63-4-

THE PERIODIC HEAVING MOTION OF A HALF-IMMERSED SPHERE: THE ANALYTIC FORM OF THE VELOCITY POTENTIAL LONG-WAVE ASYMPTOTICS OF THE VIRTUAL-MASS COEFFICIENT.*

by

F. Ursell
Department of Mathematics,
University of Manchester.

* Research supported by the Fluid Dynamics Branch, U.S. Office of Naval Research.

Research forwarded under Contract N62558-3556
(NR 062-304) with the Office of Naval Research.

ABSTRACT.

The analytic form of the velocity potential of a heaving hemisphere is studied. The potential is expanded in terms of a wave source and of wave-free potentials, and the coefficients in the expansion are studied

(1) when the dimensionless wave number Ka is small,

(2) when Ka is arbitrary.

A typical result is, that the virtual-mass coefficient is the real part of

$$\frac{(\ell n\ Ka - i\pi)A_1^*(Ka) + A_1^{**}(Ka)}{(\ell n\ Ka - i\pi)A_2^*(Ka) + A_2^{**}(Ka)} \quad .$$

where the functions $A(Ka)$ are entire functions of Ka, real for real Ka.

The argument depends on the expansion of a surface source in powers of Kr and $\ell n\ Kr$, given here for the first time.

A similar theory, not given here, can be developed for two-dimensional moving cylinders. It is believed that investigations of this kind will be helpful in studying the damped motion of freely floating bodies on still water.

1. INTRODUCTION AND SUMMARY OF RESULTS.

The heaving sphere has recently been studied by Sir Thomas Havelock (1955) and by Barakat (1962), both of whom give computations of the (dimensionless) virtual-mass coefficient and damping parameter. Their results do not agree for small values of the parameter $Ka = \sigma^2 a/g$ (where $2\pi/\sigma$ is the period, and a is the radius of the sphere). In particular, Havelock finds that the virtual mass is initially an increasing function of Ka, whereas Barakat finds that it is a decreasing function of Ka. It is the purpose of the present note to consider the analytic form of the expansion of the potential, particularly for small Ka.

It is shown that for all Ka the virtual-mass coefficient is the real part of an expression of the form

$$\frac{(\ln Ka - i\pi)A_1^*(Ka) + A_1^{**}(Ka)}{(\ln Ka - i\pi)A_2^*(Ka) + A_2^{**}(Ka)}, \tag{1.1}$$

where the A's are entire functions of Ka (i.e. power series convergent for all Ka), real for real Ka, for which explicit expressions in closed form are not known. The argument depends on the expansion of a surface source in powers of Kr and $\ln Kr$ which is here given for the first time. The leading coefficients are examined, and it is found that the virtual mass is initially an increasing function of Ka, in agreement with Havelock's result.

2. STATEMENT OF THE PROBLEM. SUPERPOSITION OF SINGULARITIES AND EXPANSION OF THE POTENTIAL.

Spherical polar coordinates r, θ are defined, with their origin in the mean free surface at the mean position of the centre of the sphere. The vertical axis of symmetry is $\theta = 0$, pointing downwards. (We shall also use coordinates $y = r \cos \theta$, $R = r \sin \theta$) The fluid is assumed inviscid, so that the motion is irrotational; the amplitude of motion is assumed so small that the equations can be linearised. Then the velocity potential $\phi(r, \theta)e^{-i\omega t}$ satisfies Laplace's equation

$$\frac{\partial^2 \phi}{\partial r^2} + \frac{2}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial}{\partial \mu}\left\{(1-\mu^2)\frac{\partial \phi}{\partial \mu}\right\} = 0 \qquad (2.1)$$

where $\mu = \cos \theta$, in the region $r > a$, $0 \leq \theta \leq \frac{1}{2}\pi$.

On the free surface the pressure is constant. It follows (Lamb, 1932, paragraph 227) that

$$K\phi + \frac{\partial \phi}{\partial y} = 0 \qquad (2.2)$$

on the mean free surface $y = 0$, $\theta = \frac{1}{2}\pi$, where $K = \sigma^2/g$.

On the sphere, the normal velocity is prescribed:

$$\frac{\partial \phi}{\partial r} = V_0 \cos \theta \quad \text{on} \quad r = a\,. \qquad (2.3)$$

At infinity the waves travel outwards:

$$r^{\frac{1}{2}}\left(\frac{\partial \phi}{\partial r} - iK\phi\right) \rightarrow 0 \quad \text{as} \quad r = r \sin \theta \rightarrow \infty\,. \qquad (2.4)$$

To solve this problem, Havelock (1955) and Barakat (1962) proceed in the same manner. The potential is expressed as the superposition of a wave source at the origin,

$$\phi(r, \theta, K) = ⨍_0^\infty \frac{k}{k - K}\, e^{-k r \cos \theta}\, J_0(k r \sin \theta)\,dk \qquad (2.5)$$

$$= \int_0^\infty \frac{k}{k - K} e^{-ky} J_0(k R)\, dk$$

where the path of integration passes below $k' = K$, together with wave-free potentials

$$\phi_n = a^{2n} \left(\frac{P_{2n}(\cos \theta)}{r^{2n+1}} + \frac{K}{2n} \frac{P_{2n-1}(\cos \theta)}{r^{2n}}\right) , \qquad (2.6)$$

$(n = 1, 2, 3, \ldots)$.

(This form of expansion is analogous to the expansion used in the plane problem of the heaving circular cylinder, Ursell 1949). It is easy to see that (2.5) and (2.6) are solutions of Laplace's equation (2.1). To verify that (2.5) and (2.6) satisfy the free-surface condition (2.2) we note the representation

$$r^{-\nu-1} P_\nu(\cos \theta) = \frac{1}{\Gamma(\nu+1)} \int_0^\infty (k)^\nu e^{-ky} J_0(kR)dk \ ; \qquad (2.7)$$

this is valid in the half-space $y > 0$, since both sides of (2.7) are clearly equal when $\theta = 0$, $R = 0$; and they are both harmonic and axially symmetric in the half-space.

$$\text{Thus} \quad (K + \frac{\partial}{\partial y})\phi_0 = - \int_0^\infty k\, e^{-ky} J_0(kR)dk$$

$$= - r^{-2} P_1(\cos \theta) ,$$

$$= 0 \quad \text{when} \quad \theta = \tfrac{1}{2}\pi . \qquad (2.8)$$

And

$$\phi_n = \frac{a^{2n}}{\Gamma(2n+1)} \int_0^\infty e^{-ky} J_0(kR) \{(k)^{2n} + K(k)^{2n-1}\}dk$$

$$= \frac{a^{2n}}{\Gamma(2n+1)} \int_0^\infty e^{-ky} J_0(kR)(k)^{2n-1}(K + k)dk ,$$

whence

$$(K + \frac{\partial}{\partial y})\phi_n = \frac{a^{2n}}{\Gamma(2n+1)} \int_0^\infty e^{-ky} J_0(kR)(k)^{2n-1}\{K^2-(k)^2\}dk$$

$$= a^{2n} \left\{\frac{K^2}{2n} \frac{P_{2n-1}(\cos \theta)}{r^{2n}} - (2n+1) \frac{P_{2n+1}(\cos \theta)}{r^{2n+2}} \right\} ,$$

$$= 0 \quad \text{when} \quad \theta = \tfrac{1}{2}\pi . \qquad (2.9)$$

Thus ϕ_o and ϕ_n satisfy the free-surface condition. Finally it must be shown that the functions ϕ_o and ϕ_n satisfy the radiation condition (2.4). Obviously the ϕ_n satisfy this trivially. As for ϕ_o, we have

$$\phi_o = \int_o^\infty \frac{k}{k-K} e^{-ky} \{\tfrac{1}{2} H_o^{(1)}(k R) + \tfrac{1}{2} H_o^{(2)}(k R)\} dk \quad (2.10)$$

$$= \pi i K e^{-Ky} H_o^{(1)}(KR)$$
$$+ \tfrac{1}{2} \int_o^{\infty \exp(1/4\pi i)} \frac{k}{k-K} e^{-ky} H_o^{(1)}(kR)dk$$
$$+ \tfrac{1}{2} \int_o^{\infty \exp(-1/4\pi i)} \frac{k}{k-K} e^{-ky} H_o^{(2)}(kR)dk \quad (2.11)$$

where the deformations in the complex k-plane are made because $H_o^{(1)}$ and $H_o^{(2)}$ are small in the first and fourth quadrants respectively. Since the line of integration in (2.10) passes below $k = K$, the integral involving $H_o^{(1)}$ (but not the integral involving $H_o^{(2)}$) contains a contribution from the pole at $k = K$. It is not difficult to see that the integrals in (2.11) tend to zero rapidly as KR tends to ∞, and satisfy the radiation condition trivially, while the term $\pi i Ke^{-Ky} H_o^{(1)}(KR)$ represents a cylindrical wave spreading outwards and satisfies the radiation condition non-trivially.

We now suppose that the potential is normalized in such a way that in suitable units

$$(a^3 \sigma)^{-1} \phi(r, \theta) = D^*(Ka)\, \phi_o(r, \theta) + \sum_1^\infty \frac{\alpha_n}{2n+1} \phi_n(r, \theta) \quad (2.12)$$

where the coefficient of ϕ_o has been arbitrarily chosen as an entire function of Ka. We shall choose $D^*(Ka) \equiv 1$ when we are interested only in small values of Ka, but we shall see later that we can cover a wider range of values of Ka by choosing for $D^*(Ka)$ the Fredholm determinant $D(Ka)$ of our system of equations. The coefficients $\alpha_n(Ka)$ are complex-valued and are to be found. The coefficient V_0 in the boundary condition (2.3) must also be treated as unknown. As we have just seen, each term in the series (2.12) satisfies all the equations except (2.3), which is also satisfied if, in the range $0 \leq \theta \leq \frac{1}{2}\pi$,

$$(a^3\sigma)^{-1} V_0 \cos\theta = D^* \frac{\partial \phi_0}{\partial r}(r, \theta) + \sum_1^\infty \frac{\alpha_n}{2n+1} \frac{\partial \phi_n}{\partial r}(r, \theta) \qquad (2.13)$$

when $r = a$. The unknowns V_0, α_n in (2.12) are to be chosen so that (2.13) is satisfied, and we are particularly interested in the solution for small values of the parameter Ka.

So far we have followed closely the treatment given for the plane problem of the circular cylinder, (Ursell 1949) except that the appropriate three-dimensional expressions have been substituted for the wave source and the wave-free potentials. Our equation (2.12) is substantially equivalent to Havelock's equation (10) and Barakat's equation (6.2) except that the source term on the right-hand side of (2.12) has been normalized rather than the velocity term on the left. This greatly simplifies the results.

Let (2.13) be rewritten in the form

$$D^*F(\mu, Ka) \equiv a^2\left(\frac{\partial \phi_0}{\partial r}(r, \theta)\right)_{r=a} D^*(Ka) = (a\sigma)^{-1} V_0 \mu$$

$$+ \sum_1^\infty \alpha_n \left\{P_{2n}(\mu) + \frac{Ka}{2n+1} P_{2n-1}(\mu)\right\} \qquad (2.14)$$

where $\mu = \cos\theta$, $0 \leq \mu \leq 1$. It is convenient first to eliminate V_0 . On integrating (2.14) from $\mu = 0$ to $\mu = 1$, and using the orthogonality of the set $\{P_{2n}(\mu)\}$ over $(0, 1)$, it is seen that

$$D^* \int_0^1 F(\mu)\, d\mu = \tfrac{1}{2} (a\sigma)^{-1} V_0$$

$$+ Ka \sum_1^\infty \frac{\alpha_n}{2n+1} \int_0^1 P_{2n-1}(\mu)\, d\mu \qquad (2.15)$$

When $(a\sigma)^{-1} V_0$ is eliminated between (2.14) and (2.15) we obtain

$$D^*\left\{F(\mu, Ka) - 2\mu \int_0^1 F(v, Ka)\, dv\right\} \equiv D^*(Ka)\, G(\mu, Ka)$$

$$= \sum_1^\infty \alpha_n(Ka) \left\{P_{2n}(\mu) + \frac{K}{2n+1}\left(P_{2n-1}(\mu)\right.\right.$$

$$\left.\left. - 2\mu \int_0^1 P_{2n-1}(v)dv\right)\right\} \qquad (2.16)$$

To solve this, (2.16) is multiplied successively by the complete set $P_{2n}(\mu)$, $(n = 0, 1, 2, \ldots)$, and integrated from $\mu = 0$ to $\mu = 1$. Using the known values for the integrals involving Legendre functions we find that the unknowns α_n satisfy the infinite system of linear equations

$$D^*(Ka)\ (4n+1) \int_0^1 G(\mu, Ka)\, P_{2n}(\mu)\, d\mu$$

$$= \alpha_n + Ka \sum_{k=1}^\infty \alpha_k \left(\frac{4n+1}{2k+1}\right) c_{kn}\ , \quad n = 1, 2, 3, \ldots \qquad (2.17)$$

where

$$c_{kn} = -\frac{k\,P_{2k}(0)P_{2n}(0)}{(k+n)(2k-2n-1)} - \frac{2\,P_{2k}(0)\,P_{2n}(0)}{(2k-1)(2n-1)(2n+2)}$$

is independent of Ka .

The equations (2.17) can be written as equations for $n^{\frac{1}{2}}\alpha_n$:

$$\zeta_n(Ka) \equiv D^*.\,n^{\frac{1}{2}}(4n+1)\int_0^1 G(\mu)\,P_{2n}(\mu)d\mu = n^{\frac{1}{2}}\alpha_n + Ka\sum_{k=1}^{\infty}(k^{\frac{1}{2}}\alpha_k)d_{kn} \qquad (2.18)$$

where $\quad d_{kn} = \frac{n^{\frac{1}{2}}(4n+1)}{k^{\frac{1}{2}}(2k+1)}c_{kn}$,

~~and can then be solved by convergent infinite processes, since it~~ can be shown that $\Sigma|\zeta_n|^2$ and $\Sigma\Sigma|d_{kn}|^2$ converge. The theory of such infinite systems is analogous to the Fredholm theory for integral equations of the second kind and shows that the solution is of the form

$$n^{\frac{1}{2}}\alpha_n = \frac{D^*(Ka)}{D(Ka)}\sum_{m=1}^{\infty} D_{nm}(Ka)\,\zeta_m(Ka) \qquad (2.19)$$

where $D_{nm}(Ka)$, $D(Ka)$ are entire functions of Ka , since the coefficients on the right-hand side of (2.18) involve only powers of Ka . (This result may be obtained, for instance, from the theory of infinite determinants, Riesz, 1913, p.36. A trivial preliminary transformation is then needed, Riesz, p.39, since $\sum_k |d_{kk}|$ diverges.) The denominator $D(Ka)$ (the Fredholm determinant of the system), may have real zeros, but clearly $D(0) = 1$. If we are interested in a range of Ka which may contain a zero of $D(Ka)$ we shall find it convenient to take $D^*(Ka) = D(Ka)$, and then

$$n^{\frac{1}{2}} \alpha_n = \sum_m D_{nm}(Ka)\, \zeta_m(Ka) \tag{2.20}$$

where the $D_{nm}(Ka)$ are entire functions of Ka. If we are interested only in small values of Ka then we normalize the source term by taking $D^*(Ka) = 1$, and we shall have

$$n^{\frac{1}{2}} \alpha_n = \sum_m \delta_{nm}(Ka)\, \zeta_m(Ka) \tag{2.21}$$

where the $\delta_{nm}(Ka)$ are power series, known to be convergent only for small Ka. In either case the general theory shows that $\Sigma n|\alpha_n|^2$ converges, and one more substitution in (2.18) shows that in fact $\alpha_n = O(n^{-3/2})$. It thence follows that the series for the potential and for the velocity components are absolutely convergent when $r \geq a$.

We are interested also in the analytic form of the coefficients $\alpha_n(Ka)$. We shall show later (eqn. 3.6) that, for all Ka,

$$\begin{aligned} G(\mu, Ka) &= (\ln Ka - i\pi)\, g^*(\mu, Ka) + g^{**}(\mu, Ka) \\ &= (\ln Ka - i\pi) \sum_{n=2}^{\infty} (Ka)^n\, g_n^*(\mu) \\ &\quad + \sum_0^{\infty} (Ka)^n\, g_n^{**}(\mu), \end{aligned} \tag{2.22}$$

where $g^*(\mu, Ka)$ and $g^{**}(\mu, Ka)$' are functions of μ, regular when $0 < \mu < 1$ and real for real Ka, and where the power series converge for all Ka. It will follow immediately that

$$\zeta_n(Ka) = (\ln Ka - i\pi)\, \zeta_n^*(Ka) + \zeta_n^{**}(Ka)$$

where the $\zeta_n{}^*$ and $\zeta_n{}^{**}$ are entire functions of Ka , real for real Ka . If we choose $D^*(Ka) = D(Ka)$, then it follows from (2.20) that

$$\alpha_n(Ka) = (\ln Ka - i\pi)\,\alpha_n{}^*(Ka) + \alpha_n{}^{**}(Ka), \quad (2.23)$$

where the $\alpha_n{}^*$ and $\alpha_n{}^{**}$ are entire functions of Ka , real for real Ka . Substitution in (2.15) now gives

$$(a\sigma)^{-1}V_o = (\ln Ka - i\pi)\; V_o{}^*(Ka) + V_o{}^{**}(Ka), \quad (2.24)$$

where $V_o{}^*$ and $V_o{}^{**}$ are entire functions of Ka , real for real Ka .

If on the other hand we are interested only in small values of Ka , then it is convenient to choose $D^*(Ka) \equiv 1$. The results (2.23) and (2.24) are then formally unaltered but the functions $\alpha_n{}^*$, $\alpha_n{}^{**}$, $V_o{}^*$, $V_o{}^{**}$ are now power series convergent for sufficiently small Ka .

We thus see that the behaviour of the unknown coefficients depends on the behaviour of the function $G(\mu, Ka)$ defined in (2.16). For this we need the expansion of the wave-source potential $\phi_o(r, \theta, K)$.

3. AN EXPANSION FOR THE POTENTIAL OF A WAVE-SOURCE IN THE FREE SURFACE.

Expansions for a submerged wave source in three dimensions have been given by Thorne (1953). Near the source the potential has the form

$$\phi = \frac{1}{r} + \sum_{0}^{\infty} A_n \, r^n \, P_n(\mu) \qquad (3.1)$$

where r is the distance from the source, and the coefficients A_n depend on the depth of the fluid, the depth of immersion of the source, and the wavelength. When the depth of immersion tends to zero the coefficients tend to infinity, and in fact the expansion for a surface source is not of the form (3.1). The simplest way of obtaining the correct expansion is perhaps the following:

Since $\phi_0(r, \theta, K)$ is axially symmetric about $\theta = 0$, it is determined by its values on $\theta = 0$.

We have

$$\begin{aligned} \phi_0(r) \equiv \phi_0(r, \theta, K) &= \int_0^\infty \frac{k}{k-K} \, e^{-kr} dk \\ &= \int_0^\infty \left(1 + \frac{K}{k-K}\right) e^{-kr} dk \\ &= \frac{1}{r} + K \int_0^\infty \frac{1}{k-K} \, e^{-kr} dk \end{aligned}$$

where the line of integration passes below the pole $k = K$.

Then the Laplace transform

$$\begin{aligned} \int_0^\infty e^{-sr}\left(\phi_0(r) - \frac{1}{r}\right) dr &= K \int_0^\infty e^{-sr}\left(\int_0^\infty \frac{1}{k-K} \, e^{-kr} dk\right) dr \\ &= K \int_0^\infty \frac{1}{k-K} \left(\int_0^\infty e^{-(s+k)r} dr\right) dk \\ &= K \int_0^\infty \frac{dk}{(k-K)(s+k)} \end{aligned}$$

, an elementary integral

$$= -\frac{K}{K+s}\,\ell n\,\frac{K}{s} + i\pi\,\frac{K}{K+s}$$

where the last term comes from the pole at $k = K$.

It follows that the Laplace transform is

$$-\sum_{0}^{\infty}(-1)^m\left(\frac{K}{s}\right)^{m+1}\ell n\,\frac{K}{s} + i\pi\sum_{0}^{\infty}(-1)^m\left(\frac{K}{s}\right)^{m+1}$$

Now
$$\int_0^\infty e^{-sr}\,\frac{(Kr)^\nu}{\Gamma(\nu+1)}\,dr = \frac{K^\nu}{s^{\nu+1}}$$

whence
$$\int_0^\infty e^{-sr}\left(\frac{\partial}{\partial\nu}\,\frac{(Kr)^\nu}{\Gamma(\nu+1)}\right)dr = \frac{K^\nu}{s^{\nu+1}}\,\ell n\,\frac{K}{s}\ .$$

Thus on $\theta = 0$

$$\phi_0(r) - \frac{1}{r} = -K\sum_{0}^{\infty}(-1)^m\left(\frac{\partial}{\partial\nu}\,\frac{(Kr)^\nu}{\Gamma(\nu+1)}\right)_{\nu=m}$$
$$+\ i\pi\,K\sum_{0}^{\infty}(-1)^m\,\frac{(Kr)^m}{m!}$$

The expansion for other values of θ can be written down at once:

$$\phi_0(r,\theta) = \frac{1}{r} - K\sum_{0}^{\infty}(-1)^m\left(\frac{\partial}{\partial\nu}\,\frac{(Kr)^\nu\,P_\nu(\cos\theta)}{\Gamma(\nu+1)}\right)_{\nu=m}$$
$$+\ i\pi\,K\sum_{0}^{\infty}(-1)^m\,\frac{(Kr)^m}{m!}\,P_m(\cos\theta)\ ,$$

whence

$$\phi_0(r,\theta) = \frac{1}{r} + K\sum_{0}^{\infty}(-1)^m(Kr)^m\,P_m(\cos\theta)\,\frac{\psi(\ +1)}{m!}$$
$$-\ K\sum_{0}^{\infty}(-1)^m\,\frac{(Kr)^m}{m!}\left(\frac{\partial}{\partial\nu}P_\nu(\cos\theta)\right)_{\nu=m}$$
$$-\ K\sum_{0}^{\infty}(-1)^m\,\frac{(Kr)^m}{m!}(\ell n\,Kr - i\pi)P_m(\cos\theta) \tag{3.2}$$

where $\psi(1) = -\gamma$, $\psi(m+1) = \frac{1}{1} + \frac{1}{2} + \ldots + \frac{1}{m} - \gamma$ are values of the logarithmic derivative of the gamma function (Erdelyi, 1953, p.16); the function $\frac{\partial}{\partial\nu}P_\nu(\cos\theta)$ is discussed by Hobson (1931, p.172); it is singular at $\theta = \pi$.*

The leading terms are

$$\phi_o(r,\theta) = \frac{1}{r} - K.\,\ell n\, Kr - K\{\gamma - i\pi + (\frac{\partial}{\partial \nu} P\ (\cos\theta))_{\nu=0}\} + O(K^2 r\ \ell n\, Kr) \qquad (3.3)$$

whence

$$\frac{\partial \phi_o}{\partial r} = -\frac{1}{r^2} - \frac{K}{r} + O\ (K^2 \ell n\, Kr)\ ; \qquad (3.4)$$

It follows that if

$$F(\mu, Ka) = a^2 (\frac{\partial \phi_o}{\partial r}\ (r,\theta))_{r=a}$$

then

$$F(\mu, Ka) = -\ (1+Ka) + (\ell n\, Ka - i\pi) \sum_0^\infty (Ka)^n f_n{}^*(\mu) + \sum_0^\infty (Ka)^n f_n{}^{**}(\mu) \qquad (3.5)$$

and that

$$\begin{aligned} G(\mu, Ka) &= F(\mu, Ka) - 2\mu \int_o F(v, Ka) dv \\ &= (2\mu - 1)(1 + Ka) \\ &\quad + (\ell n\, Ka - i\pi) \sum_{n=2}^\infty (Ka)^n g_n{}^*(\mu) \\ &\quad + \sum_2^\infty (Ka)^n g_n{}^{**}(\mu) \qquad (3.6) \end{aligned}$$

as was stated in equation (2.22) above.

A different expression for the source function, used by Havelock and by Barakat, appears to be convenient for computation but leads to difficulties near $\theta = 0$. Havelock has also given the first few terms in the expansion in powers of Ka, but the last term in his equation (26) is a non-analytic function of θ near the axis of symmetry $\theta = 0$ where in fact the potential must be regular.

* Similar expansions, not given here, can be obtained for surface singularities of higher order.

4. THE VIRTUAL-MASS COEFFICIENT.

Since the hydrodynamic pressure is $-\rho\frac{\partial}{\partial t}(\phi\, e^{-i\sigma t})$ $= i\rho\sigma\phi\, e^{-i\sigma t}$, the hydrodynamic force on the sphere is

$$-2\pi i\rho\sigma a^2 e^{-i\sigma t}\int_0^{\frac{1}{2}\pi}\phi(a,\theta)\sin\theta\cos\theta\, d\theta$$

$$= -2\pi i\rho\sigma^2 a^5 e^{-i\sigma t}\int_0^1 \{D^*(Ka)\phi_o(\mu,Ka)$$

$$+\sum_1^\infty \frac{\alpha_n}{a(2n+1)}(P_{2n}(\mu)+\frac{Ka}{2n}P_{2n-1}(\mu))\}\mu\, d\mu \qquad (4.1)$$

Equation (3.2) shows that

$$a\,\phi_o(\mu,Ka) = (\ln Ka - i\pi)\sum_1^\infty (Ka)^n h_n^*(\mu)$$

$$+\sum_0^\infty (Ka)^n h_n^{**}(\mu) \qquad (4.2)$$

and the α_n's are of the form (2.23) . Thus the force (4.1) is of the form

$$-2\pi i\rho\sigma^2 a^4 e^{-i\sigma t}\{(\ln Ka - i\pi)\, a_1^*(Ka) + a_1^{**}(Ka)\}, \qquad (4.3)$$

where the a_1's are entire functions of Ka , real when Ka is real.

The velocity of heaving is given by (2.15)

$$V_o e^{-i\sigma t} = 2e^{-i\sigma t} a\sigma\{D^*(Ka)\int_0^1 F(v,Ka)dv$$

$$- Ka\sum_1^\infty \frac{\alpha_n}{2n+1}\int_0^1 P_{2n-1}(v)dv\}, \qquad (4.4)$$

the acceleration is $-i\sigma V_o\, e^{-i\sigma t}$. Thus, from (3.5) , the velocity is of the form

$$2\,a\sigma\, e^{-i\sigma t}\{(\ln Ka - i\pi)a_2^*(Ka) + a_2^{**}(Ka)\} \qquad (4.5)$$

where the a_2's are entire functions, real when Ka is real.

By definition the virtual-mass coefficient is

$$\Re\,\frac{(\text{force})}{\frac{2}{3}\pi\varrho a^3 \times \text{acceleration}} \tag{4.6}$$

i.e. the real part of

$$3\,\frac{(\ln Ka - i\pi)\, a_1{*}(Ka) + a_1{**}(Ka)}{(\ln Ka - i\pi)\, a_2{*}(Ka) + a_2{**}(Ka)}, \tag{4.7}$$

where the functions $a_1(Ka)$ and $a_2(Ka)$ are entire functions of Ka, real when Ka is real. Except for the notation this is the result stated in (1.1).

It remains to find the value of this expression for small Ka. We now put $D^*(Ka) \equiv 1$. For small Ka we have, from (3.3) that

$$\phi_0(\mu, Ka) = \frac{1}{a} - K\,\ln Ka + O(K) \tag{4.7}$$

and from (2.17) that

$$\begin{aligned}\alpha_n &= (4n+1)\int_0^1 G(\mu, Ka)\, P_{2n}(\mu)\,d\mu + O(Ka)\\ &= (4n+1)\int_0^1 (2\mu-1)\, P_{2n}(\mu)\,d\mu + O(Ka), \quad \text{from (3.6)}\\ &= 2(4n+1)\int_0^1 P_{2n}(\mu)\,\mu\,d\mu + O(Ka)\,.\end{aligned} \tag{4.8}$$

Thus the hydrodynamic force (4.1) is

$$-2\pi i\varrho\sigma^2 a^5 e^{-i\sigma t}\int_0^1 \Big\{\frac{1}{a} - K\,\ln Ka + \frac{1}{a}\sum_0^\infty \frac{2(4n+1)}{2n+1} P_{2n}(\mu)\int_0^1 P_{2n}(v)v\,dv + O(K)\Big\}\mu\,d\mu$$

$$= -\pi i\varrho\sigma^2 a^4 e^{-i\sigma t}\Big\{1 - Ka\ln Ka + \sum_1^\infty \frac{4n+1}{2n+1}\Big(\int_0^1 P_{2n}(v)v\,dv\Big)^2 + O(Ka)\Big\} \tag{4.9}$$

$$= -\pi i\varrho\sigma^2 a^4 e^{-i\sigma t}\,\{A - Ka\,\ln Ka + O(Ka)\} \tag{4.10}$$

where A is a real <u>positive</u> constant.

Similarly the velocity of heaving (4.4) is, from (3.5), equal to

$$2a\sigma\, e^{-i\sigma t} \{- \int_0^1 (1 + Ka)dv + O(Ka)\}$$

$$= - 2a\sigma\, e^{-i\sigma t}(1 + O(Ka)) \tag{4.11}$$

The acceleration is thus

$$2ia\sigma^2\, e^{-i\sigma t}(1 + O(Ka)) , \tag{4.12}$$

and the virtual-mass coefficient, defined by (4.6) is

$$k(Ka) = \frac{3}{4}\,(A - Ka\,\ell n\,Ka + O(Ka)) ,$$

where $\frac{3}{4}A$ is the (positive) virtual-mass coefficient when $Ka = 0$. It is evident that the gradient $d\,k(Ka)/d(Ka)$ is positive infinite when $Ka = 0$, as was found by Havelock.

Barakat's results must therefore be incorrect near $Ka = 0$. At $Ka = 0$ the series can be summed in terms of tabulated constants:

$$k(0) = \frac{17}{2\pi} - 1 - \frac{3}{\pi}G$$

$$= 0.8309..$$

where $G = \frac{1}{1^2} - \frac{1}{3^2} + \frac{1}{5^2}\; ..... = 0.915966..$ is Catalan's constant of the theory of elliptic integrals. This is in reasonable agreement with Barakat's result.

REFERENCES

BARAKAT, R. 1962. Vertical motion of a floating sphere in a sine-wave sea. J.Fluid Mech., 13, 540-556.

ERDÉLYI, A (ed.)1953. Higher Transcendental Functions, Vol. 1. New York: McGraw-Hill.

HAVELOCK, Sir Thomas 1955. Waves due to a floating sphere making periodic heaving oscillations. Proc.Roy.Soc. London, A 231, 1-7.

HOBSON, E. W. 1931. Spherical and Ellipsoidal Harmonics. Cambridge University Press.

LAMB, Sir Horace 1932. Hydrodynamics, 6th Ed. Cambridge University Press.

RIESZ, F. 1913. Les systèmes d'équations linéaires à une infinité d'inconnues. Paris: Gauthier-Villars.

THORNE, R.C. 1953. Multipole expansions in the theory of surface waves. Proc.Cambridge Phil.Soc. 49, 707-716.

URSELL, F. 1949. On the heaving motion of a circular cylinder on the surface of a fluid. Quart.J.Mech.Appl.Math. 2, 218-231.