The position of $s_2$ with respect to $s_1$ can be defined by means of two rectangular Cartesian systems created as follows:

(1) The origin of system $\begin{Bmatrix}[X_1]\\ [X_2]\end{Bmatrix}$ is at the center of spheroid $\begin{Bmatrix}s_1\\ s_2\end{Bmatrix}$

(2) The $\begin{Bmatrix}X_1^3\\ X_2^3\end{Bmatrix}$ axis coincides with the semi-minor axis of $\begin{Bmatrix}s_1\\ s_2\end{Bmatrix}$ and is positive to the north.

(3) The $\begin{Bmatrix}X_1^1\\ X_2^1\end{Bmatrix}$ axis lies in the plane of the zero meridian of $\begin{Bmatrix}s_1\\ s_2\end{Bmatrix}$ and is positive in the direction of Greenwich. Then $s_2$ is located with respect to $s_1$ as follows. $\left[X_1\right]$ is translated parallel to itself until the origins of $\left[X_1\right]$ and $\left[X_2\right]$ coincide. This translation defines the vector

$$\left[\Delta X_1^1 , \Delta X_1^2, \Delta X_1^3\right]$$

The translated $\left[X_1\right]$ is now rotated counter-clockwise about $\left[X_1^3\right]$ until $\left[X_1^1\right]$ lies in the plane $\left[X_2^1 , X_2^2\right]$, defining the angle $\psi_1$.

This new (translated and once-rotated) system $\left[X_{11}\right]$ is now rotated counter-clockwise about $X_{11}^1$ (into which $X^1$ went) until $X_{11}^3$ coincides with $X_2^3$, defining the angle $\psi_2$. Finally, this translated and twice rotated system is rotated counter-clockwise about $X_2^3$ until its remaining axes coincide with the corresponding axes of $\left[X_2\right]$, defining an angle $\psi_3$. Denoting these rotations by $\left[R_1\right]\left[R_2\right]$ respectively, we have

$$\left[X_2\right] = \left[R_3\right]\left[R_2\right]\left[R_1\right]\left\{\left[X_1\right] - \left[\Delta X_1\right]\right\} \tag{1}$$

or

$$\left[X_2\right] = \left[R\right]\left\{\left[X_1\right] - \left[\Delta X_1\right]\right\}. \tag{2}$$

The solution to the problem is then given by the following steps:

1. $P(\lambda_1, \phi_1, h_1) \longrightarrow P(X_1^1, X_1^2, X_1^3)$

thru the equations

$$\begin{bmatrix} X_1^1 \\ X_1^2 \\ X_1^3 \end{bmatrix} = \begin{bmatrix} (N_1 + h_1)\cos\phi_1 \cos\lambda_1 \\ (N_1 + th_1)\cos\phi_1 \sin\lambda_1 \\ \left\{N_1(1 - e_1^2)th_1\right\} \sin\phi_1 \end{bmatrix} \tag{3}$$

where

$$N_1 \equiv a_1(1 - e_1^2 \sin^2 \phi_1)^{-\frac{1}{2}} \tag{4}$$

and

$$e_1^2 \equiv \frac{a_1^2 - b_1^2}{a_1^2} \tag{5}$$

2. $P\left[X_1\right] \longrightarrow P\left[X_2\right]$ by the equations

$$\left[X_2\right] = \left[R\right]\left\{\left[X_1\right] - \left[\Delta X_1\right]\right\} \tag{6}$$

3. $P\left[X_2\right] \longrightarrow P(\lambda_2, \phi_2, h_2)$ by solving the equations

$$\begin{bmatrix} X_2^1 \\ X_2^2 \\ X_2^3 \end{bmatrix} = \begin{bmatrix} (N_2 + h_2) \cos \phi_2 \cos \lambda_2 \\ (N_2 + h_2) \cos \phi_2 \sin \lambda_2 \\ \left\{ N_2(1 - e_2^2) + h_2 \right\} \sin \phi_2 \end{bmatrix} \tag{7}$$

for $\lambda_2$, $\phi_2$, and $h_2$ ($N_2$ and $e_2$ are defined similarly to $N_1$ and $e_1$). It is step #3 which offers the greatest difficulty, since it involves the solution of a system of three simultaneous non-linear equations. The longitude $\lambda_2$ can be found immediately from

$$\begin{Bmatrix} \tan \lambda_2 \\ \cot \lambda_2 \end{Bmatrix} = \begin{Bmatrix} X_2^2 / X_2^1 \\ X_2^1 / X_2^2 \end{Bmatrix} \tag{8}$$

The latitude $\phi_2$ and height $h_2$ may be found by linearizing the equations to give (9)

$$\begin{bmatrix} X_2^1 \\ X_2^2 \\ X_2^3 \end{bmatrix} = \begin{bmatrix} 0 & \left[ -(N_j + h_j) + \dfrac{N_j^3 e_o^3 \cos^3 \phi_j}{a^2} \right] \sin \phi_j \cos \lambda_o & \cos \phi_j \cos \lambda_o \\ 0 & \left[ -(N_j + h_j) + \dfrac{N_j^3}{a_o^2} e_o^3 \cos^3 \phi_j \right] \sin \phi_j \sin \lambda_o & \cos \phi_j \sin \lambda_o \\ 0 & \left\{ \left[ N_j(1-e_o^2) + h_j \right] + \dfrac{N_j^3}{a_o^2} e_o^2 (1 - e_o^2 \sin^2 \phi_o) \right\} \cos \phi_j & \sin \phi_j \end{bmatrix} \begin{bmatrix} \Delta\lambda \\ \Delta\phi \\ \Delta h \end{bmatrix}$$

and iterating to convergence, using $\lambda_2$, $\phi_1$, $h_1$ for starting values. Since the $\lambda$, $\phi$, $h$ will be very close together, the process converges rapidly. It is sometimes simpler to proceed, when an <u>exact</u> solution is not needed, by solving the equations

$$\begin{bmatrix} X_2^1 \\ X_2^2 \\ X_2^3 \end{bmatrix} = \begin{bmatrix} N & \cos\lambda_2 & \cos\phi \\ N & \sin\lambda_2 & \cos\phi \\ N(1-e_2^2) & & \sin\phi \end{bmatrix} \tag{10}$$

for $\lambda_2$ and $\phi$. Then

$$\begin{bmatrix} \lambda_2 \\ \phi \end{bmatrix} \equiv \begin{bmatrix} \lambda_2 \\ \phi_2 \end{bmatrix}$$

and $h_2$ is computed from equation D-7. While the latitudes $\underline{\phi}$ and heights $\underline{h}$ that result are not exact, since $\phi$ is for an ellipsoid homothetic to the ellipsoid actually wanted, the difference in most cases is substantially less than 1 meter.

In practice, the angles will either be identically zero by definition, since it is customary in determining "best" fitting ellipsoids to enforce the parallelism of that ellipsoid's axes to the axes of the "best-fitting" ellipsoid at Greenwich, or so small that they may be considered zero. Then only the translation vector remains. Furthermore, in many applications, it is merely a case of from one ellipsoid to another, not completely from one datum to another. The two ellipsoids are then assumed tangent at the common datum point, and the transformation vector is given by

$$P\begin{bmatrix} X_2 \end{bmatrix} - P\begin{bmatrix} X_1 \end{bmatrix} = \begin{bmatrix} \Delta X_1 \end{bmatrix} .$$

## APPENDIX E: THE ELLIPSE OF ERROR

A non-singular symmetric matrix can be brought into the diagonal canonical form by a similarity transformation. In the present case, this involves the diagonalization of a series of 2 x 2 submatrices of $\triangle E^3 = 0$. Hence the transformation to principal axes is

$$\begin{bmatrix} V \end{bmatrix} = \begin{bmatrix} R_1 & 0 & 0 \;\ldots & 0 \\ 0 & R_2 & & \cdot \\ \cdot & & \ddots & \vdots \\ \cdot & & & \cdot \\ \cdot & \cdot & \ldots & R_{n-1} \end{bmatrix}^T \begin{bmatrix} \\ S \\ \\ \end{bmatrix} \begin{bmatrix} R_1 & 0 \;\ldots & & 0 \\ 0 & R_2 & & \cdot \\ \cdot & & \ddots & \vdots \\ \cdot & & & \cdot \\ \cdot & \cdot & \ldots & R_{n-1} \end{bmatrix}$$

where each rotation submatrix is of the form (61.2) or (61.3), depending on wheter or not the $\triangle E^3 = 0$ or not.

The derivation of the ellipse of error equation for the simple case $R_i = 0$, $i \geqslant 2$, is given by Grossmann (reference 9). Interpretation of the ellipse in this way is not easy, however, but the extension of the ellipse of error to an n-dimensional case is quite easy. A more general presentation is given by Cramer (reference 10).

In dealing with the definition of the n-dimensional ellipsoid of error, it is simplest to consider the axes of the ellipsoid to be defined by the eigen-values of the covariance matrix, so that

$$q^2 = \left[\triangle E^i - \overline{\triangle E}^i\right]^T \left[S_i^i\right]^{-1} \left[\triangle E^i - \overline{\triangle E}^i\right]$$

defines the ellipsoid. $\left[\overline{\triangle E}^i\right]$ here denotes the least-squares estimation of $\left[\triangle E^i\right]$, and q2 is the parameter defining the size of the ellipsoid (reference 19). The value of $q^2$ is the same as the value of $x^2$, this value is chosen to represent the $1-\alpha$ level of confidence.

The confidence ellipsoid, in n-dimensional space, is generally defined by the axes of the coordinate system in which the measurements are carried out and by the standard deviation values gotten from the variance matrix. This is not strictly analogous to the ellipse of error used in 2-dimensional error analysis (reference 9, Grossman). An extension of the ellipse of error concept is desirable, and is provided in the same way.

As in the 2-dimensional case, a rotation matrix, this time of 3 or more dimensions, is sought which will convert the existing coordinate system to one in which the variance matrix is diagonal and for which the variances are a maximum over any other rotation. The required rotation matrix $\left[ R_{ij} \right]_2$ and the corresponding variance matrix $\left[ \Sigma_{ij} \right]$ are derived from the initial variance matrix $\left[ \sigma_{ij} \right]$ by solving equation

$$\left\{ \left[ \sigma_{ij} \right] - \Sigma_{ii} \left[ I \right] \right\} \left[ R_i \right] = 0$$

for the eigen vectors $\left[ R_i \right]$ corresponding to the eigen values $\Sigma_{ii}$ of the original matrix $\left[ \sigma_{ij} \right]$. The usual method of computation permits an easy ordering of the $\Sigma_{ii}$ and rotated $\Delta X^i$ so that

$$\Sigma_{ij} = 0 \quad , \; i \neq j$$

$$\Sigma_{ii} \geqslant \Sigma_{i+1,\, i+1} \geqslant 0$$

## APPENDIX F: TRANSFORMATION BETWEEN DATUMS

The problem is to convert corrections to the observation site into corrections to the corresponding datum. Occultations give corrections $\left[\Delta E^i\right]$ to the observation site. These corrections are made up of two portions

$$\left[\Delta E^i\right] = \left[E^i\right]_s + \left[T\right]\left[\Delta E^i\right]_d$$

The $\left[\Delta E^i\right]_s$ arise from errors in the actual survey connecting the site to the triangulation system, while the $\left[\Delta E^i\right]_d$ arise from the deflection of the vertical at the datum. Since $\left[\Delta E^i\right]_s$ and $\left[\Delta E^i\right]_d$ enter into the equations in different ways, the latter being subjected to a linear transformation, it is theoretically possible to solve for both. Practically, however, the number of occultations available is not sufficient to permit solving for a large number of unknowns, and attention must be focused on the $\left[\Delta E^i\right]_d$, which are the more important. $\left[E^i\right]_s$ can usually be set equal to 0, since the occultation sites are connected to the existing triangulation by third-order survey and the resulting error is less than two meters at the worst.

The formulae of Hristow (reference 8) permit adjustment of a large number of parameters simultaneously. They can be written as

$$\begin{bmatrix} d\lambda \cos \phi \\ d\phi \\ ds \\ dA \\ d\alpha \\ da \end{bmatrix}_{obs.} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{16} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{26} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{36} \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ a_{61} & a_{62} & a_{63} & \cdots & a_{66} \end{bmatrix} \begin{bmatrix} d\lambda \cos \phi \\ d\phi \\ ds \\ dA \\ d\alpha \\ da \end{bmatrix}_d$$

where $\underline{s}$ is the geodetic distance, $\underline{A}$ is the azimuth (from the datum), $\underline{\alpha}$ is the flattening, and $\underline{a}$ is the semi-major axis of the ellipsoid. For the present purposes

$$d\alpha = 0$$

$$da = 0$$

Furthermore, the fact that $P_{obs}$ and $P_d$ are connected through a triangulation net implies that

$$ds = 0$$

While the condition $dA = 0$ is not necessary, it is desirable because of the paucity of occultation data. Its enforcement merely keeps the internal orientation of the net unchanged. Hence, the transformation equations can be written as

$$\begin{bmatrix} \Delta E^1 \\ \Delta E^2 \end{bmatrix}_{obs} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} E^1 \\ E^2 \end{bmatrix}_{datum}$$

The individual elements of the $a_{ij}$ matrix are

$$a_{11} = \cos \phi_d \sec \phi_s$$

$$a_{12} = \sec \phi_s \left\{ \left[ t(1-\eta^2 + \eta^4)\Delta\lambda + (1+t^2-\eta^2-2t^2\eta^2) \Delta\phi\Delta\lambda + t(1+t^2)\Delta\phi^2\lambda - \frac{1}{6}\cos^2\phi t(1+t^2)\Delta\lambda^3 \right] \right\}$$

$$a_{21} = 0$$

$$a_{31} = \left[ 1-3t(\eta^3 - \eta^4)\Delta\phi - \frac{3}{2}(\eta^2 - t^2\eta^2)\Delta\phi^2 - - \frac{1}{2}\cos^2\phi\,(1+t^2+\eta^2)\Delta\lambda^2 \right]$$

Here $\Delta\phi = \phi_{obs} - \phi_{datum}$

$$\Delta\lambda = \lambda_{obs} - \lambda_{datum}$$

$$t = \tan\phi$$

$$\eta^2 = e'^2 \cos^2\phi$$

$$e'^2 = \frac{a^2 - b^2}{b^2}$$

## APPENDIX G: THE ADDITION OF CORRECTIONS FOR LUNAR LIMB INEQUALITIES

The equal-limb-line method depends for a great deal of its effect upon the assumption that at points 1 and 2 the occultation is observed at the same point on the moon or that, at any rate, in the equation (the Taylor Series expansion with quadrates and higher terms dropped)

$$\Delta h_{moon} = \frac{\partial \rho_{moon}}{\partial t} \Delta t + \frac{\partial \rho_{moon}}{\partial \lambda_2} \Delta \lambda + \frac{\partial \rho_{moon}}{\partial \phi_2} \Delta \phi^2$$

( $\rho$ moon is the lunar radius observed from point 1; $\Delta t$, $\Delta \lambda_2$ $\Delta \phi_2$ are the variations in time, longitude, and latitude causing the occultation point on the moon to change; and $\Delta h_{moon}$ is the corresponding change in lunar height), the partial derivatives are small. This assumption may, in cases where the quantities $\Delta t$, $\Delta \lambda_2$, $\Delta \phi_2$ are large, but quite wrong, and a method of correcting for this error is desirable.

One such method is to use a chart of the lunar limb in the region being considered, and from the chart to estimate the partials in the Taylor Series linear terms. The definitive charts produced by Dr. C. B. Watts of the U.S. Naval Observatory provide a test of the procedure. These charts, with a putative accuracy of ± 200 meters in height, may be used if we assume that, although

$$h_{Watts,\ 1} - h_{true,\ 1} \approx 200 \text{ m}$$

$$h_{Watts,\ 2} - h_{true,\ 2} \approx 200 \text{ m}$$

yet $h_{Watts,2} - h_{Watts,1} \neq h_{true,2} - h_{true,1}$

consequently, we may proceed as follows:

a. For the time of occultation at point-1, compute from the charts the lunar radius at the occultation point.

b. Resolve the point-2 shift in longitude and latitude into components parallel to and perpendicular to the lunar profile at the instant of occultation at point-1.

c. Convert the parallel component into limits which can be used on the charts, and compute the height of profile at the shifted point on the profile.

d. Add the difference between the two profiles to the $\Delta\sigma$ of the survey.

e. For the predicted position angle corresponding to the occultation at point-2, and for the profile at the instant of occultation at point-2 as seen from point-2, compute the lunar radius.

f. Add the difference between the two radii to the $\Delta\sigma$ computed in step d. above.

g. Use the new $\Delta\sigma$ to recompute the position of site 2.

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SUBMITTED:

S. W. HENRIKSEN
Geodesist
Research and Analysis Division

RECOMMEND APPROVAL:

D. L. MILLS
Chief, Department of Geodesy

APPROVED:

ROBERT C. MILLER
Colonel, Corps of Engineers
Commanding