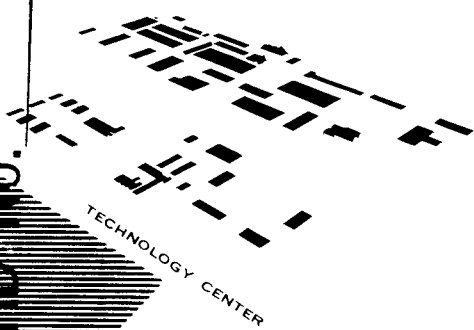


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ARF Project No. K274  
FINAL REPORT

DYNAMICS OF FLEXIBLE ROTORS

for

Bureau of Ships  
Department of the Navy  
Washington 25, D. C.

**NOTE: Effective June 1, 1963,  
the name of Armour Research  
Foundation of Illinois Institute  
of Technology will change to  
IIT RESEARCH INSTITUTE.**

ARMOUR RESEARCH FOUNDATION  
of  
ILLINOIS INSTITUTE OF TECHNOLOGY  
Technology Center  
Chicago 16, Illinois

ARF Project No. K274

FINAL REPORT

DYNAMICS OF FLEXIBLE ROTORS

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May 1963

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Department of the Navy  
Washington 25, D. C.

under

Contract No. NObs-86805

FOREWORD

This is the final report of Armour Research Foundation Project No. K274 entitled, "Dynamics of Flexible Rotors". This project was conducted for the Department of the Navy, Bureau of Ships, Washington 25, D. C. under Contract No. NObs-86805, during the period 29 May 1962 through 28 May 1963.

The project was monitored by Mr. F. F. Vane, Code 377, Bureau of Ships. Contributing ARF personnel include Dr. J. E. Panarelli, Dr. R. A. Eubanks, Mr. R. B. Lambert, Mr. W. J. Wheeler and Mr. L. Bujalski.

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## ABSTRACT

The determination of the critical speeds of a flexible rotor is usually carried out on the assumptions of negligible bearing mass and stiffness. In this report the effect of bearing mass and stiffness on the natural frequencies of a uniform flexible shaft is found. Two specific cases are considered: the shaft of symmetric cross-section rotating in bearings of different horizontal and vertical stiffness, and the unsymmetric shaft rotating in symmetrically stiff bearings. The frequency equations for various types of end constraints are solved by means of a simple graphical procedure. It is found that bearing mass and stiffness usually have considerable influence on all but the first or second critical speeds.

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## NOMENCLATURE

$S_1, S_2$  principal flexural rigidities of shaft ( $S_2 > S_1$ )

$S=(S_1+S_2)/2$  mean flexural rigidity

$L$  length of shaft

$\gamma$  mass per unit length of shaft

$\Omega$  angular speed

$\lambda$  natural frequency (radians per second)

$k$  flexibility of bearing

$M$  mass of bearing

$V_1$  displacement of shaft in first principal direction

$V_2$  displacement of shaft in second principal direction

$Z$  distance along shaft

### Non-Dimensional Variables and Parameters

$\tau$  =  $t$  time

$u$  =  $V_1/L$  displacement

$v$  =  $V_2/L$  displacement

$z$  =  $Z/L$  variable distance along shaft

$\epsilon$  =  $(S_2 - S_1)/(S_2 + S_1)$  measure of shaft anisotropy

$p$  =  $\Omega \sqrt{\gamma L^4/S}$  angular speed

$h$  =  $\gamma g L^3/S$  reduced gravity force

$K$  =  $kL^3/S$  ratio of bearing and shaft flexibilities

$m$  =  $M/\gamma L$  ratio of mass of bearing to mass of shaft

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ARF Project K274

DYNAMICS OF FLEXIBLE ROTORS

OBJECTIVES

Because of its importance in rotating machinery, the dynamics of flexible shafting has been the object of much theoretical and experimental study. This research has not only attempted to determine the vibrational behavior of various systems as a function of operating speeds, shaft parameters and support conditions, but also devised methods of minimizing these motions. For a review of previous analysis and balancing techniques, the reader is referred to Armour Research Foundation Report No. K206 and the extensive bibliography contained therein.

Despite the most exacting balancing procedures, there usually exists a small deviation of the center of gravity of the shaft from the axis of rotation. As the shaft rotates, this mass eccentricity produces a periodic centrifugal force; it is by this excitation that flexible shafts are brought into unwanted vibration. Generally, these oscillations are negligible but at certain speeds of rotation, the system may vibrate at a level which is intolerable for proper operation of the machinery or its environment. In addition to causing structural damage, excessive vibration may interfere with delicate instruments, be of discomfort to humans, or generate noise. The problem of noise reduction is of particular importance in naval construction since sound detectors are a primary means of locating ships above and below the surface.

The initial step in developing intelligent means of vibration attenuation is an analysis of the motion. Once an understanding of the interrelation of various system parameters has been made, the designer can rationally proceed to a system of desired smoothness.

In standard analyses of rotating shafts, it is assumed that the bearing mass itself has negligible effect on the characteristics of the vibrations. This is certainly the case in a rigid bearing; in practice, however, this

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idealization is sometimes impossible to achieve because of practical limitations on the size of the supporting structure. Indeed, it is possible that a bearing is rigid in one direction and quite flexible in the other direction.

The Bureau of Ships, Department of the Navy, Washington 25, D. C., established this research project under Contract No. NObs-86805 in order to investigate the effects of heavy flexible bearings on the dynamic behavior of flexible rotors. It was believed that this information would be of value to anyone concerned with rotating machinery as a means to understand and avoid unwanted vibrations.

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## INTRODUCTION

We shall investigate the dynamics of a flexible shaft rotating at constant speed in heavy flexible bearings. To keep the problem as general as possible without obscuring it with mathematical complexities we shall assume that the shaft has different stiffnesses in two mutually perpendicular directions but is otherwise uniform along its length. The bearing masses and flexibilities may all be different.

Several authors have considered the dynamics of non-symmetric rotors<sup>(1-5)\*</sup>; all of these, however, have not included the effect of bearing mass and/or flexibility. Foote, Poritsky, and Slade<sup>(6)</sup> have treated a system comprised of a single mass on a weightless non-symmetric shaft symmetrically mounted in massless flexible bearings. Their investigation involved two ordinary coupled differential equations with periodic coefficients. The results indicate that shaft anisotropy alone will cause instability over a range of speeds while shaft and bearing anisotropy can cause unstable operation over as many as three ranges of speed. Furthermore, they noted that the inclusion of damping does not necessarily remove these instabilities. Because of the nature of the differential equations, the mathematics is quite lengthy and it is doubtful whether the approach could be modified to include the effect of massive bearings. Moreover, since they considered only a one-lump parameter system, there can be no information as to stability of the continuous system at super-critical speeds.

Kellenberger<sup>(7)</sup> has treated the uniform non-circular shaft rotating in immovable bearings. The partial differential equations of motion, when referred to a rotating coordinate system, are easily solved. His method, however, is applicable only to systems whose boundary conditions are invariant under rotation. It is found that there occur an infinity of critical speed ranges that vary with the amount of anisotropy of the shaft. Secondary critical speeds induced by gravity, are also determined.

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\* Numbers in brackets denote references collected at end of this report.

In this report, the vibration of an undamped shaft in heavy self-aligning flexible bearings is studied. A method of characteristic exponents is used to investigate the stability of the system. Expressions are found in closed form for the dividing surfaces between stable and unstable domains. The secondary critical speeds are found to bear a simple relation with one of these expressions. The method, being sufficiently general, is applicable to other types of end constraints.

As a special case of the above analysis the frequency equations for various practical round shaft systems are found and the critical speeds are presented in graphical form as functions of bearing-shaft mass and stiffness ratios.

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## PROBLEM STATEMENT

Let the shaft of Fig. 1 have unequal principal stiffnesses  $S_1$  and  $S_2$ . As the shaft rotates, its stiffness in any fixed coordinate system will be a periodic function of time completing two cycles for every revolution of the shaft. Consequently, the equations of motion in any fixed coordinate system will have periodic time-dependent coefficients. When referred to rotating coordinates fixed in the shaft along the principal directions of stiffness all coefficients in the equations of motion are constant. If the horizontal and vertical flexibilities at the bearings are unequal, the flexibilities in the rotating system are also periodic functions of time. In order to avoid periodic coefficients in the boundary conditions, we shall assume that the bearings possess equal flexibility in two mutually perpendicular directions. This assumption facilitates the analysis of the non-symmetric shaft and is unnecessary in the treatment of the isotropically stiff shaft. We are thus able to investigate two cases; the anisotropic shaft in isotropic bearings and the isotropic shaft in anisotropic bearings.

So that the results may be applicable to any system, the analysis is carried out non-dimensionally. In terms of  $u$  and  $v$ , the dimensionless displacements in the rotating system, and other dimensionless variables and system parameters defined elsewhere, the equations of motion for the anisotropic shaft are

$$(1-\epsilon) \frac{\partial^4 u}{\partial z^4} + p^2 \left( \frac{\partial^2 u}{\partial \tau^2} - u - 2 \frac{\partial v}{\partial \tau} \right) = -h \sin \tau + p^2 f(z) \quad (1)$$

$$(1+\epsilon) \frac{\partial^4 v}{\partial z^4} + p^2 \left( \frac{\partial^2 v}{\partial \tau^2} - v + 2 \frac{\partial u}{\partial \tau} \right) = -h \cos \tau + p^2 g(z) \quad (2)$$

Here the parameter  $\epsilon$  ( $0 \leq \epsilon \leq 1$ ) is a measure of the shaft asymmetry with vanishing  $\epsilon$  implying an isotropic shaft. The angular speed of the shaft is characterized by  $p$  while  $h$  measures the gravity force. The functions  $f(z)$  and  $g(z)$  represent the mass eccentricity of the shaft.

We shall investigate the motion of a shaft mounted in heavy, self-aligning bearings. These lead to boundary conditions commonly referred

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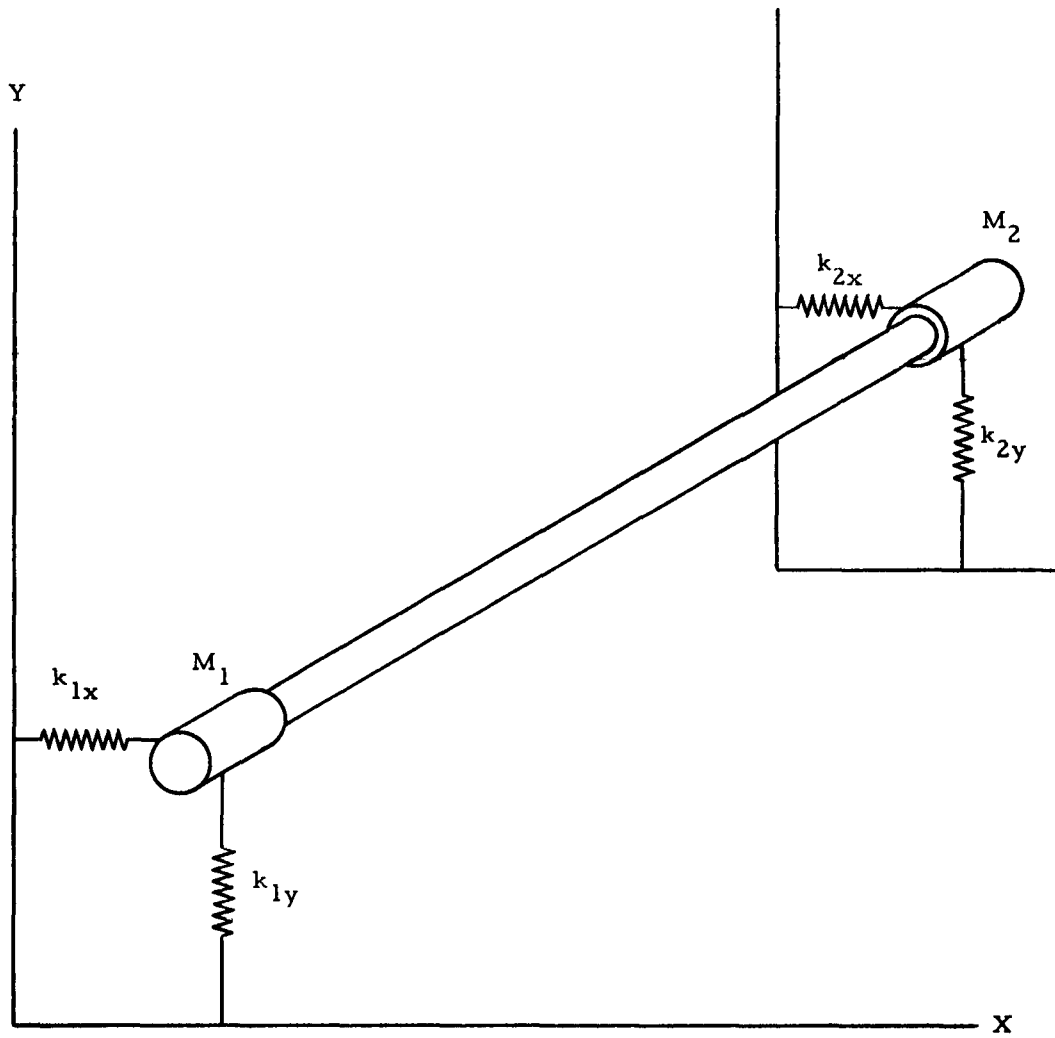


Fig. 1 SHAFT AND BEARING CONFIGURATION

to as pinned-pinned ends; the analysis that follows could apply equally as well to any other type of end constraint. The equations of motion of the bearings are

$$\begin{aligned}
 \text{at } z = 0 \quad (1-\epsilon) \frac{\partial^3 u}{\partial z^3} + K_1 u + m_1 p^2 \left( \frac{\partial^2 u}{\partial \tau^2} - u - 2 \frac{\partial v}{\partial \tau} \right) &= 0 \\
 (1+\epsilon) \frac{\partial^3 v}{\partial z^3} + K_1 v + m_1 p^2 \left( \frac{\partial^2 v}{\partial \tau^2} - v + 2 \frac{\partial u}{\partial \tau} \right) &= 0 \\
 \text{at } z = 1 \quad -(1-\epsilon) \frac{\partial^3 u}{\partial z^3} + K_2 u + m_2 p^2 \left( \frac{\partial^2 u}{\partial \tau^2} - u - 2 \frac{\partial v}{\partial \tau} \right) &= 0 \\
 +(1+\epsilon) \frac{\partial^3 v}{\partial z^3} + K_2 v + m_2 p^2 \left( \frac{\partial^2 v}{\partial \tau^2} - v + 2 \frac{\partial u}{\partial \tau} \right) &= 0
 \end{aligned} \tag{3}$$

The additional conditions for pinned-pinned end constraints are

$$\text{at } z = 0, 1 \quad \frac{\partial^2 u}{\partial z^2} = 0 \quad \frac{\partial^2 v}{\partial z^2} = 0 \tag{4}$$

Equations (1) - (3) are derived in Appendix A.

The problem is to find the displacements as functions of the bearing masses,  $m_1$ ,  $m_2$ , stiffnesses  $K_1$ ,  $K_2$ , speed  $p$ , and asymmetry factor  $\epsilon$ . For certain values of speed and asymmetry we should expect a steady-state solution while at other speeds the displacements will increase without limit. While infinite amplitudes will not occur physically, the system will operate with excessive vibrations at these so-called critical speeds. The first task, then, is to determine these critical speeds.

## THE HOMOGENEOUS SOLUTION

Because of the nature of the non-symmetric shaft, the homogeneous solution is of considerable importance. Even though damping may be present in the shaft, the homogeneous part of the solution, while being transient, can still become unbounded. This seemingly paradoxical situation is explained in Appendix B.

Since the equations of motion (1), (2) have constant coefficients, the homogeneous solution can be expressed as

$$u = A e^{\alpha z + i \mu \tau} \quad v = B e^{\alpha z + i \mu \tau} \quad (5)$$

where  $A$ ,  $B$ ,  $\alpha$  and  $\mu$  are, in general, complex constants to be determined by the end conditions. When equations (5) are substituted into (1) and (2), it is found that these constants are related by

$$\begin{aligned} \left[ (1 - \epsilon) \alpha^4 - p^2 (1 + \mu^2) \right] A - 2 p^2 \mu i B &= 0 \\ 2 p^2 \mu i A + \left[ (1 + \epsilon) \alpha^4 - p^2 (1 + \mu^2) \right] B &= 0 \end{aligned} \quad (6)$$

For a non-trivial solution, the determinant of coefficients of (6) must vanish. Performing this operation, we obtain a relation between  $\alpha$  and  $\mu$  which is quadratic in  $\alpha^4$ . Since the time exponent  $\mu$  determines whether the motion is stable, it is advisable to eliminate  $\alpha$  in favor of  $\mu$ . Thus

$$\alpha^4 = \left[ p^2 / (1 - \epsilon^2) \right] \left[ (1 + \mu^2) \pm \sqrt{\epsilon^2 (1 + \mu^2)^2 + 4 \mu^2 (1 - \epsilon^2)} \right] \quad (7)$$

For each possible  $\mu$  there are eight values of  $\alpha$  and sixteen constants of integration  $A_j$ ,  $B_j$  which, from (6) and (7) are related by

$$B_j = -i \theta_j A_j \quad (j = 1, 2, 3, \dots, 8) \quad (8)$$

where

$$\theta_j = \left[ -\epsilon (1 + \mu^2) \pm \sqrt{\epsilon^2 (1 + \mu^2)^2 + 4 \mu^2 (1 - \epsilon^2)} \right] / \left[ 2 \mu (1 + \epsilon) \right]$$

The upper sign is to be taken for  $j = 1, 2, 3, 4$  and the lower sign for  $j = 5, 6, 7, 8$ . The solution (5) may now be written as

$$u = \sum_{\text{root } \mu} \sum_{j=1}^{\infty} A_j e^{\alpha_j(\mu) z + i\mu \tau}$$

$$v = -i \sum_{\text{root } \mu} \sum_{j=1}^{\infty} \theta_j A_j e^{\alpha_j(\mu) z + i\mu \tau}$$
(9)

Here the summation is extended over all characteristic values  $\mu$  which shall be determined from the boundary conditions. For each as yet unknown  $\mu$  the constants of integration  $A_j$  are found from the boundary conditions (3) and (4), which in terms of the notation (8) and (9) become

$$\sum_{j=1}^8 A_j \left[ (1-\epsilon) \alpha_j^3 + K_1 + m_1 p^2 (-\mu^2 - 1 + 2i\theta_j) \right] = 0$$

$$\sum_{j=1}^8 A_j \left[ -(1+\epsilon) i\theta_j \alpha_j^3 - K_1 i\theta_j + m_1 p^2 (i\theta_j \mu^2 + i\theta_j + 2i\mu) \right] = 0$$

$$\sum_{j=1}^8 A_j e^{\alpha_j} \left[ -(1-\epsilon) \alpha_j^3 + K_2 + m_2 p^2 (-\mu^2 - 1 + 2i\theta_j) \right] = 0$$

$$\sum_{j=1}^8 A_j e^{\alpha_j} \left[ (1+\epsilon) i\theta_j \alpha_j^3 - K_2 i\theta_j + m_2 p^2 (i\theta_j \mu^2 + i\theta_j + 2i\mu) \right] = 0$$
(10)

and

$$\sum_{j=1}^8 \alpha_j^2 A_j = 0 \quad \sum_{j=1}^8 \theta_j \alpha_j^2 A_j = 0$$

$$\sum_{j=1}^8 \alpha_j^2 e^{\alpha_j} A_j = 0 \quad \sum_{j=1}^8 \theta_j \alpha_j^2 e^{\alpha_j} A_j = 0$$
(11)



The frequency equation for  $\mu$  is obtained by setting the determinant of coefficients of (10) and (11) equal to zero. In this manner, a non-trivial solution for  $A_j$  will be obtained. For any system configuration, all parameters except  $\mu$ , of course, are known. When all frequencies are real, the vibrations are steady-state; when the frequencies are complex, the motion is transient. The transient will decay or become unbounded depending upon the imaginary part of the complex frequency.

It is evident from Eq. (9) that the system is stable for given system parameters  $p$ ,  $\epsilon$ ,  $m$ , and  $K$  whenever real ( $i\mu$ ) is non-positive. It is a simple matter to show that the frequency equation is an even function of  $\mu$  so that both  $\mu^*$  and  $-\mu^*$  are roots. If  $\mu^*$  is a complex root and real ( $i\mu^*$ ) is non-positive then real ( $-i\mu^*$ ) is positive so that the amplitudes may increase exponentially with time. Clearly, stability is assured whenever all frequencies are real. Since instability occurs whenever the frequencies become complex, the bounding surfaces between stable and unstable domains are the loci of parameters  $p$ ,  $\epsilon$ ,  $m$ , and  $K$  for which  $\mu^2 = 0$  is a root of the frequency equation. Physically  $\mu = 0$  implies there is no time variation of the displacements in the rotating coordinate system. The shaft, then, is performing a "frozen whirl" at the onset of instability.

As indicated in Appendix C when  $\mu$  approaches zero, the eighth order determinant reduces to the product of two determinants of fourth order. The bounding surfaces are found from

$$\Delta_1 \Delta_2 = 0$$

$$\Delta_1 = \begin{vmatrix} K_1 - m_1 p^2 & K_1 - m_1 p^2 & (1 - \epsilon) \alpha_1^3 & (1 - \epsilon) \alpha_1^3 \\ \left[ (K_2 - m_2 p^2) \cosh \alpha_1 \right. & \left. (K_2 - m_2 p^2) \cos \alpha_1 \right. & \left. (K_2 - m_2 p^2) \sinh \alpha_1 \right. & \left. -(K_2 - m_2 p^2) \sin \alpha_1 \right] \\ \left. - (1 - \epsilon) \alpha_1^3 \sinh \alpha_1 \right] & \left. - (1 - \epsilon) \alpha_1^3 \sin \alpha_1 \right] & \left. - (1 - \epsilon) \alpha_1^3 \cosh \alpha_1 \right] & \left. - (1 - \epsilon) \alpha_1^3 \cos \alpha_1 \right] \\ \alpha_1^2 & -\alpha_1^2 & 0 & 0 \\ \alpha_1^2 \cosh \alpha_1 & -\alpha_1^2 \cos \alpha_1 & \alpha_1^2 \sinh \alpha_1 & \alpha_1^2 \sin \alpha_1 \end{vmatrix} \quad (12a)$$

where  $\alpha_1^4 = p^2 / (1 - \epsilon)$

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$\Delta_2 =$

$$\begin{array}{cccc}
K_1 - m_1 p^2 & K_1 - m_1 p^2 & (1 + \epsilon) \alpha_5^3 & (1 + \epsilon) \alpha_5^3 \\
\left[ (K_2 - m_2 p^2) \cosh \alpha_5 \right] & \left[ (K_2 - m_2 p^2) \cos \alpha_5 \right] & \left[ (K_2 - m_2 p^2) \sinh \alpha_5 \right] & \left[ -(K_2 - m_2 p^2) \sin \alpha_5 \right] \\
-(1 + \epsilon) \alpha_5^3 \sinh \alpha_5 & -(1 + \epsilon) \alpha_5^3 \sin \alpha_5 & -(1 + \epsilon) \alpha_5^3 \cosh \alpha_5 & -(1 + \epsilon) \alpha_5^3 \cos \alpha_5 \\
\alpha_5^2 & -\alpha_5^2 & 0 & 0 \\
\alpha_5^2 \cosh \alpha_5 & -\alpha_5^2 \cos \alpha_5 & \alpha_5^2 \sinh \alpha_5 & \alpha_5^2 \sin \alpha_5
\end{array}$$

where  $\alpha_5^4 = p^2 / (1 + \epsilon)$

(12b)

The loci of parameters for which  $\Delta_1$  or  $\Delta_2$  vanishes represents the beginning or end, respectively, of an unstable domain (as the speed  $p$  is increased). As  $\epsilon$  approaches zero, these two curves in the  $p$ - $\epsilon$  plane converge to the natural frequencies of vibration of the symmetric shaft. The determinants (12) can be expressed as

$$\begin{aligned} \Delta_1 &= (1-\epsilon)^2 \alpha_1^6 (1 - \cos \alpha_1 \cosh \alpha_1) + \alpha_1^3 (1-\epsilon) \left[ (K_1 + K_2) - (m_1 + m_2)(1-\epsilon) \alpha_1^4 \right] \times \\ &\quad \times \left[ \sinh \alpha_1 \cos \alpha_1 - \cosh \alpha_1 \sin \alpha_1 \right] + 2 \left[ m_1 (1-\epsilon) \alpha_1^4 - K_1 \right] \left[ m_2 (1-\epsilon) \alpha_1^4 - K_2 \right] \sin \alpha_1 \sinh \alpha_1 = 0 \\ \Delta_2 &= (1+\epsilon)^2 \alpha_5^6 (1 - \cos \alpha_5 \cosh \alpha_5) + \alpha_5^3 (1+\epsilon) \left[ (K_1 + K_2) - (m_1 + m_2)(1+\epsilon) \alpha_5^4 \right] \times \\ &\quad \times \left[ \sinh \alpha_5 \cos \alpha_5 - \cosh \alpha_5 \sin \alpha_5 \right] + 2 \left[ m_1 (1+\epsilon) \alpha_5^4 - K_1 \right] \left[ m_2 (1+\epsilon) \alpha_5^4 - K_2 \right] \sin \alpha_5 \sinh \alpha_5 = 0 \end{aligned}$$

For the special case of symmetric mounts,  $m_1 = m_2 = m$ ,  $K_1 = K_2 = K$  the loci reduce to

$$\begin{aligned} (1-\epsilon)^2 \alpha_1^6 (1 - \cos \alpha_1 \cosh \alpha_1) + 2(1-\epsilon) \alpha_1^3 (K - mp^2) (\cos \alpha_1 \sinh \alpha_1 - \sin \alpha_1 \cosh \alpha_1) \\ + 2 (K - mp^2)^2 \sin \alpha_1 \sinh \alpha_1 = 0 \end{aligned} \quad (13a)$$

and

$$\begin{aligned} (1+\epsilon)^2 \alpha_5^6 (1 - \cos \alpha_5 \cosh \alpha_5) + 2(1+\epsilon) \alpha_5^3 (K - mp^2) (\cos \alpha_5 \sinh \alpha_5 - \sin \alpha_5 \cosh \alpha_5) \\ + 2 (K - mp^2)^2 \sin \alpha_5 \sinh \alpha_5 = 0 \end{aligned} \quad (13b)$$

The unstable speed ranges for a pinned-pinned shaft in immovable bearings, found previously by Kellenberger, are obtained from Eqs. (13) when  $K$  approaches infinity. This results in

$$\sin \alpha_1 \sinh \alpha_1 = 0$$

$$\sin \alpha_5 \sinh \alpha_5 = 0$$

$$\alpha_1, \alpha_5 = n\pi$$

or

$$(n\pi)^2 \sqrt{1 - \epsilon} < p < (n\pi)^2 \sqrt{1 + \epsilon} \quad (n = 1, 2, \dots) \quad (14)$$

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## EFFECT OF GRAVITY - SECONDARY CRITICAL SPEEDS

The gravity solution can be treated completely and independently of the previous solution for the homogeneous case. Let us assume a particular solution of Eqs. (1) and (2) in the form

$$u = x(z) \sin \tau \quad v = y(z) \cos \tau \quad (15)$$

With this formulation, the time dependence can be factored out of the equations of motion leaving the following equations in the amplitudes  $x(z)$  and  $y(z)$ :

$$(1-\epsilon) \frac{d^4 x}{dz^4} + 2p^2(-x + y) = -h$$
$$(1+\epsilon) \frac{d^4 y}{dz^4} + 2p^2(x - y) = -h$$
(16)

Eliminating  $y(z)$  from Eqs. (16) we obtain

$$y(z) = - \left[ (1 - \epsilon^2)/2p^2 \right] \frac{d^4 x}{dz^4} + x - h/2p^2$$
$$\frac{d^8 x}{dz^8} - \left[ 4p^2/(1-\epsilon^2) \right] \frac{d^4 x}{dz^4} = 4p^2 h/(1-\epsilon^2)$$
(17)

A complete solution of equation (17) is

$$x(z) = A \sin \sigma z + B \cos \sigma z + C \sinh \sigma z + D \cosh \sigma z + Ez^3 + Fz^2 + Gz + H - hz^2/24$$
$$\sigma^4 = 4p^2/(1-\epsilon^2)$$
(18)

The eight constants of integration are obtained from the boundary conditions (3) and (4). Once again, the time dependency factors out; this would not have happened had damping been present so that a solution more general than (15) would have to have been used. These reduced boundary conditions in  $x(z)$  and  $y(z)$  are

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$$\text{at } z = 0, (1-\epsilon) \frac{d^3x}{dz^3} + K_1x - 2m_1p^2(x-y) = 0$$

$$(1+\epsilon) \frac{d^3y}{dz^3} + K_1y + 2m_1p^2(x-y) = 0 \quad (19)$$

$$\frac{d^2x}{dz^2} = \frac{d^2y}{dz^2} = 0$$

$$\text{at } z = 1, -(1-\epsilon) \frac{d^3x}{dz^3} + K_2x - 2m_2p^2(x-y) = 0$$

$$-(1+\epsilon) \frac{d^3y}{dz^3} + K_2y + 2m_2p^2(x-y) = 0 \quad (20)$$

$$\frac{d^2x}{dz^2} = \frac{d^2y}{dz^2} = 0$$

When (18) is put into (19) there results eight simultaneous linear nonhomogeneous equations for the constants of integration. For general values of  $p$ ,  $\epsilon$ ,  $K$ , and  $m$ , the constants will be bounded functions of these system parameters. However, for certain combinations of parameters, the amplitudes will become unbounded; this happens when the determinant of the coefficients of the linear equations vanishes. These values of  $p$  for fixed  $\epsilon$ ,  $m$ , and  $K$  are the secondary or gravity-induced critical speeds and are given by

$$\begin{aligned} & \sigma^6 (1-\epsilon^2)^2 (\cos\sigma \cosh\sigma - 1) + \sigma^3 (1-\epsilon^2) \left[ (m_1+m_2)(1-\epsilon^2) \sigma^4 - (K_1+K_2) \right] x \\ & x \left[ \sinh\sigma \cos\sigma - \cosh\sigma \sin\sigma \right] - 2 \left[ m_1(1-\epsilon^2) \sigma^4 - K_1 \right] \left[ m_2(1-\epsilon^2) \sigma^4 - K_2 \right] x \\ & x \left[ \sin\sigma \sinh\sigma \right] = 0 \end{aligned} \quad (21)$$

It can be seen from comparison of Eq. (21) and the first of (12a) that these curves are identical if  $\alpha_1$  is replaced by  $\sigma$  and  $\epsilon$  is replaced by  $\epsilon^2$ . Thus from the plot of  $\Delta_1$ , in the  $p$ - $\epsilon$  plane, we can obtain the secondary critical speed by replacing  $p$  with  $2p$  and  $\epsilon$  with  $\epsilon^2$ . Thus the gravity critical speeds for the round shaft are half the natural frequencies of the shaft.

## EFFECT OF MASS ECCENTRICITY

It is a practical impossibility to manufacture a shaft which is perfectly straight and uniform throughout its cross section. Consequently, there exists some small mass eccentricity along the length of the shaft even after it has been balanced. As the shaft rotates, the small centrifugal forces may be sufficient to cause vibrations which render the system unserviceable. If  $f(z)$  and  $g(z)$  are the eccentricities along the principal directions, the equations of motion become

$$(1-\epsilon) \frac{\partial^4 u}{\partial z^4} + p^2 \left( \frac{\partial^2 u}{\partial \tau^2} - u - 2 \frac{\partial v}{\partial \tau} \right) = p^2 f(z) \quad (22)$$

$$(1+\epsilon) \frac{\partial^4 v}{\partial z^4} + p^2 \left( \frac{\partial^2 v}{\partial \tau^2} - v + 2 \frac{\partial u}{\partial \tau} \right) = p^2 g(z)$$

Note that in the rotating coordinate system, the centrifugal forces are constant while in a fixed system these forces are periodic with a frequency equal to that of the shaft. A possible motion of the shaft is a "frozen whirl" where it is deformed but does not vibrate. This implies that there is no time dependency of the displacements along the principal directions; therefore we try a solution of the form

$$u = u(z) \quad v = v(z) \quad (23)$$

With the formulation (23) the equations (22) become uncoupled. The actual form of the eccentricities insofar as stability is concerned is immaterial. The solution of these equations is

$$u = A \sin \alpha_1 z + B \cos \alpha_1 z + C \sinh \alpha_1 z + D \cosh \alpha_1 z + \phi(z)$$

$$v = E \sin \alpha_5 z + F \cos \alpha_5 z + G \sinh \alpha_5 z + H \cosh \alpha_5 z + \psi(z) \quad (24)$$

$$\alpha_1^4 = p^2 / (1-\epsilon) \quad \alpha_5^4 = p^2 / (1+\epsilon)$$



where  $\phi(z)$  and  $\psi(z)$  are particular integrals of Eqs. (22). The constants of integration are found from the boundary conditions (3) and (4). This results in two sets of four non-homogeneous simultaneous linear equations. The determinants of coefficients of these sets is identical with those given by equation (12). This should have been expected since (12) was developed on the basis of vanishing frequency which is precisely the "frozen whirl" of Eqs. (23).

It is apparent, then, that the flattened shaft possesses two sets of natural frequencies like a non-symmetric beam which vibrates in two mutually perpendicular directions. Moreover, all frequencies between these adjacent critical speeds are also critical. For the symmetric shaft, these natural frequencies will coincide.

## ROUND SHAFT IN UNSYMMETRIC BEARINGS

In most practical cases, the shaft will be symmetric while the supports possess unequal flexibility in two mutually perpendicular directions. In determining the natural frequencies of the system, we do not need to use the previous notation. Since the equations of motion and the boundary conditions are uncoupled it is necessary to consider motion only in one direction. The equation of motion for the undamped system is

$$EI \frac{\partial^4 U}{\partial Z^4} + \gamma \frac{\partial^2 U}{\partial t^2} = 0 \quad (25)$$

while the boundary conditions are

$$\text{at } Z = 0 \quad EI \frac{\partial^3 U}{\partial Z^3} + k_1 U + m_1 \frac{\partial^2 U}{\partial t^2} = 0 \quad (26a)$$

$$\frac{\partial^2 U}{\partial Z^2} = 0$$

$$\text{at } Z = L \quad -EI \frac{\partial^3 U}{\partial Z^3} + k_2 U + m_2 \frac{\partial^2 U}{\partial t^2} = 0 \quad (26b)$$

$$\frac{\partial^2 U}{\partial Z^2} = 0$$

Here  $EI$  is the flexural rigidity of the shaft while the constants  $k_1$  and  $k_2$  are the stiffnesses of the supports at  $Z = 0$  and  $Z = L$  respectively.

A solution of Eq. (25) is

$$U = (A \cosh \beta \frac{Z}{L} + B \cos \beta \frac{Z}{L} + C \sin \beta \frac{Z}{L} + D \sin \beta \frac{Z}{L}) e^{i\lambda t} \quad (27a)$$

where

$$\beta^4 = \frac{\gamma \lambda^2 L^4}{EI} \quad (27b)$$

The constants of integration are found from the boundary conditions (26). A non-trivial solution for these constants results in the frequency equation

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$$\begin{array}{r}
 0 = \Delta = \\
 \hline
 \begin{array}{cccc}
 \frac{k_1 L^3}{EI} - \frac{M_1 \lambda^2 L^3}{EI} & \frac{k_1 L^3}{EI} - \frac{M_1 \lambda^2 L^3}{EI} & \beta^3 & -\beta^3 \\
 \left[ \frac{k_2 L^3}{EI} - \frac{M_2 \lambda^2 L^3}{EI} \right] \cosh \beta & \left[ \frac{k_2 L^3}{EI} - \frac{M_2 \lambda^2 L^3}{EI} \right] \cos \beta & \left[ \frac{k_2 L^3}{EI} - \frac{M_2 \lambda^2 L^3}{EI} \right] \sinh \beta & \left[ \frac{k_2 L^3}{EI} - \frac{M_2 \lambda^2 L^3}{EI} \right] \sin \beta \\
 -\beta^3 \sin \beta & -\beta^3 \sin \beta & -\beta^3 \cosh \beta & +\beta^3 \cos \beta \\
 \beta^2 & -\beta^2 & 0 & 0 \\
 \beta^2 \cosh \beta & -\beta^2 \cos \beta & \beta^2 \sinh \beta & -\beta^2 \sin \beta
 \end{array} \\
 \hline
 \end{array} \tag{28}$$

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In terms of the redefined non-dimensional parameters

$$\begin{aligned} K_1 &= \frac{k_1 L^3}{EI} & m_1 &= \frac{M_1}{\gamma L} & p^2 &= \lambda^2 \frac{\gamma L^4}{EI} = \beta^4 \\ K_2 &= \frac{k_2 L^3}{EI} & m_2 &= \frac{M_2}{\gamma L} \end{aligned} \quad (29)$$

the frequency equation (28) becomes

$$\begin{aligned} &\beta^6 (1 - \cos \beta \cosh \beta) + \beta^3 [(K_1 + K_2) - (m_1 + m_2) \beta^4] [\sinh \beta \cos \beta - \cosh \beta \sin \beta] \\ &+ 2 [m_1 \beta^4 - K_1] [m_2 \beta^4 - K_2] \sin \beta \sinh \beta = 0 \end{aligned} \quad (30)$$

For the special case of identical bearings  $K_1 = K_2 = K$ ,  $m_1 = m_2 = m$  and

$$K = m\beta^4 + \frac{\beta^3 \cosh \beta \sin \beta - \sinh \beta \cos \beta \pm (\sin \beta - \sinh \beta)}{2 \sin \beta \sinh \beta} \quad (31)$$

Taking the upper and lower signs separately, we obtain the frequency equations

$$\begin{aligned} K &= m\beta^4 + \frac{\beta^3}{2} (\tanh \frac{\beta}{2} + \tan \frac{\beta}{2}) \\ K &= m\beta^4 + \frac{\beta^3}{2} (\coth \frac{\beta}{2} - \cot \frac{\beta}{2}) \end{aligned} \quad (32)$$

The natural frequencies  $\lambda = \sqrt{\frac{EI \beta^4}{\gamma L^4}}$  for specified values of the system parameters  $m$  and  $K$  can be found by a simple graphical procedure. The functions

$$\begin{aligned} F^+ &= \frac{\pi^3 \sigma^3}{2} (\tanh \frac{\pi \sigma}{2} + \tan \frac{\pi \sigma}{2}) \\ F^- &= \frac{\pi^3 \sigma^3}{2} (\coth \frac{\pi \sigma}{2} - \cot \frac{\pi \sigma}{2}) \quad \sigma = \frac{\beta}{\pi} = \frac{L}{\pi} \left[ \frac{\gamma \lambda^2}{EI} \right]^{1/4} \end{aligned} \quad (33)$$

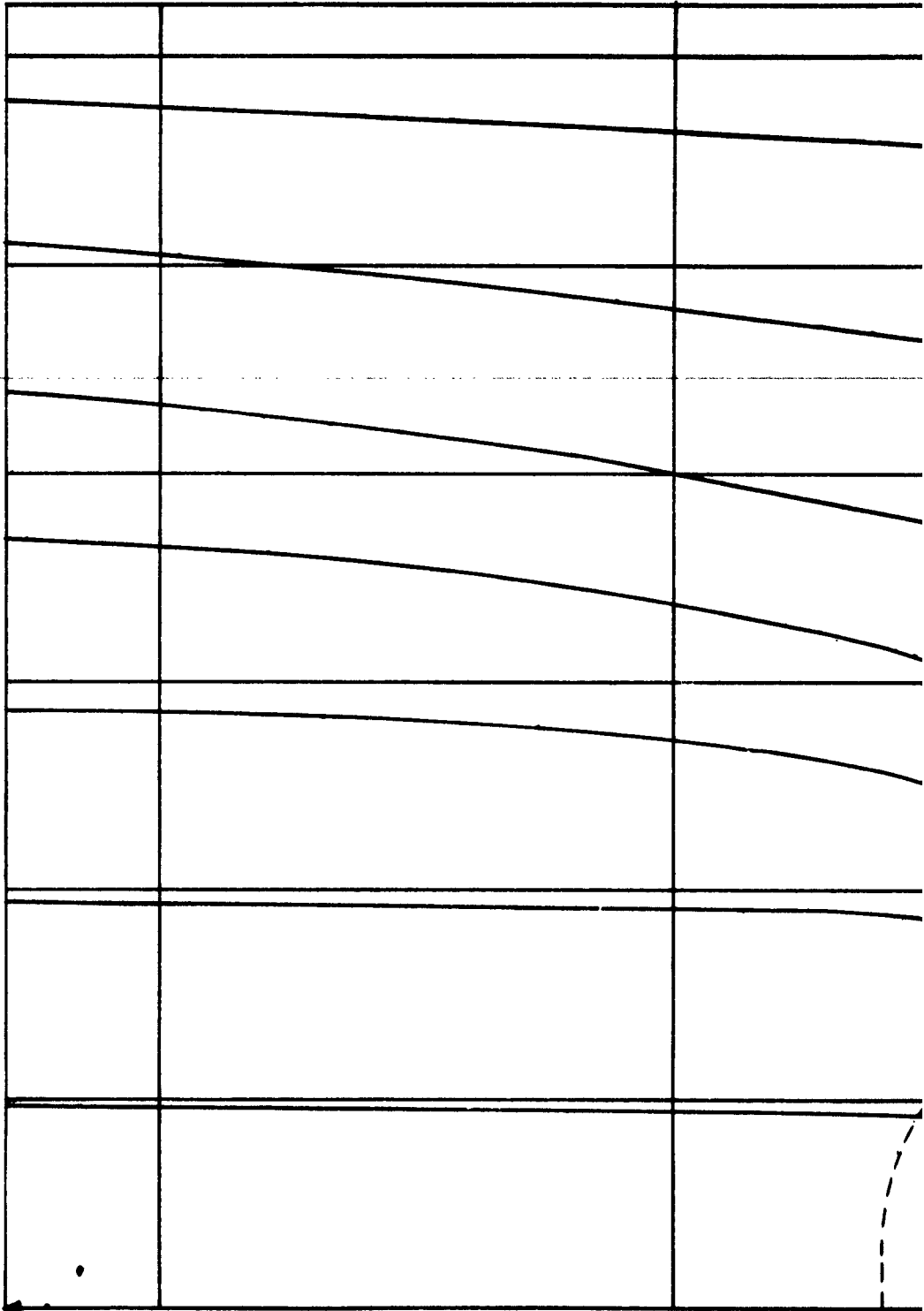
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1

$$\frac{KL^3}{EI}$$

2000

1000



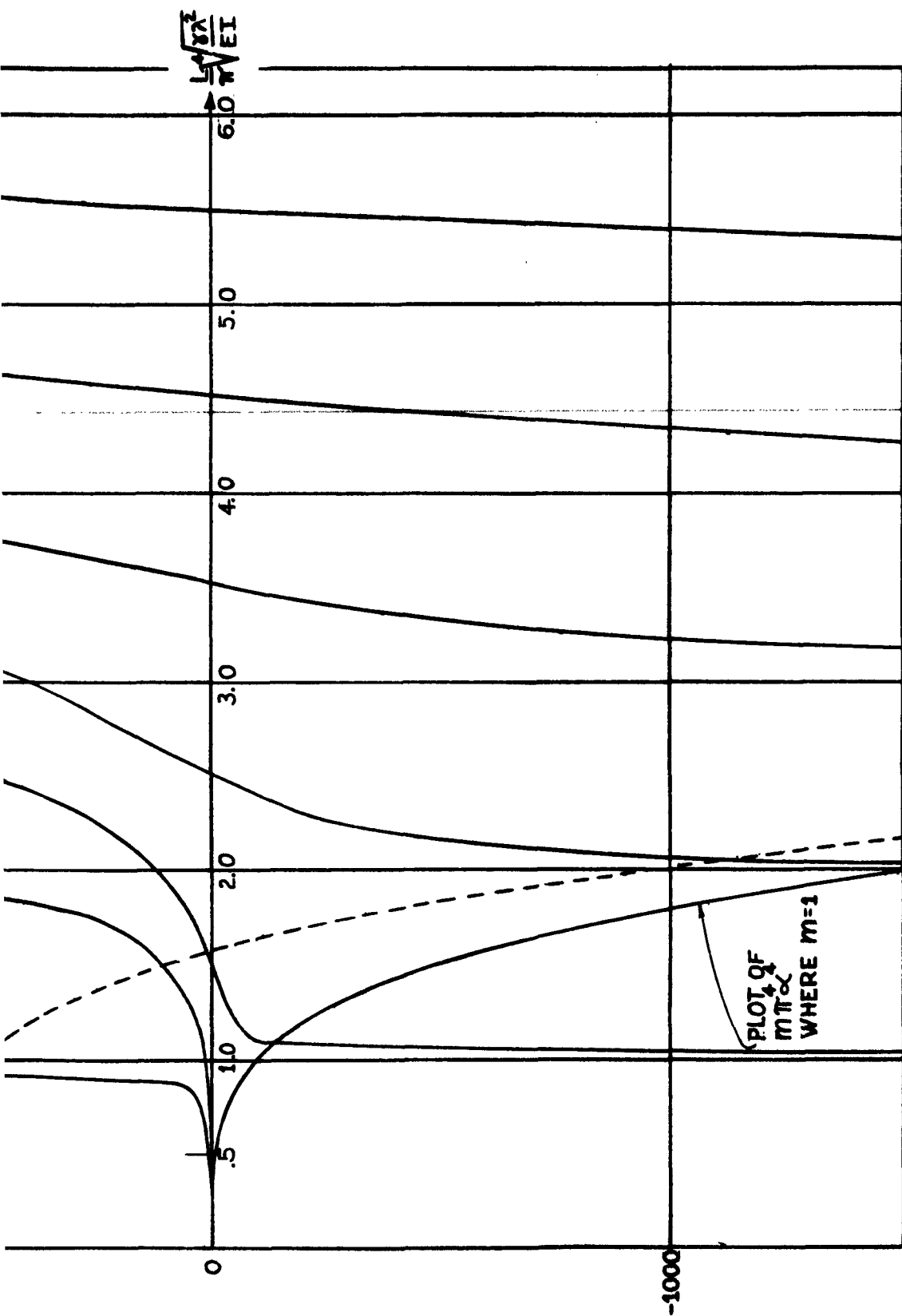


Fig. 2 GRAPHICAL SOLUTION FOR NATURAL FREQUENCIES



are plotted in Fig. 2. For particular values of  $K$  and  $m$  we need only plot  $m\pi^4\sigma^4$  as indicated and shift this curve upward through a distance  $K$  (dashed curve in Fig. 2). The intersections of this curve with the curves  $F^+$  and  $F^-$  give those values of  $\beta$  which satisfy the frequency equation. Note that these curves asymptotically approach  $\beta = n\pi$  ( $n = 1, 2, \dots$ ) as  $K$  approaches infinity. Thus, as expected,  $\lambda = \left(n^2\pi^2/L^2\right)(EI/\gamma)$  are the natural frequencies for a pinned-pinned shaft in rigid self-aligning bearings. As a further check, we note that the intersections of these curves with the  $\alpha$ -axis ( $m = 0, K = 0$ ) correspond to the natural frequencies of the free-free shaft. The expressions, of course, also provide the natural frequencies of a uniform beam under the same type of end constraints. A numerical example will be found in Appendix D.

The natural frequencies for various mass and stiffness ratios are given in Appendix E, Figures (E1) to (E11). This appendix also contains a treatment of six other types of bearing configurations which are of practical importance.

### UNSYMMETRIC SHAFT IN IDENTICAL BEARINGS

The graphical procedure developed in the last section can be used to solve the frequency equations (13). The first of these equations is solved for  $K$  in terms of  $\alpha$  to obtain

$$\frac{K}{1-\epsilon} = m\alpha_1^4 + \frac{\alpha_1^3}{2} \left( \tanh \frac{\alpha_1}{2} + \tan \frac{\alpha_1}{2} \right) \quad (34a)$$

$$\frac{K}{1-\epsilon} = m\alpha_1^4 + \frac{\alpha_1^3}{2} \left( \coth \frac{\alpha_1}{2} - \cot \frac{\alpha_1}{2} \right) \quad (34b)$$

The frequency equation (13b) is identical to the above when  $-\epsilon$  and  $\alpha_1$  are replaced by  $+\epsilon$  and  $\alpha_5$  respectively. In terms of the physical constants of the system, the parameters in Eqs. (34) are

$$\frac{K}{1-\epsilon} = \frac{kL^3}{S(1-\epsilon)} = \frac{kL^3}{EI_1}, \quad \alpha_1^4 = \frac{\lambda^2 \gamma L^4}{S(1-\epsilon)} = \frac{\lambda^2 \gamma L^4}{EI_1}$$

$$m = \frac{M}{\gamma L}$$

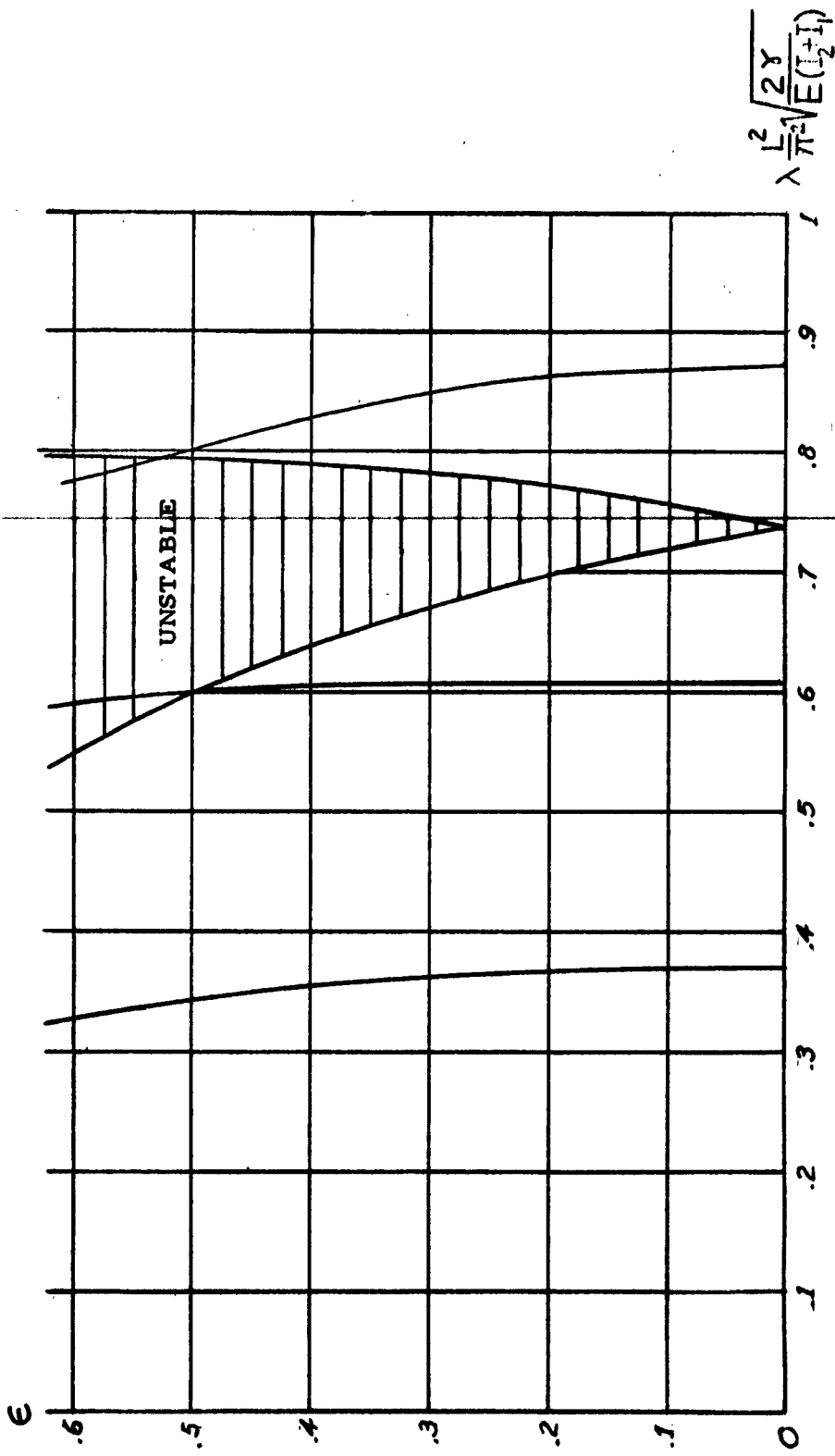
$$\frac{K}{1+\epsilon} = \frac{kL^3}{S(1+\epsilon)} = \frac{kL^3}{EI_2}, \quad \alpha_5^4 = \frac{\lambda^2 \gamma L^4}{S(1+\epsilon)} = \frac{\lambda^2 \gamma L^4}{EI_2} \quad (35)$$

Comparison of equations (35) and (34) with (29) and (32), respectively, indicates that, aside from subscripts on the flexural rigidities, the two sets of frequency equations are identical.

The calculations for a particular rectangular shaft are given in Appendix D. The dependence of critical speeds on the shaft stiffness anisotropy  $\epsilon$  is shown in Fig. 3. The solid lines represent the first three gravity critical speeds. The first two natural frequencies enclose a speed range for which the transient solution is unstable. The effect of damping on this critical speed range is indicated schematically in Fig. (C1).

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$$\lambda \frac{L^2 \sqrt{2\gamma}}{\pi^2 \sqrt{E(I_2^2 I_1)}}$$

Fig. 3 CRITICAL SPEEDS RECTANGULAR SHAFT  $A = 1.3125 \text{ in.}^2$ ,  $m = 1/3$ ,  $kL^3/E = 10.4 \text{ in.}^4$

## EXPERIMENTATION

The test fixture used in this program may be considered as consisting of three parts: the drive section, which was simply a source of uniform rotational motion; the experimental section, which consists of a very stiff beam upon which the shaft is installed in spring-mounted bearings, and the instrumentation, by means of which the rotational speed and forces on the test shaft bearings are measured. The mechanical parts may be seen in Fig. 4, and the electrical and electronic parts in Fig. 5.

The drive system has as its principle element a 28 v. d. c. motor rated at 33 amps, 10,000 rpm. It was selected for its smoothness of operation and wide running speed range. Power is supplied to it by a 220 v transformer seen on the floor behind the oscilloscope cart in Fig. 5 and by two rectifiers and two variacs shown at the left of Fig. 5. (It was determined that fluctuations in the 110 volt line were too great to allow accurate speed control.) The voltmeter and ammeter on the front of the large rectifier case are in the armature circuit. Speed is regulated by adjustment of the larger variac on top of the case at the front. The motor is coupled to a flywheel whose polar moment of inertia is  $59.64 \text{ in.} \cdot \text{lb} \cdot \text{sec}^2$ . (The polar moment of inertia of the flat shaft is  $0.00505 \text{ in.} \cdot \text{lb} \cdot \text{sec}^2$ .) The flywheel is coupled to the test shaft by a connecting shaft and two universal joints. A portion of the connecting shaft and one of the universal joints may be seen in Fig. 6. The drive section is mounted on a stiff, cross-braced table which has incorporated in it a vibration damping pad, which reduces any vibration transmitted to the experimental section.

The experimental section has as a base a welded box beam 11-1/2 in. wide by 19-1/2 in. deep, with four stiffening baffles, all made from 3/4 in. steel plates. Two A-frames are bolted to the beam for mounting the springs and bearings. The stiffnesses of the A-frames in both horizontal and vertical directions are about 100 times as great as that of the springs used in the program, thereby insuring that the springs constitute the sole significant flexibility at the bearings. Adjustable bolts through the A-frames have cups at one end into which the ends of the springs are fitted. Similar cups on the bearing blocks accept the other ends of the springs, thus allowing motion in either the horizontal or vertical direction.

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Self-aligning anti-friction bearings are used to minimize moments applied to the bearing block by rotation or bending of the shaft. Axial motion is prevented by the fixity of the connecting universal joint on the test shaft and the held bearing on the outboard flywheel journal. The test shaft is not held in either bearing. The bearing blocks provide mounts for the accelerometers.

Speed sensing was effected by means of a light source and an electric eye directed toward a strip of aluminum tape on the flywheel rim (see Fig. 4). Since the rest of the rim was covered with black paint, the electric eye emitted a voltage which rose and fell once during each revolution. The instrument at the top of the rack at the right in Fig. 5 is a tachometer which was used for rough measurements of the speed. In addition, the output of the electric eye was fed into the oscilloscope where, by adjustment of the sweep time base and measurement of the trace length for one revolution, accurate speed measurements were made. Furthermore, the oscilloscope was used for vibration phase measurement, through observation of the relative positions of the electric eye trace and the spike in the accelerometer trace.

Bearing forces were sensed by C. R. L. accelerometers on the bearing blocks, and measured by the Ballantine Model 300 voltmeter on the shelf in the rack. Other instrumentation in the rack includes selector switching for location and direction of accelerometers used, calibration circuitry, and cathode follower isolation amplifiers for the accelerometers.

The experimental procedure was directed solely toward the determination of the critical speeds of rotation. Displacements were not found at this time since analytic results were not available for comparison.

Tests were conducted on two shaft systems: a 50-in. long 1-in. diameter round shaft and a 50-in. long  $7/8$  by  $1-1/2$ -in. rectangular shaft. The bearing flexibilities, limited by the availability of commercial coil springs, were 2500 and 5300 lbs/in. An infinite amount of stiffness, i. e. rigid bearings, was obtained by compressing the springs to the bottoming point. The balancing fixtures on the round shaft were not needed since the amplitudes of vibration near resonance were not severe as long as the rotor

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passed quickly through this critical speed. No attempt was made to balance the flat shaft because unstable speed ranges and gravity critical speeds exist independently of any mass eccentricity of the shaft.

The results for the round shaft are presented in Fig. 7 and 8 in terms of the horizontal and vertical accelerations at the bearing remote from the driving end. The observed first resonant speed was at 1640, 1720, and 1770 rpm for bearings of 2500 lb/in., 5300 lb/in. and infinite stiffness, respectively. The secondary gravity critical speeds were not noticeable. The resonant speeds were apparent from the large vibrations and noise levels.

The theoretical predictions of Appendix D are 1720, 1870, and 1910 rpm. The calculations, however, did not include the 1.15 lb weight of the balancing fixtures. The corrected frequencies, based on a new mass of shaft and elastic modulus  $E = 29 \times 10^6$  psi, are 1610, 1750, and 1790 rpm, which are in very good agreement with the experimental results.

The results for the flat shaft with bearing stiffness of 2500 lb/in. are presented in Fig. 9. In this case the apparent secondary critical speeds, occurring at 950, 1450 rpm were quite severe. In the rough running range from 1600 to 2100 rpm it was difficult to maintain constant speed because of the large vibrations of the rotor. At approximately 2100 rpm the shaft became disengaged from the bearing mounts causing some minor damage to the apparatus.

The analysis of Appendix D predicted gravity critical speeds at 910, 1500, and 2140 rpm and a critical speed range from 1600 to 2120 rpm. (For  $E = 29 \times 10^6$  psi the predictions would be 900, 1470, and 2100 rpm for the gravity critical speeds and 1570 to 2080 rpm for the critical speed range.) It is clear that the destructive vibrations near 2100 rpm were caused when the effects of gravity, mass eccentricity, and shaft stiffness inequality were felt at the same time.

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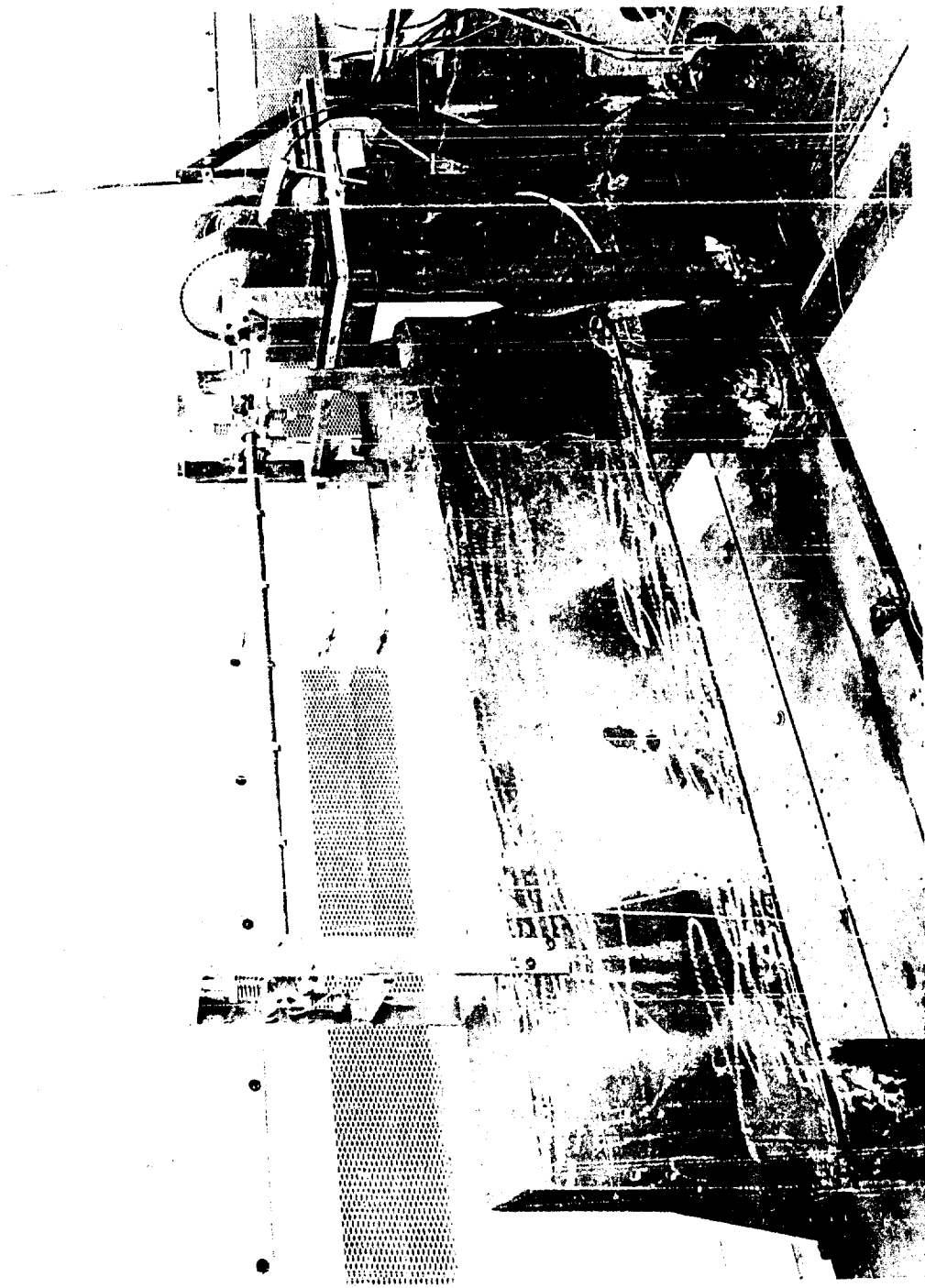


Fig. 4 ROTOR IN PLACE

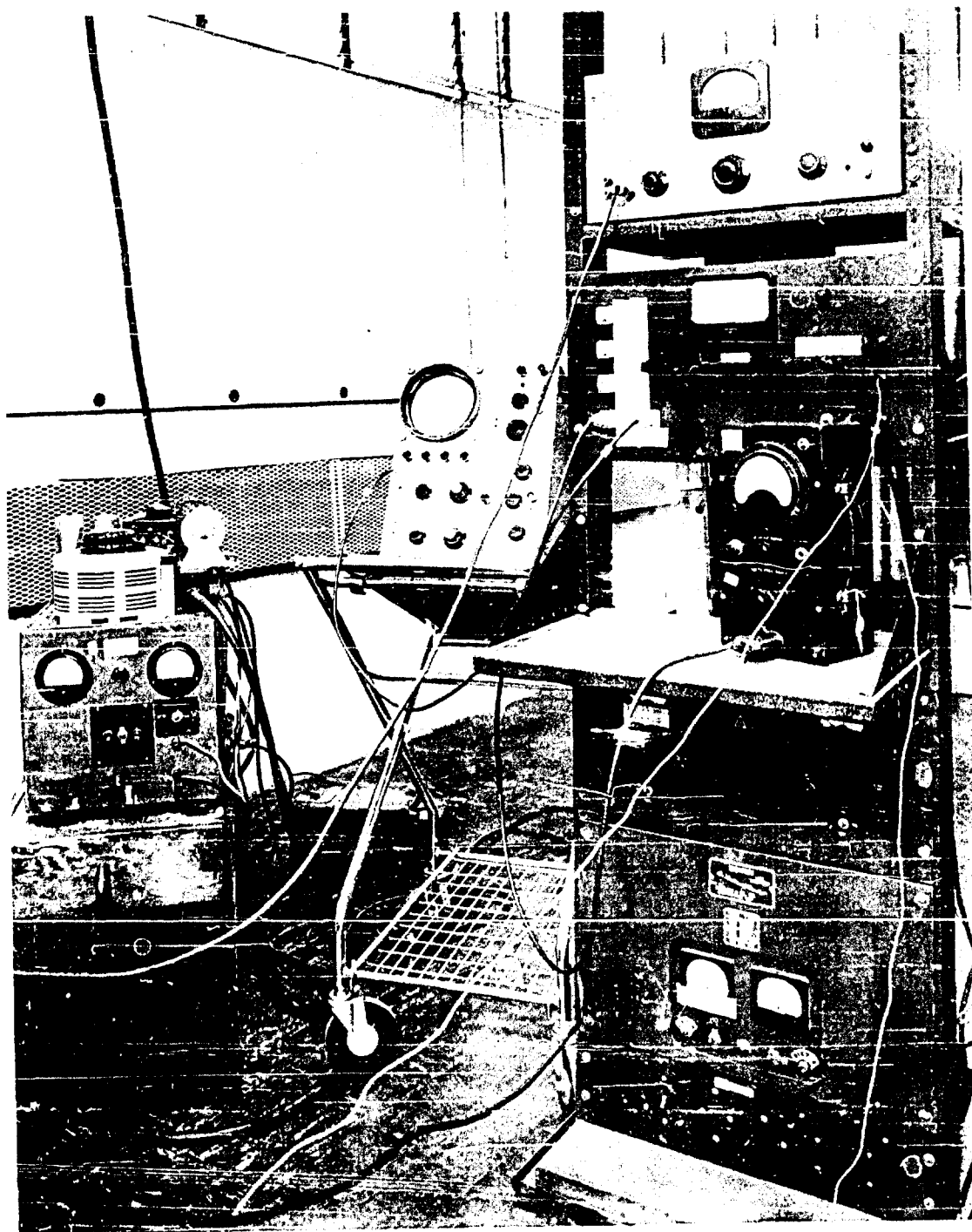


Fig. 5 INSTRUMENTATION

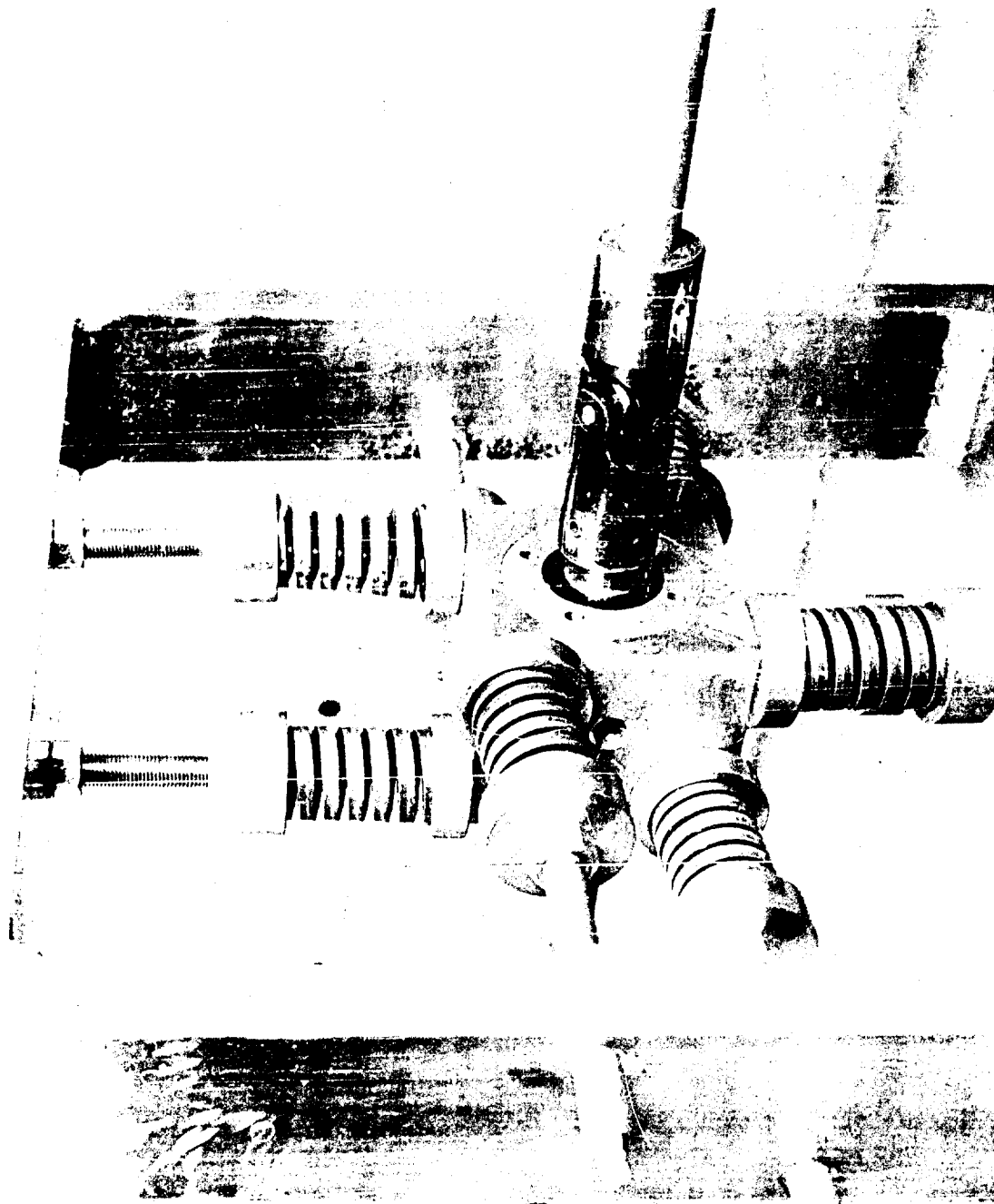


Fig. 6 BEARING CONFIGURATION

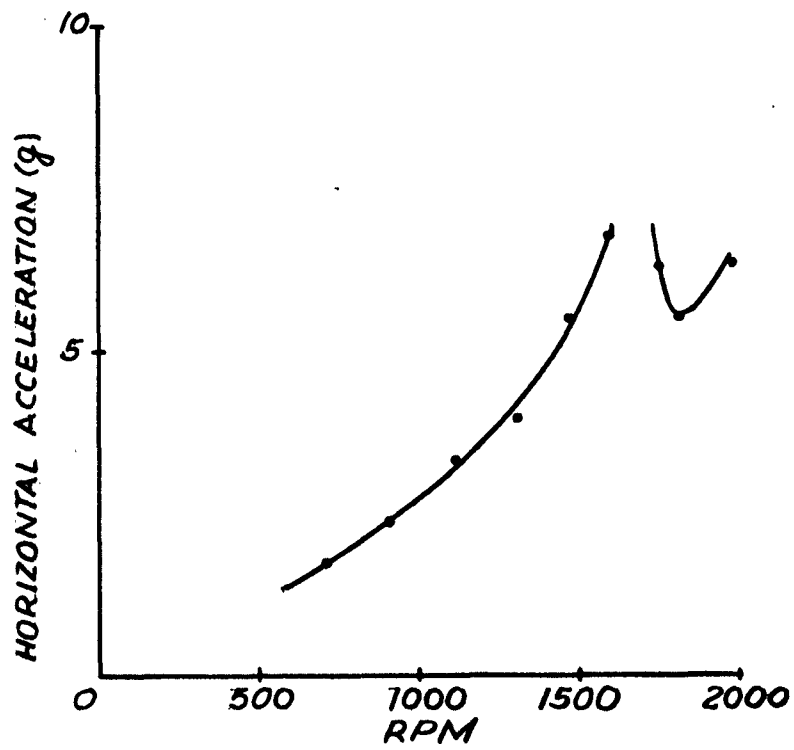
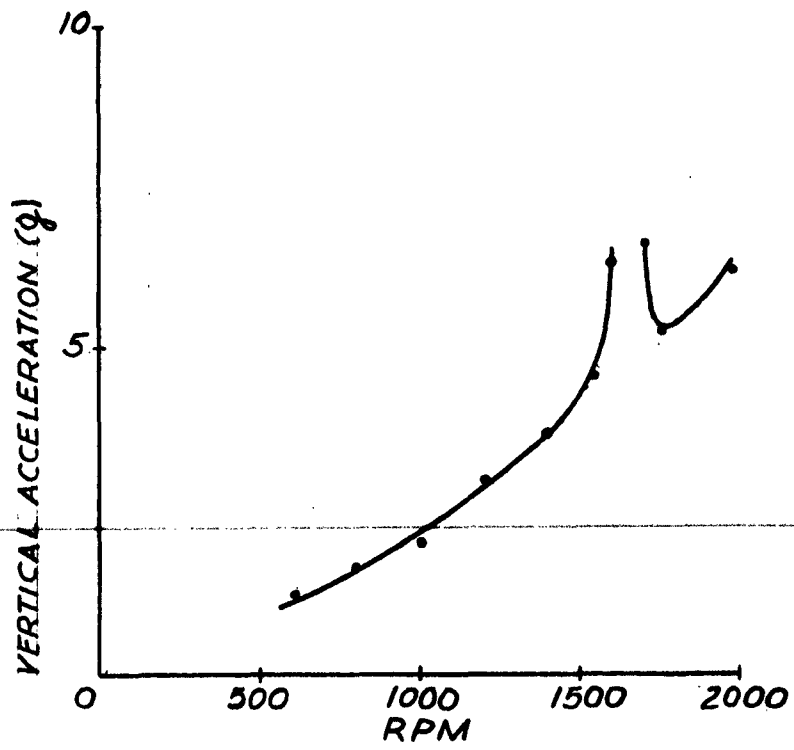


Fig. 7 BEARING ACCELERATION ROUND SHAFT  $k = 2500 \text{ lb/in.}$



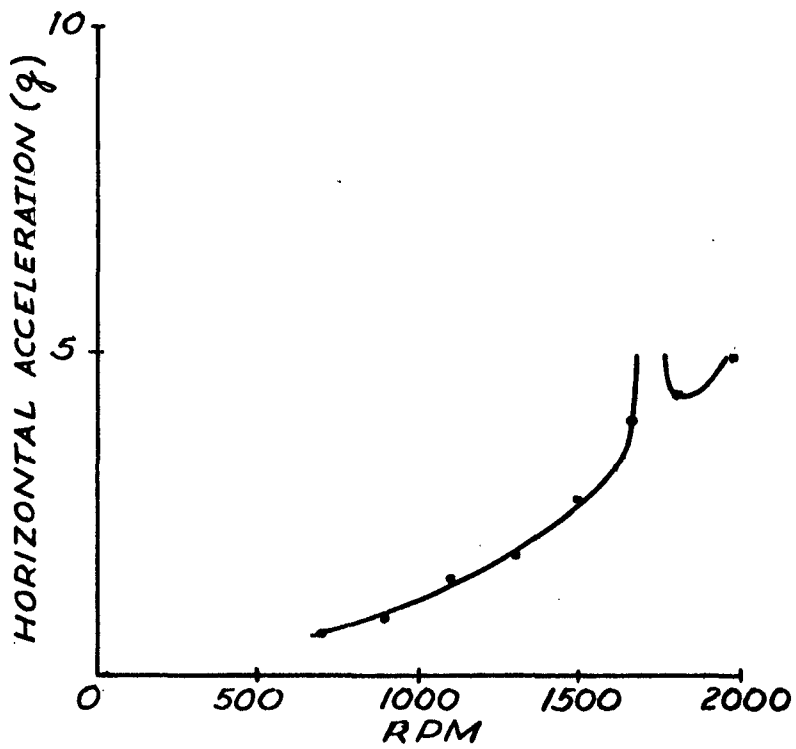
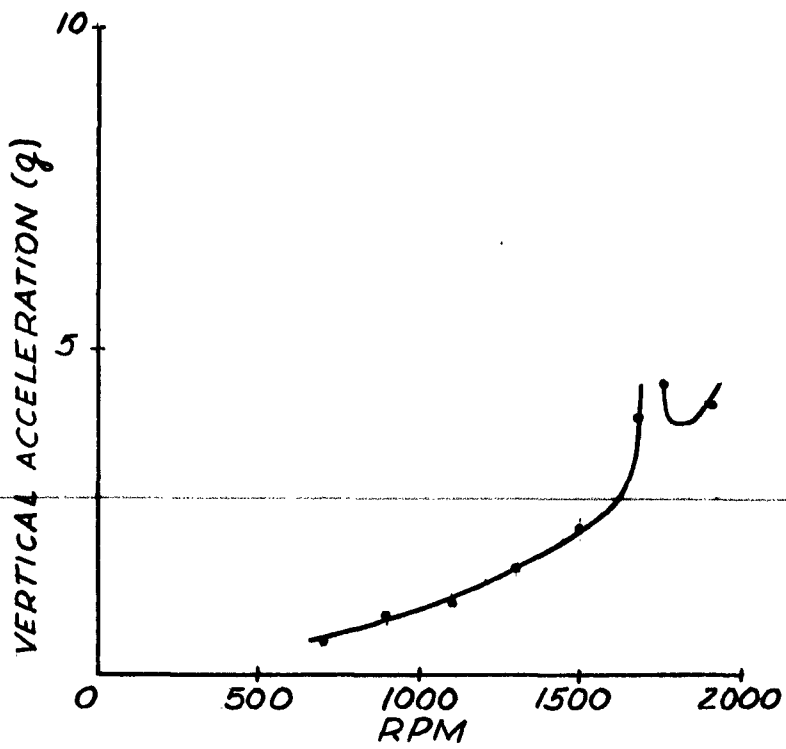


Fig. 8 BEARING ACCELERATION ROUND SHAFT  $k = 5300 \text{ lb/in.}$

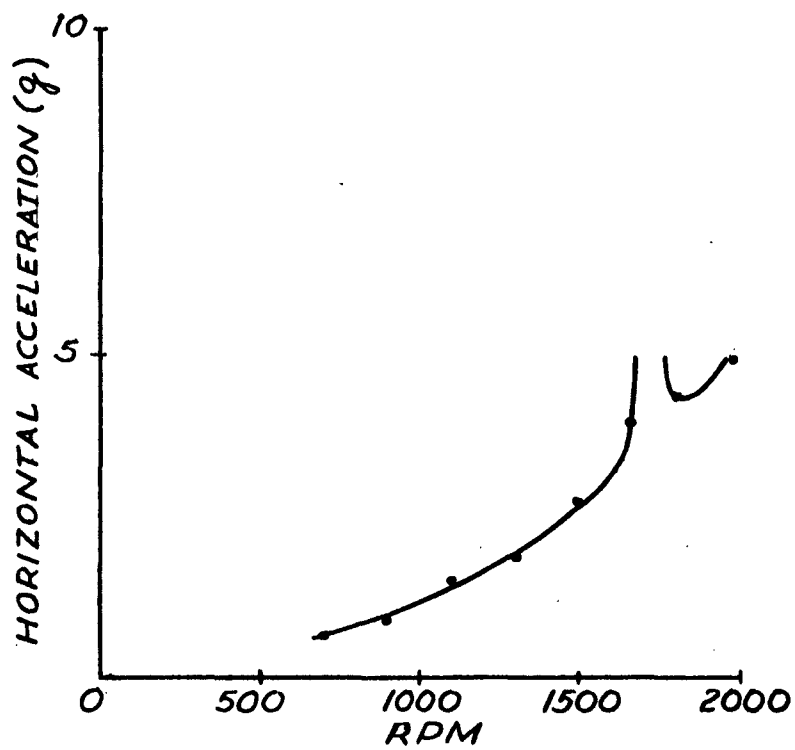
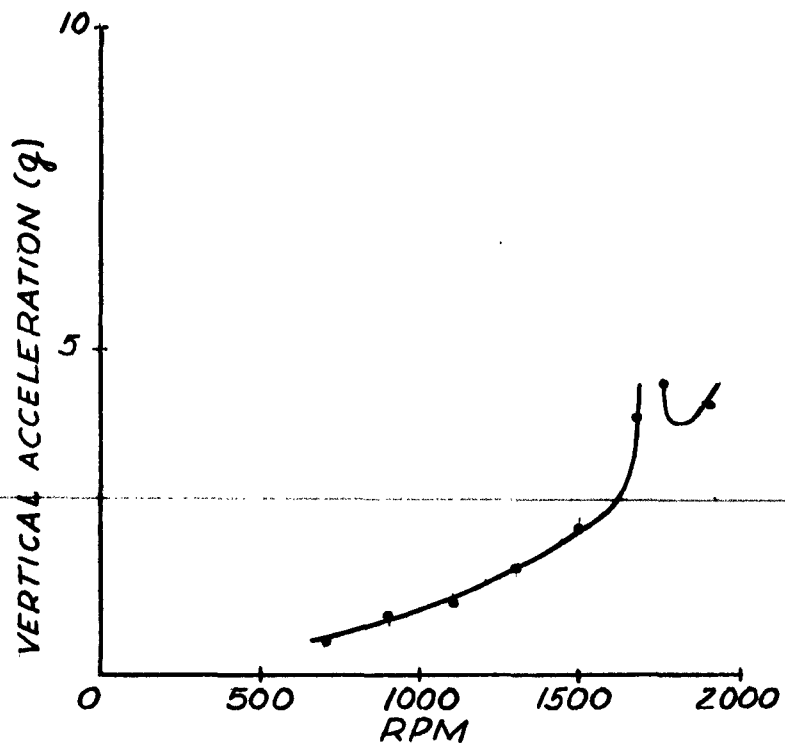


Fig. 8 BEARING ACCELERATION ROUND SHAFT  $k = 5300 \text{ lb/in.}$

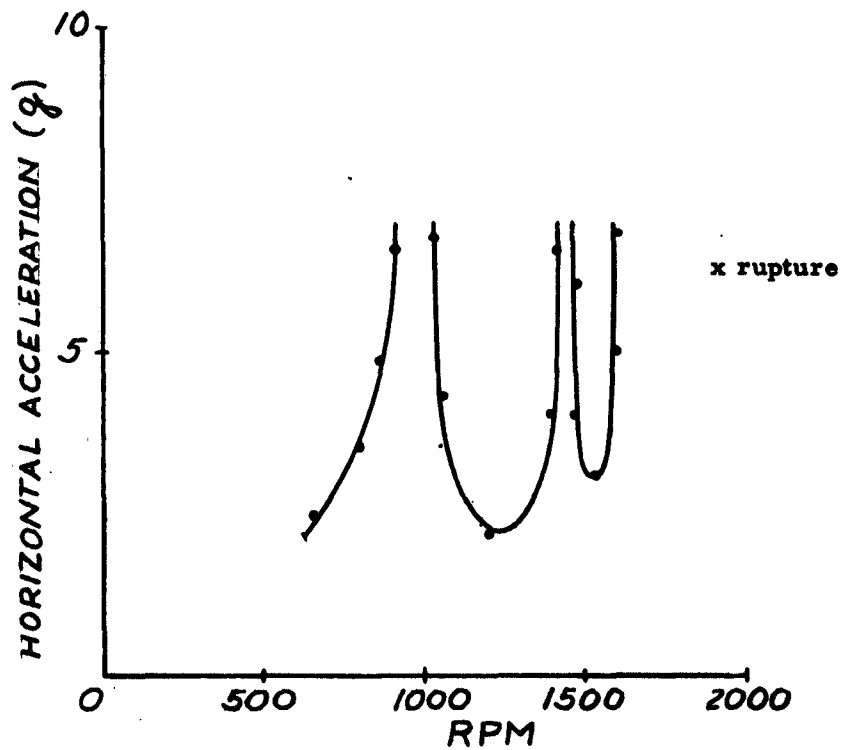
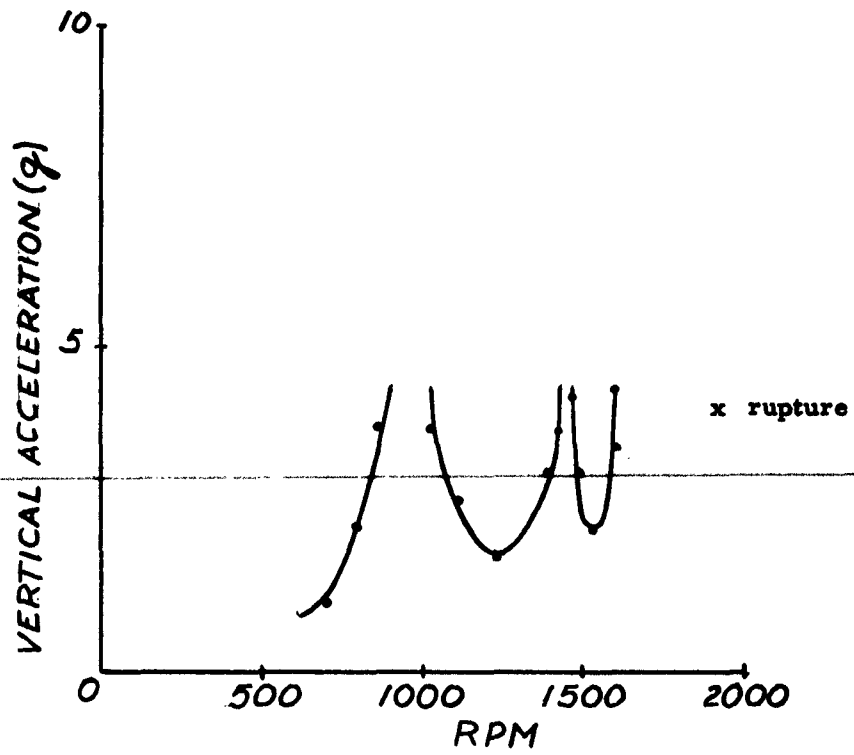


Fig. 9 BEARING ACCELERATION FLAT SHAFT  $k = 2500 \text{ lb/in.}$

## CONCLUSIONS

The natural frequencies of a uniform undamped shaft rotating in heavy undamped flexible bearings have been found for two specific cases: 1) a shaft with non-symmetric cross section rotating in heavy, self-aligning bearings whose principal flexibilities are equal, and 2) a symmetric shaft rotating in heavy bearings of unequal principal flexibilities. Secondary critical speeds induced by gravity are also determined. The general case of non-symmetry in both shaft and bearings is not amenable to solution.

In the first case, a doubly infinite set of natural frequencies are found. The spread between corresponding frequencies of each set increases from zero with increasing shaft anisotropy. Centrifugal forces induced by mass eccentricity excite a steady state response at all speeds except the natural frequencies where the response becomes unbounded. The transient solution, however, will become unbounded at any speed which lies between corresponding natural frequencies. Damping in the system will not completely eliminate this range of unstable operating speeds. The unstable nature of this transient makes it imperative that this type of shaft design be avoided wherever possible.

For the case treated above, the secondary gravity critical speeds can be found with little extra effort from the data on the natural frequencies. For slight shaft asymmetry these critical speeds are approximately one-half the natural frequencies.

For the special case of identical bearings, the transcendental frequency equation is sufficiently reduced so that by means of a simple graphical construction natural frequencies may be found for any combination of the remaining system parameters.

When the symmetric shaft rotates in unsymmetric bearings, there are again two sets of natural frequencies, but no ranges of critical speeds. In this case, there is a direct analogy to the beam whose end supports possess different horizontal and vertical flexibilities; the natural frequencies of each system are identical. These frequencies can be found from the same graphical construction mentioned above.

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It is interesting to compare the system considered in this report with the idealized case of immovable pinned-pinned bearings. In terms of the bearing-shaft flexibility ratio  $K$  and mass ratio  $m$  the following conclusions can be made: 1) for large  $K$  and small  $m$  the first two or three natural frequencies are approximately equal; 2) for large  $K$  and  $m$  or medium  $K$  small  $m$  only the first natural frequency corresponds to the idealized case; 3) for small  $K$  the frequencies are very sensitive to the mass ratio. Moreover, since there is another set of natural frequencies corresponding to the different bearing flexibility in the perpendicular direction, the similarities to the idealized case are further obscured. If the principal bearing flexibilities are nearly equal the natural frequencies may occur in clusters.

It is apparent that, aside from rotating systems whose supports are very inflexible, the vibrations of flexible rotors operating at supercritical speeds will be sensitive to the bearing characteristics. It appears that the bearing with adjustable stiffness may be suitable as a resonance changer, thereby effecting a decrease in noise and vibration levels.

In summary,

1. Bearing mass and flexibility have considerable effect on the critical speeds of flexible rotors. Generally only the first and second natural frequencies of flexural vibrations are accurately predicted by the idealized assumption of rigid (immovable) bearings.
2. When the bearing stiffnesses in two mutually perpendicular directions are unequal, the occurrence of critical speeds is doubled. It is possible for these speeds to occur in clusters thereby giving the appearance of a range of unstable operating speeds.
3. Shafts of non-symmetric cross section should be avoided whenever possible because of instabilities which, in general, cannot be removed by damping.
4. Half critical speeds in symmetric shafts are unimportant.
5. Bearings with adjustable stiffness offer a means of changing resonance thereby allowing smooth operation in between critical speeds.

## RECOMMENDATIONS FOR FURTHER WORK

In this report only the effects of bearing mass and stiffness on the natural frequencies of a uniform shaft were found. Account should also be taken of shafts of non-uniform cross section which may be carrying thin heavy disks on interior and/or overhanging sections. It is also of practical interest to determine the magnitude of vibrations for various combinations of shaft system parameters.

In addition, the possibility of using flexible bearings as a means of controlling vibrations should be pursued. The chief cause of excessive vibration is an unbalanced rotor operating near a critical frequency. Precision balancing and conservative design are two common methods of vibration control. While it is standard procedure to balance all critical rotating elements, these rotors may eventually operate poorly at or near the natural frequencies of the system. On the other hand, conservative design in many cases would be too costly and impractical especially if the system operates over a wide range of speeds. Adjustable bearings would perform the same task as a redesign since they would relocate the natural frequencies, i. e., they act as a resonance changer. This method may not reduce vibrations, but simply avoid them by adjusting the troublesome critical speeds to be sufficiently removed from the operating speed.

A program to accomplish the above recommendations would contain the following elements:

1. Develop the dynamics of non-uniform shafts carrying thin heavy disks. This would include gyroscopic effects and the effects of bearing mass, damping, and stiffness.
2. Determine the amplitudes of vibration for various shaft system parameters with particular emphasis on changes of bearing stiffness.
3. Perform experiments which will verify the analytical results.
4. Evaluate the feasibility of adjustable bearings as a resonance changer.

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APPENDIX A  
THE EQUATIONS OF MOTION

It is convenient to derive the equations of motion in the rotating  $V_1, V_2$  coordinate system (Fig. A1). In determining the boundary conditions, we shall make use of the fixed  $X, Y$  coordinate system. We develop the equations in their most general form and then specialize to the problem at hand.

Let  $S_1(Z)$  and  $S_2(Z)$  be the principal stiffnesses in the  $V_1$  and  $V_2$  directions, respectively. Classical beam theory yields the following relations between shear force  $F$ , bending moment  $B$  and displacements  $V_1, V_2$  of the neutral axis:

$$F_1 = -\frac{\partial B_2}{\partial Z}, \quad F_2 = -\frac{\partial B_1}{\partial Z}, \quad B_2 = S_1 \frac{\partial^2 V_1}{\partial Z^2}, \quad B_1 = S_2 \frac{\partial^2 V_2}{\partial Z^2} \quad (\text{A-1})$$

We recall that the components of acceleration referred to a coordinate system which rotates at a constant speed  $\Omega$  are

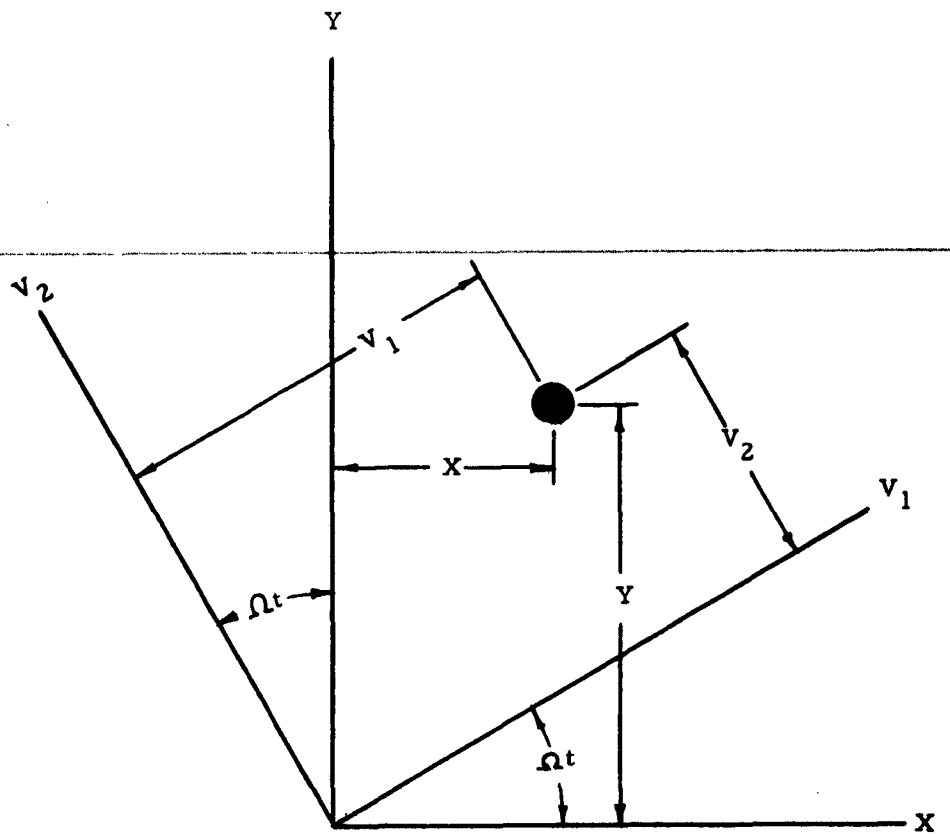
$$a_1 = \frac{\partial^2 V_1}{\partial t^2} - \Omega^2 V_1 - 2\Omega \frac{\partial V_2}{\partial t}$$

$$a_2 = \frac{\partial^2 V_2}{\partial t^2} - \Omega^2 V_2 + 2\Omega \frac{\partial V_1}{\partial t} \quad (\text{A-2})$$

For a horizontal shaft of density  $\gamma(z)$  per unit length, the equations of motion are

$$-\gamma(Z) g \cos \Omega t + \frac{\partial F_1}{\partial Z} = \gamma(Z) \left( \frac{\partial^2 V_1}{\partial t^2} - \Omega^2 V_1 - 2\Omega \frac{\partial V_2}{\partial t} \right)$$

$$-\gamma(Z) g \sin \Omega t + \frac{\partial F_2}{\partial Z} = \gamma(Z) \left( \frac{\partial^2 V_2}{\partial t^2} - \Omega^2 V_2 + 2\Omega \frac{\partial V_1}{\partial t} \right) \quad (\text{A-3})$$



**Fig. A1 FIXED AND ROTATING COORDINATE SYSTEM**

where the first term of each of equations (A-3) represents the gravity force. Substituting relations (A-1) into (A-3) we obtain

$$\frac{\partial^2}{\partial Z^2} \left[ S_1(Z) \frac{\partial^2 V_1}{\partial Z^2} \right] + \gamma(Z) \left[ \frac{\partial^2 V_1}{\partial t^2} - \Omega^2 V_1 - 2\Omega \frac{\partial V_2}{\partial t} \right] = -\gamma(Z) g \cos \Omega t$$

$$\frac{\partial^2}{\partial Z^2} \left[ S_2(Z) \frac{\partial^2 V_2}{\partial Z^2} \right] + \gamma(Z) \left[ \frac{\partial^2 V_2}{\partial t^2} - \Omega^2 V_2 + 2\Omega \frac{\partial V_1}{\partial t} \right] = -\gamma(Z) g \sin \Omega t$$

(A-4)

When the shaft is uniform,  $S_1$ ,  $S_2$ , and  $\gamma$  are constants and, after suitable non-dimensionalization, equations (1) and (2) result.

In the rotating system, the component of force exerted by the shaft on the bearing is

$$-S \frac{\partial^3 V}{\partial Z^3} \quad \text{at } Z = 0 \quad \text{and} \quad +S \frac{\partial^3 V}{\partial Z^3} \quad \text{at } Z = L . \quad (\text{A-5})$$

The displacements in the fixed coordinate system are related to  $V_1$  and  $V_2$  by

$$X = V_1 \cos \Omega t - V_2 \sin \Omega t$$

$$Y = V_1 \sin \Omega t + V_2 \cos \Omega t$$

(A-6)

$F_x$  and  $F_y$ , the components of a force in the fixed system, and  $F_1$  and  $F_2$ , its components in the rotating system, are related by

$$F_1 = F_x \cos \Omega t + F_y \sin \Omega t$$

$$F_2 = -F_x \sin \Omega t + F_y \cos \Omega t$$

(A-7)

Now the force exerted by the springs on the mass is

$$F_x = -k_x X; \quad F_y = k_y Y \quad (\text{A-8})$$

where  $k_x$ ,  $k_y$  are the flexibilities of the springs in the X and Y directions, respectively. Substituting (A-6) and (A-8) into (A-7) we obtain

$$F_1 = - \left[ \frac{(k_x + k_y)}{2} + \frac{(k_x - k_y)}{2} \cos 2\Omega t \right] V_1 + \left[ \frac{(k_x - k_y)}{2} \sin 2\Omega t \right] V_2$$

$$F_2 = \left[ \frac{(k_x - k_y)}{2} \sin 2\Omega t \right] V_1 - \left[ \frac{(k_x + k_y)}{2} - \frac{(k_x - k_y)}{2} \cos 2\Omega t \right] V_2 \quad (A-9)$$

as the components of the force of the spring on the bearings.

Finally, when (A-2), (A-5), and (A-9) are taken into account, the equations of motion of the bearings become

at  $Z = 0$

$$S_1 \frac{\partial^3 V_1}{\partial Z^3} + \left[ \frac{k_{1x} + k_{1y}}{2} + \frac{k_{1x} - k_{1y}}{2} \cos 2\Omega t \right] V_1 - \left[ \frac{k_{1x} - k_{1y}}{2} \sin 2\Omega t \right] V_2$$

$$+ M_1 \left[ \frac{\partial^2 V_1}{\partial t^2} - \Omega^2 V_1 - 2\Omega \frac{\partial V_2}{\partial t} \right] = 0$$

$$S_2 \frac{\partial^3 V_2}{\partial Z^3} - \left[ \frac{k_{1x} - k_{1y}}{2} \sin 2\Omega t \right] V_1 + \left[ \frac{k_{1x} + k_{1y}}{2} - \frac{k_{1x} - k_{1y}}{2} \cos 2\Omega t \right] V_2$$

$$+ M_1 \left[ \frac{\partial^2 V_2}{\partial t^2} - \Omega^2 V_2 - 2\Omega \frac{\partial V_1}{\partial t} \right] = 0$$

at  $Z = L$

$$-S_1 \frac{\partial^3 V_1}{\partial Z^3} + \left[ \frac{k_{2x} + k_{2y}}{2} + \frac{k_{2x} - k_{2y}}{2} \cos 2\Omega t \right] V_1 - \left[ \frac{k_{2x} - k_{2y}}{2} \sin 2\Omega t \right] V_2$$

$$+ M_2 \left[ \frac{\partial^2 V_1}{\partial t^2} - \Omega^2 V_1 - 2\Omega \frac{\partial V_2}{\partial t} \right] = 0$$

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$$\begin{aligned}
& -S_2 \frac{\partial^3 v_2}{\partial z^3} - \left[ \frac{k_{2x} - k_{2y}}{2} \sin 2\Omega t \right] v_1 + \left[ \frac{k_{2x} + k_{2y}}{2} - \frac{k_{2x} - k_{2y}}{2} \cos 2\Omega t \right] v_2 \\
& + M_2 \left[ \frac{\partial^2 v_2}{\partial t^2} - \Omega^2 v_2 + 2\Omega \frac{\partial v_1}{\partial t} \right] = 0
\end{aligned}$$

We see that these end conditions have time dependent periodic coefficients when the flexibilities  $k_x$  and  $k_y$  at the bearings are not equal. When referred to the fixed coordinate system, the boundary conditions have periodic terms arising from the shaft anisotropy. Moreover, the equations of motion in the fixed coordinate system also have time-dependent periodic coefficients. For the special case of equal bearing flexibilities  $k_{1x} = k_{1y} = k_1$  and  $k_{2x} = k_{2y} = k_2$  the nondimensionalized equations (3) result.

## APPENDIX B

### AN EXAMPLE OF AN UNSTABLE TRANSIENT OF A ROTOR CONTAINING DAMPING

The transient solution of the damped non-symmetric shaft systems considered in this report can be shown to be unstable for certain ranges of operating speed; however, for the sake of clarity we shall demonstrate this behavior for a simplified lumped parameter system. Consider an unsymmetric shaft of negligible weight supporting a mass  $m$ . In terms of an  $x$ ,  $y$  coordinate system which rotates with the shaft, the equations of motion are

$$\begin{aligned} m(\ddot{y} - \Omega^2 y + 2\Omega \dot{x}) + C_1 \dot{y} + k_1 y &= 0 \\ m(\ddot{x} - \Omega^2 x - 2\Omega \dot{y}) + C_2 \dot{x} + k_2 x &= 0 \end{aligned} \tag{B-1}$$

Here  $\Omega$  is the constant angular speed of the shaft,  $C$  is the damping factor, and  $k$  is the shaft stiffness.

Since the differential equations (B-1) have constant coefficients, the solution has the form

$$y = A e^{\lambda t} \quad x = B e^{\lambda t} \tag{B-2}$$

The frequency equation for  $\lambda$  is found from the requirements that the determinant of coefficients of  $A$  and  $B$  vanish. This results in

$$0 = \begin{vmatrix} m\lambda^2 + C_1\lambda + (k_1 - m\Omega^2) & 2\Omega\lambda m \\ -2\Omega\lambda m & m\lambda^2 + C_2\lambda + (k_2 - m\Omega^2) \end{vmatrix} \tag{B-3}$$

or

$$\begin{aligned} m^2 \lambda^4 + m(C_1 + C_2) \lambda^3 + [(k_1 + k_2)m + C_1 C_2 + 2m\Omega^2] \lambda^2 \\ + [C_1(k_2 - m\Omega^2) + C_2(k_1 - m\Omega^2)] \lambda + (k_1 - m\Omega^2)(k_2 - m\Omega^2) = 0 \end{aligned}$$

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In general, the solution of the frequency equation (B-3) has two sets of complex conjugate roots. If the solution (B-2) is bounded, the real part of the complex frequency  $\lambda$  must be less than or equal to zero. When  $\text{Real } (\lambda) > 0$  the transient is unstable since it increases exponentially with time.

While it is possible to obtain a relation among the shaft system parameters for which  $\text{Real } (\lambda) > 0$ , it would be too cumbersome to be informative. It is found that critical speeds occur in a range whose width depends on the stiffness difference  $(k_2 - k_1) / (k_2 + k_1)$  and the damping factor  $C$ . This critical speed dependence is indicated schematically in Fig. B1.

Since the right-hand sides of Eqs. (B-1) are zero, the casual observer may conclude that the system is in free vibration. One may wonder why the homogeneous solution of a damped system can be unstable when it is known that any freely vibrating dissipative system will eventually come to rest. However, the equations of motion (B-1) are really forced since the angular speed is kept constant. At the critical speeds, more energy is added to the shaft than is absorbed by dissipative elements. This can happen only when there exists a shaft stiffness inequality which some authors have considered apparently equivalent to "negative damping".

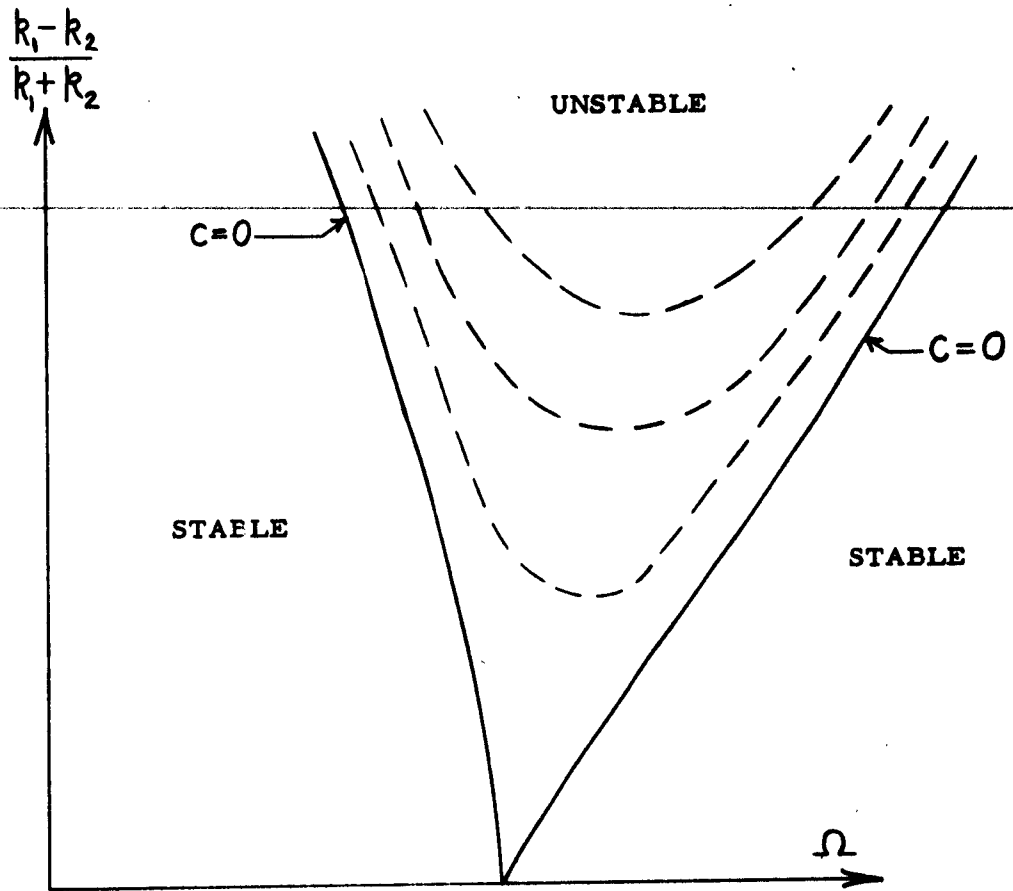


Fig. B1 EFFECT OF DAMPING AND STIFFNESS ANISOTROPY ON CRITICAL SPEEDS



**APPENDIX C**  
**DERIVATION OF CRITICAL SPEED EQUATION**  
**FOR A NON-SYMMETRIC SHAFT**

As  $\mu$  approaches zero, the conditions (10) and (11) become

$$\sum_{j=1}^4 A_j \left[ (1-\epsilon)\alpha_j^3 + K_1 - m_1 p^2 \right] + \sum_{j=1}^5 \left[ A_j (1-\epsilon)\alpha_j^3 + K_1 - m_1 p^2 \left( \frac{1-\epsilon}{1+\epsilon} \right) \right]$$

$$\sum_{j=1}^4 A_j e^{\alpha_j} \left[ -(1-\epsilon)\alpha_j^3 + K_2 - m_2 p^2 \right] + \sum_{j=1}^5 A_j e^{\alpha_j} \left[ -(1-\epsilon)\alpha_j^3 + K_2 - m_2 p^2 \left( \frac{1-\epsilon}{1+\epsilon} \right) \right] =$$

$$\sum_{j=1}^8 \alpha_j^2 A_j = 0$$

$$\sum_{j=1}^8 \alpha_j^2 e^{\alpha_j} A_j = 0 \tag{C-1}$$

$$\frac{-\epsilon}{1+\epsilon} \sum_{j=5}^8 A_j \left[ (1+\epsilon)\alpha_j^3 + K_1 - m_1 p^2 \right] = 0$$

$$\frac{-\epsilon}{1+\epsilon} \sum_{j=5}^8 A_j e^{\alpha_j} \left[ -(1+\epsilon)\alpha_j^3 + K_2 - m_2 p^2 \right] = 0$$

$$\frac{-\epsilon}{1+\epsilon} \sum_{j=5}^8 \alpha_j^2 A_j = 0$$

$$\frac{-\epsilon}{1+\epsilon} \sum_{j=5}^8 \alpha_j^2 e^{\alpha_j} A_j = 0$$

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where

$$\alpha_j = (i)^{j-1} \left[ p^2 / (1 - \epsilon) \right]^{1/4} \quad j = 1, 2, 3, 4$$

$$\alpha_j = (i)^{j-5} \left[ p^2 / (1 + \epsilon) \right]^{1/4} \quad j = 5, 6, 7, 8$$

If  $\Delta$ , the determinant of coefficients of equations (C-1), were written out explicitly, we would note that the elements of the lower left four by four sub-determinant vanish. Consequently, the eighth order determinant is expressible as the product of two fourth order determinants; that is (see page C-3)

$$\Delta = - \frac{256 \epsilon}{1 + \epsilon} \delta_1 \delta_2$$

The expression for  $\delta_2$  is obtained from  $\delta_1$  when  $(1 - \epsilon)$  and  $\alpha_1$  are replaced by  $(1 + \epsilon)$  and  $\alpha_5$  respectively. By a simple rearrangement of columns of (C-2), we can show that  $\delta_1$  and  $\delta_2$  are respectively equivalent to  $\Delta_1$  and  $\Delta_2$  of equations (12).

16  $\delta_1 =$

$$\begin{array}{l}
 \left[ (1-\epsilon)\alpha_1^3 + K_1 - m_1 p^2 \right] \cdot \left[ -i(1-\epsilon)\alpha_1^3 + K_1 - m_1 p^2 \right] \cdot \left[ -(1-\epsilon)\alpha_1^3 + K_1 - m_1 p^2 \right] \cdot \left[ i(1-\epsilon)\alpha_1^3 + K_1 - m_1 p^2 \right] \\
 \left[ -(1-\epsilon)\alpha_1^3 + K_2 - m_2 p^2 \right] \alpha_1 \cdot \left[ i(1-\epsilon)\alpha_1^3 + K_2 - m_3 p^2 \right] i\alpha_1 \cdot \left[ (1-\epsilon)\alpha_1^3 + K_2 - m_3 p^2 \right]^{-\alpha_1} \cdot \left[ -i(1-\epsilon)\alpha_1^3 + K_2 - m_2 p^2 \right]^{-i\alpha_1} \\
 \alpha_1^2 \qquad \qquad \qquad -\alpha_1^2 \qquad \qquad \qquad \alpha_1^2 \qquad \qquad \qquad -\alpha_1^2 \\
 \alpha_1^2 e \qquad \qquad \qquad -\alpha_1^2 e \qquad \qquad \qquad i\alpha_1 \qquad \qquad \qquad -\alpha_1^2 e^{-i\alpha_1}
 \end{array}$$

(C-2)

APPENDIX D  
NUMERICAL EXAMPLES

I. ROUND SHAFT

We are to find the natural frequencies of a 50-in. long 1-in. diameter steel shaft mounted in self-aligning bearings. The 6.2 lb bearing blocks are connected to a rigid foundation by means of springs of 2500 lbs/in. stiffness. The total mass of the springs is negligible in comparison to the mass of the bearing. The physical constants of the system are

- L = 50 in. length of shaft
- $\gamma gL$  = 11.11 lb weight of shaft
- Mg = 6.2 lb weight of bearing
- k = 2500 lb/in. spring stiffness
- E =  $30 \times 10^6$  lb/in.<sup>2</sup> Young's modulus
- $\delta$  =  $5.75 \times 10^{-4}$  lb sec<sup>2</sup>/in.<sup>2</sup> mass per unit length of shaft
- I =  $\pi r^4/4 = 4.91 \times 10^{-2}$  in.<sup>4</sup> area moment of inertia
- K =  $kL^3/EI = 212$  bearing-shaft stiffness ratio
- m =  $6.2/11.11 = 0.56$  bearing-shaft mass ratio

The equations for the natural frequencies  $\lambda$  are given by

$$K - m \pi^4 \sigma^4 = \frac{\pi^3 \sigma^3}{2} \left( \tanh \frac{\pi \sigma}{2} + \tan \frac{\pi \sigma}{2} \right) \quad (D-1)$$

$$K - m \pi^4 \sigma^4 = \frac{\pi^3 \sigma^3}{2} \left( \coth \frac{\pi \sigma}{2} - \cot \frac{\pi \sigma}{2} \right)$$

where

$$\sigma = \frac{L}{\pi} \sqrt[4]{\frac{\gamma \lambda^2}{EI}}$$

In Fig. D-1 the functions given by Eqs. (D-1) and the dashed curve  $212 - 0.56 \pi^4 \sigma^4$  are drawn. The intersections of these curves, points A, B, C, and D, represent the first four dimensionless natural frequencies  $\sigma = 0.95, 1.31, 1.43, \text{ and } 2.09$ , respectively. The frequencies  $\lambda$  (in radians per second) are given by

$$\lambda = \frac{\sigma^2 \pi^2}{L^2} \sqrt{\frac{EI}{\delta}} = 200\sigma^2$$

thus

$$\lambda_1 = 180 \text{ rad/sec} = 1720 \text{ rpm}$$

$$\lambda_2 = 342 \text{ rad/sec} = 3270 \text{ rpm}$$

$$\lambda_3 = 409 \text{ rad/sec} = 3900 \text{ rpm}$$

$$\lambda_4 = 874 \text{ rad/sec} = 8350 \text{ rpm}$$

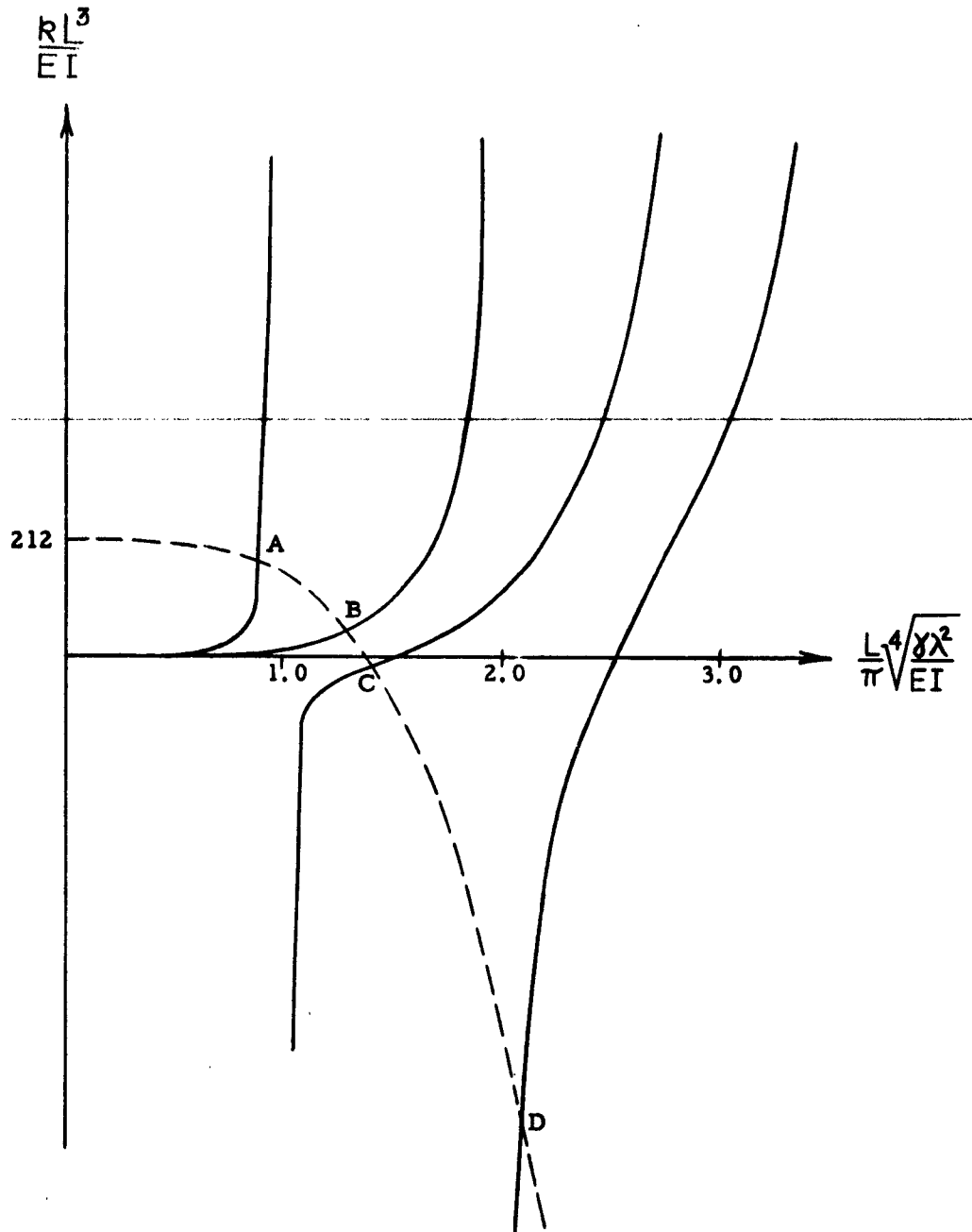


Fig. D-1 NATURAL FREQUENCIES - K=212 and m= .56

## II. RECTANGULAR SHAFT

We determine the rough running speeds of a rectangular shaft mounted in the flexible bearings of the previous case. The constants of the system are

cross section	=	0.875 in. x 1.5 in.
$\gamma gL$	=	18.6 lb
$I_1$	=	0.0837 in. <sup>4</sup>
$I_2$	=	0.246 in. <sup>4</sup>
$S = E(I_1 + I_2)/2$	=	$4.45 \times 10^6$ lb.in. <sup>2</sup>
$\epsilon = (I_2 - I_1)/(I_2 + I_1)$	=	0.49
$K$	=	$kL^3/S = 61.3$
$K/(1-\epsilon)$	=	124
$K/(1+\epsilon)$	=	42.3
$K/(1-\epsilon^2)$	=	83.2
$m$	=	0.333

The frequency equation for an anisotropic shaft is given by

$$K' - m\alpha^4 = \frac{\alpha^3}{2} \left( \tanh \frac{\alpha}{2} + \tan \frac{\alpha}{2} \right)$$

(D-2)

$$K' - m\alpha^4 = \frac{\alpha^3}{2} \left( \coth \frac{\alpha}{2} - \cot \frac{\alpha}{2} \right)$$

when

$$K' = \frac{K}{1-\epsilon}, \quad \alpha^4 = \frac{\lambda^2 \gamma L^4}{S(1-\epsilon)} = \frac{\lambda^2 \gamma L^4}{EI_1}$$

when

$$K' = \frac{K}{1+\epsilon}, \quad \alpha^4 = \frac{\lambda^2 \gamma L^4}{S(1+\epsilon)} = \frac{\lambda^2 \gamma L^4}{EI_2}$$

A graphical solution of these equations is obtained by constructing the functions

$$f^+(\eta) = \frac{\pi^3 \eta^3}{2} \left( \tanh \frac{\pi \eta}{2} + \tan \frac{\pi \eta}{2} \right)$$

$$f^-(\eta) = \frac{\pi^3 \eta^3}{2} \left( \coth \frac{\pi \eta}{2} - \cot \frac{\pi \eta}{2} \right)$$

and

$$K' = m \pi^4 \eta^4.$$

The intersections of the dashed curves  $K' = 124$  and  $42.3$  with the full curves in Fig. (D-2) determine the dimensionless natural frequencies

$$\eta = \frac{L}{\pi} \sqrt[4]{\frac{\gamma \lambda^2}{EI_1}} \quad \text{and}$$

$$\eta = \frac{L}{\pi} \sqrt[4]{\frac{\gamma \lambda^2}{EI_2}} \quad \text{respectively.}$$

Taking  $K' = 124$  we obtain

$$\eta = 0.91 \quad \text{or} \quad \lambda = 167 \text{ rad/sec} = 1600 \text{ rev/min}$$

$$\eta = 1.25 \quad \text{or} \quad \lambda = 315 \text{ rad/sec} = 3010 \text{ rev/min}$$

$$\eta = 1.43 \quad \text{or} \quad \lambda = 413 \text{ rad/sec} = 3940 \text{ rev/min}$$

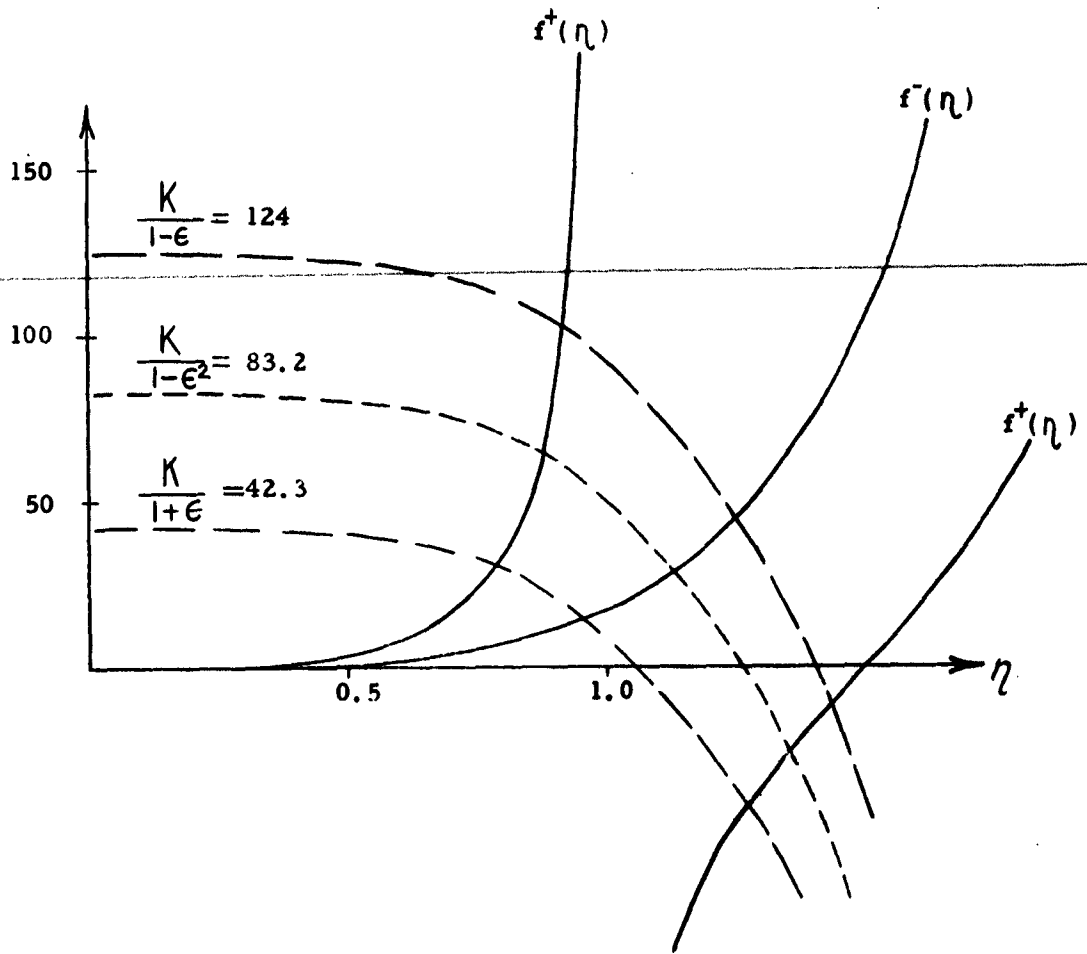
For  $K' = 42.3$  we find

$$\eta = 0.79 \quad \text{or} \quad \lambda = 222 \text{ rad/sec} = 2120 \text{ rev/min}$$

$$\eta = 0.95 \quad \text{or} \quad \lambda = 312 \text{ rad/sec} = 2980 \text{ rev/min}$$

$$\eta = 1.27 \quad \text{or} \quad \lambda = 558 \text{ rad/sec} = 5330 \text{ rev/min}$$





**Fig. D-2 GRAPHICAL SOLUTION FOR CRITICAL SPEEDS**  
**OF A RECTANGULAR SHAFT  $m = 1/3$**

In obtaining the gravity critical speeds, we can easily show that the frequency equation is identical with Eq. (D-2) where

$$K' = K/(1 - \epsilon^2) \quad \text{and} \quad \alpha = \frac{4\gamma\lambda^2 L^4}{8(1 - \epsilon^2)}$$

Thus from Fig. (D-2) with  $K' = 83.2$  and

$$\eta = \frac{L}{\pi} \sqrt[4]{\frac{4\gamma\lambda^2}{8(1 - \epsilon^2)}}$$

we obtain

$$\eta = 0.88 \quad \text{or} \quad \lambda = 95 \text{ rad/sec} = 910 \text{ rev/min}$$

$$\eta = 1.13 \quad \text{or} \quad \lambda = 156 \text{ rad/sec} = 1500 \text{ rev/min}$$

$$\eta = 1.35 \quad \text{or} \quad \lambda = 224 \text{ rad/sec} = 2140 \text{ rev/min}$$

This particular shaft system, then, has the following characteristics: discrete rough running speeds at 910, 1500, 2140 rpm, etc. and critical speed ranges 1600-2120 rpm, 2980-3010 rpm, 3940-5330 rpm, ....

### MODE SHAPES FOR CASE I

The three lowest modes of vibration corresponding to the pinned-pinned shaft of case I are

$$u_1 = 0.5 \cosh 2.970 z + 0.5 \cos 2.970 z - 0.449 \sinh 2.970 z + 5.964 \sin 2.970 z$$

$$u_2 = 0.5 \cosh 4.102 z + 0.5 \cos 4.102 z - 0.518 \sinh 4.102 z + 0.260 \sin 4.102 z$$

$$u_3 = 0.5 \cosh 4.478 z + 0.5 \cos 4.478 z - 0.488 \sinh 4.478 z - 0.663 \sin 4.478 z$$

where  $z = Z/L$ .

These modes are constructed in Fig. D-3 where the bearing displacement is taken as unity. Although the first and third modes are orthogonal to the second, they are not orthogonal to one another.

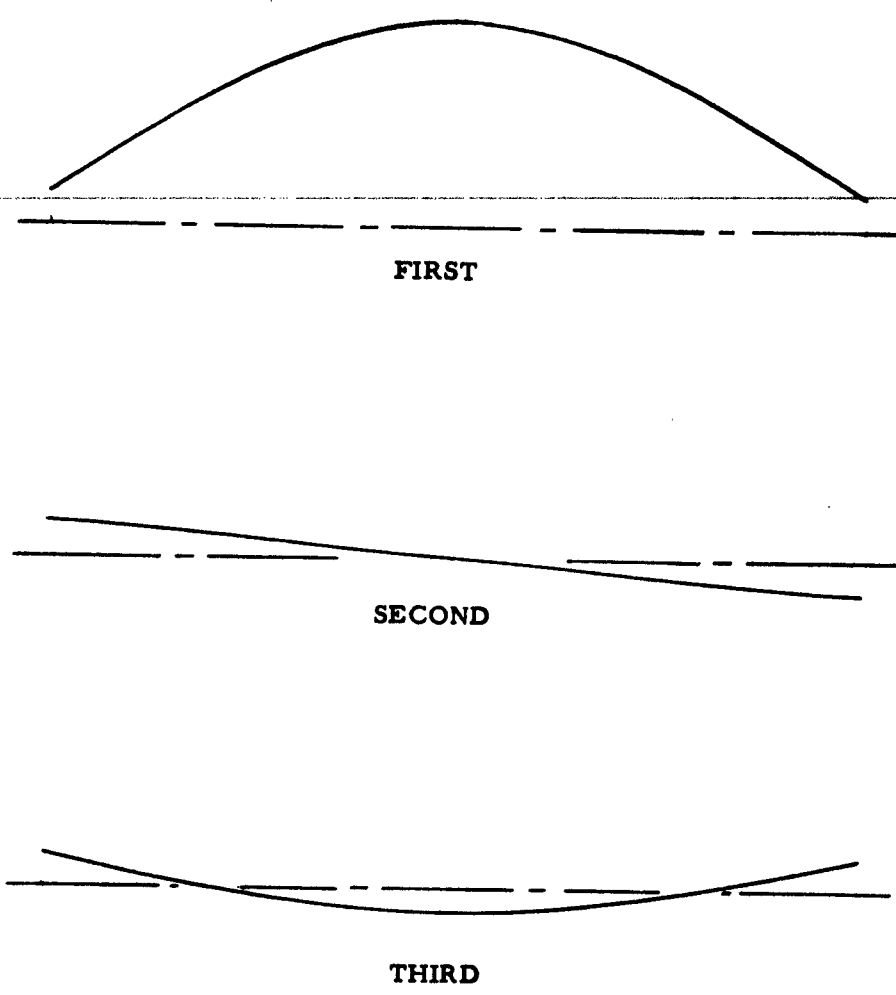


Fig. D3 MODES OF VIBRATION  $K = 212, m = .56$

APPENDIX E

THE FREQUENCY EQUATIONS FOR VARIOUS  
BEARING CONFIGURATIONS

The frequency equations for uniform round shaft systems can be obtained in reasonably simple forms. The various bearing configurations can be characterized as follows:

I Both Bearings Flexible ( $K_1 = K_2, M_1 = M_2$ )

- a) Pinned-Pinned (already treated)
- b) Clamped-Clamped
- c) Clamped-Pinned

II Left Bearing Flexible, Right Bearing Rigid

- a) Pinned-Pinned
- b) Pinned-Clamped
- c) Clamped-Pinned
- d) Clamped-Clamped

The natural frequencies for a free type of end constraint may be found as the limiting case  $K \rightarrow 0$ , while the immovable constraint is found from the limit  $K \rightarrow \infty$

### Case Ib Clamped-Clamped

The boundary conditions (26) now become

$$\text{at } Z = 0 \quad EI \frac{\partial^3 U}{\partial Z^3} + k_1 U + M_1 \frac{\partial^2 U}{\partial t^2} = 0 \quad (\text{E1})$$

$$\frac{\partial U}{\partial Z} = 0 \quad (\text{E2})$$

$$\text{at } Z = L \quad -EI \frac{\partial^3 U}{\partial Z^3} + k_2 U + M_2 \frac{\partial^2 U}{\partial t^2} = 0 \quad (\text{E3})$$

$$\frac{\partial U}{\partial Z} = 0 \quad (\text{E4})$$

Using expression (27) as the solution at the equation of motion (25), we obtain the frequency equation (E6) given on the next page. For the case of identical bearings the frequency equation takes on the simple form

$$K = m\pi^4 \alpha^4 + 2\pi^3 \alpha^3 / (\tanh \frac{\pi\alpha}{2} - \tan \frac{\pi\alpha}{2})$$

$$\text{or } K = m\pi^4 \alpha^4 + 2\pi^3 \alpha^3 / (\coth \frac{\pi\alpha}{2} + \cot \frac{\pi\alpha}{2}) \quad (\text{E5})$$

$$\text{where } \alpha = \frac{\beta}{\pi} = \frac{L}{\pi} \left[ \frac{\gamma \lambda^2}{EI} \right]^{1/4}$$

The first four natural frequencies for various mass and stiffness ratios are given in Figures (E12) to (E17). The asymptotes correspond to the natural frequencies of immovable bearings which are found from

$$\tanh^2 \frac{\pi\alpha}{2} = \tan^2 \frac{\pi\alpha}{2}$$

### Case Ib Clamped-Clamped

The boundary conditions (26) now become

$$\text{at } Z = 0 \quad EI \frac{\partial^3 U}{\partial Z^3} + k_1 U + M_1 \frac{\partial^2 U}{\partial t^2} = 0 \quad (\text{E1})$$

$$\frac{\partial U}{\partial Z} = 0 \quad (\text{E2})$$

$$\text{at } Z = L \quad -EI \frac{\partial^3 U}{\partial Z^3} + k_2 U + M_2 \frac{\partial^2 U}{\partial t^2} = 0 \quad (\text{E3})$$

$$\frac{\partial U}{\partial Z} = 0 \quad (\text{E4})$$

Using expression (27) as the solution at the equation of motion (25), we obtain the frequency equation (E6) given on the next page. For the case of identical bearings the frequency equation takes on the simple form

$$K = m\pi^4 \alpha^4 + 2\pi^3 \alpha^3 / (\tanh \frac{\pi\alpha}{2} - \tan \frac{\pi\alpha}{2})$$

$$\text{or } K = m\pi^4 \alpha^4 + 2\pi^3 \alpha^3 / (\coth \frac{\pi\alpha}{2} + \cot \frac{\pi\alpha}{2}) \quad (\text{E5})$$

$$\text{where } \alpha = \frac{\beta}{\pi} = \frac{L}{\pi} \left[ \frac{\gamma\lambda^2}{EI} \right]^{1/4}$$

The first four natural frequencies for various mass and stiffness ratios are given in Figures (E12) to (E17). The asymptotes correspond to the natural frequencies of immovable bearings which are found from

$$\tanh^2 \frac{\pi\alpha}{2} = \tan^2 \frac{\pi\alpha}{2}$$

$$\begin{array}{c}
 0 = \frac{
 \begin{array}{c}
 K_1 - m_1 \beta^4 \\
 0 \\
 (K_2 - m_2 \beta^4) \cosh \beta - \beta^3 \sinh \beta \\
 \beta \sinh \beta
 \end{array}
 }{
 \begin{array}{c}
 K_1 - m_1 \beta^4 \\
 0 \\
 (K_2 - m_2 \beta^4) \cos \beta - \beta^3 \sin \beta \\
 - \beta \sin \beta
 \end{array}
 }
 \frac{
 \begin{array}{c}
 \beta^3 \\
 \beta \\
 (K_2 - m_2 \beta^4) \sinh \beta - \beta^3 \cosh \beta \\
 \beta \cosh \beta
 \end{array}
 }{
 \begin{array}{c}
 \beta^3 \\
 \beta \\
 (K_2 - m_2 \beta^4) \sin \beta + \beta^3 \cos \beta \\
 \beta \cos \beta
 \end{array}
 }
 \end{array}$$

(E6)



Case Ic Clamped-Clamped

Here the boundary conditions (E1), (E2) and (E3) are unchanged while the remaining condition becomes

$$\text{at } Z = L \quad \frac{\partial^2 U}{\partial Z^2} = 0 \quad (\text{E7})$$

For identical bearings, the frequency equation (E9) given on the next page reduces to

$$K = m\omega^4 \alpha^4 + \frac{\pi^3 \alpha^3}{2} \frac{[1 + 3 \cosh \pi \alpha \cos \pi \alpha + \sqrt{(1 - \cos \pi \alpha \cosh \pi \alpha)^2 + 4(\cos \pi \alpha + \cosh \pi \alpha)^2}]}{2(\sinh \pi \alpha \cos \pi \alpha - \cosh \pi \alpha \sin \pi \alpha)}$$

The first few natural frequencies for various mass and stiffness ratios are given in Figures (E18) to (E23). The asymptotes correspond to the natural frequencies of the ideal clamped-pinned shaft which are found from

$$\tanh \pi \alpha = \tan \pi \alpha$$

$$\begin{array}{l}
 0 = \frac{(K_1 - m_1 \beta) \quad (K_1 - m_1 \beta^4) \quad \beta^3}{\left[ (K_2 - m_2 \beta^4) \cosh \beta - \beta^3 \sinh \beta \right] \left[ (K_2 - m_2 \beta^4) \cos \beta - \beta^3 \sin \beta \right] \left[ (K_2 - m_2 \beta^4) \sinh \beta - \beta^3 \cosh \beta \right] \left[ (U_2 - m_2 \beta^4) \sin \beta + \beta^3 \cos \beta \right]} \\
 \begin{array}{l}
 0 \qquad \qquad \qquad 0 \qquad \qquad \qquad \beta \\
 \beta^2 \cosh \beta \quad - \beta^2 \cos \beta \quad \beta^2 \sinh \beta \quad - \beta^2 \sin \beta
 \end{array}
 \end{array}$$

(E9)

### Case IIa Pinned Flexible-Pinned Rigid

Any number of frequency equations may be presented by varying the shaft system parameters  $K_1$ ,  $K_2$ ,  $M_1$  and  $M_2$ . However, we shall restrict ourselves to the case where one bearing is practically rigid (at  $Z = L$ ) while the other bearing remains flexible (at  $Z = 0$ ). Accordingly the boundary conditions (26a) remain unchanged while the boundary conditions at the other end become

$$\text{at } Z = L \quad U = 0, \quad \frac{\partial^2 U}{\partial Z^2} = 0 \quad (\text{E10})$$

Again assuming the solution (27), we obtain the frequency equation

$$K = m\pi^4 \alpha^4 + \frac{\pi^3 \alpha^3}{2} (\coth \pi \alpha - \cot \pi \alpha) \quad (\text{E11})$$

$$\text{where } \alpha = \frac{\beta}{\pi} = \frac{L}{\pi} \left[ \frac{\gamma \lambda^2}{EI} \right]^{1/4}$$

The first few natural frequencies for various  $K$  and  $m$  are given in Figures (E24) to (E29). Note that the idealized case of pinned-pinned immovable bearings ( $K \rightarrow \infty$ ) furnishes the natural frequencies

$$\alpha = 1, 2, 3, \dots$$

**Case IIb Pinned Flexible-Clamped Rigid**

In addition to the boundary conditions (26a) we have

at  $Z = L$   $U = 0, \frac{\partial U}{\partial Z} = 0$  (E12)

The frequency equation corresponding to the solution (27) is

$$K = m\pi^4 \alpha^4 + \frac{\pi^3 \alpha^3 (1 + \cos \pi \alpha \cosh \pi \alpha)}{\sinh \pi \alpha \cos \pi \alpha - \cosh \pi \alpha \sin \pi \alpha} \quad (\text{E13})$$

$$\alpha = \frac{L}{\pi} \left[ \frac{\gamma \lambda^2}{EI} \right]^{1/4}$$

The dependence of the natural frequencies  $\lambda$  on  $K$  and  $m$  is illustrated in Figures (E30) to (E35).

**Case IIc Clamped Flexible-Pinned Rigid**

The conditions for a clamped flexible bearing are given by conditions (E1), (E2), while a pinned-rigid bearing has the conditions

$$\text{at } Z = L \quad U = 0, \quad \frac{\partial^2 U}{\partial Z^2} = 0 \quad (\text{E14})$$

The frequency equation is

$$K = m\omega^4 \alpha^4 + 2\pi^3 \alpha^3 / (\tanh \pi\alpha - \tan \pi\alpha) \quad (\text{E15})$$

The first few solutions of this equation are given graphically in Figures (E36) to (E41).

Case II d Clamped Flexible-Clamped Rigid

In addition to conditions (E1), (E2), we have

$$\text{at } Z = L \quad U = 0, \quad \frac{\partial U}{\partial Z} = 0 \quad (\text{E16})$$

The frequency equation is

$$K = m\pi^4 \alpha^4 + \frac{\pi^3 \alpha^3 (\sin \pi \alpha \cosh \pi \alpha + \cos \pi \alpha \sinh \pi \alpha)}{(\cos \pi \alpha \cosh \pi \alpha - 1)} \quad (\text{E17})$$

The first few solutions of this equation are given graphically in Figures (E42) to (E47).

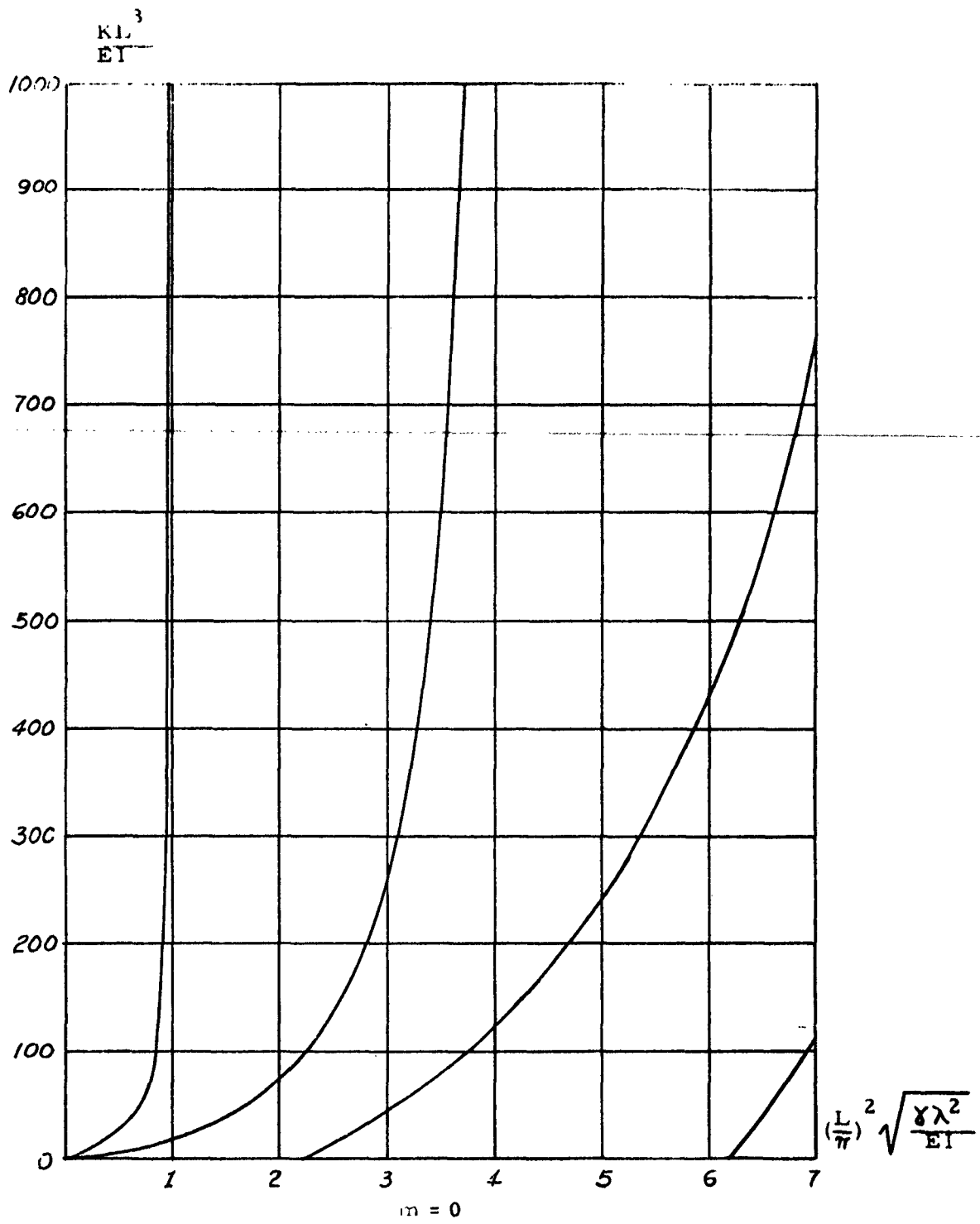


Fig. E1 NATURAL FREQUENCIES - CASE Ia

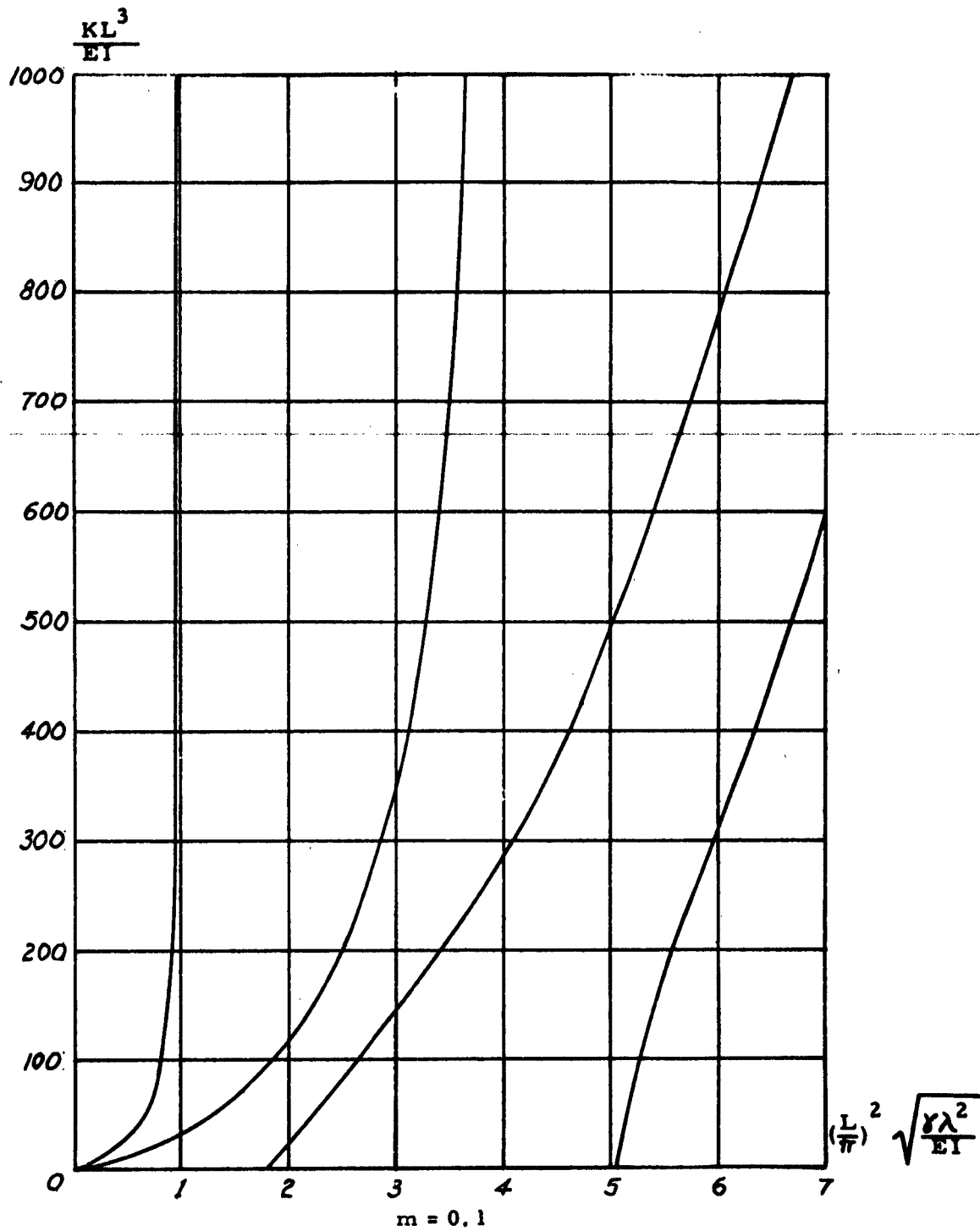


Fig. E2 NATURAL FREQUENCIES - CASE Ia



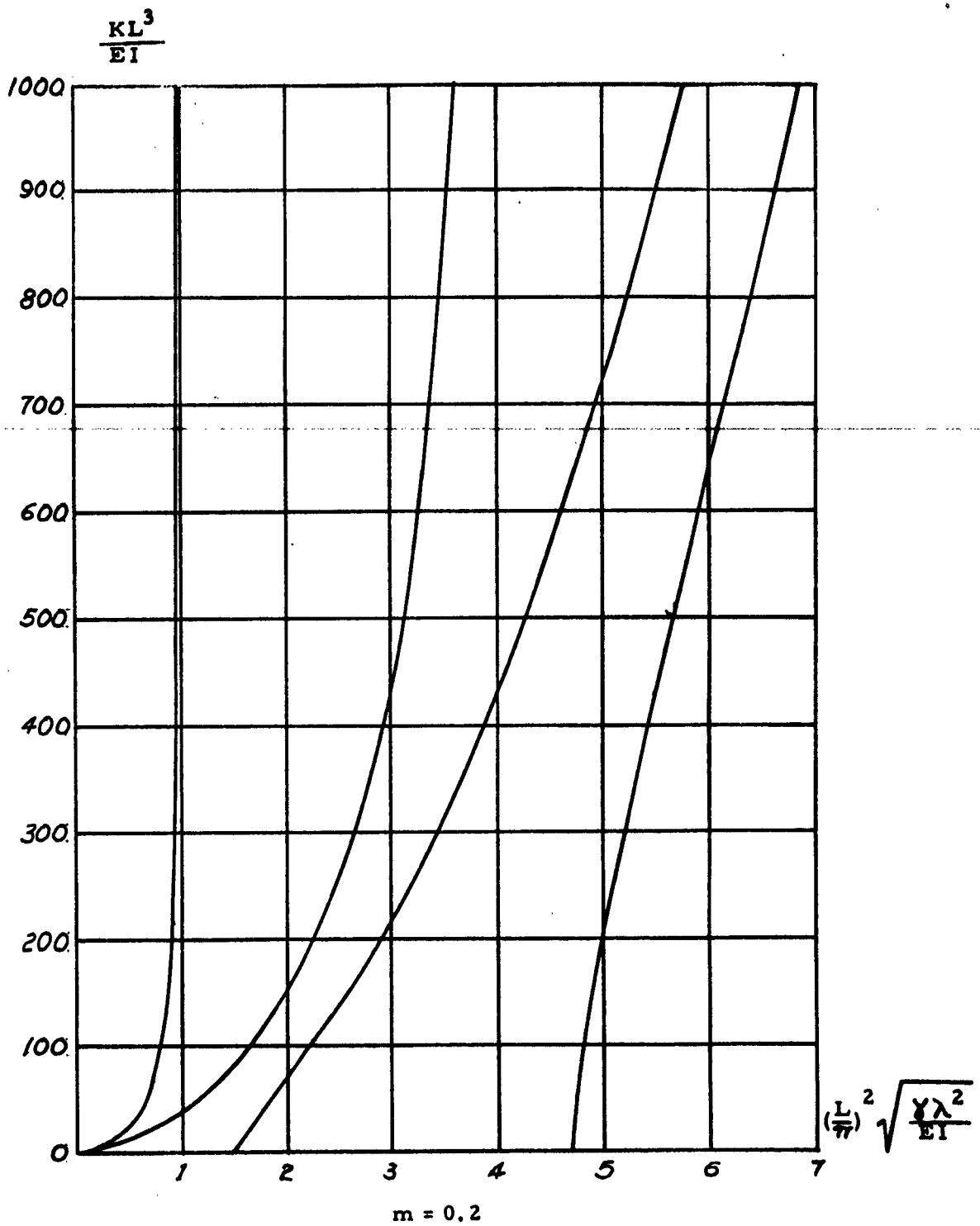


Fig. E3 NATURAL FREQUENCIES - CASE 1a

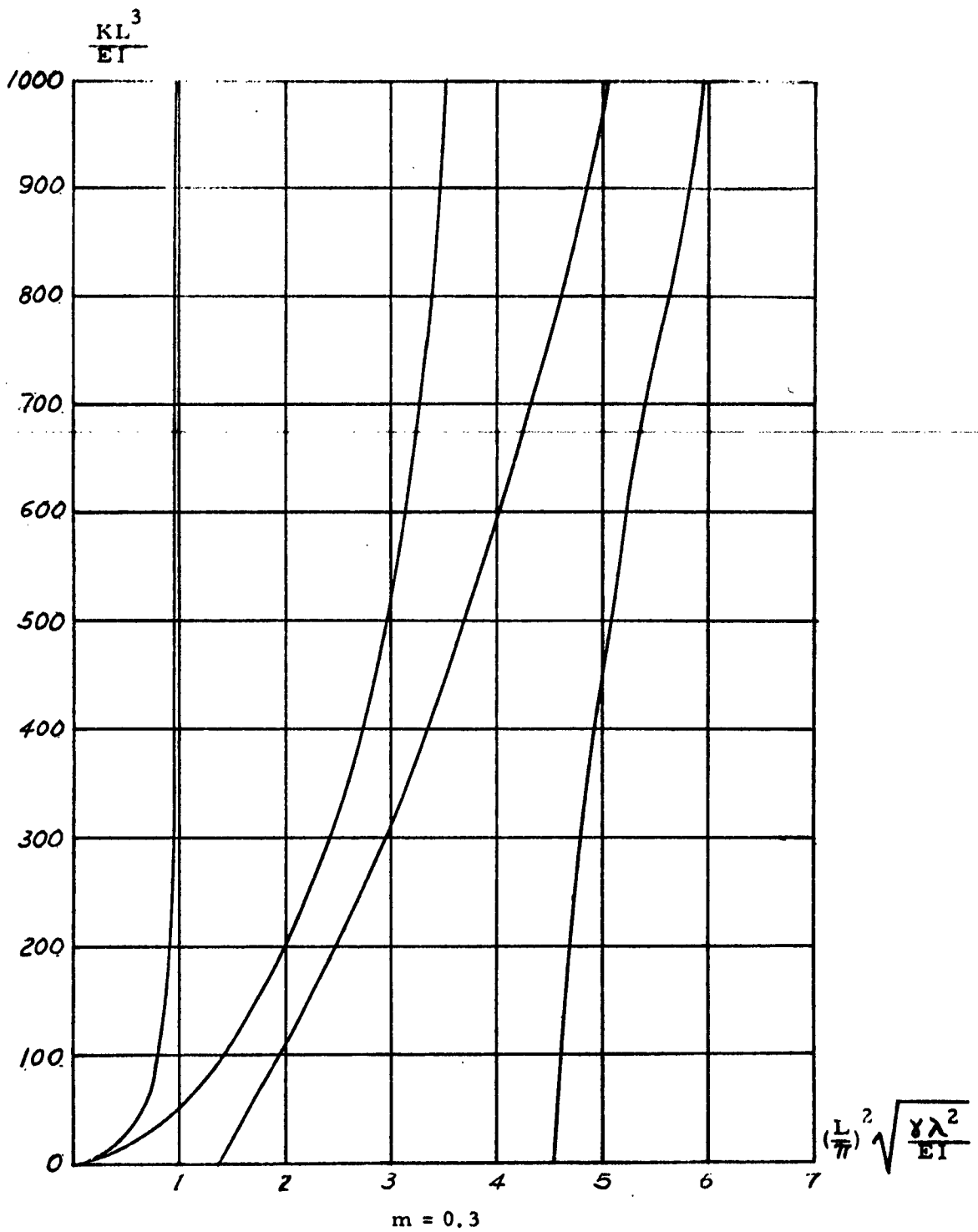


Fig. E4 NATURAL FREQUENCIES - CASE 1a

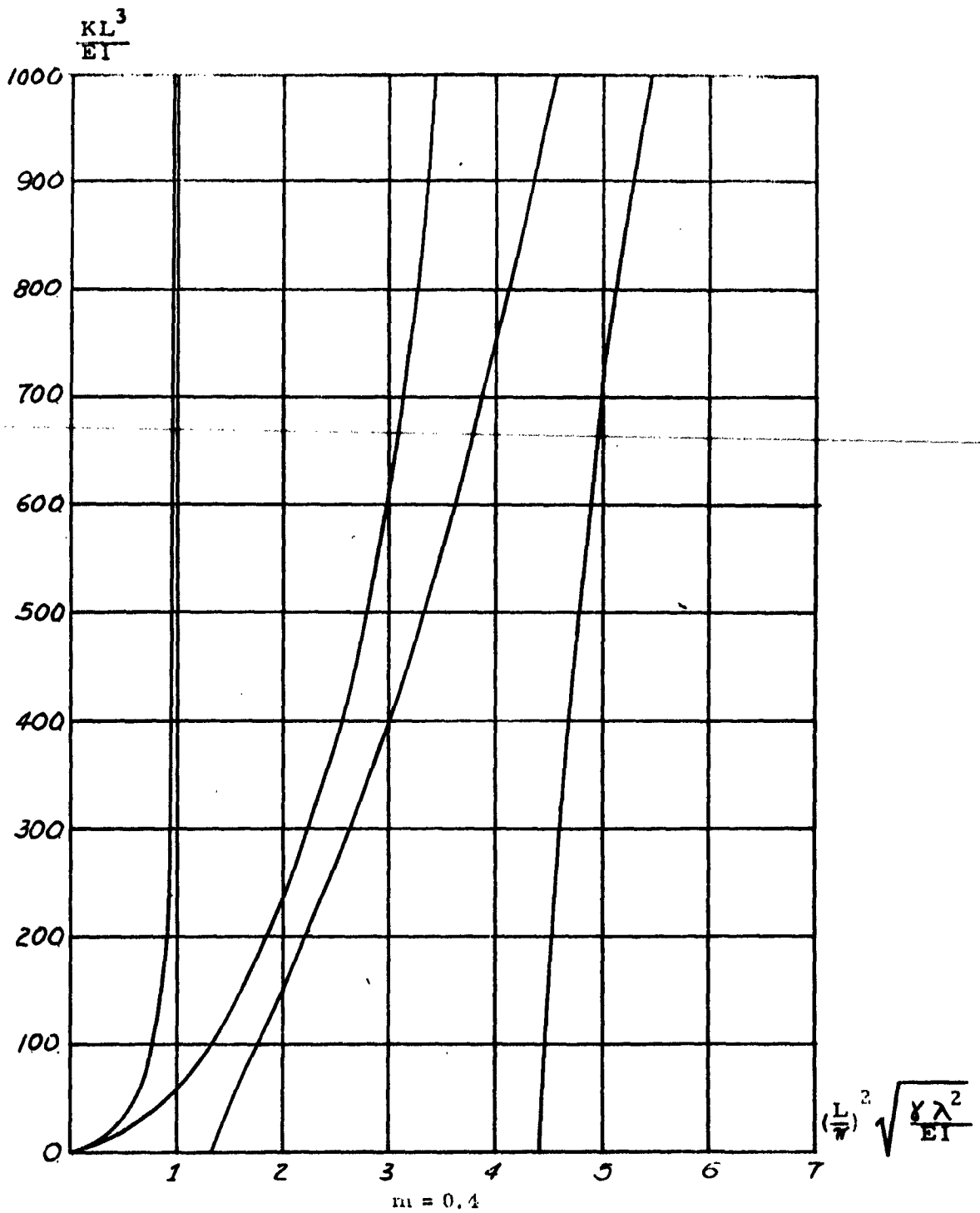


Fig. E5 NATURAL FREQUENCIES - CASE 1a

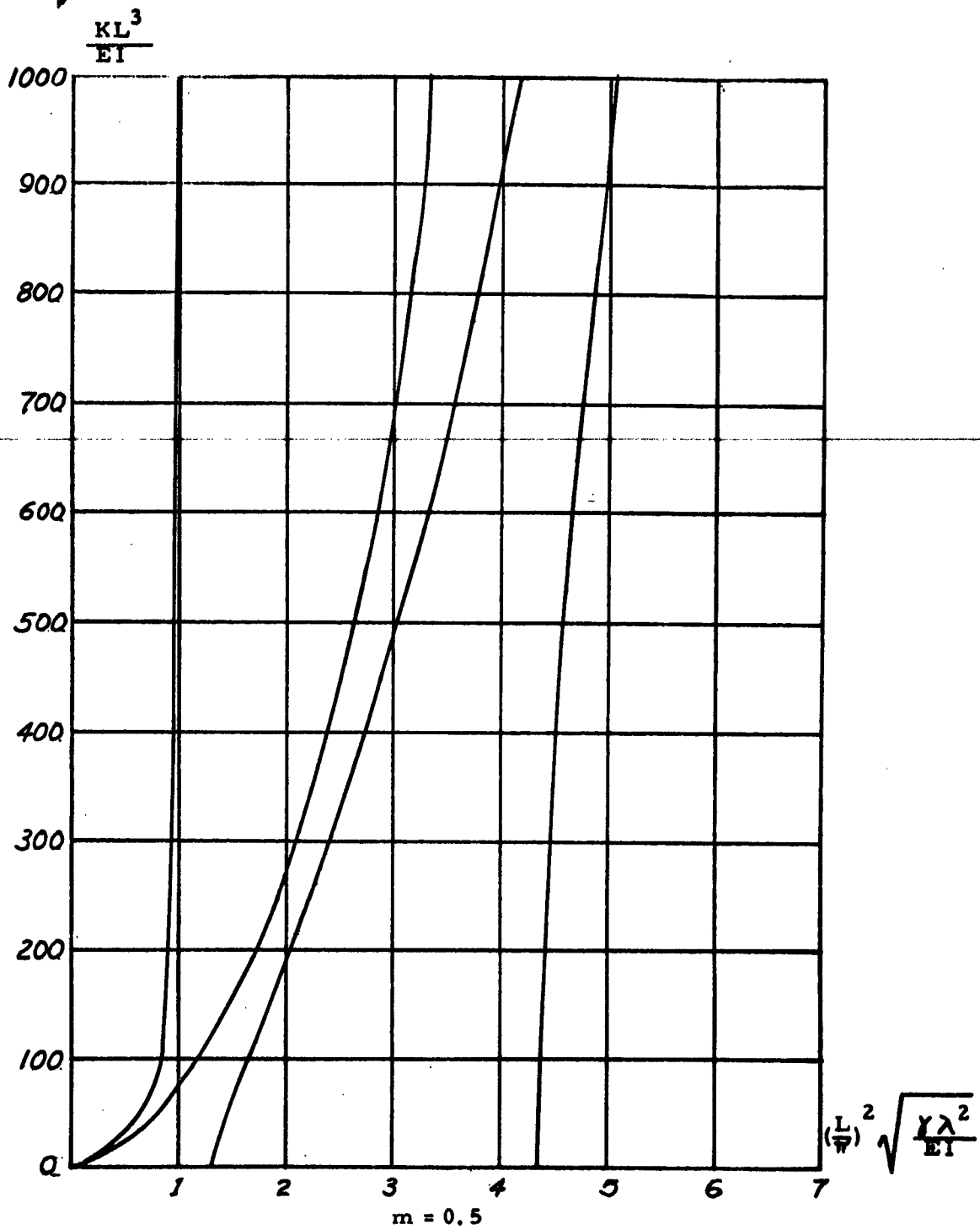


Fig. E6 NATURAL FREQUENCIES - CASE 1a

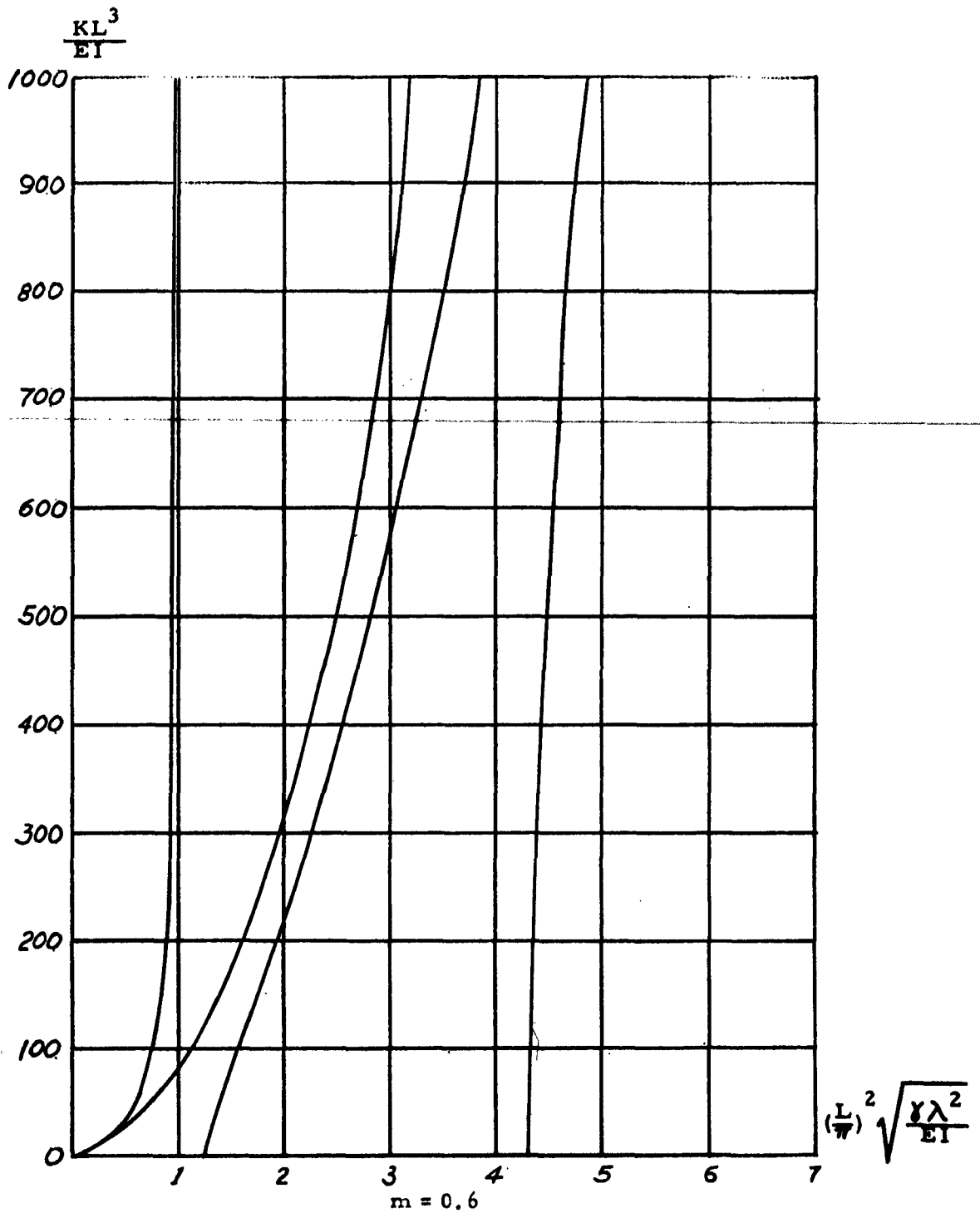


Fig. E7 NATURAL FREQUENCIES - CASE Ia

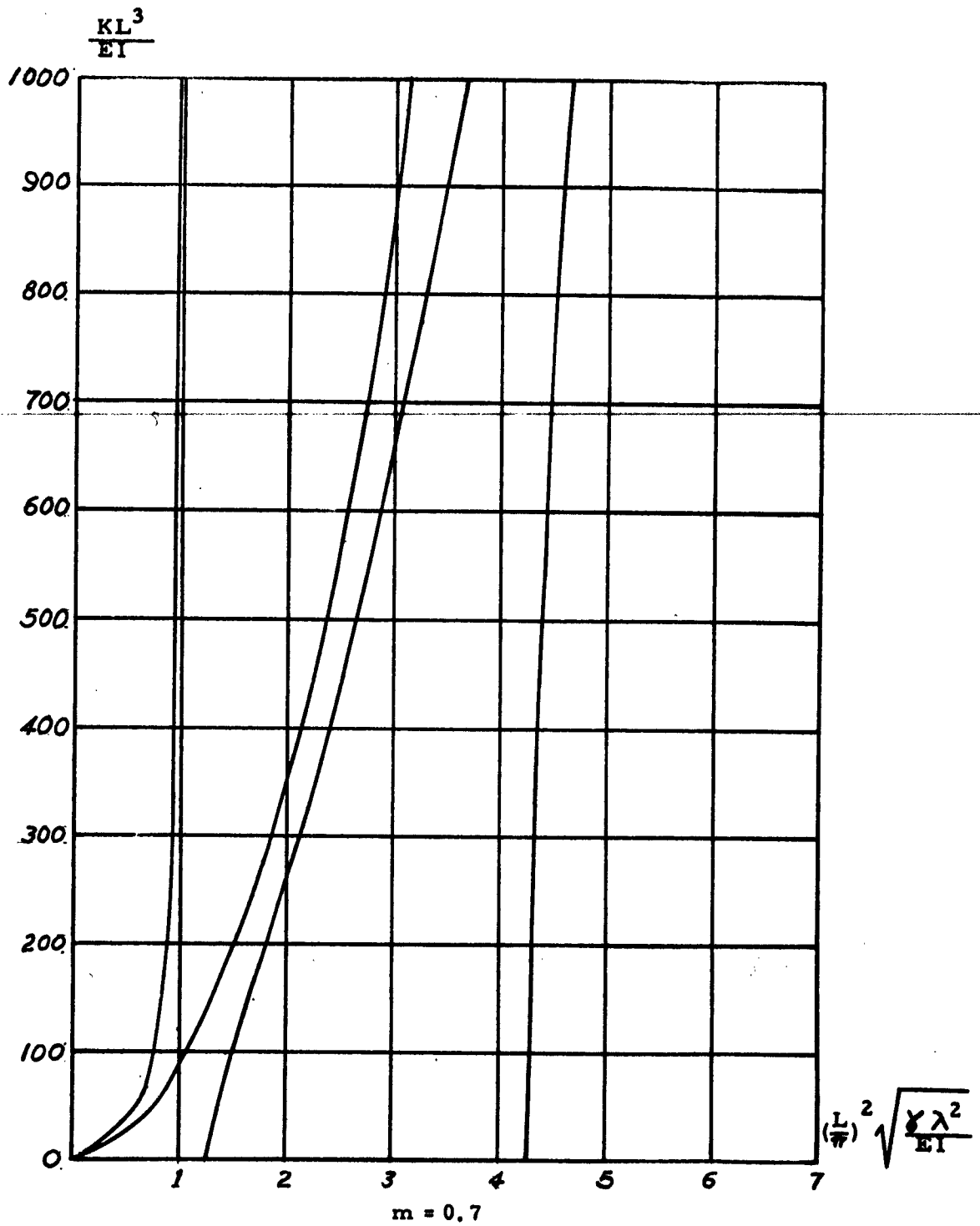


Fig. E8 NATURAL FREQUENCIES - CASE Ia

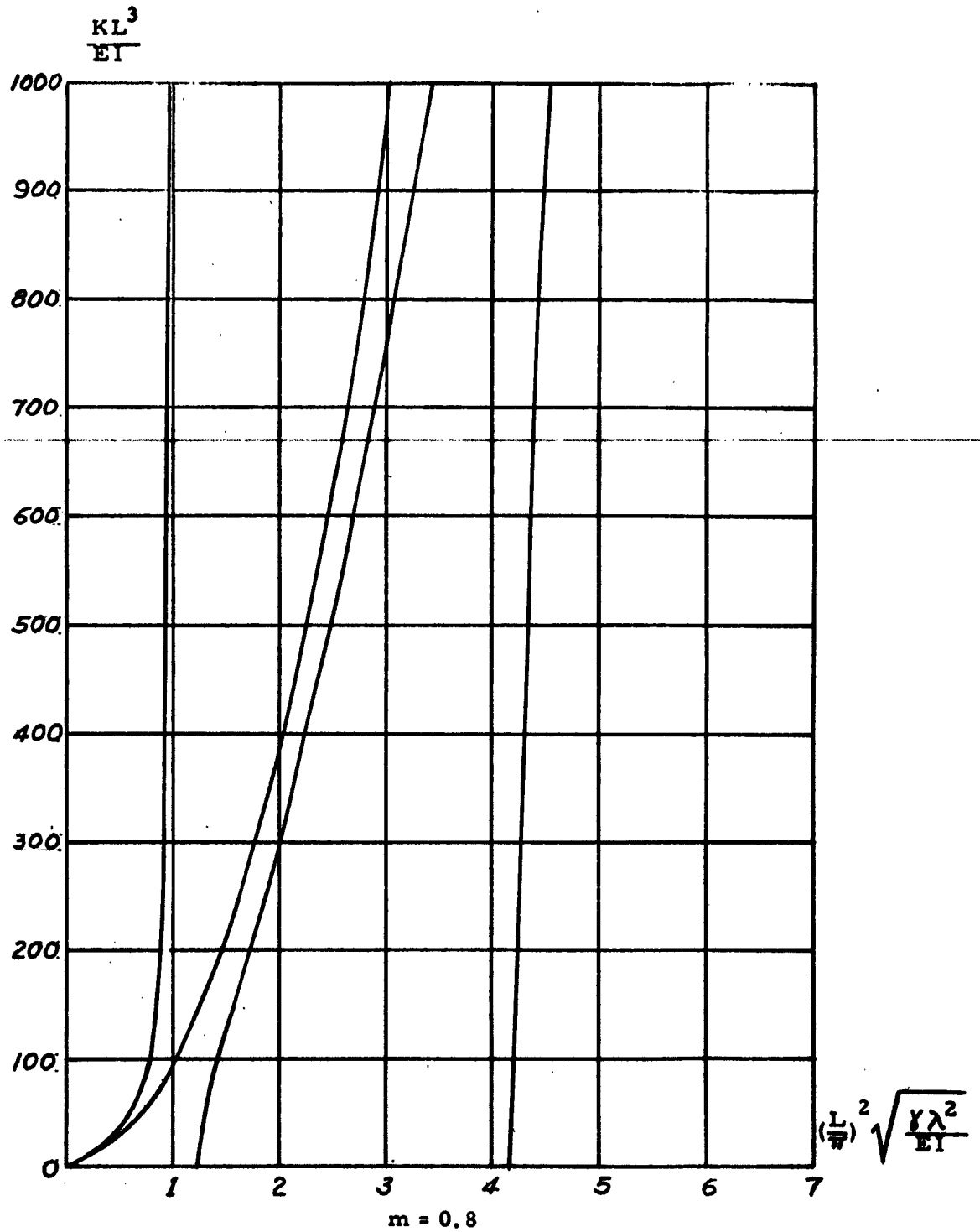


Fig. E9 NATURAL FREQUENCIES - CASE 1a

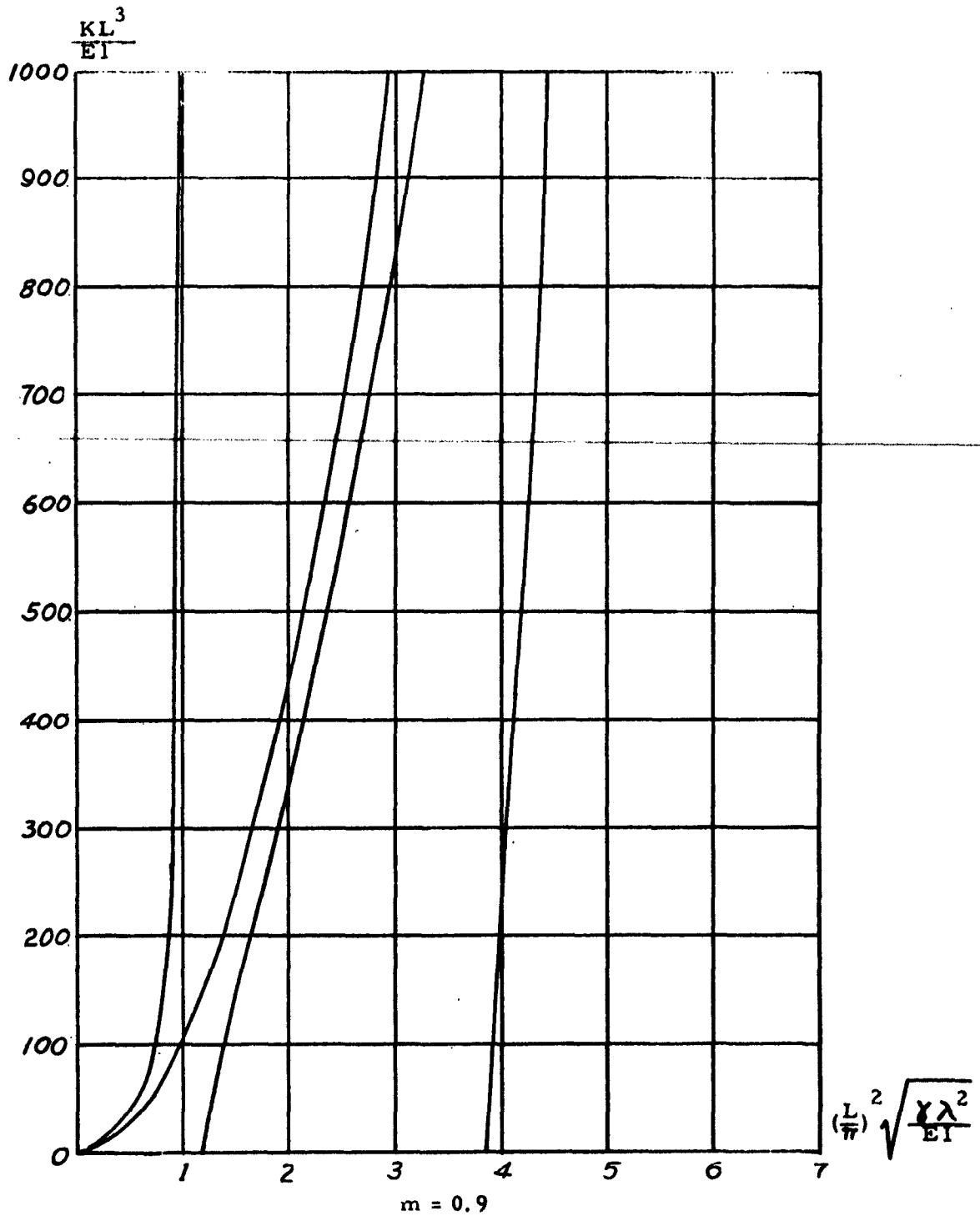


Fig. E10 NATURAL FREQUENCIES - CASE Ia



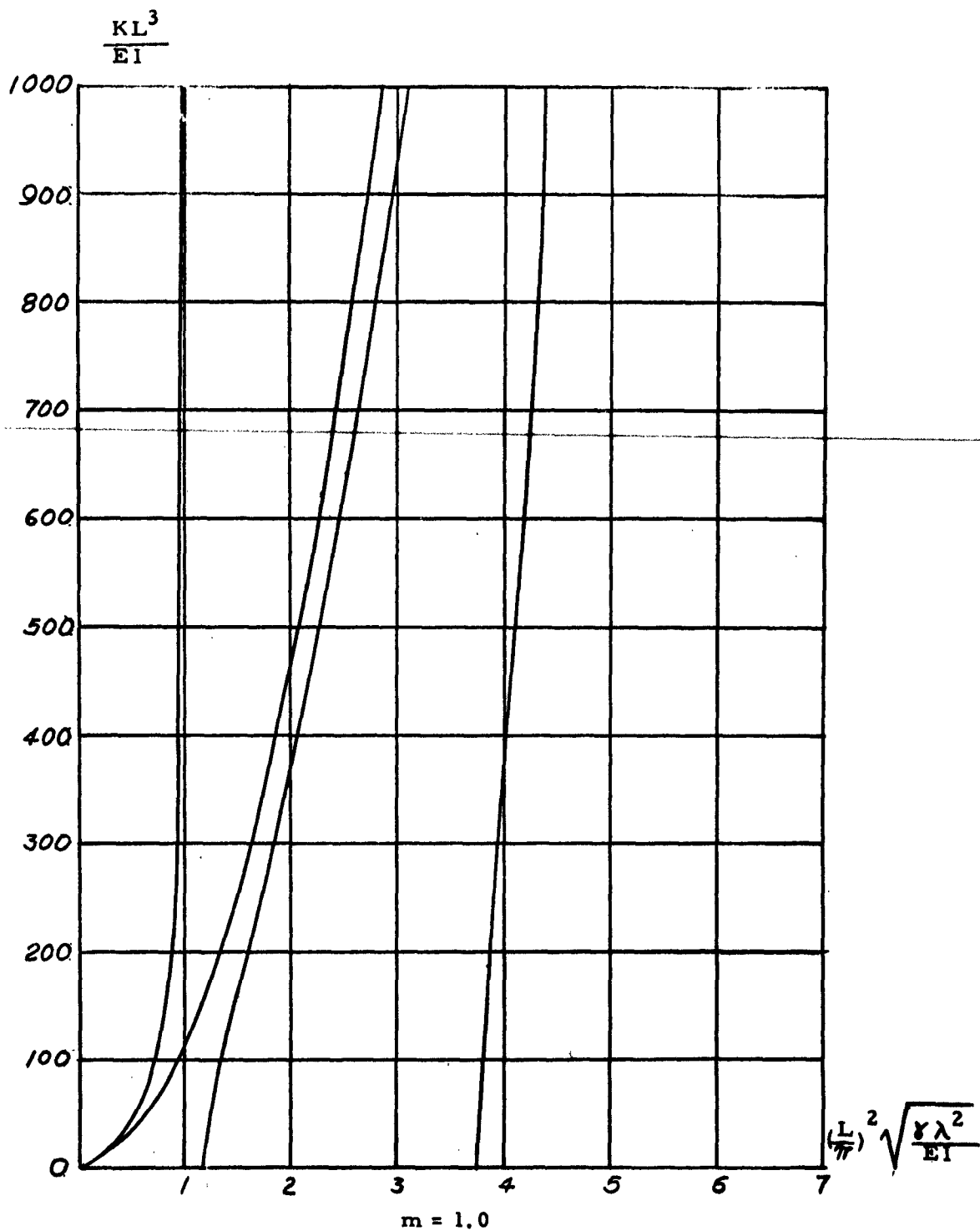


Fig. E11 NATURAL FREQUENCIES - CASE Ia

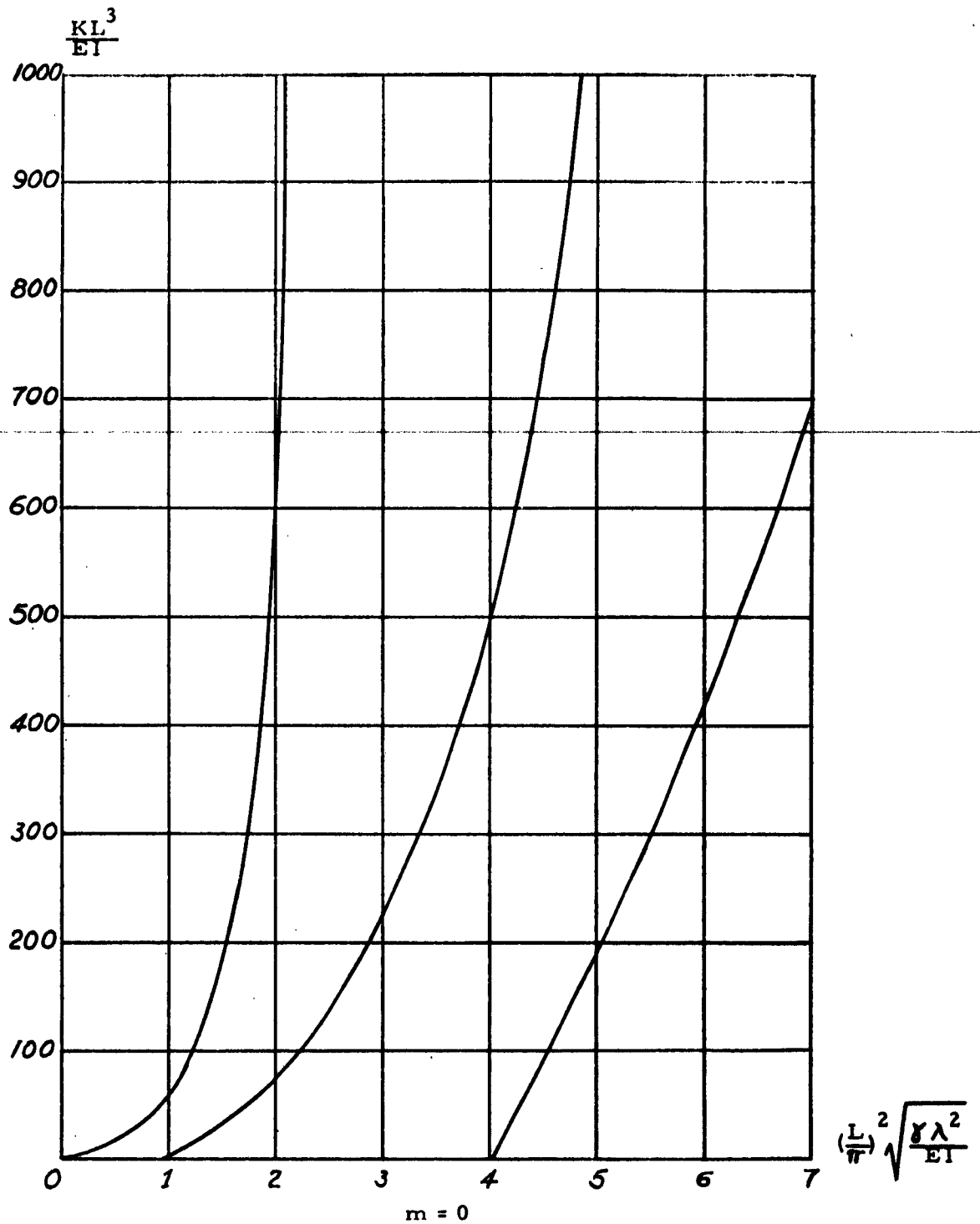


Fig. E12 NATURAL FREQUENCIES - CASE 1b

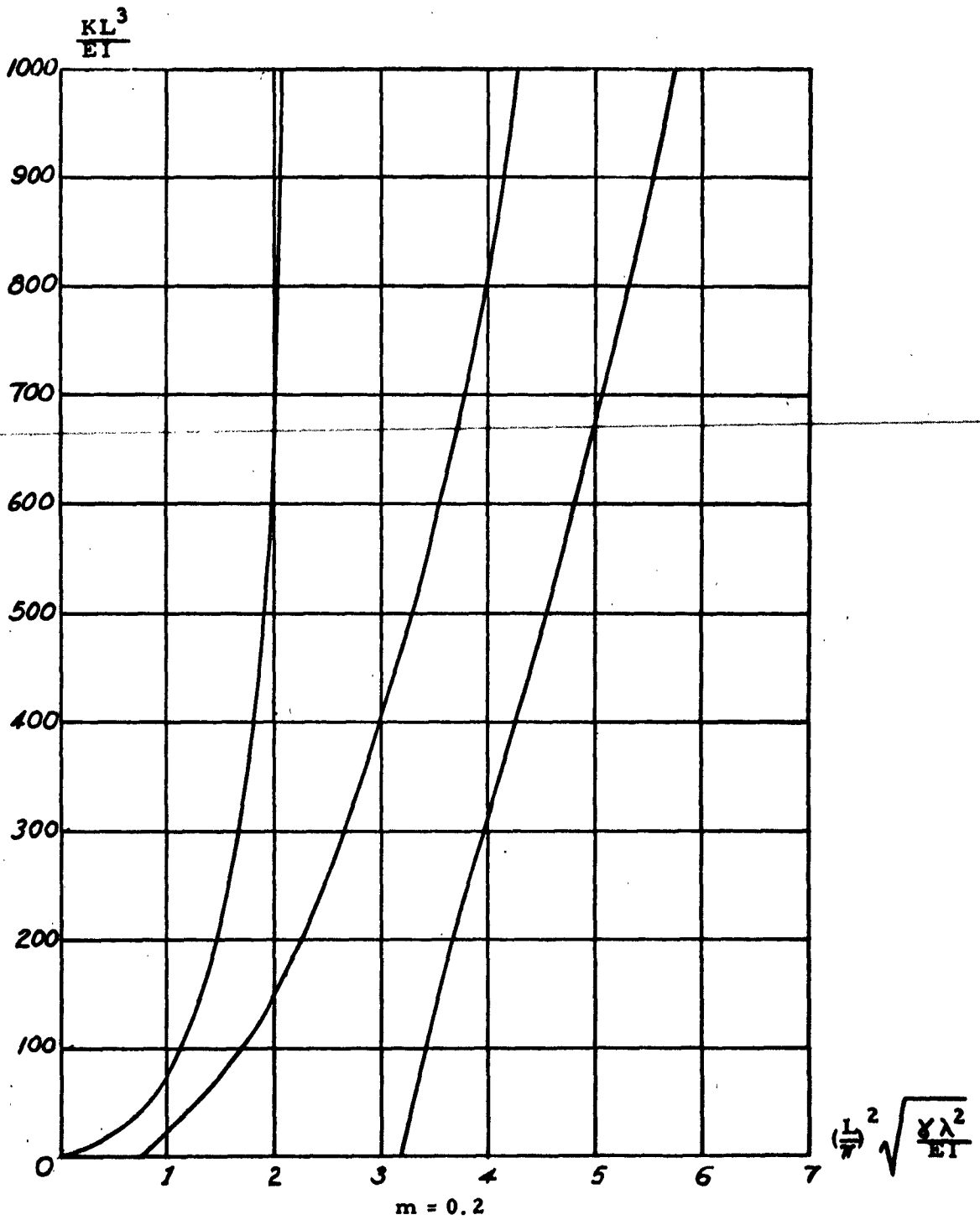


Fig. E13 NATURAL FREQUENCIES - CASE 1b

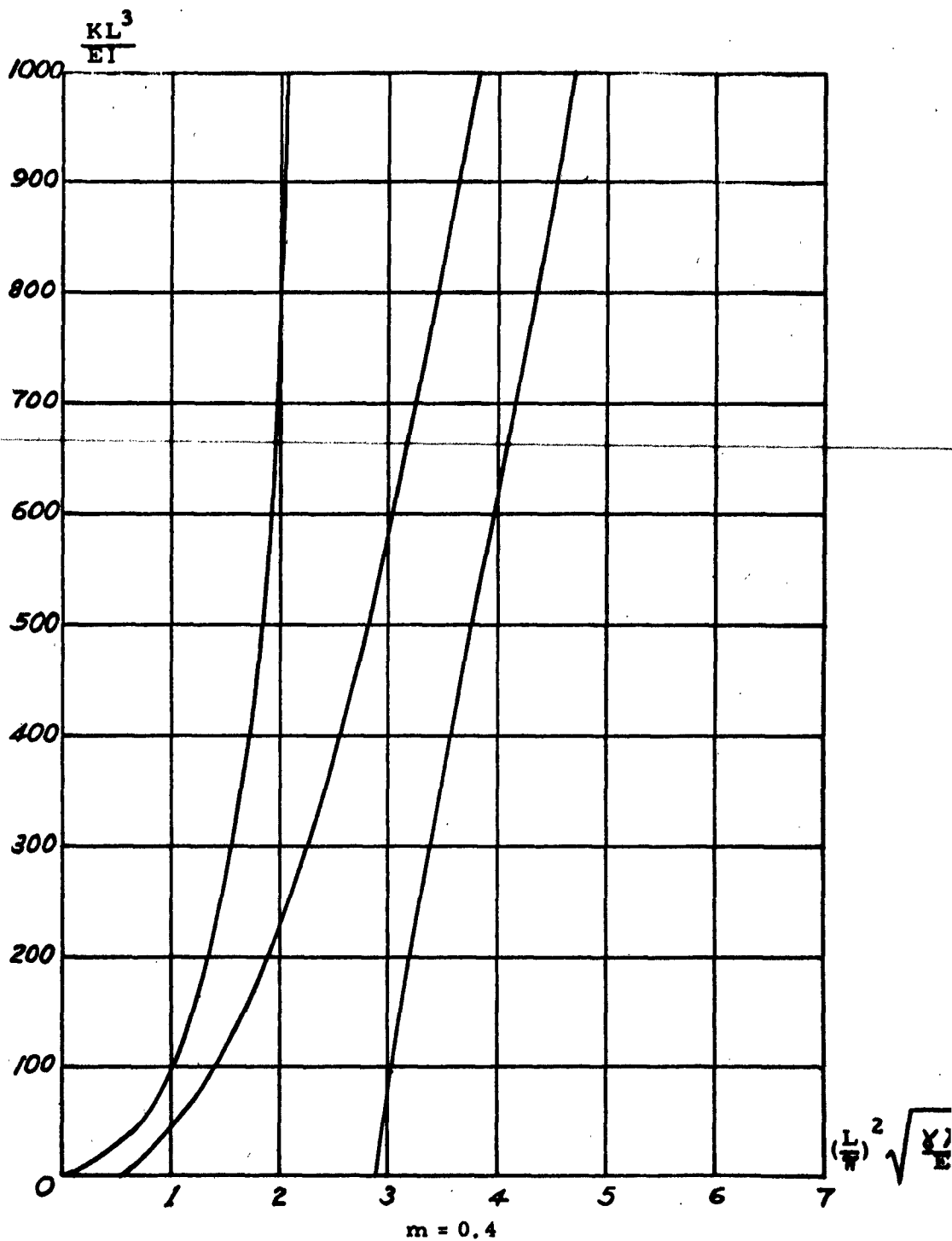


Fig. E14 NATURAL FREQUENCIES - CASE 1b

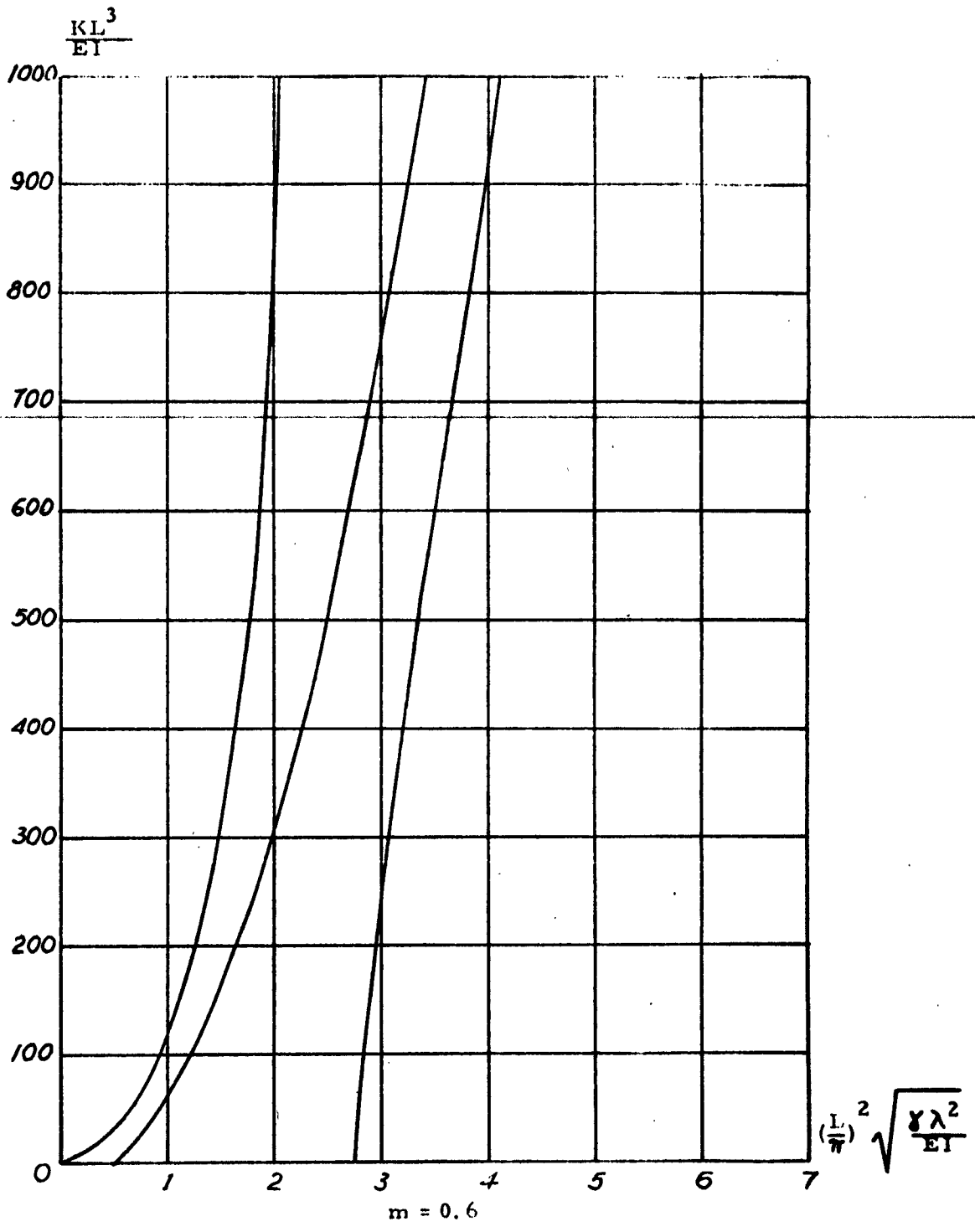


Fig. E15 NATURAL FREQUENCIES - CASE 1b

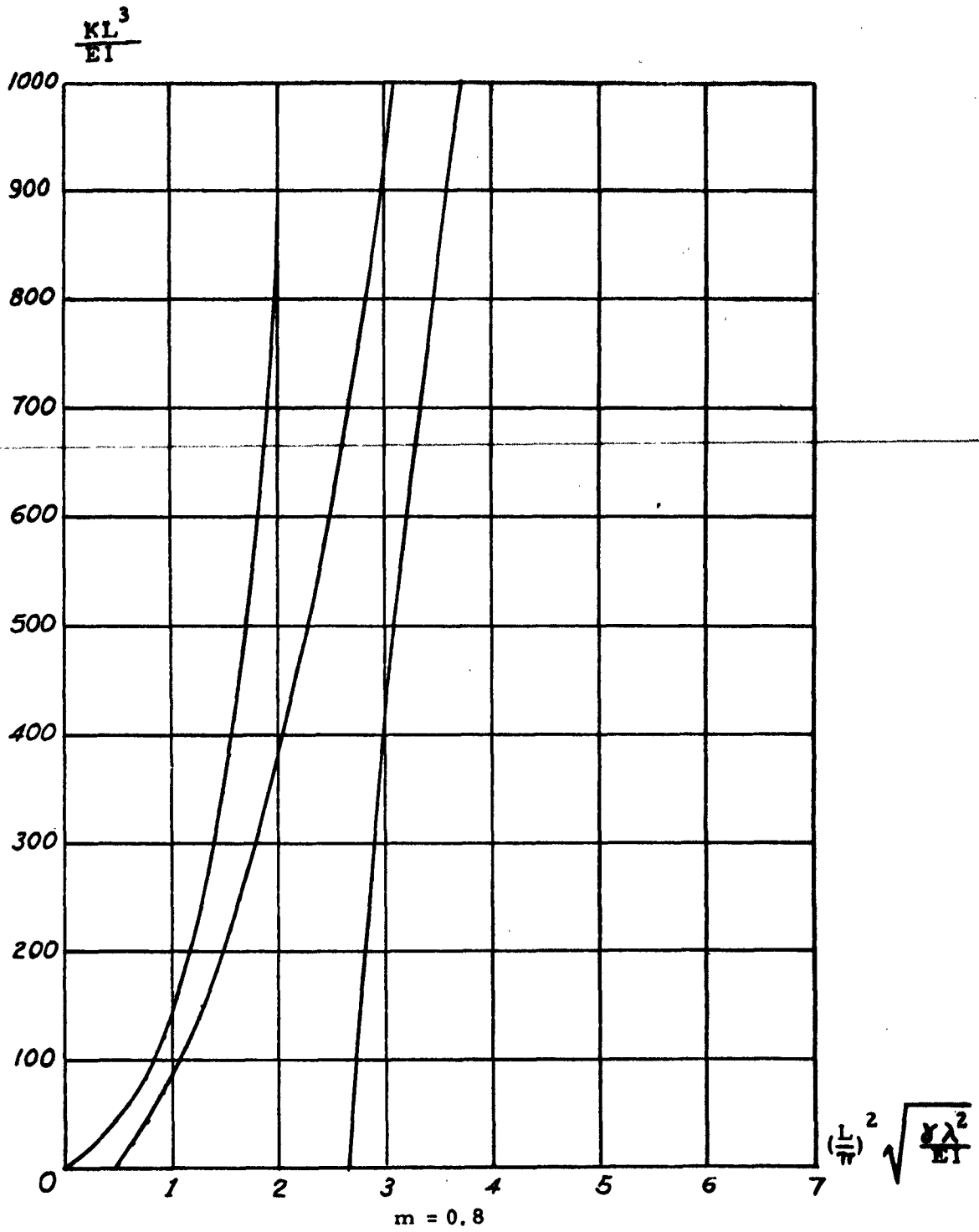


Fig. E16 NATURAL FREQUENCIES - CASE 1b

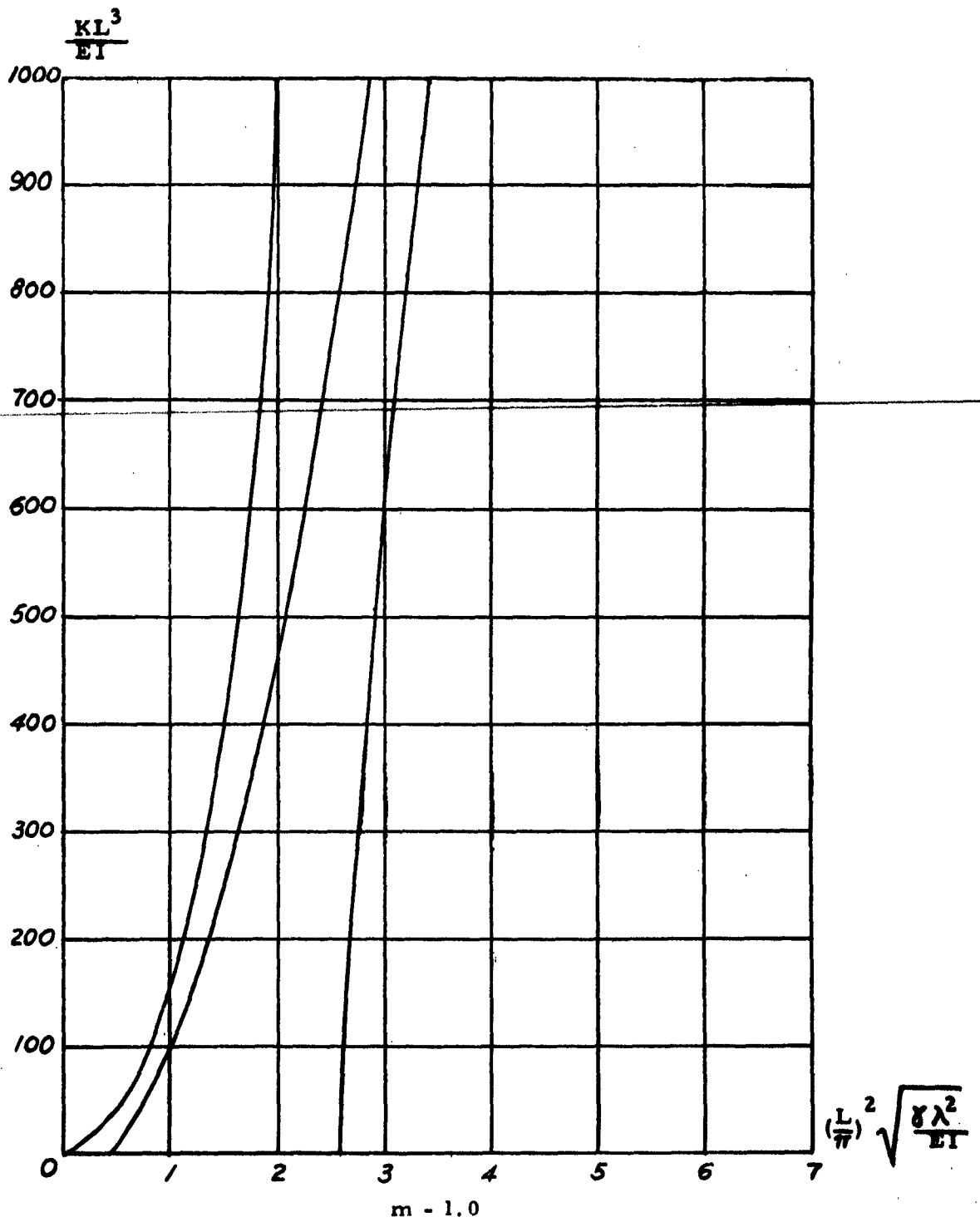


Fig. E17 NATURAL FREQUENCIES - CASE 1b

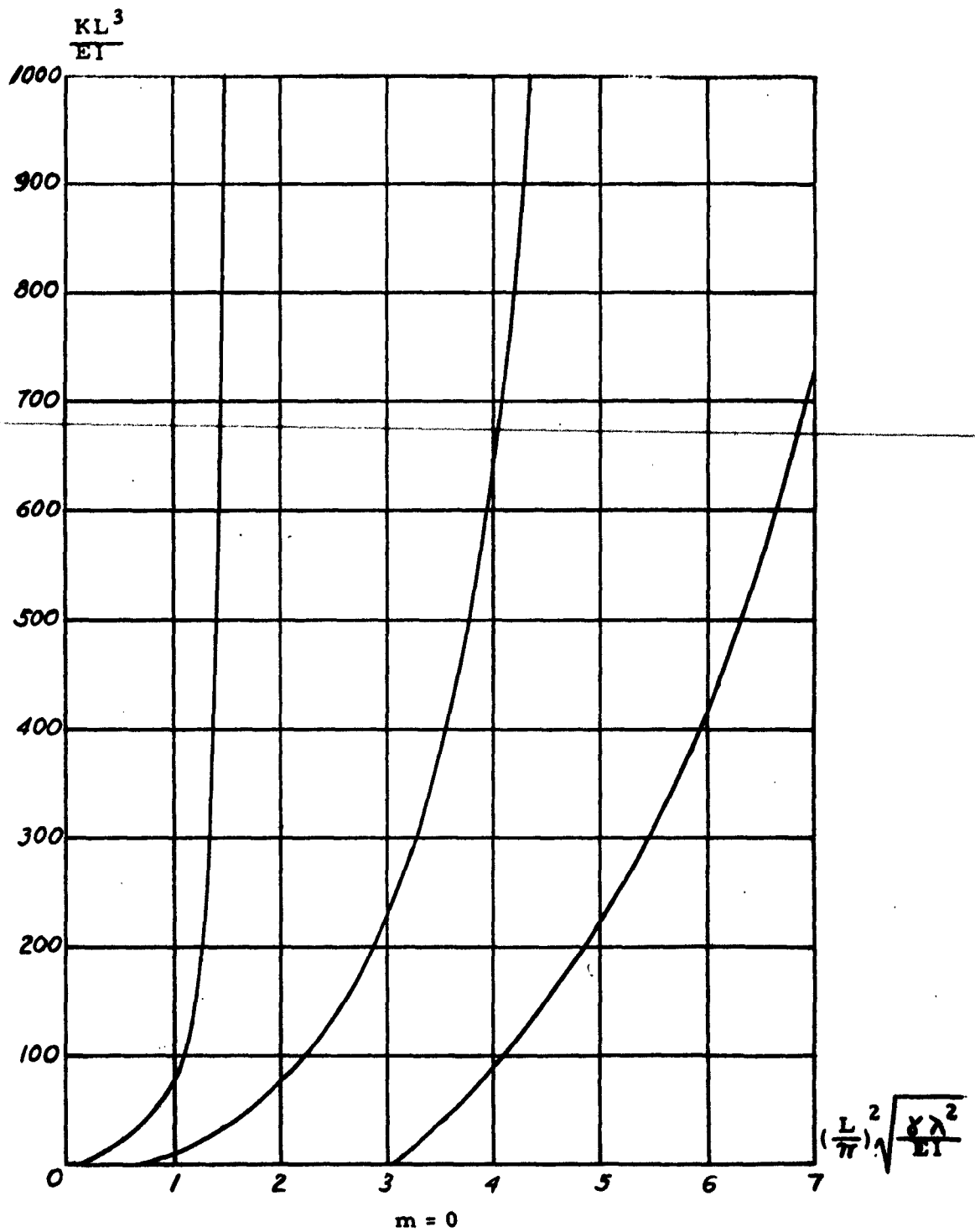


Fig. E18 NATURAL FREQUENCIES - CASE 1c



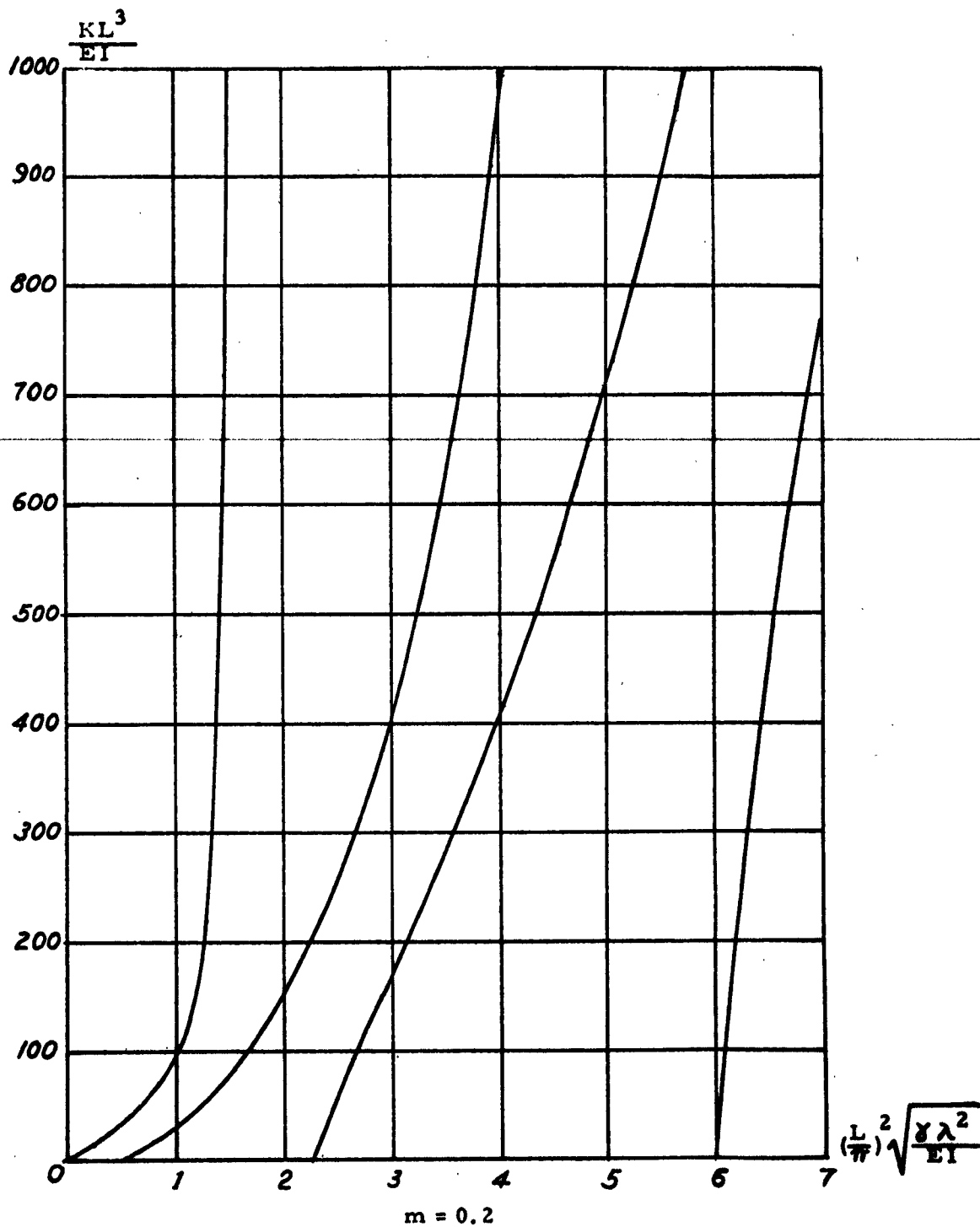


Fig. E19 NATURAL FREQUENCIES - CASE 1c

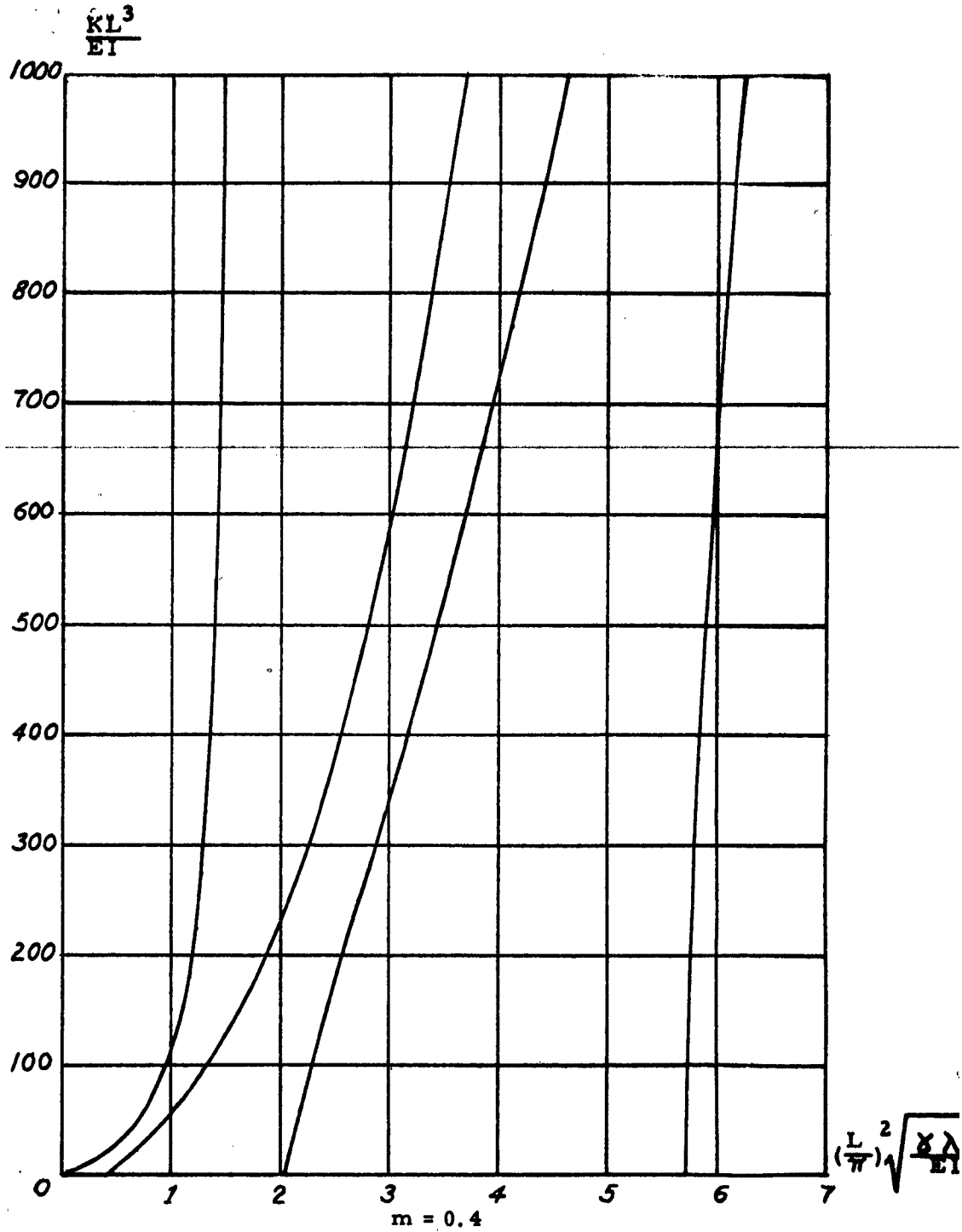


Fig. E20 NATURAL FREQUENCIES - CASE Ic

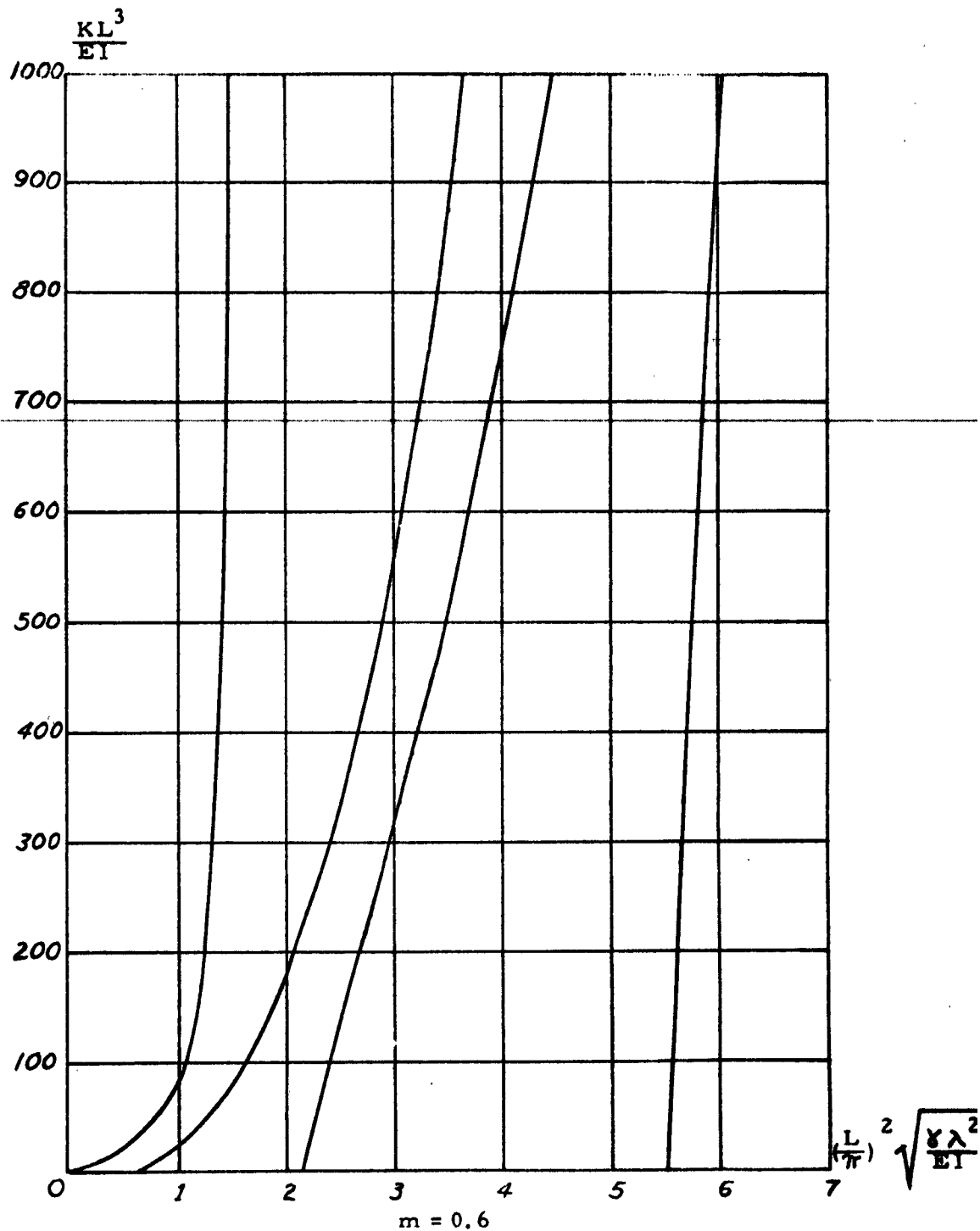


Fig. E21 NATURAL FREQUENCIES - CASE 1c

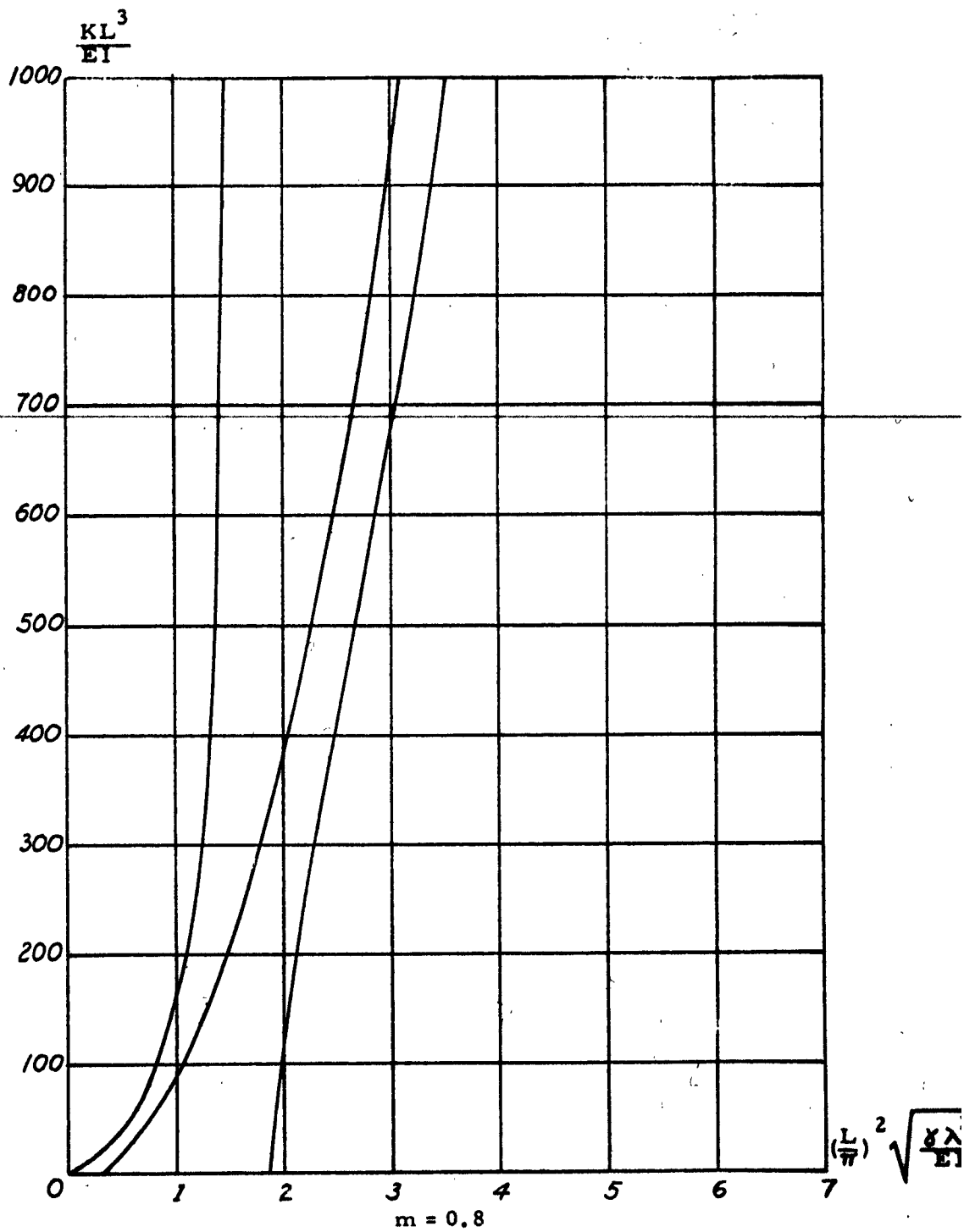


Fig. E22 NATURAL FREQUENCIES - CASE 1c

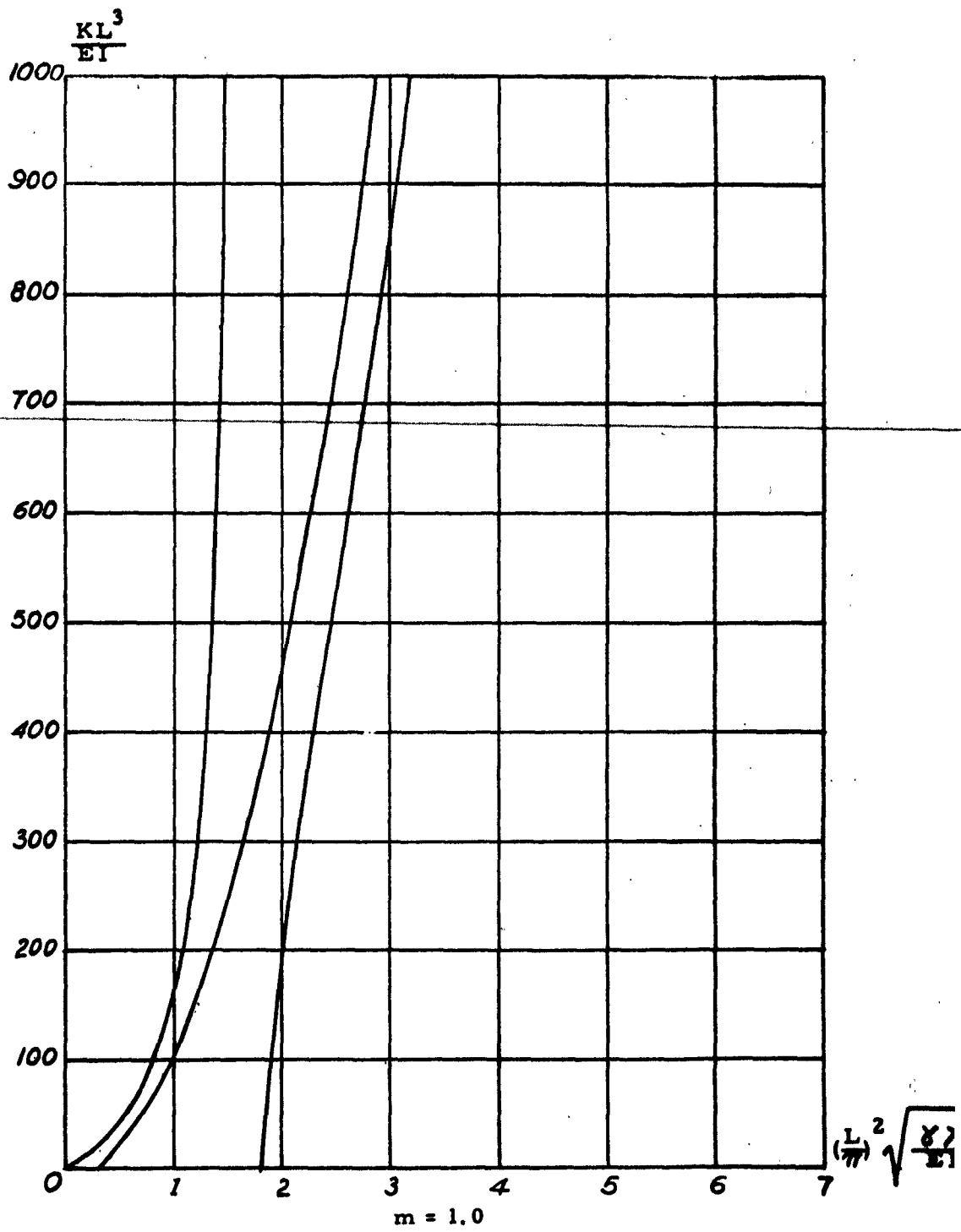


Fig. E23 NATURAL FREQUENCIES - CASE 1c

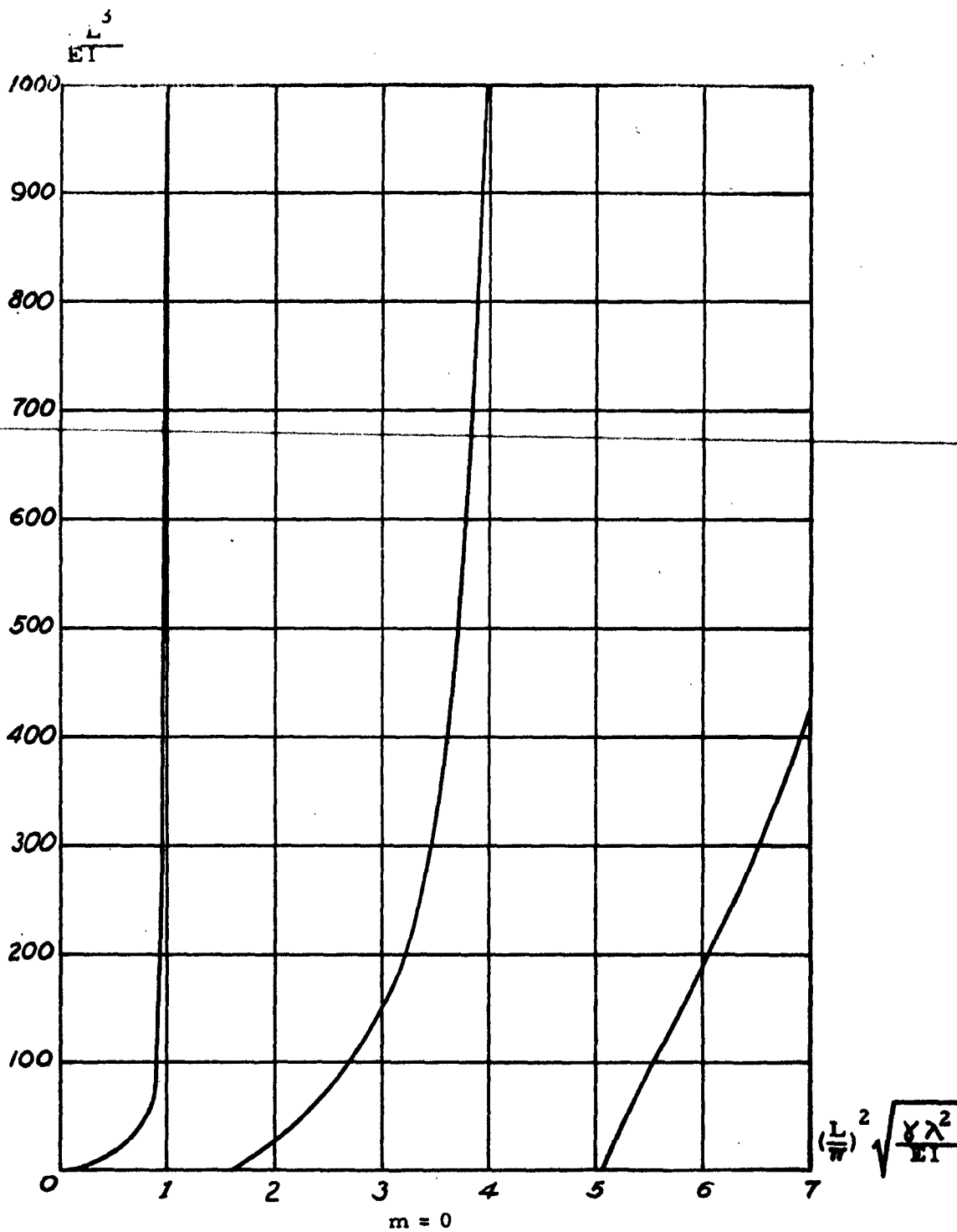


Fig. E24 NATURAL FREQUENCIES - CASE IIa

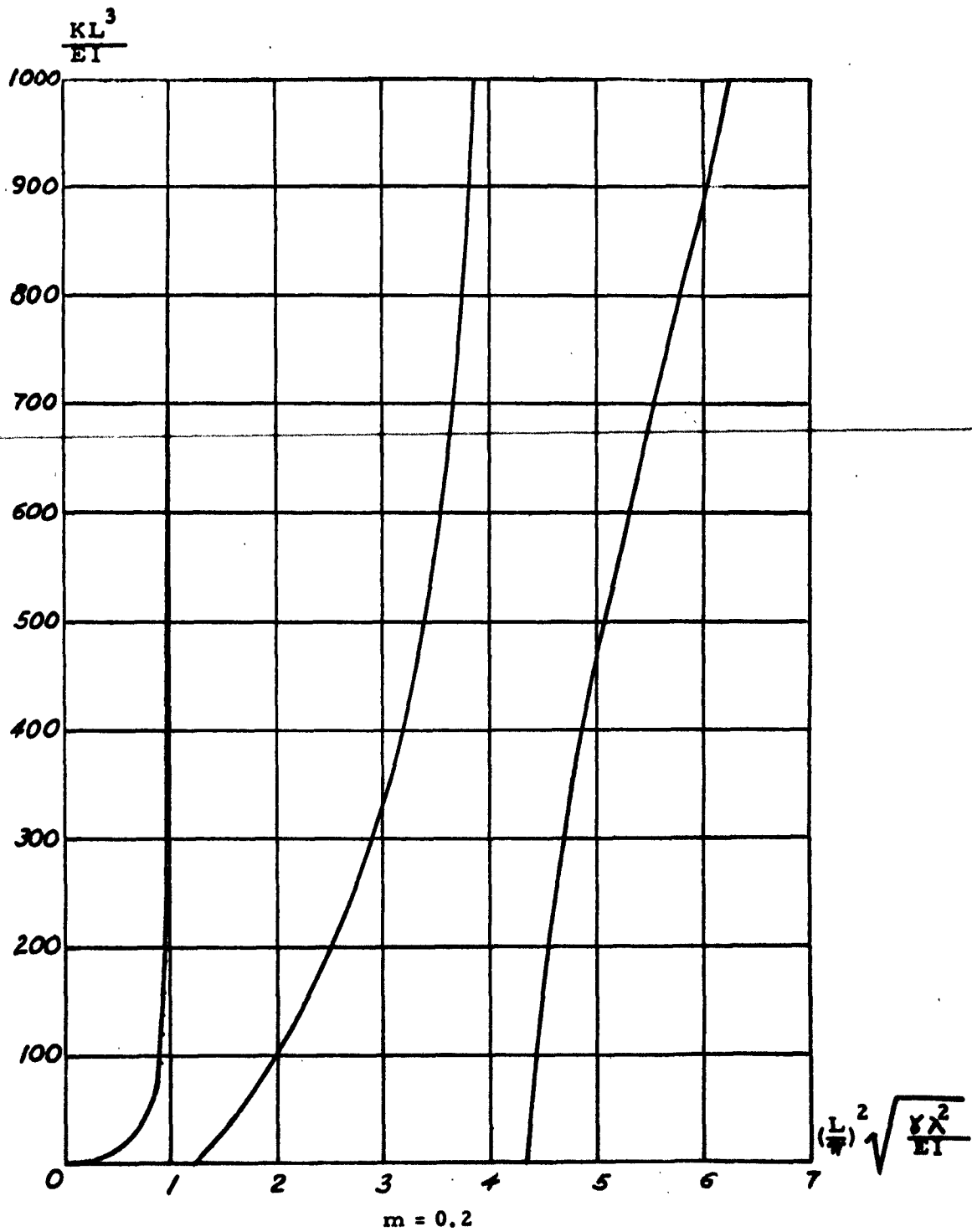


Fig. E25 NATURAL FREQUENCIES - CASE IIa

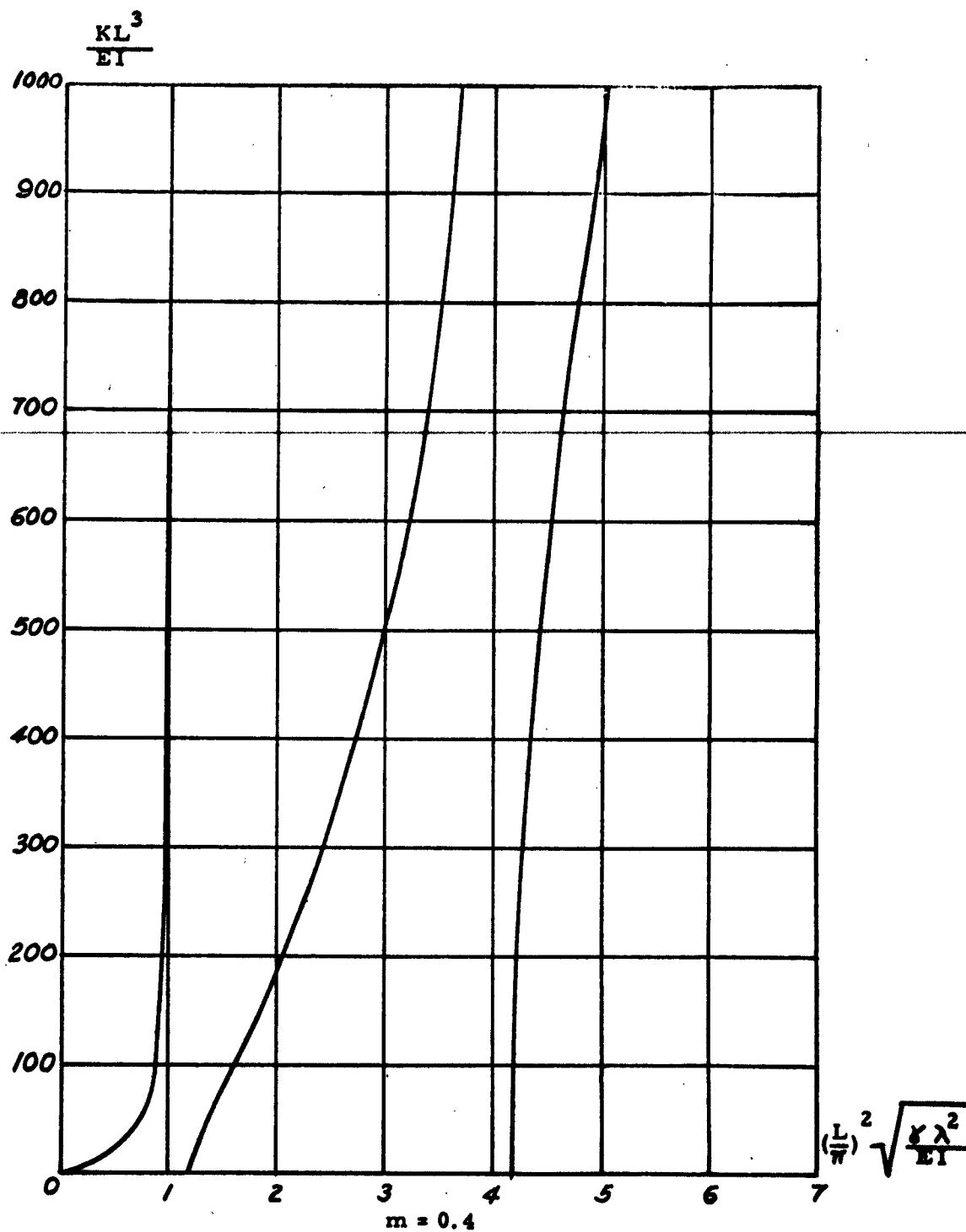


Fig. E26 NATURAL FREQUENCIES - CASE IIa



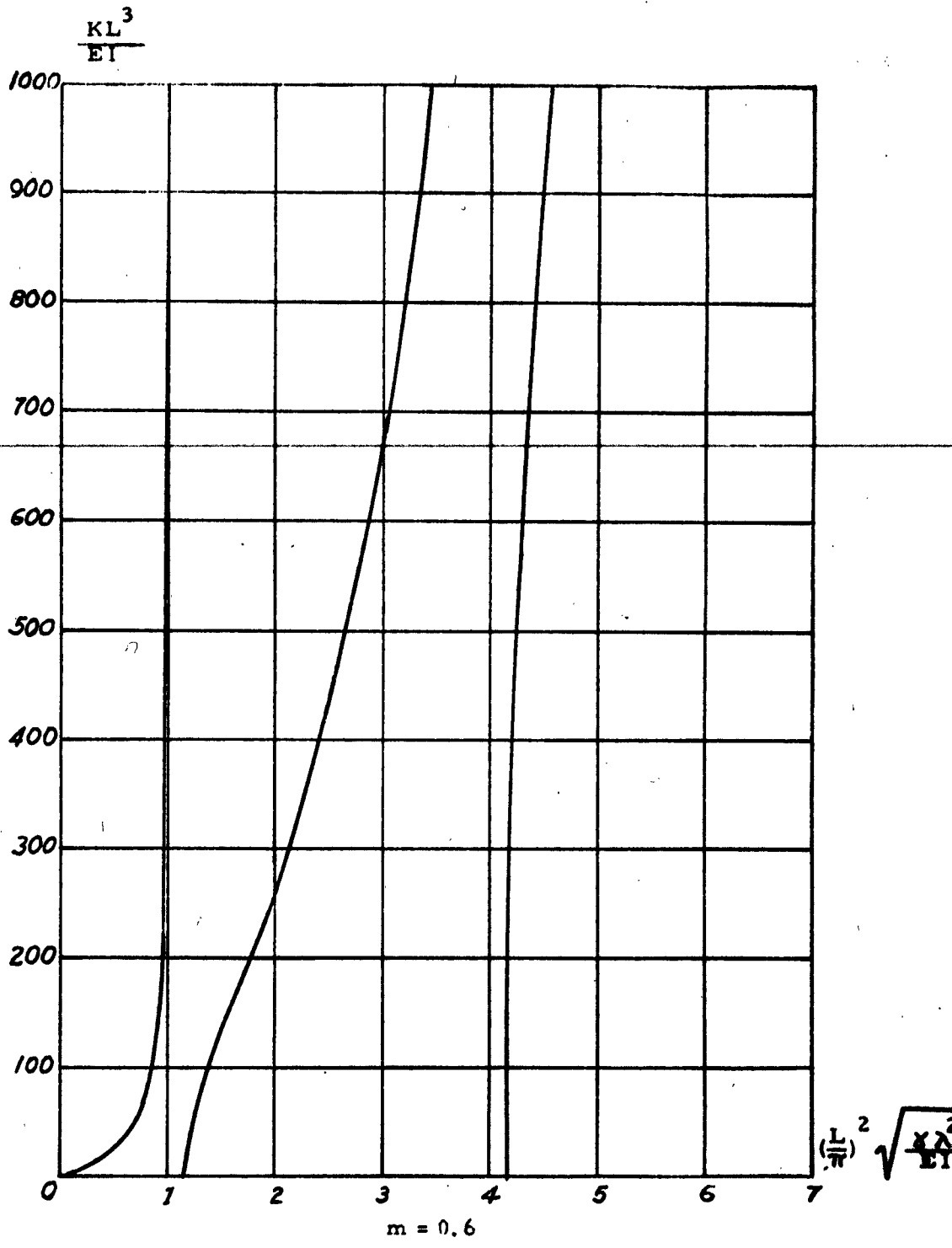


Fig. E27 NATURAL FREQUENCIES - CASE IIa

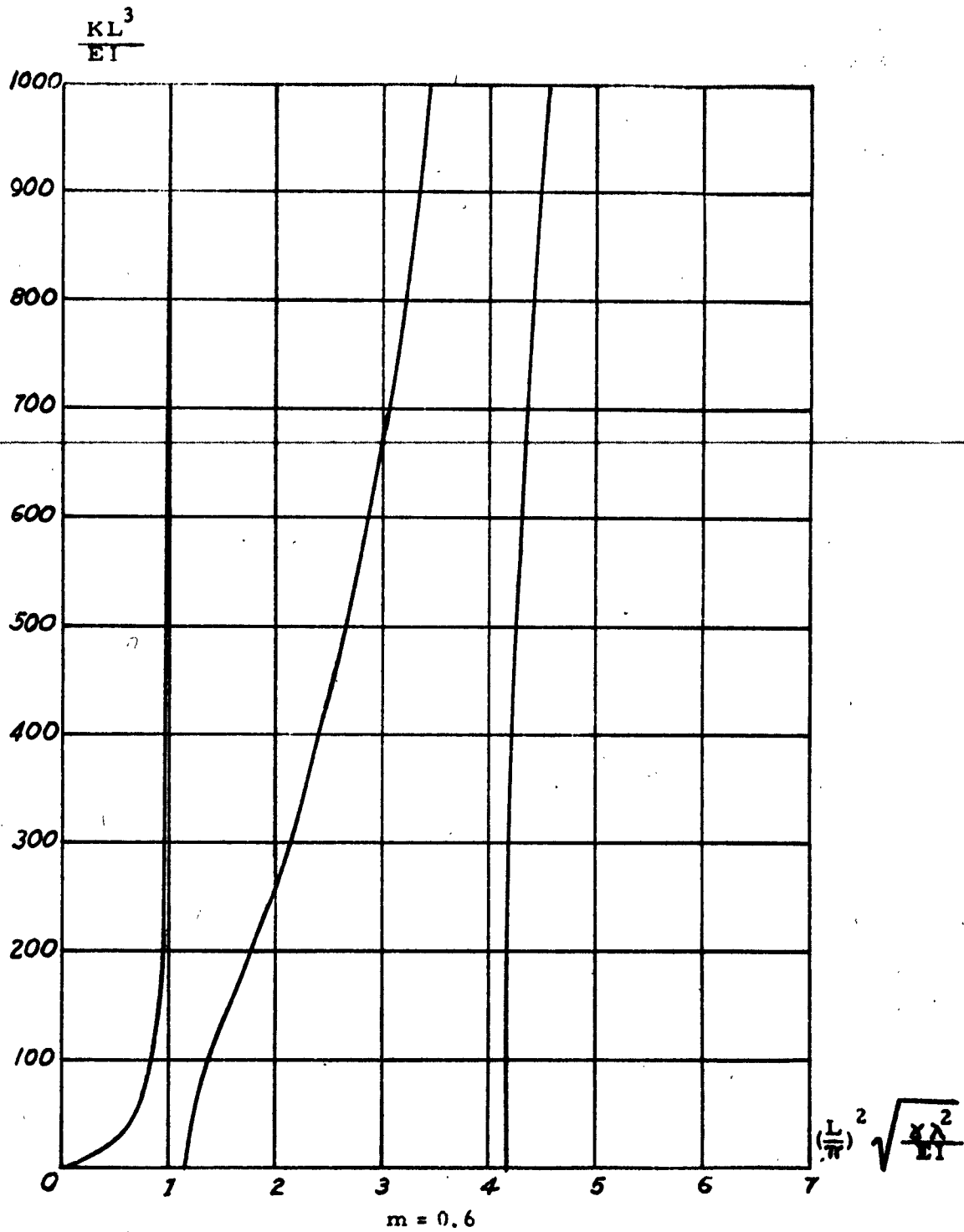


Fig. E27 NATURAL FREQUENCIES - CASE IIa

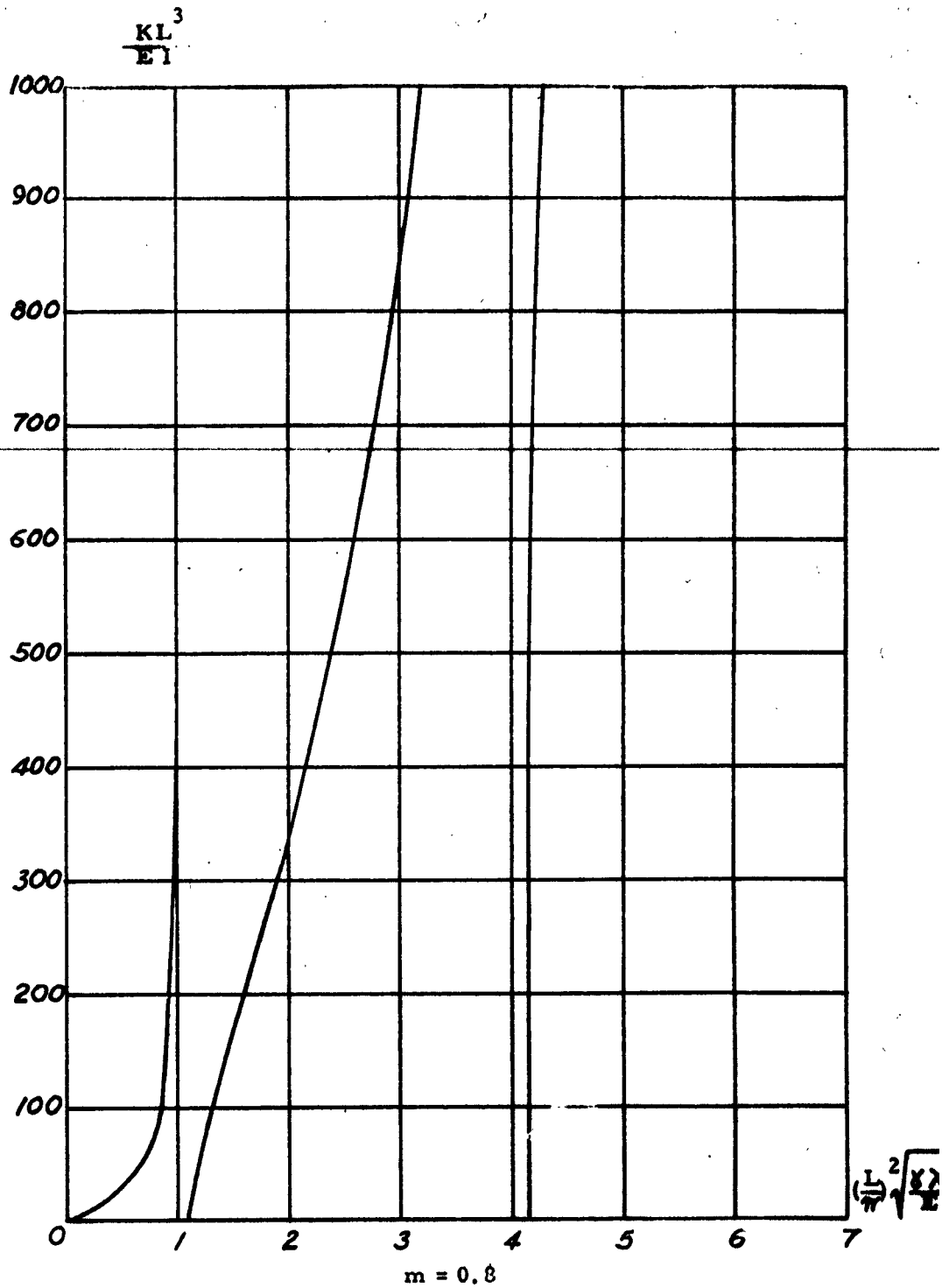


Fig. E28 NATURAL FREQUENCIES - CASE IIa

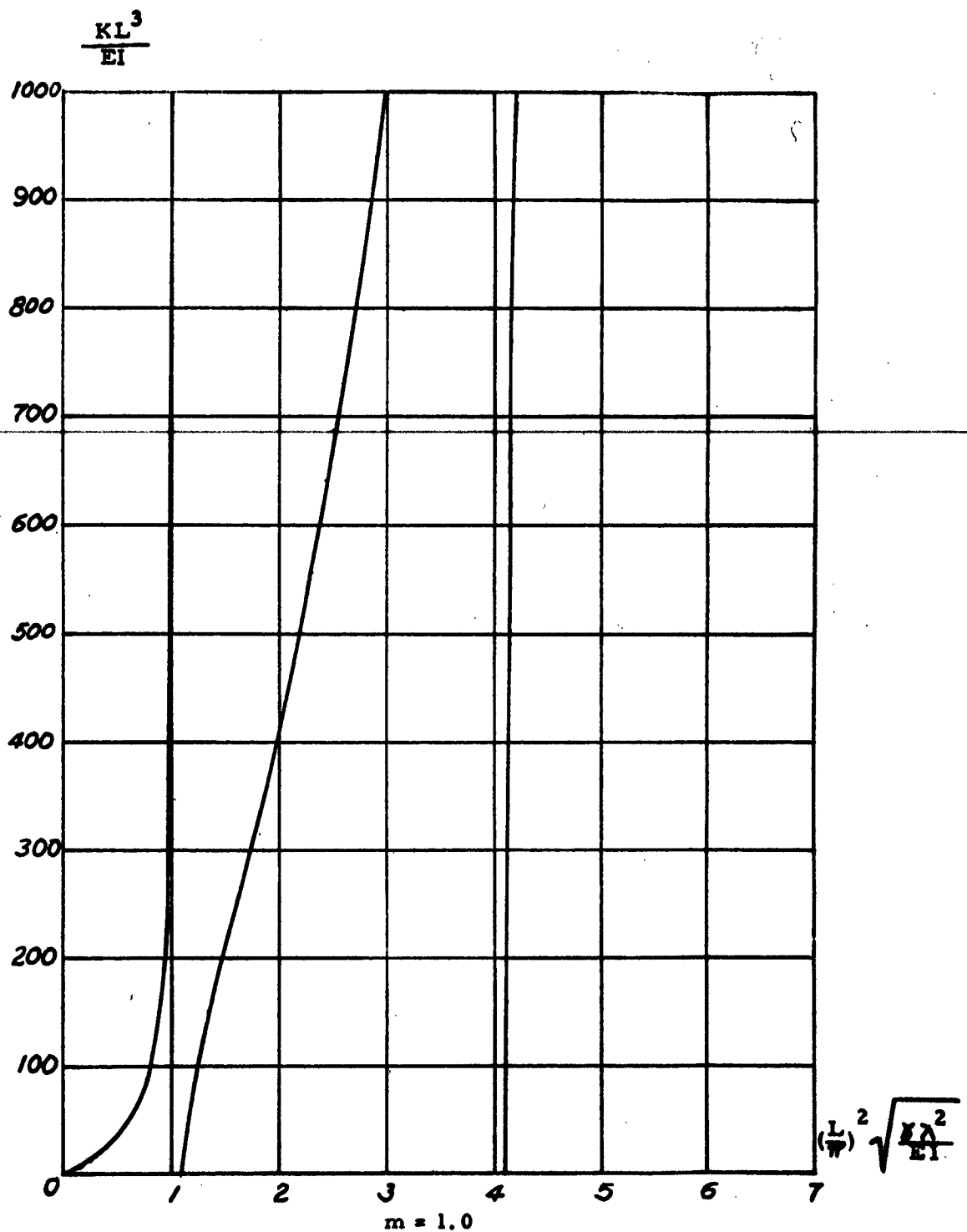


Fig. E29 NATURAL FREQUENCIES - CASE IIa

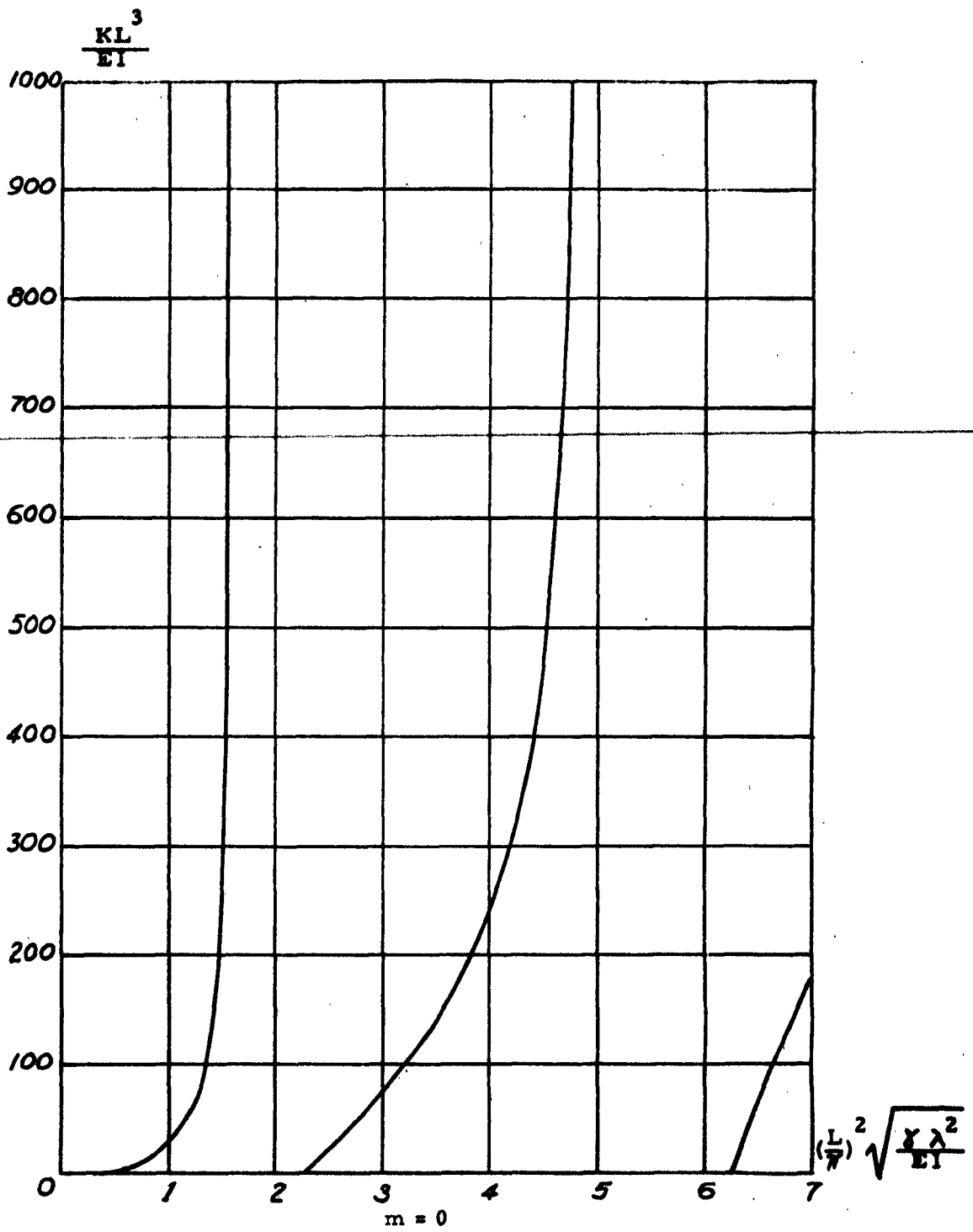


Fig. E30 NATURAL FREQUENCIES - CASE IIb

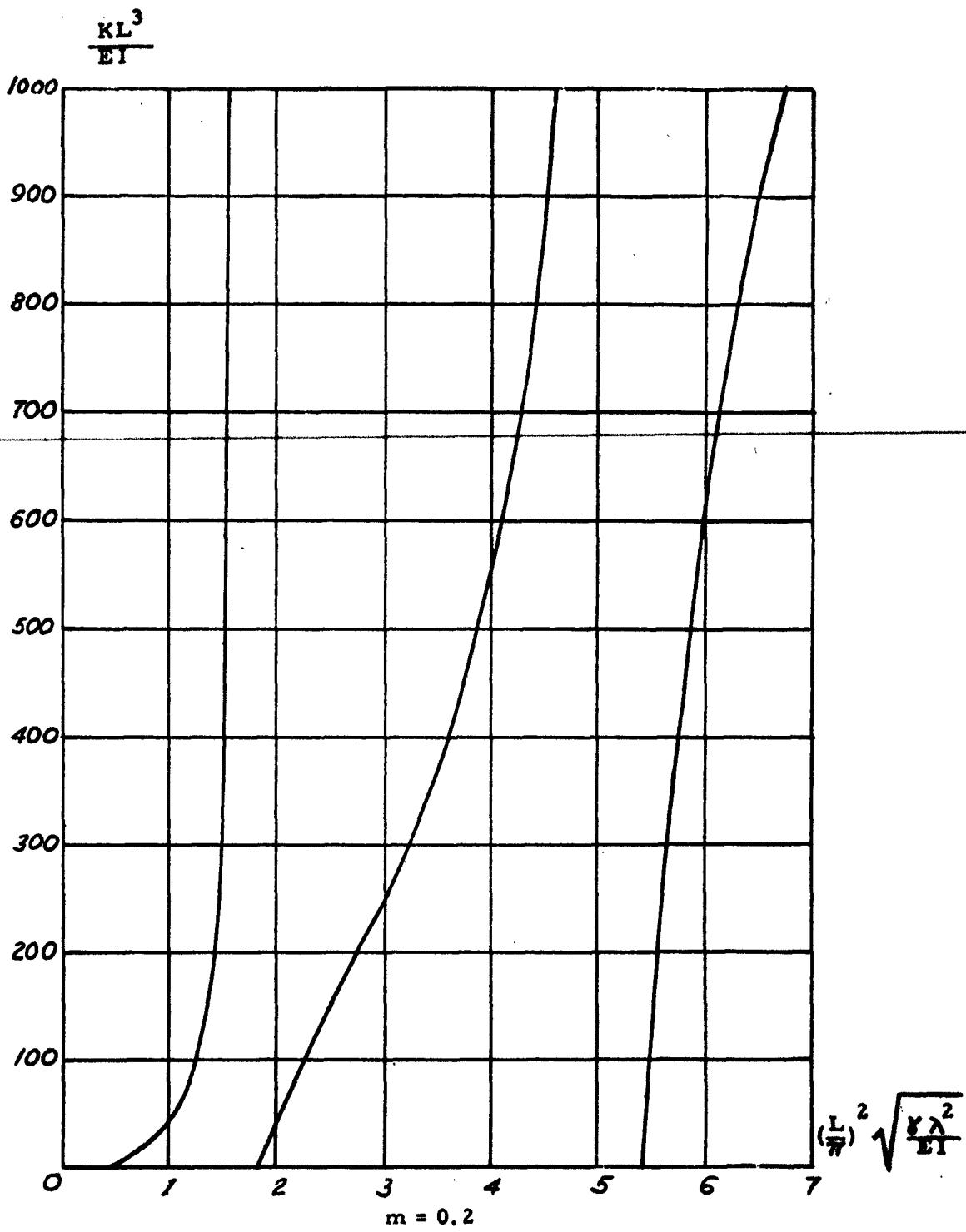


Fig. E31 NATURAL FREQUENCIES - CASE IIb

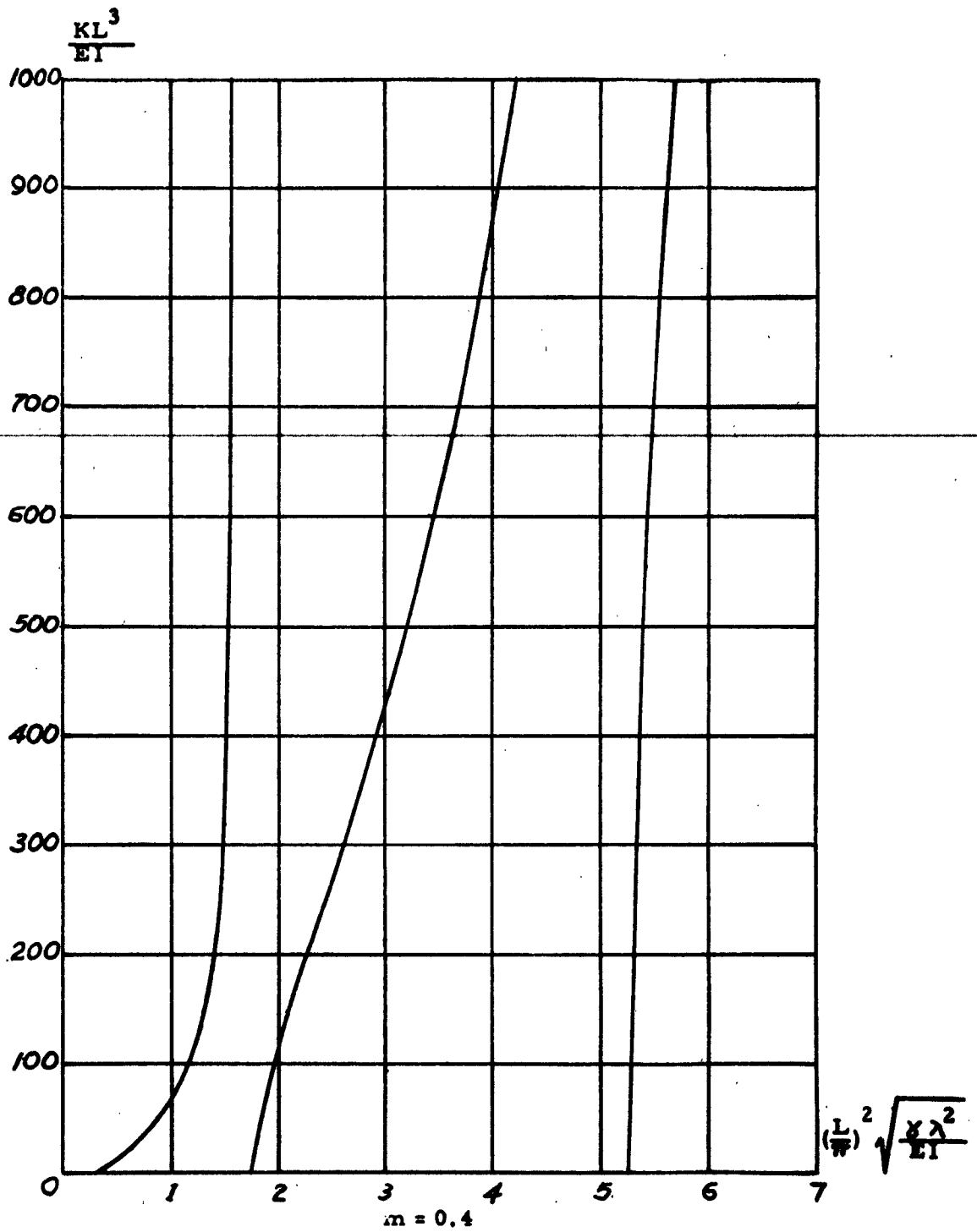


Fig. E32 NATURAL FREQUENCIES - CASE IIb

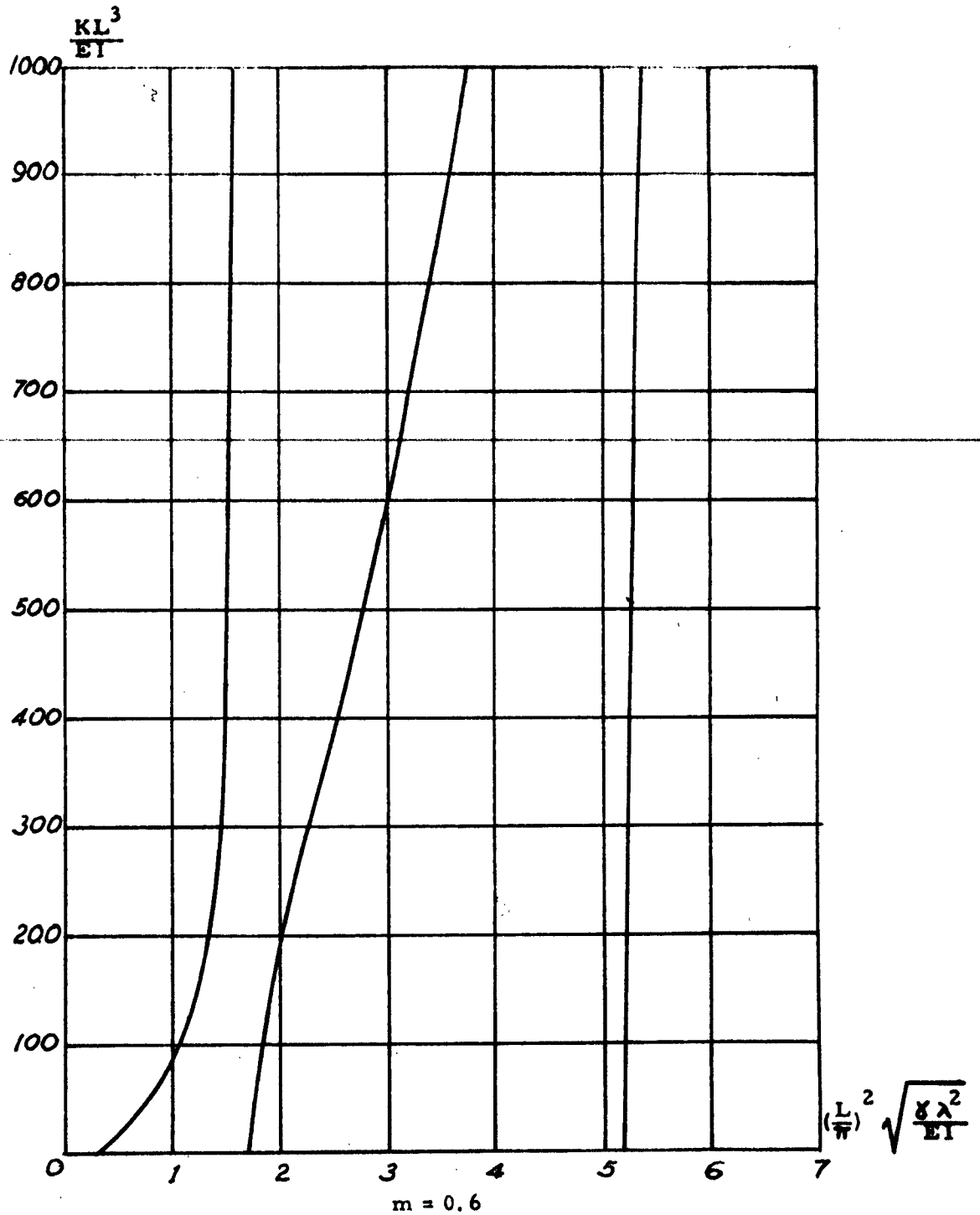


Fig. E33 NATURAL FREQUENCIES - CASE IIb



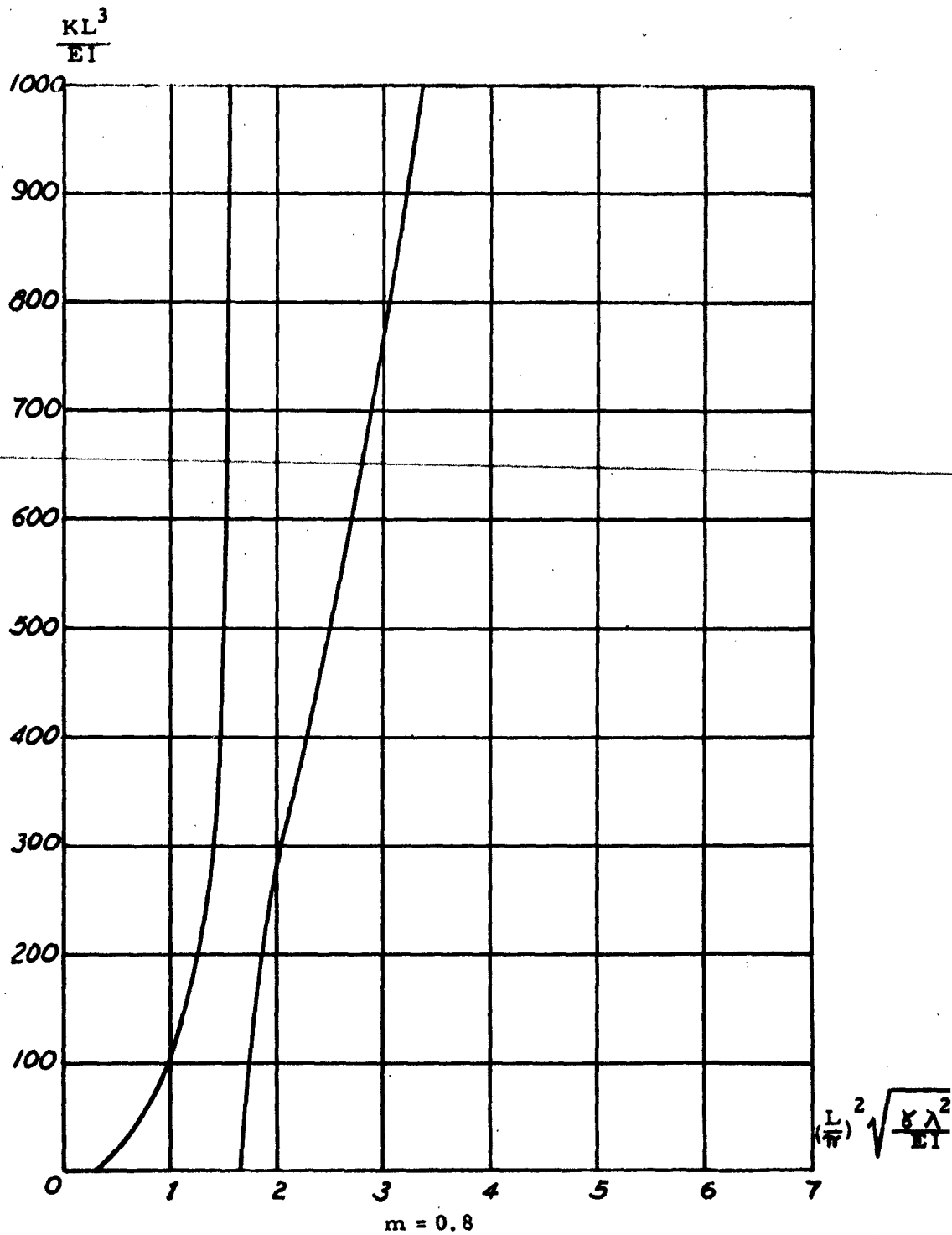


Fig. E34 NATURAL FREQUENCIES - CASE IIb

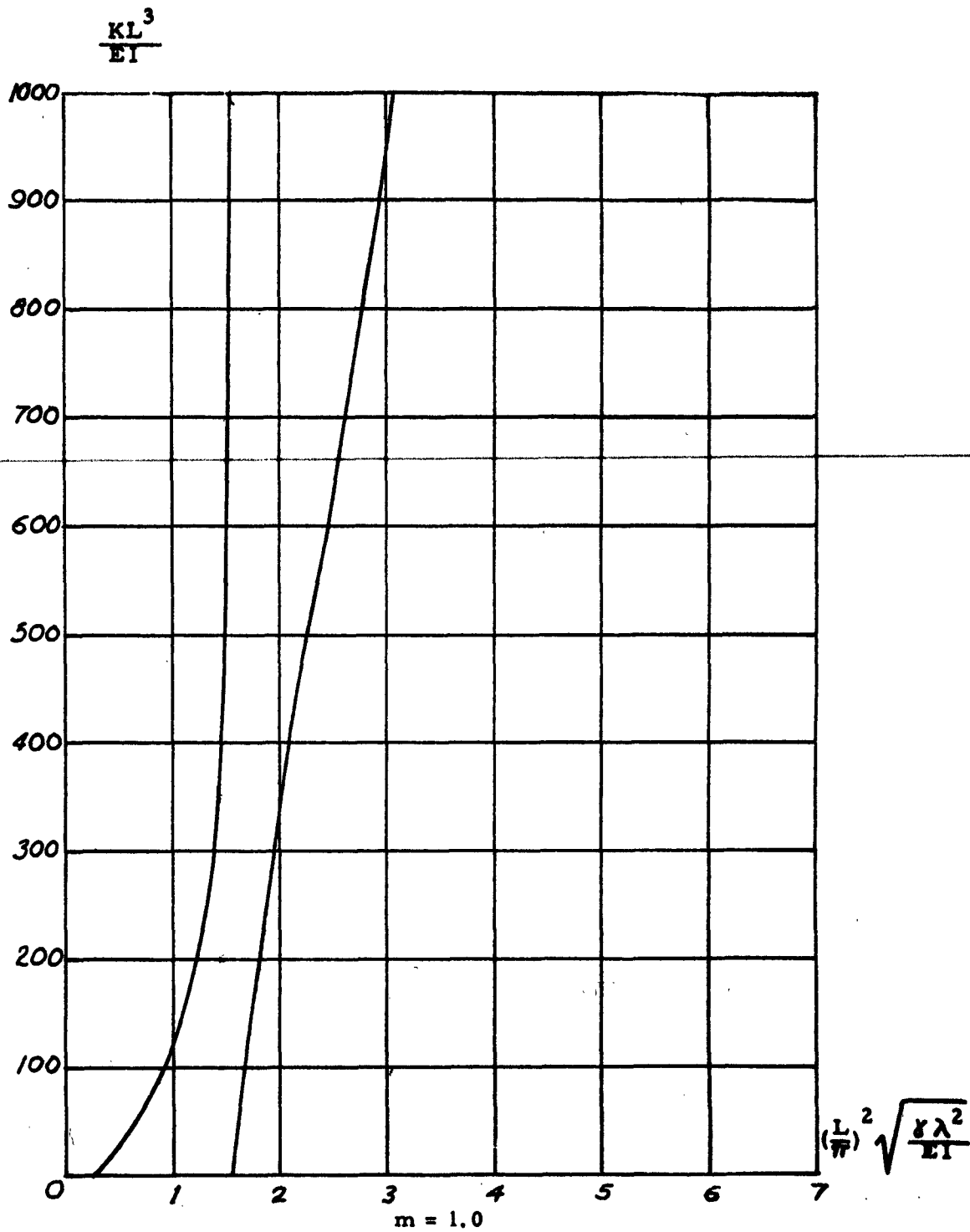


Fig. E35 NATURAL FREQUENCIES - CASE IIb

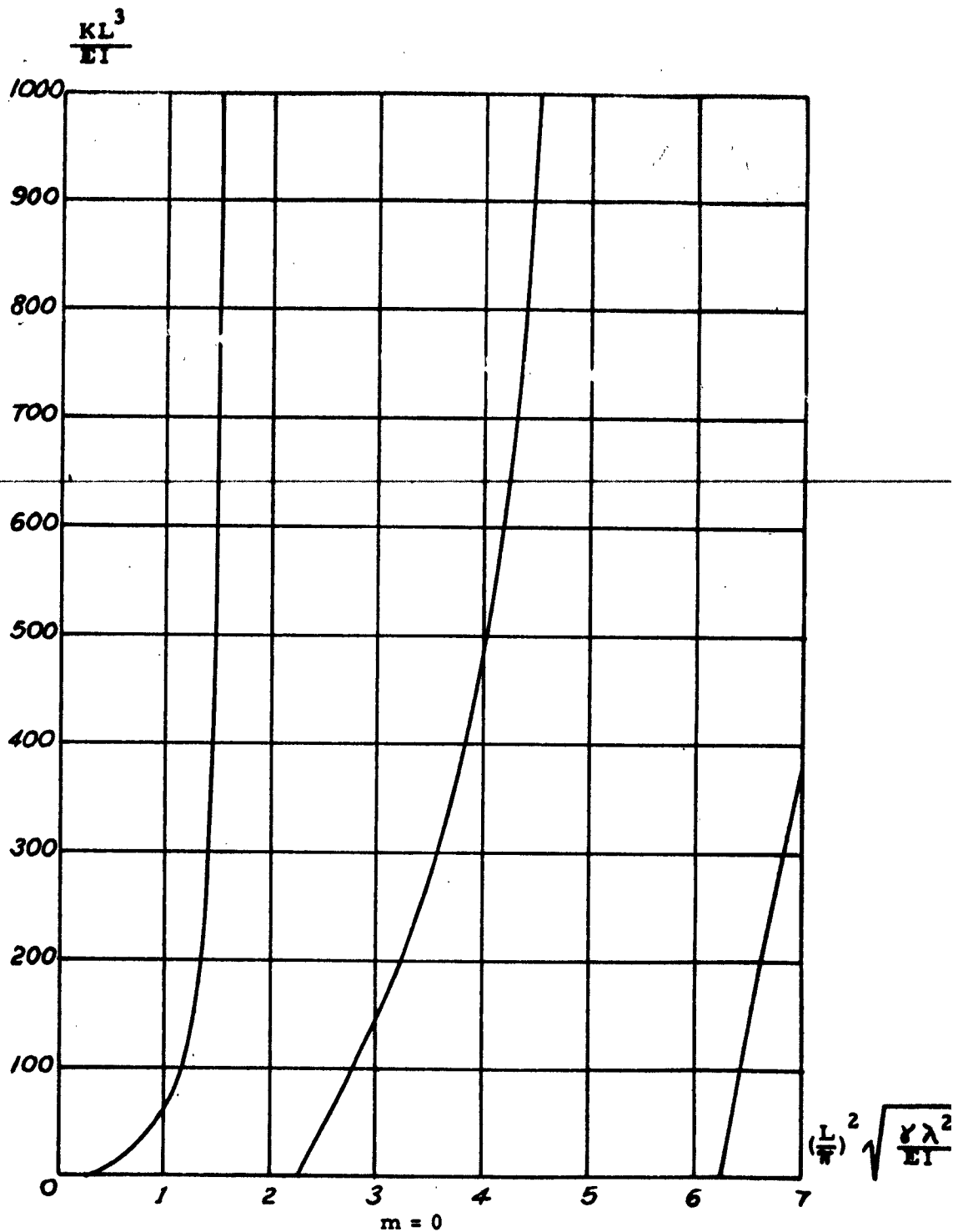


Fig. E36 NATURAL FREQUENCIES - CASE IIc

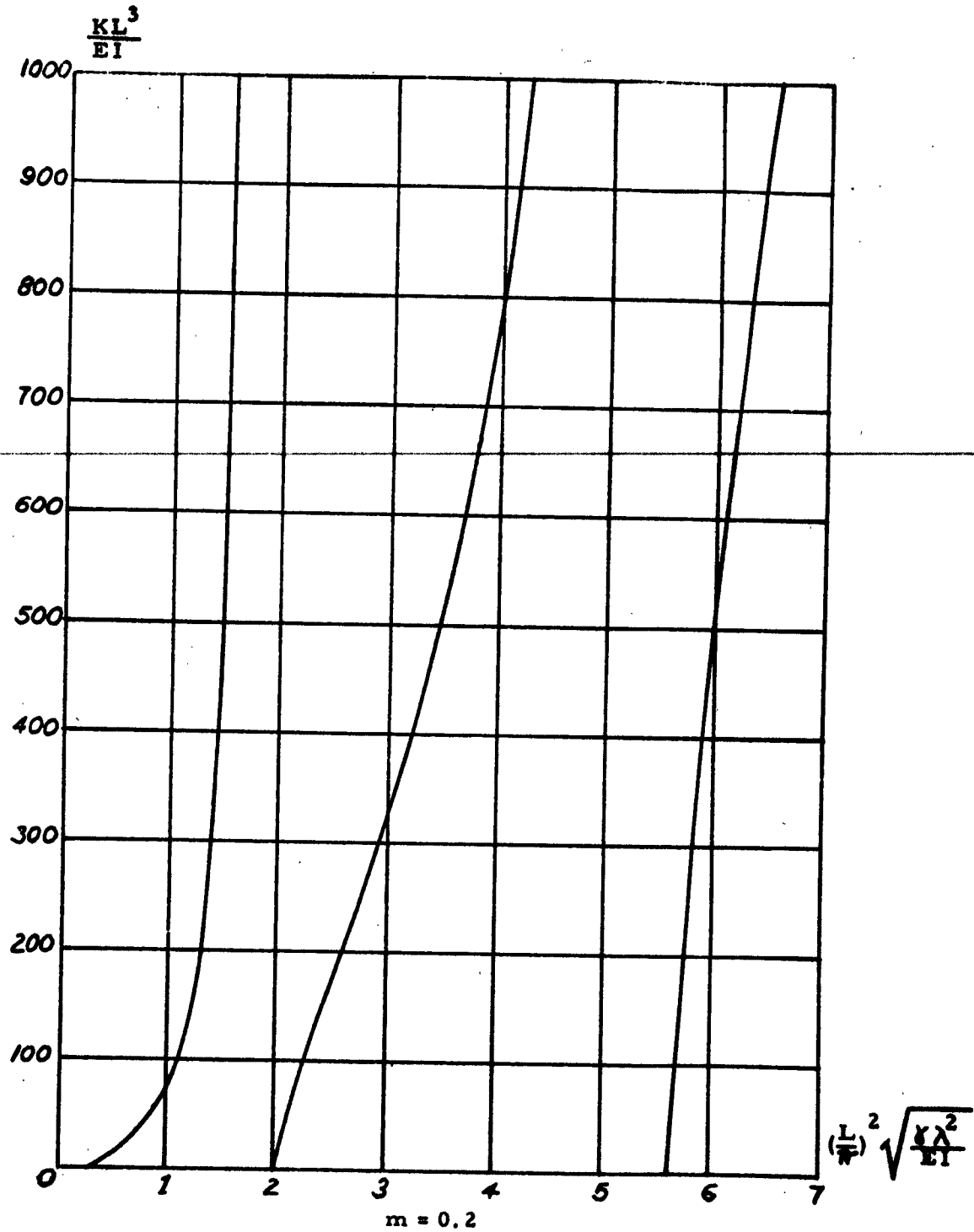


Fig. E37 NATURAL FREQUENCIES - CASE IIc

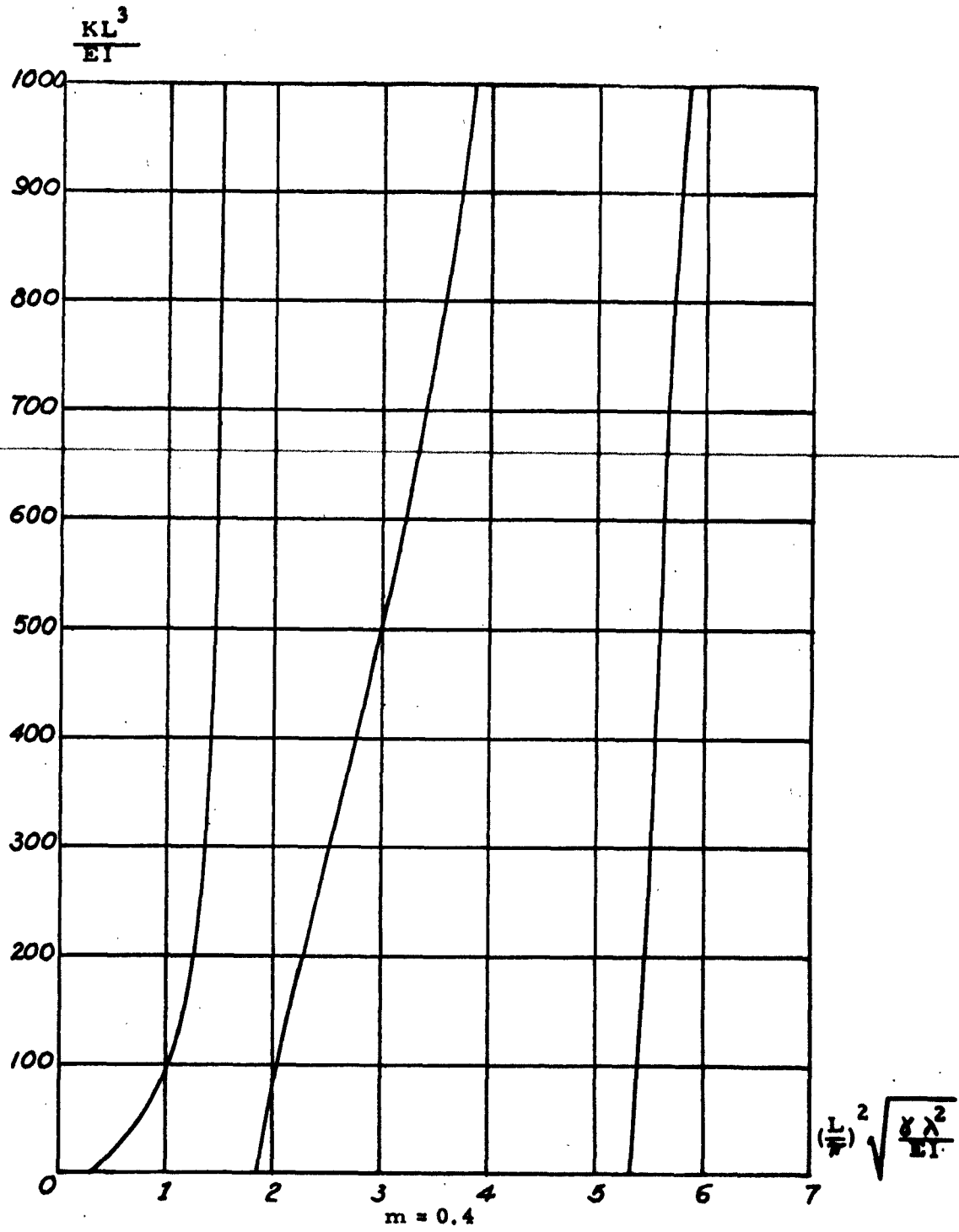


Fig. E38 NATURAL FREQUENCIES - CASE IIc

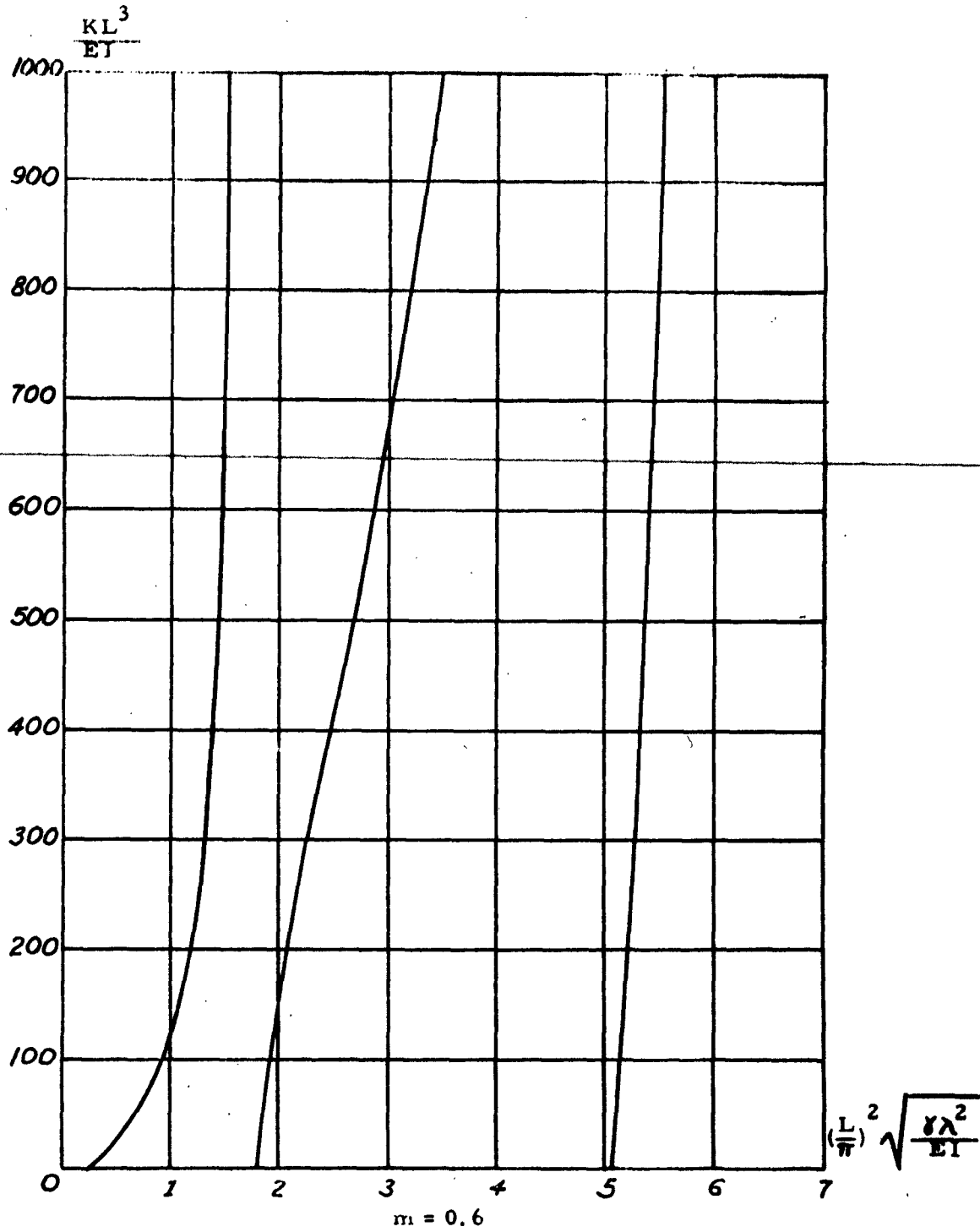


Fig. E39 NATURAL FREQUENCIES - CASE 11c

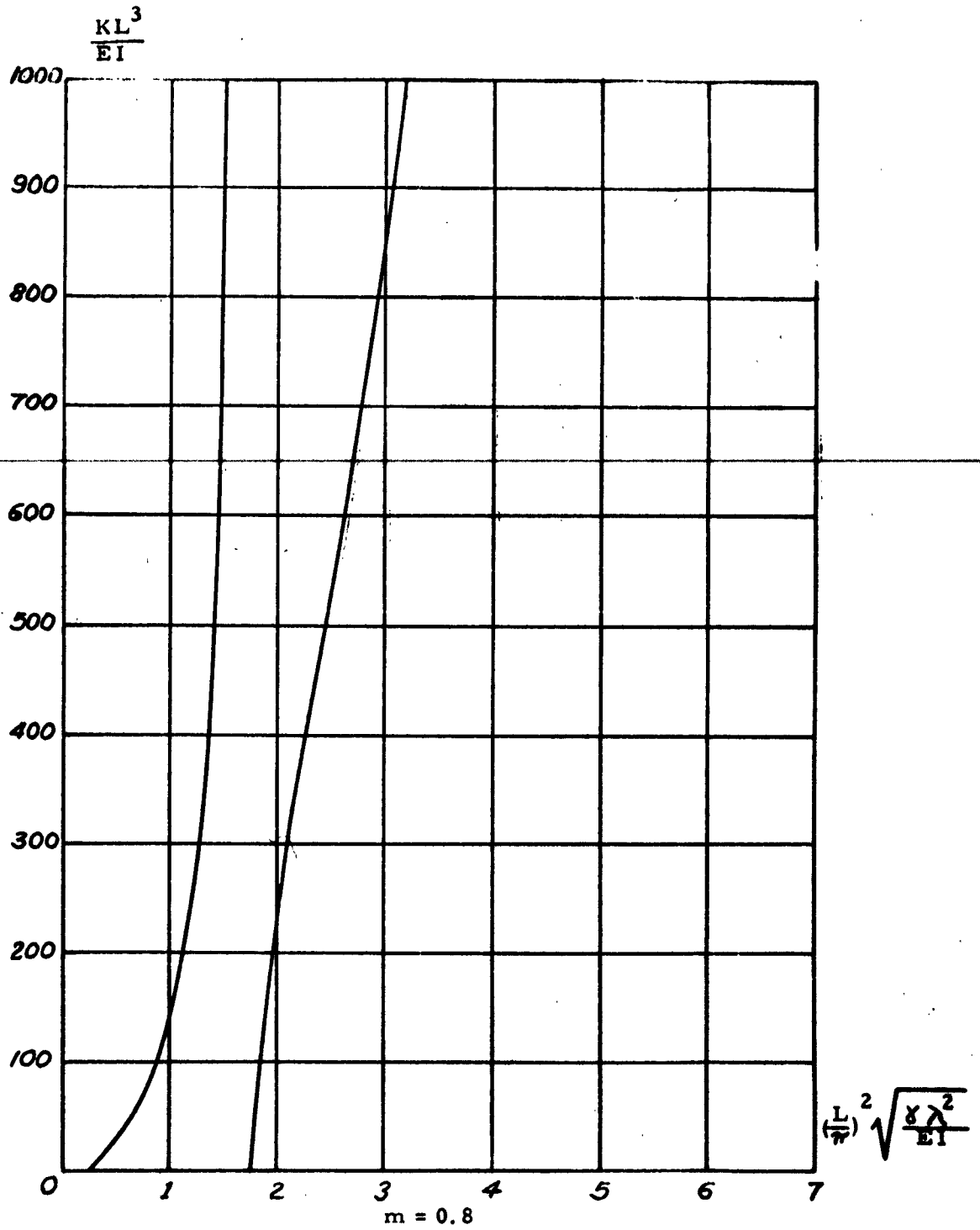


Fig. E40 NATURAL FREQUENCIES - CASE IIc

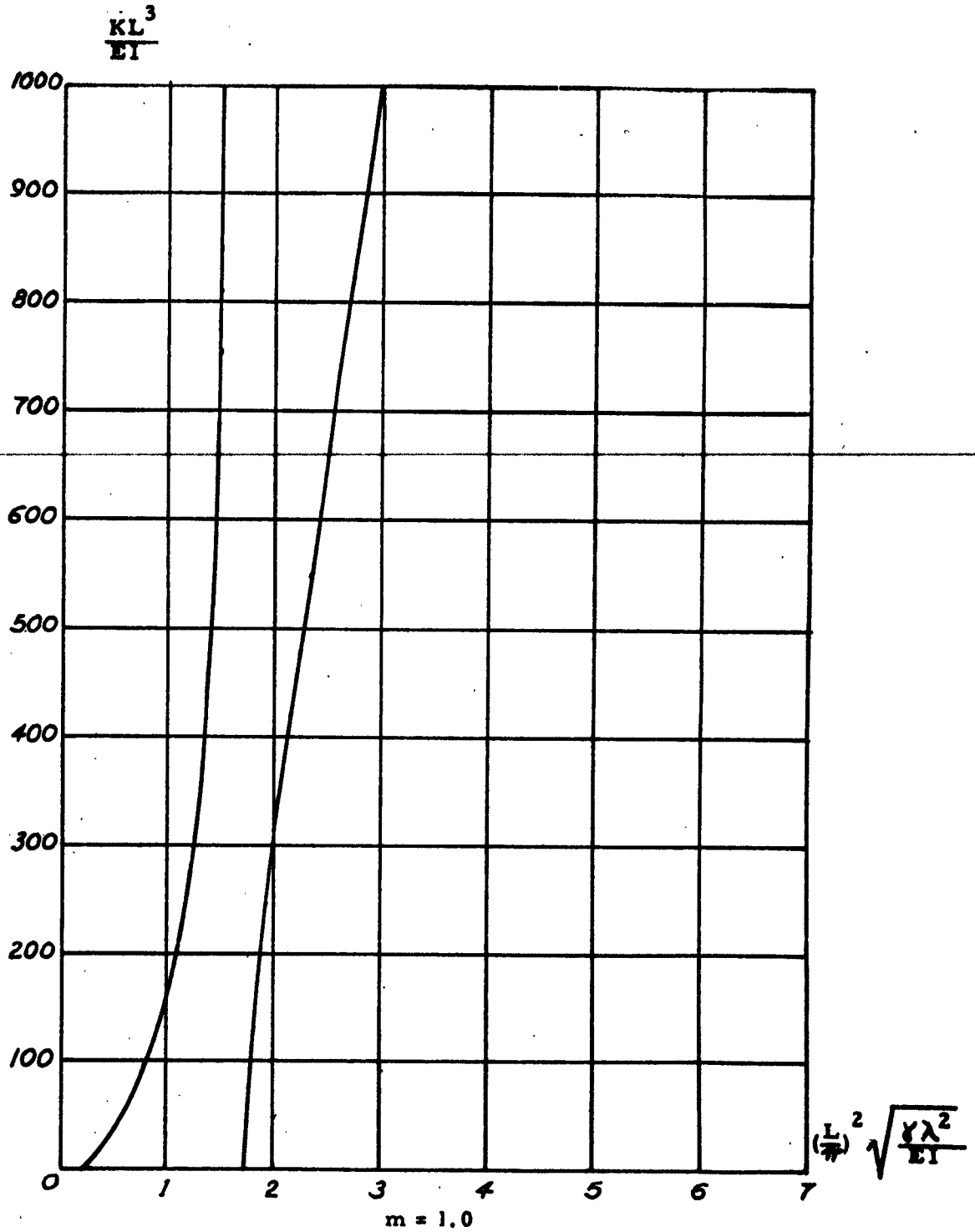


Fig. E41 NATURAL FREQUENCIES - CASE IIc



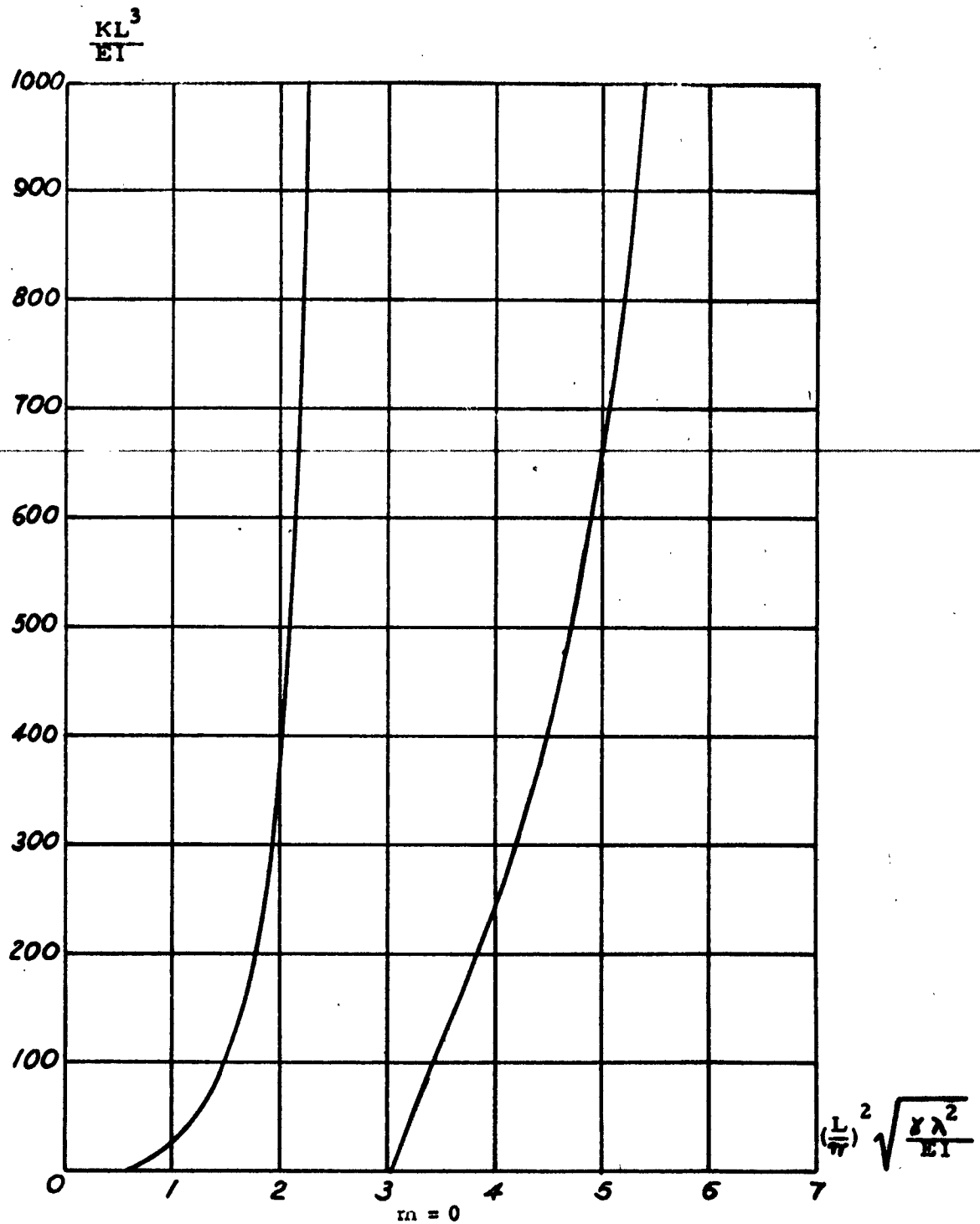


Fig. E42 NATURAL FREQUENCIES - CASE IIa

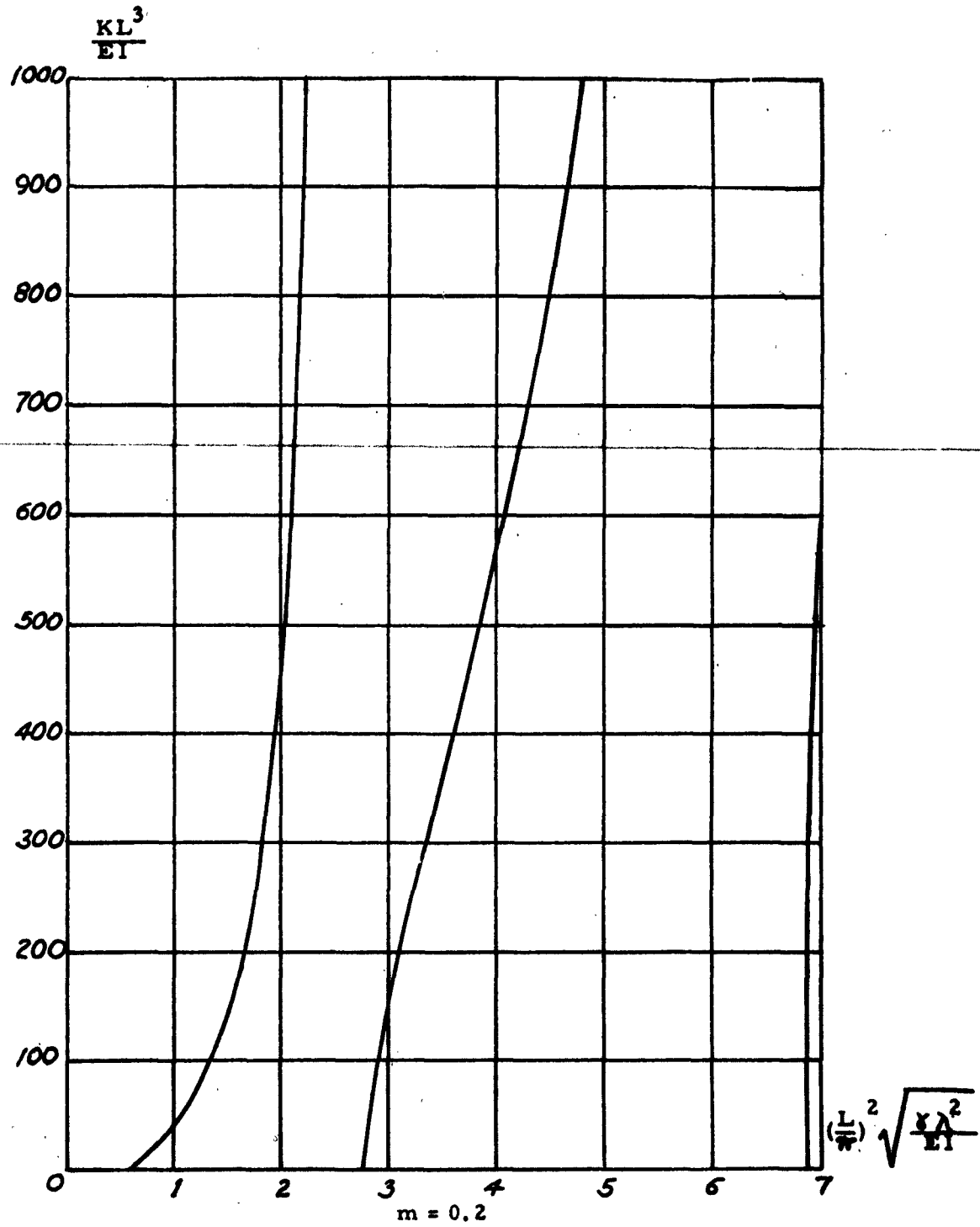


Fig. E43 NATURAL FREQUENCIES - CASE IIa

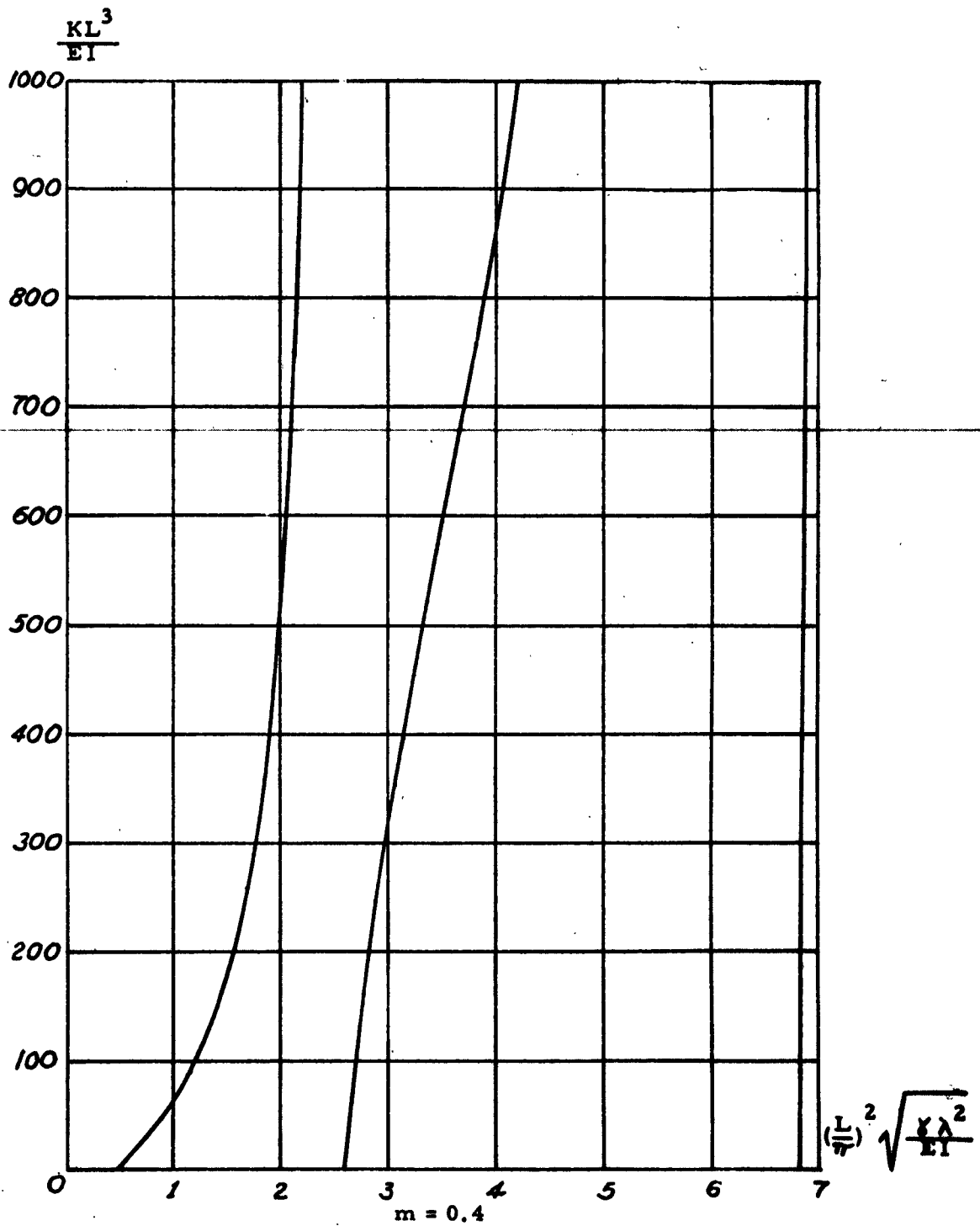


Fig. E44 NATURAL FREQUENCIES - CASE IIId

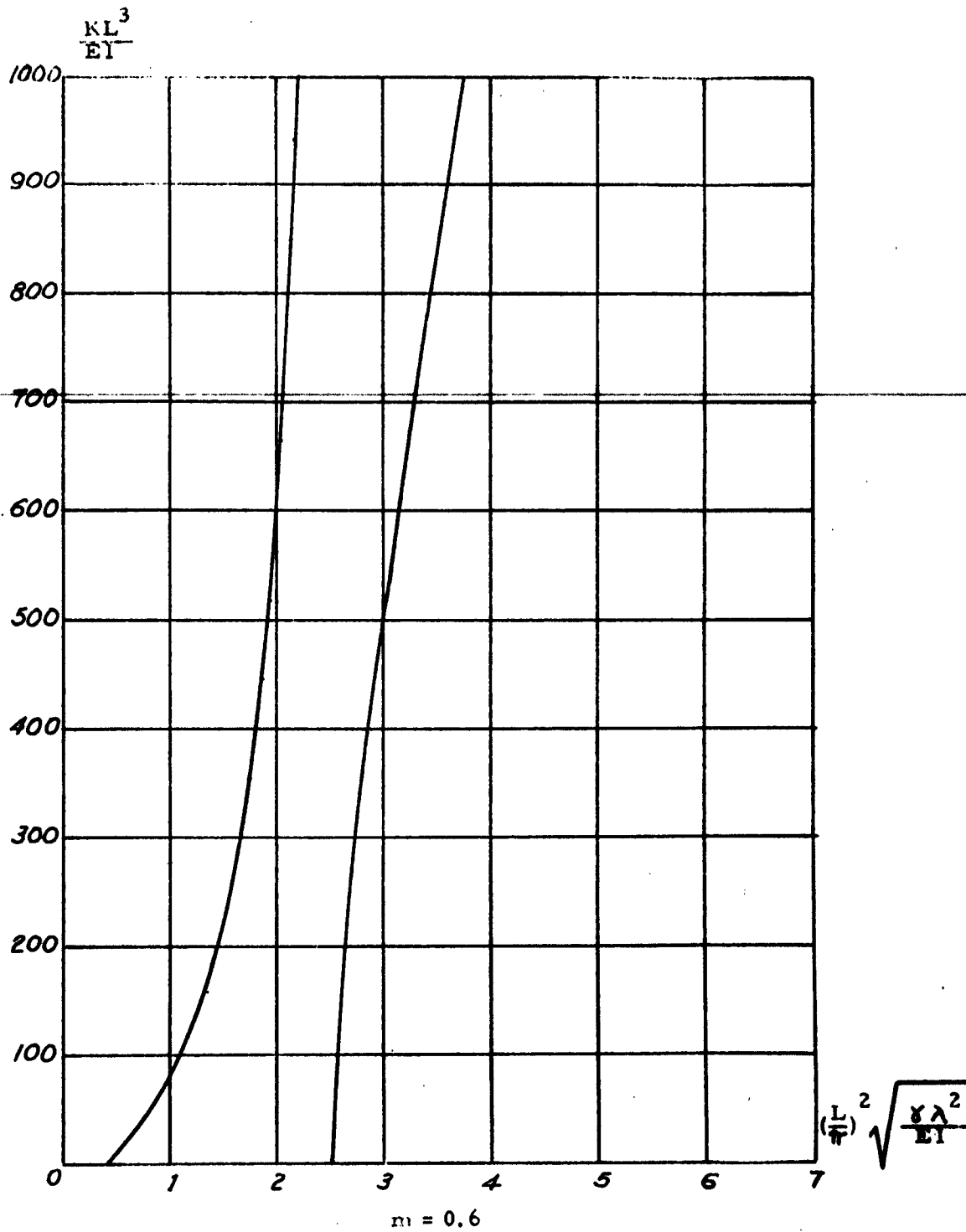


Fig. E45 NATURAL FREQUENCIES - CASE II d

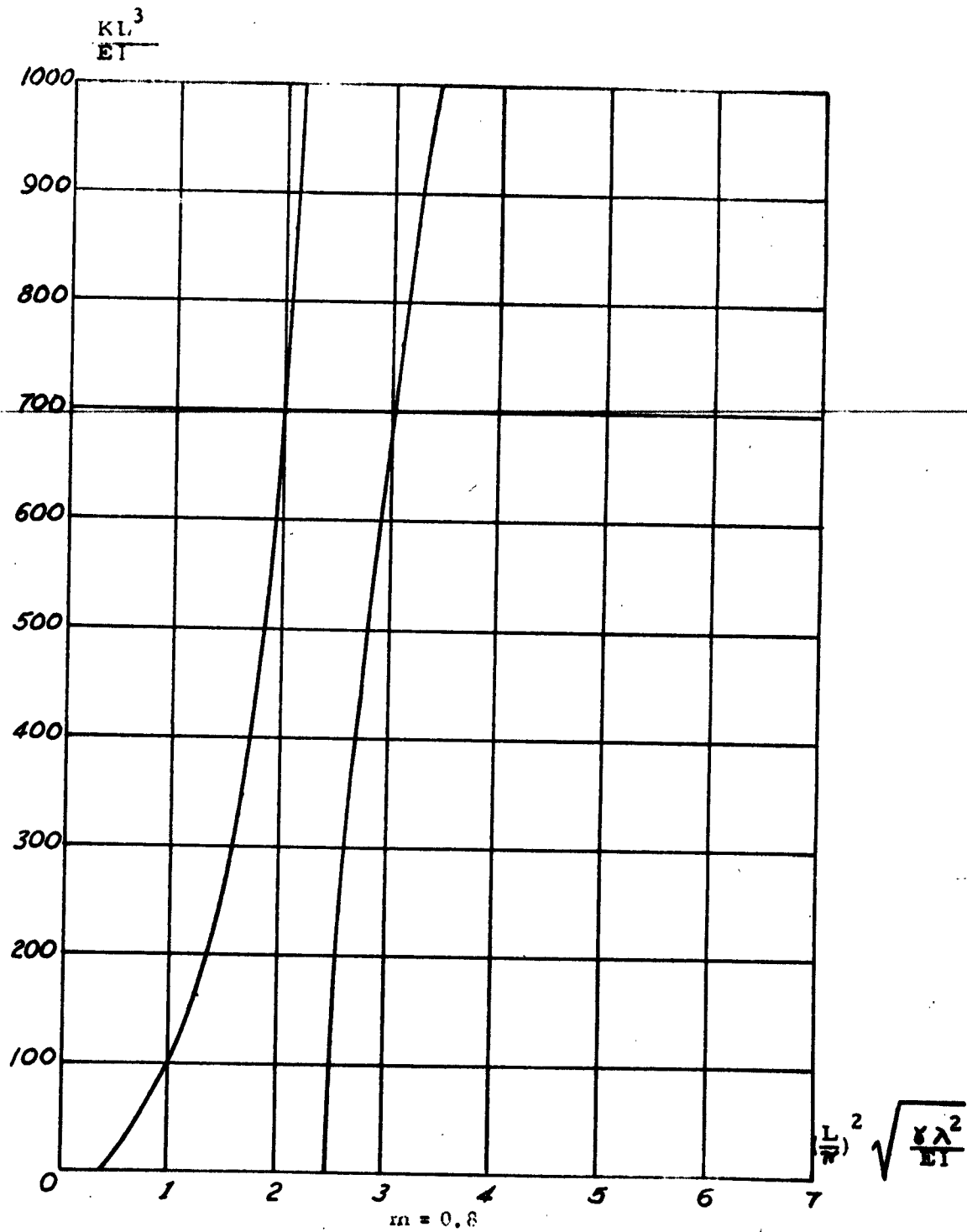


Fig. E46 NATURAL FREQUENCIES - CASE IIc

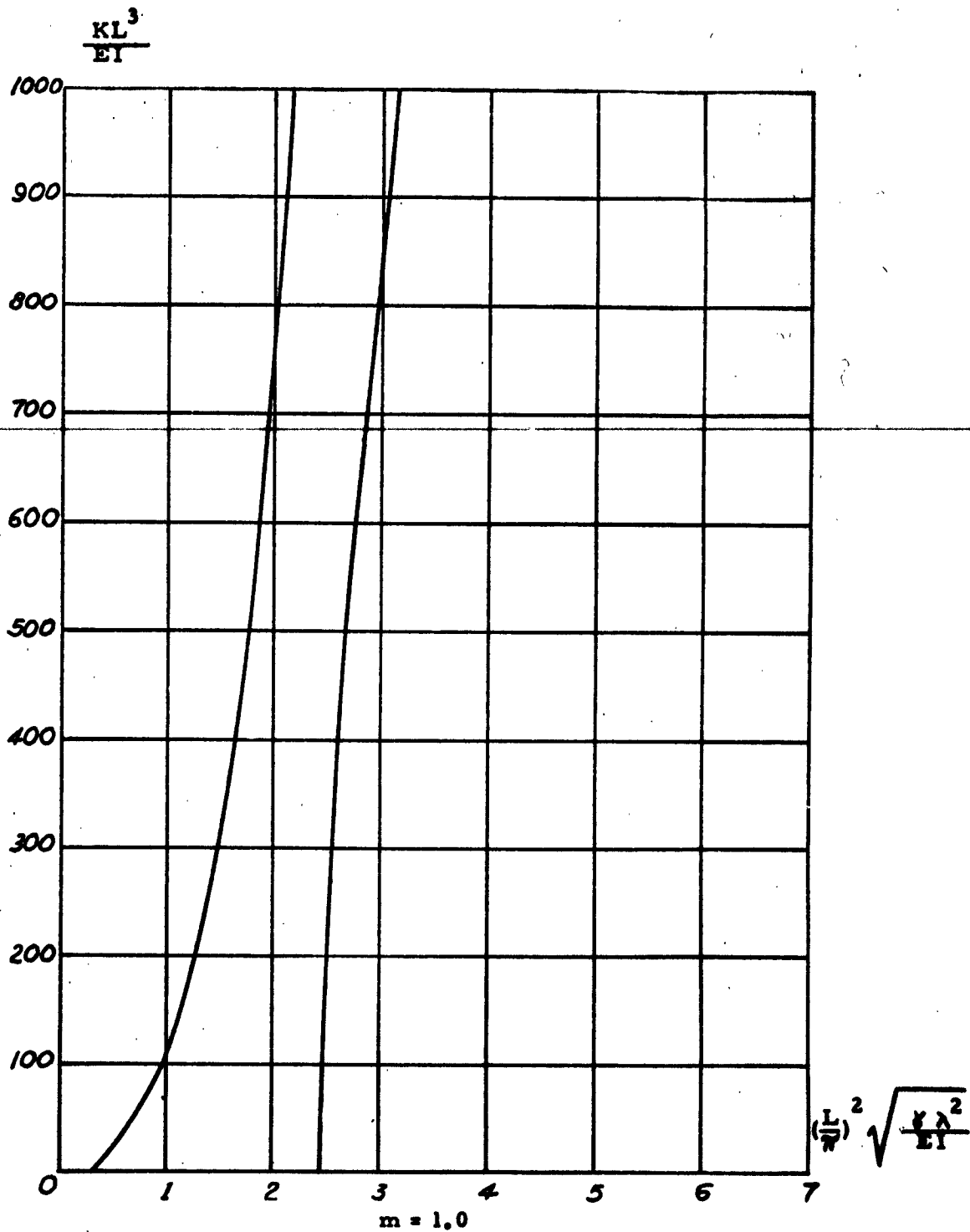


Fig. E47 NATURAL FREQUENCIES - CASE IIa

APPENDIX F

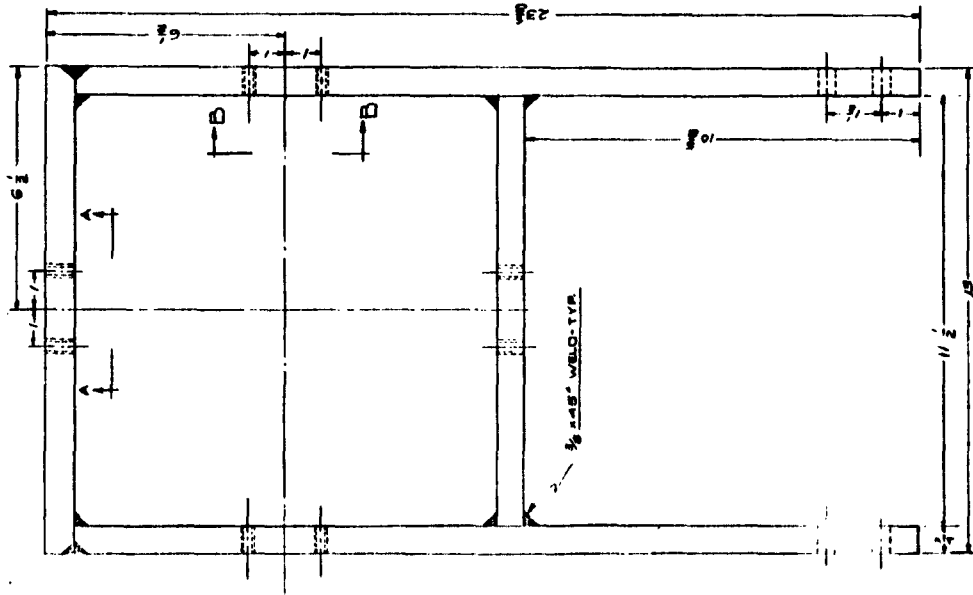
DETAIL DRAWINGS OF TEXT FIXTURE

APPENDIX F

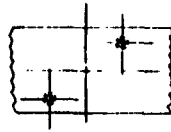
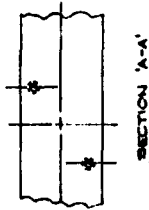
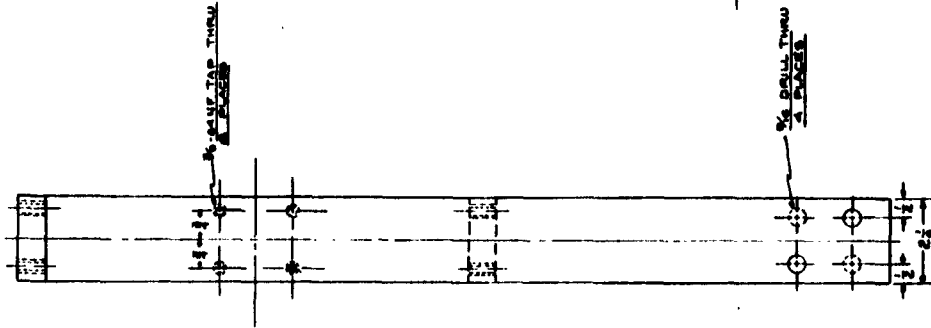
DETAIL DRAWINGS OF TEXT FIXTURE



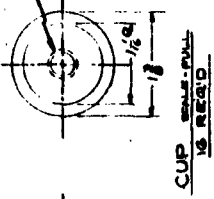




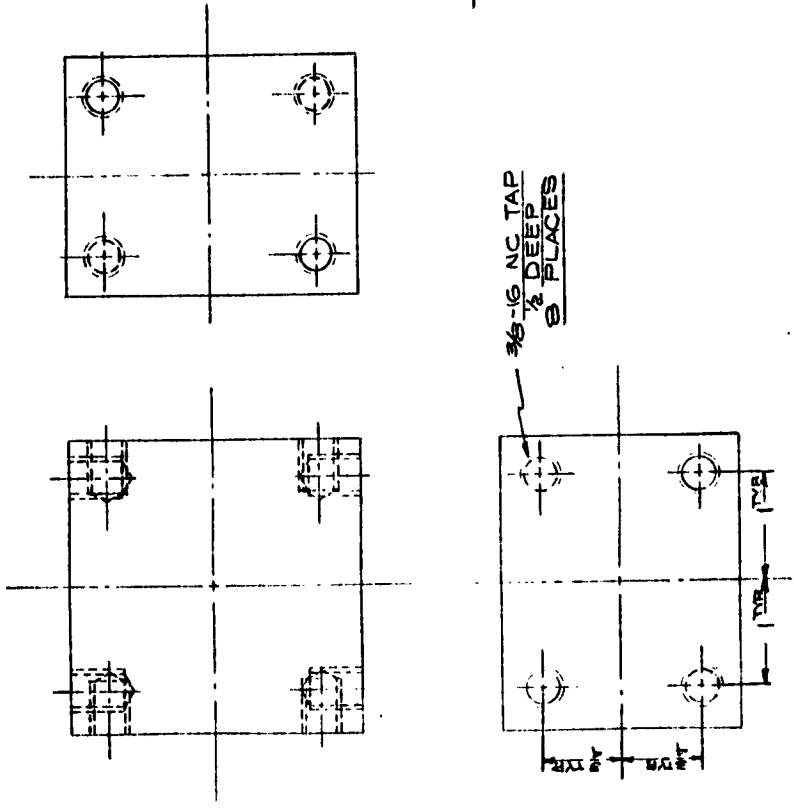
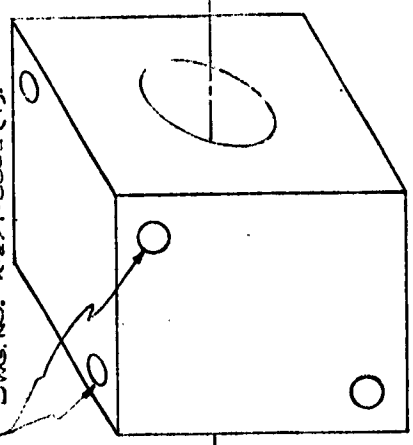
FRAME SCALE- $\frac{1}{2}$ "=1"  
2 REGID



NOTE: TAPPED HOLES ADJOINING A  
TAPED CORNER SHALL BE POSITIONED  
ON THE CENTER LINE OF THE CORNER  
DRAWING NO. 4571-200-6 (C)



NOTE: TAPPED HOLES ADJOINING THE SAME CORNER MUST BE ON OPPOSITE SIDES. SEE ALSO DWS. NO. K-271-000.1(C).



PROVIDE ALSO 16 EACH OF THE FOLLOWING SPRINGS:  
 SAMUEL HARRIS CO. CAT. NO. S-50  
 " " " S-64  
 " " " S-63  
 " " " S-79

EXPERIMENTAL FIXTURE ARRANGEMENT  
ARF PROJECT NO. K-274

ANGLE ENCODING  
TRANSDUCER

250 W. DC MOTOR

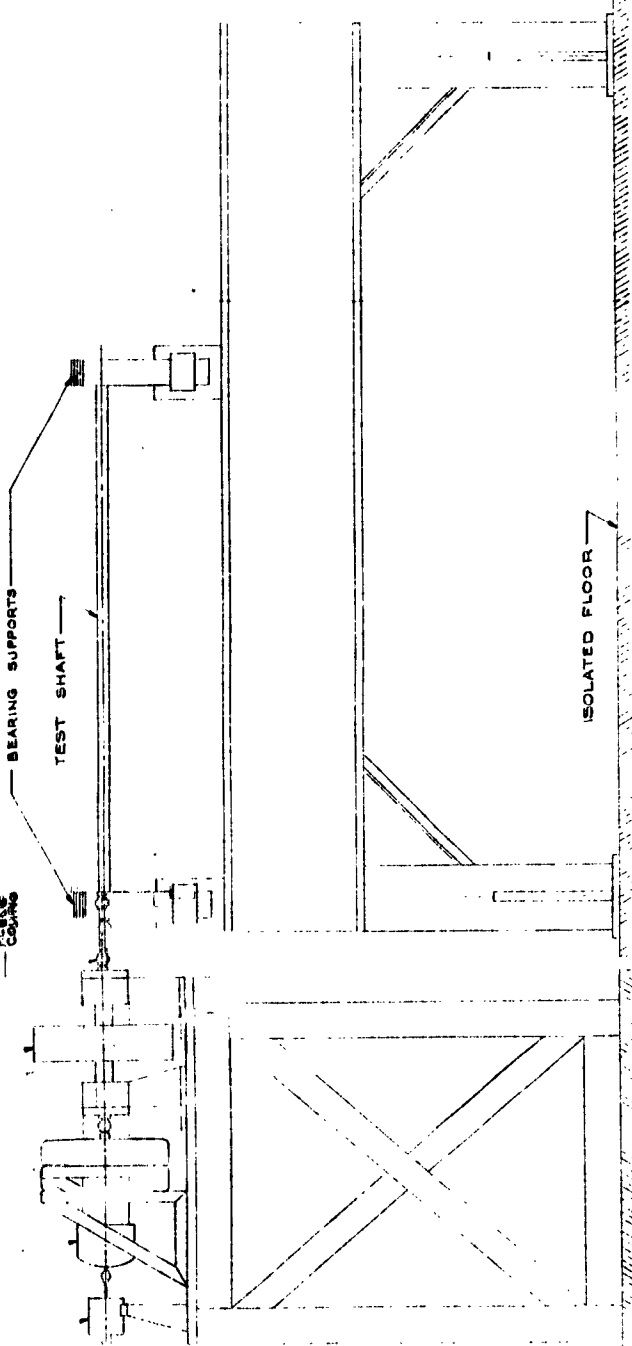
FLYWHEEL

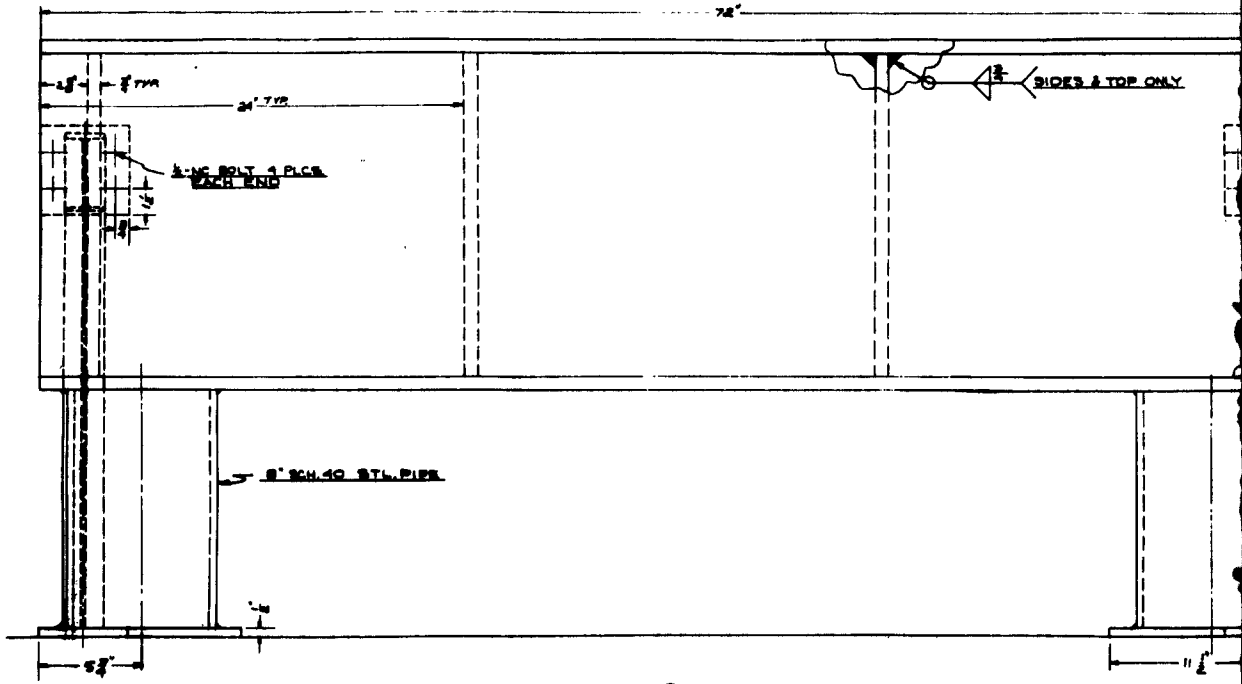
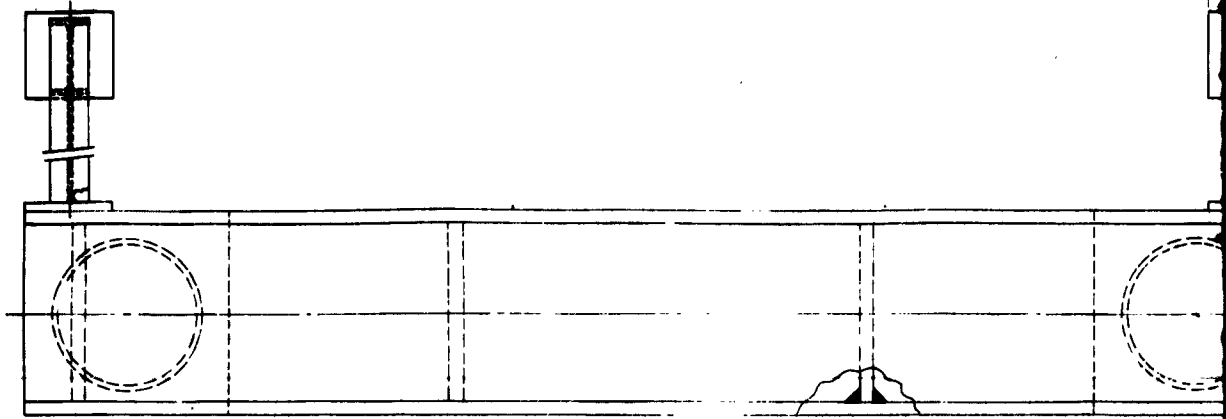
FLANG  
COUPLER

BEARING SUPPORTS

TEST SHAFT

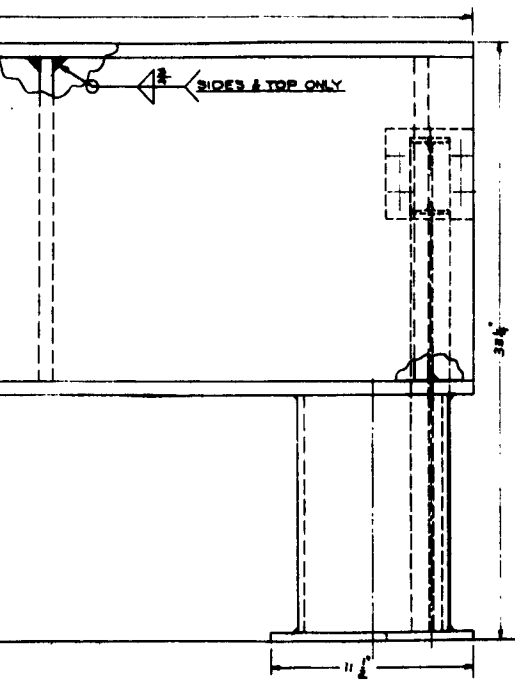
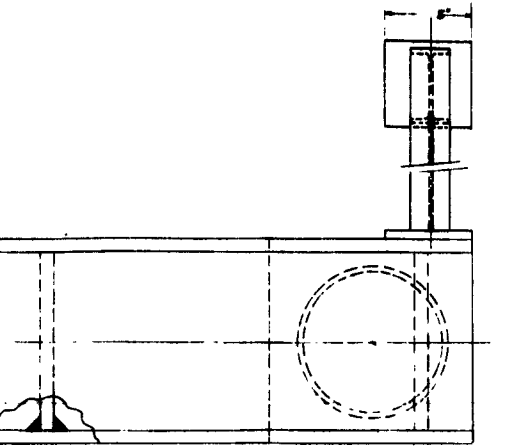
ISOLATED FLOOR



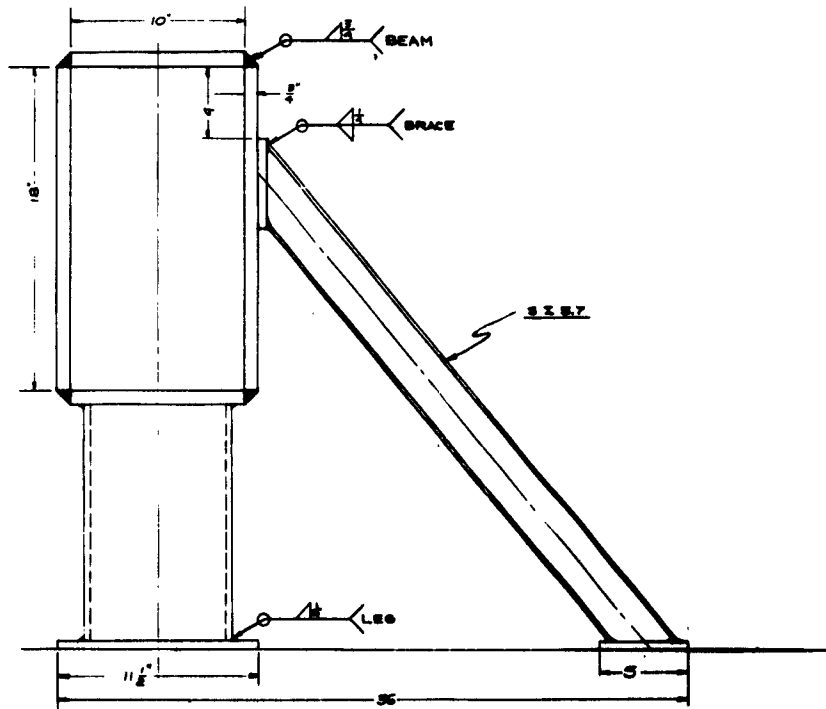


⑬ EXPERIMENTAL SECTION FRAME  
MAYL-STL 1 REQ'D

1

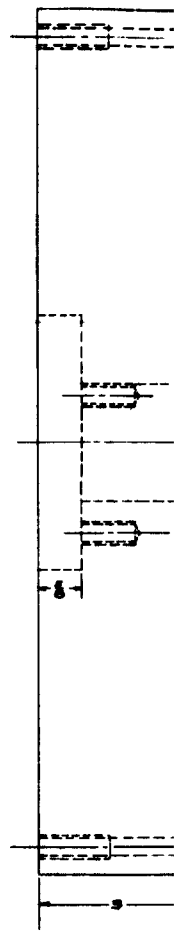
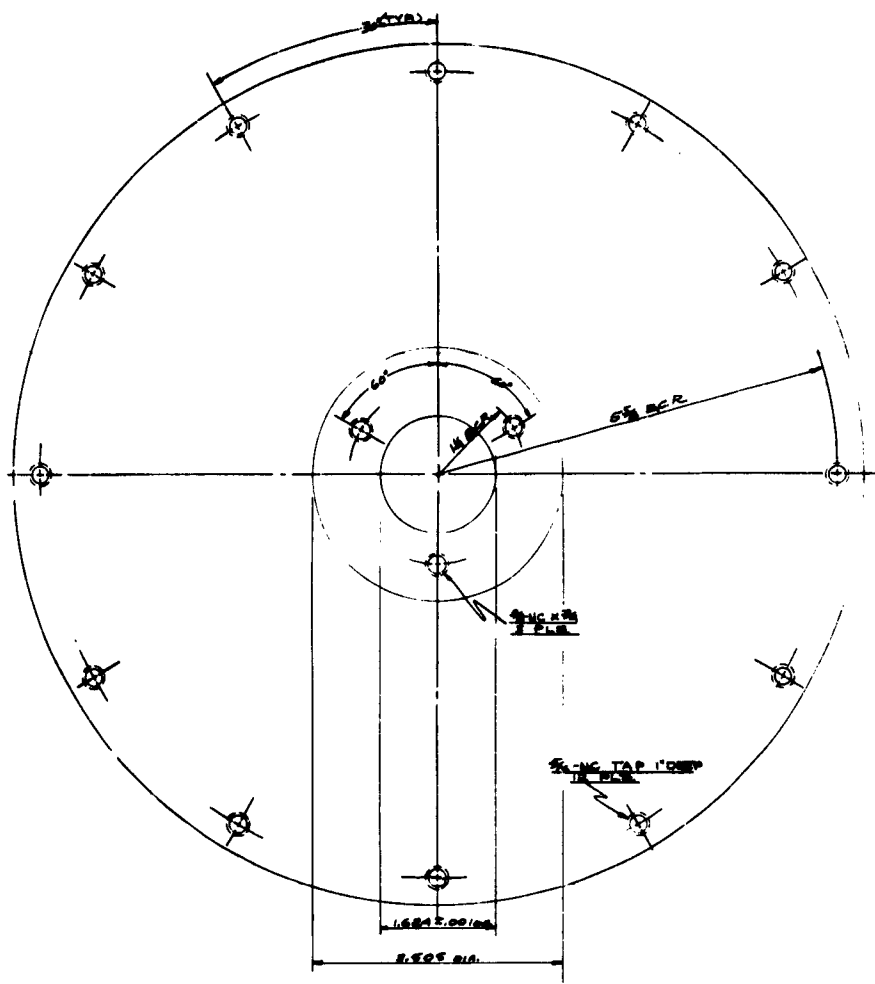


PARTIAL SECTION FRAME  
1 REGION

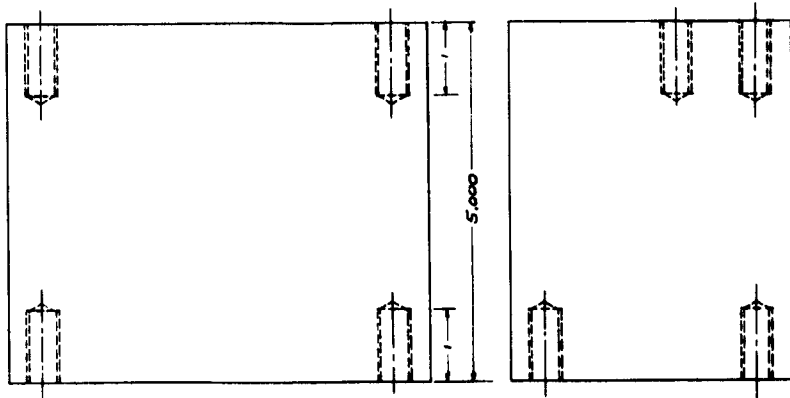
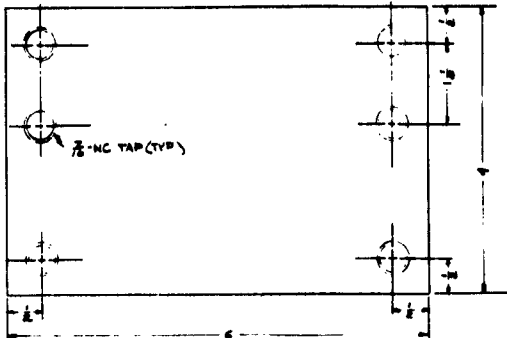


AS NOTED	TEST FIXTURE DETAILS
DESIGNED BY	DATE
CHECKED BY	DATE
APPROVED BY	DATE
PROJECT NO.	TEST NO.
TEST DATE	TEST TIME
TEST LOCATION	TESTER
TESTER'S SIGNATURE	TESTER'S TITLE
TESTER'S PHONE	TESTER'S ADDRESS
TESTER'S CITY	TESTER'S STATE
TESTER'S ZIP	TESTER'S COUNTRY
K-274	WET-028 (N)

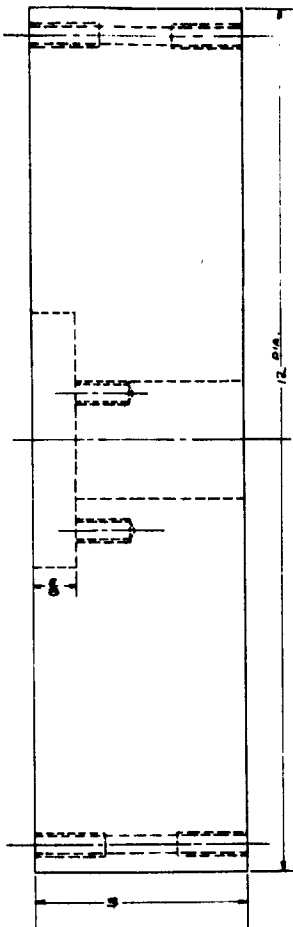
1



© FLYWHEEL  
MAY 19 1964  
MILWAUKEE, WISCONSIN  
MILWAUKEE, WISCONSIN  
MILWAUKEE, WISCONSIN



9 BEARING PEDESTAL  
MAT'L - STL. 2 REQ'D



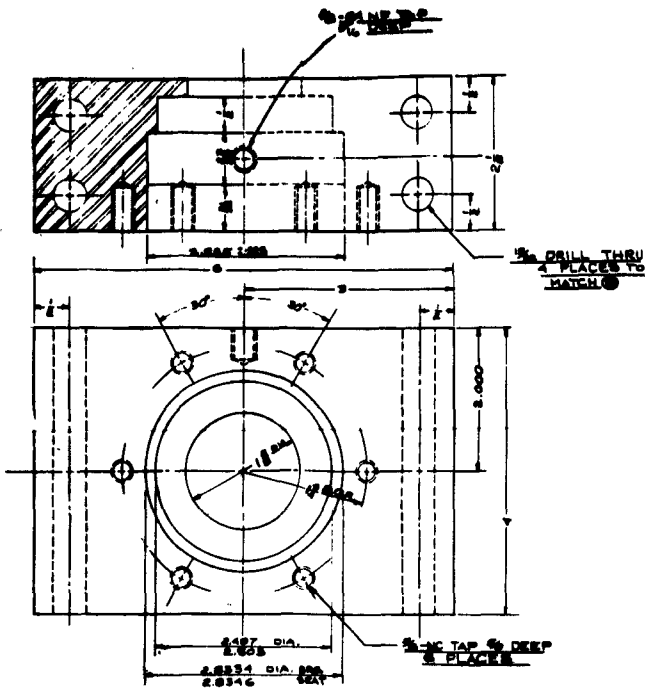
ALL DIMENSIONS IN INCHES

REV. NO.		REV. DATE		REV. BY		REV. REASON	
AS NOTED						REF. FIGURE DETAILS	
DESIGNED BY	DATE	DRAWN BY	DATE	CHECKED BY	DATE		
APPROVED BY							
MATERIAL SPECIFICATIONS							
K-271		K-271-2008(9)					

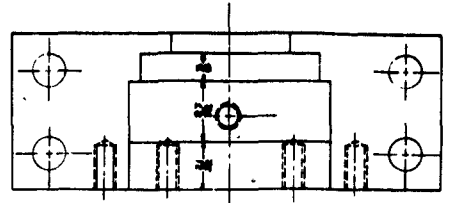
FLY  
TITLE  
SYMBOL  
SCALE



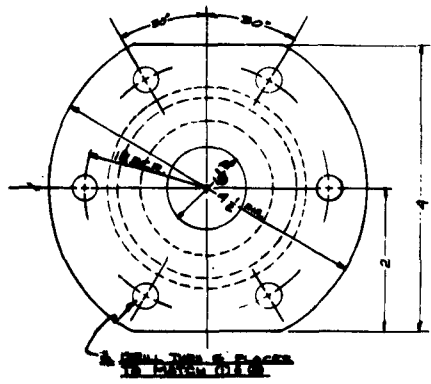
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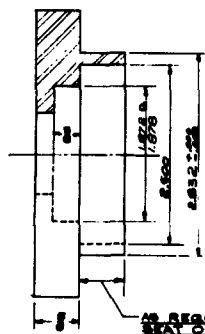
① FLYWHEEL BEARING HOUSING  
HELD END  
MAT'L - MILD STL 1 REQ'D



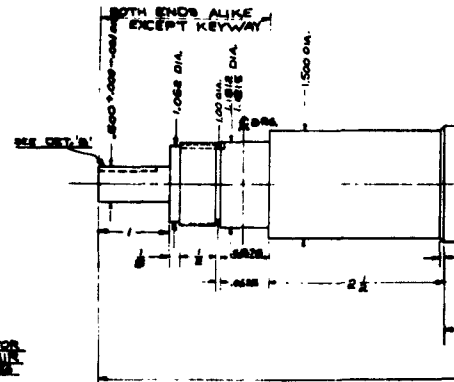
② FLYWHEEL BEARING HOUSING  
FREE END  
SAME AS ① EXCEPT AS NOTED  
REQ'D



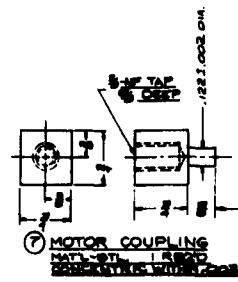
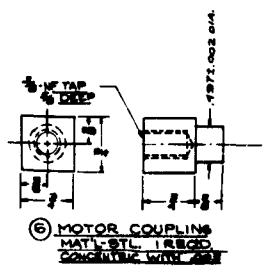
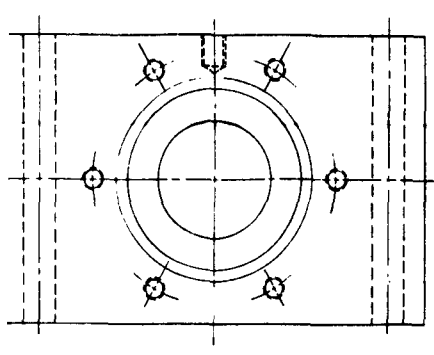
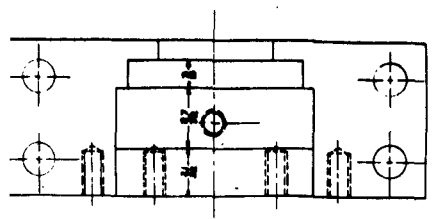
③ BEARING HOUSING COVER PLATE  
MAT'L - MILD STEEL 1 EACH REQ'D  
① - AS SHOWN - HELD END  
② - WITHOUT SLEEVE - FREE END



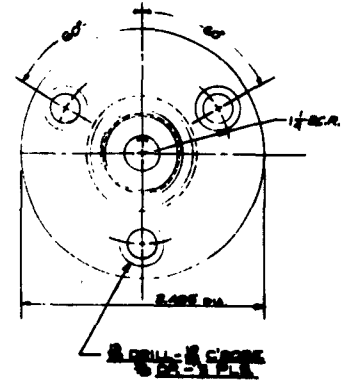
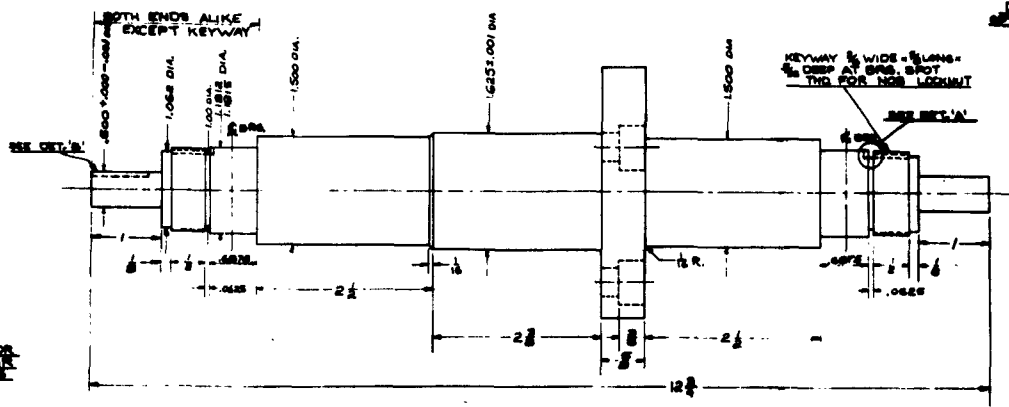
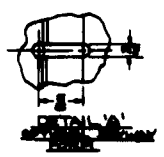
AS REQUIRED FOR  
MATCH ON FACED  
OF KEY BEARING  
ONLY



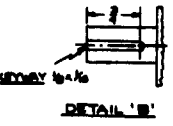
④ FL  
NOTE:  
MAT'L - 1  
STL MAT



② FLYWHEEL BEARING HOUSING  
FREE END  
SAME AS ① EXCEPT AS NOTED  
REQ'D



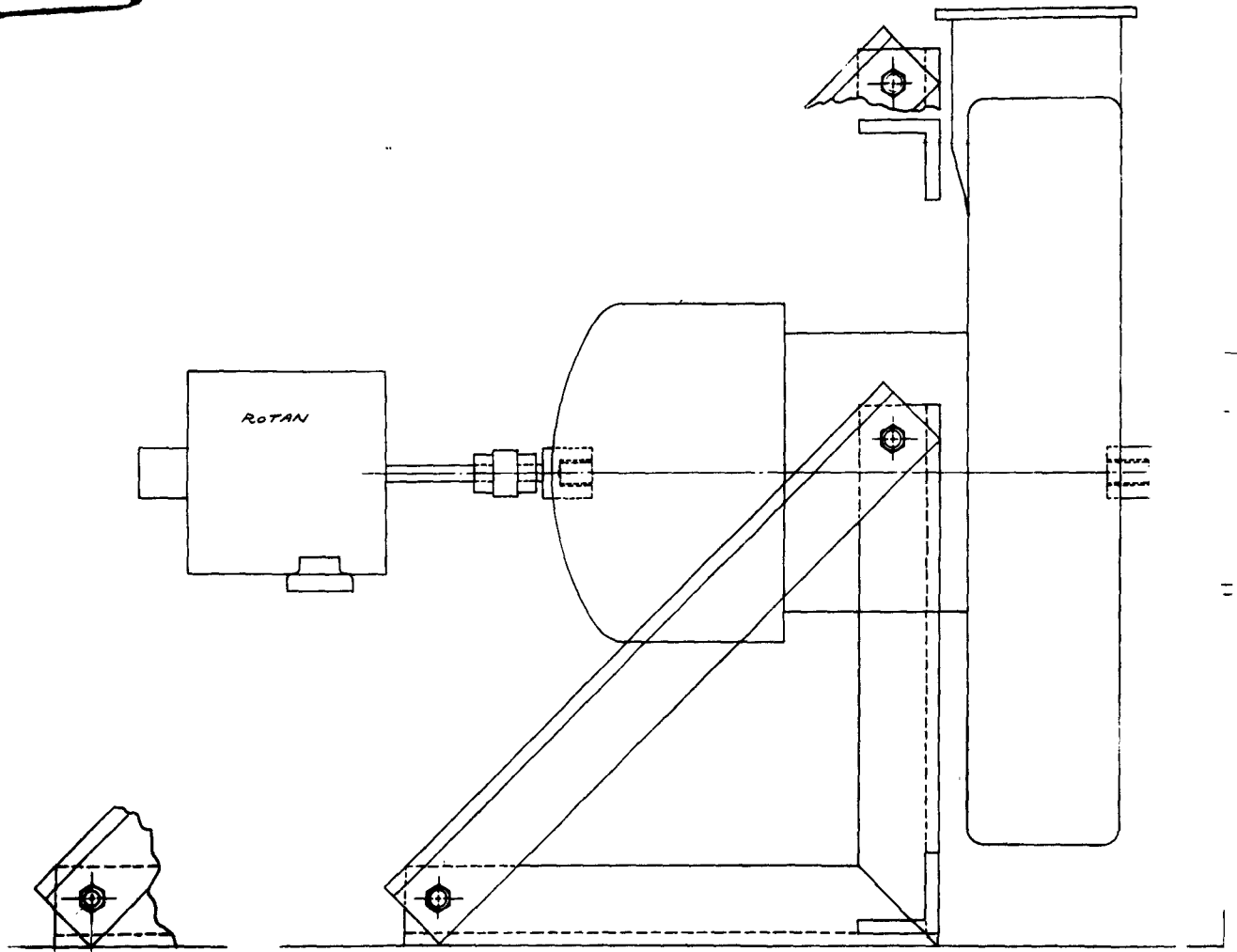
⑤ FLYWHEEL SHAFT  
NOTE: BEARING MOUNTS ALIKE.  
MATERIAL - MILD ST. REQ'D  
20' MAX. T.S. - 80A - PARTIAL 800 W

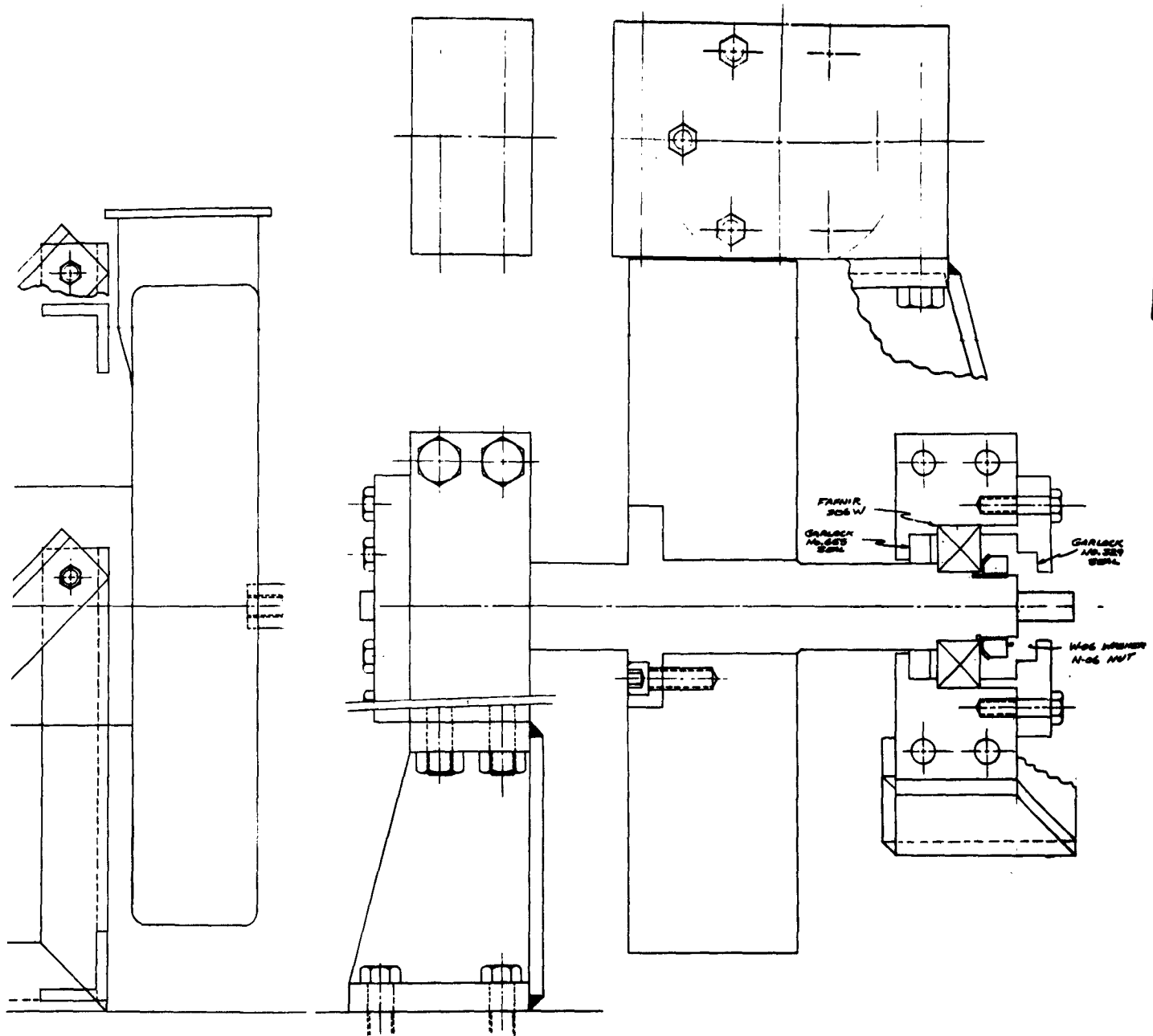


ALL DIMENSIONS IN INCHES  
UNLESS SPECIFIED OTHERWISE

AS NOTED	TEST FIXTURE DRAWING
MANUFACTURED BY	
DESIGNED BY	
CHECKED BY	
DATE	
SCALE	
QUANTITY	
APPROVED BY	
DATE	
	K-274 K-274-000(10)

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REV	DATE	BY	CHKD	APP'D	DESCRIPTION
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