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Basic Features Common to Some
Combinatorial Covering Problems

by

J. A. Riley

Project 4608

Task 460804

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ABSTRACT

An abstract covering problem is formulated which includes as special cases the prime implicant problem, the usual set-theoretic covering and systems of representatives problems, and the problems of externally stable sets, 1-widths of matrices and minimal including sums. The known methods of solution are extended to the general case.

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Basic Features Common to Some
Combinatorial Covering Problems

by

J. A. Riley

1. Introduction. The most basic problem in the theory of minimization of Boolean formulas is that of finding the minimal normal formulas representing a given Boolean function. The known methods for solving this problem all consist essentially of the following two steps: first, find the irredundant normal formulas representing the function, and then, second, use the preassigned criterion of minimality to extract

the minimal formulas from the list of irredundants. It is well known that the components of an irredundant normal formula are prime implicants; the problem of finding the irredundant equivalents of a given function breaks up, then, into two further problems: that of computing the prime implicants of the function, and that of finding those unions of prime implicants which are irredundant equivalents. The problem of finding the prime implicants of a given function has been solved in more or less satisfactory fashion, depending on the data in terms of which the function is specified. Of course, in practice, enormous difficulties may be encountered, the point being that there are functions in not too many variables which have literally thousands of prime implicants. Nevertheless, there does not seem to be too much room for improvement in the existing algorithms. A much more serious problem is encountered in the second step of the program, that of computing the irredundant equivalents. The bulk of the succeeding sections will be taken up with descriptions of the known methods of solution of this problem. These are essentially three in number, (if we except the McCluskey-Pyne attempt to use the methods of linear programming; see [16]), a 'naive' trial and error process, the 'Boolean algebra' or 'multiplying-out' method, and the 'branching' technique. In one form or another the basic ideas of these methods underlie all known procedures and algorithms for finding irredundant normal formulas.

In the course of our work on these problems, we had devised several procedures which, at first sight, appeared to be quite promising. On closer examination, however, these turned out to differ only trivially from one or the other of the methods just mentioned.

The fundamental difficulty in applying each of these methods is that they each require the generation of many more formulas than are actually found to be irredundant, that is, none of these three methods, it seems, goes to the heart of the matter in as direct a fashion as would be desirable. It should be mentioned that if minimal formulas

are what are desired, the minimality criterion may be applied at various intermediate points in the computations; this has the effect of weeding out more quickly those formulas which would eventually be found to be non-minimal. Even so, the problem still remains - to devise a procedure which will, in some sense, avoid the necessity of considering a host of superfluous formulas, and give the desired irredundant and/or minimal formulas in a straight forward, effective fashion.

This is obviously a vague objective; it may even have been attained. By this I mean that it could be maintained, without obvious objection, that the Boolean algebra and branching methods are themselves optimal or near-optimal, at least as far as the computation of irredundants is concerned. However, we feel that there is merit in attempting to improve and extend these algorithms. At the very least, we may discover various improvements in detail which would make the concrete applications of the methods more tractable.

In view of the preceding remarks, therefore, we have undertaken to survey the published literature on these subjects, and to try to extract the essence of what has been accomplished. The following sections constitute a first draft of our findings. Very early in our work, we discovered the already more or less well known fact that our problems are abstractly identical with other combinatorial problems, currently under study, such as set-theoretic covering problems, the "prime implicant" problem, the theory of minimal including sums, and the theory of "1-widths" of matrices of 0's and 1's. This abstract identity consists in the fact that each of these problems is a special case of a more general abstract "covering" problem, which we present in section 2. The methods used by various authors for solving the concrete problems are themselves capable of being abstracted to give methods for solving our general problem. In sections 4, 5 and 6 we

give our abstract version of these. In section 4 we give what we feel is a useful mathematical framework for the discussion of such problems. In section 7 we turn to a special case - the problem of minimal including sums. We show how the multiplying out method may be combined with Gazalé's method of "fractions" to yield an interesting method of finding minimal including sums.

. Our presentation is necessarily very sketchy, our main purpose being simply to present what we feel is the essence of these concrete covering problems. For the purpose of maintaining contact with the literature we give in section 3 a list of these concrete problems, together with an indication of how they fit into our general framework. We have not attempted to write a complete "comparative anatomy" of the literature, but have limited ourselves here to an occasional remark at the appropriate point in our discussion.

The list of bibliographical references at the end is not known to be complete; we present it simply as an adequate and useful collection of source-material discussed here.

2. The Abstract Covering Problem. Let R, C be finite sets, and $\rho: R \rightarrow C$ a relation between R and C , i.e. a multiple valued mapping from R to C . We will assume that ρ is onto, i.e. that each element c of C is contained in $\rho(r)$, for some $r \in R$. We will also assume that the domain of ρ is the whole of R . If $r \in R$ and $c \in C$, we will say that c covers r if $c \in \rho(r)$. A subset C' of C is a cover of R if for each $r \in R$, $\rho(r) \cap C'$ is not empty, i.e. if each element of R is covered by at least one element of C' . If C'', C' are covers of R , and $C'' \subseteq C'$, then C'' is called a sub-cover of C' . It is obvious that C is a cover of R and that any cover of R is a sub-cover of C .

A cover C' of R is said to be irredundant if no proper subset of C' is a cover. It is easy to see that any cover of R contains an irredundant sub-cover.

Our first covering problem is: (I) Find the irredundant covers of R .

Conceptually, of course, this problem has a trivial solution, viz. list, in some order, all of the covers of R , and then pick out those which are irredundant. In practice, however, such a procedure is almost never practicable; what is needed is a method of circumventing this exhaustive listing.

A second problem, possibly even more important for the applications, is that of finding minimal, weighted covers.

A cost function, γ on \mathcal{C} , is a map which associates to each subset S of \mathcal{C} a non-negative real value, $\gamma(S)$: the "cost" of S . We assume in addition that γ satisfies the axiom: if $S' \subseteq S$, then $\gamma(S') \leq \gamma(S)$, with equality only if $S' = S$. A cover, C' , of R is said to be γ -minimal if for any other cover C'' , $\gamma(C') \leq \gamma(C'')$. It is easy to see that a γ -minimal cover is necessarily irredundant (although the converse is not true).

Our second covering problem is then: (II) Find the γ -minimal covers of R .

The remarks made above concerning the impracticality of starting with an exhaustive list of all covers also apply here. What is needed here also is a more refined method for finding minimal covers - one which does not require at the outset a listing of all covers.

A possible solution consists in first computing the irredundant covers; we have noted above that any γ -minimal cover is irredundant. The minimal ones may then be located by inspection.

In many cases of practical interest, the number of irredundant covers is uncomfortably large, so that, again, it would be desirable to find methods of computing minimal covers directly.

It is possible to combine both problems (I) and (II) into a single more general one by introducing an abstract order relation into the collection of subsets of \mathcal{C} . Precisely, we will assume the existence of a "pre-order" relation, which we will denote by " \preceq " on the subsets of \mathcal{C} , and which satisfies the following axioms:

- a) $S \leq S$ for all subsets S of G' ;
- b) if $S \leq T$ and $T \leq U$, then $S \leq U$;
- c) if $S' \leq S$, then $S' \leq S$, and, in this case, $S \leq S'$ only if $S = S'$.

It is easy to see that the relations of set-inclusion, and of "cost less than or equal to" are concrete instances of the pre-order relation. It is important to notice that we have deliberately not assumed that $S' \leq S$ and $S \leq S'$ together imply that $S = S'$. This last property is one of the usual requirements for a so-called partial order relation. The relation, \subseteq of set-inclusion does, of course, possess this property of 'exclusivity;' the cost relation, however, does not, since it is possible for distinct subsets to have the same cost.

In terms of \leq , a \leq -minimal cover C' will be one which satisfies the property : if C'' is any cover of R such that $C'' \leq C'$, then in addition, $C' \leq C''$.

Specialization of this notion yields immediately the notions of irredundant, and of \forall -minimal cover. We point out that in view of axiom c), any \leq -minimal cover is also an irredundant cover of R .

The general covering problem with which we will be concerned here is then the following, and includes (I) and (II) as special cases:
(III) Find the \leq -minimal covers of R .

Since \leq -minimal covers are irredundant, we again have a first tentative solution : find the irredundant covers, and then weed out those which are not \leq -minimal.

In the following sections we will adapt the known methods for finding irredundant and minimal covers to our more general situation.

We point out here that Roth [2] has given an algorithm, the so-called "extraction" algorithm, for solving a very general covering problem. His problem is essentially the same, in another terminology, as our abstract version, the only difference being that he allows for the presence of "don't-care" conditions; we may express this by saying

that the domain of ρ is not necessarily the whole of R .

The algorithm itself, translated into our framework, is exactly the branching technique, to be discussed in section 6.

3. Examples. In this section we collect a number of examples from the current literature which may be interpreted as covering problems and which fall within our abstract framework.

Example A. Set theoretic coverings. If $R = \{r_1, \dots, r_m\}$ is a finite set, and $C = \{c_1, \dots, c_n\}$ is a collection of subsets of R such that $R = c_1 \cup \dots \cup c_n$, then C is called a cover of R . A subset, C' , of C is a sub-cover of C if $R = \cup \{c \mid c \in C'\}$, i.e. if each element of R belongs to at least one set in C' . Denote by $|C'|$ the cardinality of C' , i.e. the number of sets which it contains. Then the minimal covering problem is: find the minimal sub-covers of C , i.e. those sub-covers of smallest cardinality. To show that this problem is a special case of our general one is easy: we define $\rho(r)$ to be the collection of sets c_i in C which contain r , and we define $S' \leq S$ to mean that S' has fewer sets than S .

Edmonds [3] gives an interesting extension to these set-theoretic covering problems of the Berge-Norman-Rabin theory (see [4] and the references given there) for minimal coverings of a graph. It seems that any covering problem may be transformed into a variant of the covering problem for graphs; the Berge-Norman-Rabin technique is then generalized to obtain some information concerning the structure of minimal covers. We have not included any discussion of this theory here both because it lies in somewhat different direction and because a definite algorithm for finding minimal covers has not yet been obtained (for the general problem, that is; the Berge-Norman-Rabin theorem (cf. [4]) yields a more-or-less explicit algorithm for the "ordinary" covering problem for graphs).

Example B. Systems of Representatives. Let $R = \{r_1, \dots, r_m\}$ be a finite collection of finite sets r_1, \dots, r_m and let G denote the union of the sets in R , $G = \cup \{r_i \mid r_i \in R\}$. A subset $G' = \{c_1, \dots, c_r\}$ is a system of representatives of R if each set r_i in R contains at least one c_j from G' . An irredundant system of representatives is one such that no proper subset of it is also a system of representatives. One problem which may be posed is to find the irredundant systems of representatives of R . If a cost function is defined by attaching a cost or weight to each element of G , we may pose the problem of finding the systems of representatives of smallest cost; here by the cost of a system of representatives we mean the sum of the costs of its elements. It is easy to see how this problem also fits our general pattern.

Example C. The Prime Implicant Problem. Let F be a Boolean function, π_1, \dots, π_n its prime implicants, and p_1, \dots, p_m its canonical terms or states. The prime implicant table for F is defined to be the matrix A whose entry a_{ij} is 1 if p_i implies π_j , and 0 otherwise. The prime implicant problem is to find the sets of columns covering A having as few columns in them as possible - i.e. letting the cost, $\gamma(S)$, of a set of columns be the number of columns in it, the problem is to find the γ -minimal covers of A .

Evidently, these γ -minimal covers correspond to those normal formulas representing F which consist of a minimal number of summands.

Variations in the problem are possible; we can ask for the irredundant covers, for example. These correspond to the irredundant normal formulas representing F . In his thesis [1a] McCluskey gives several other cost functions which may be used.

The prime implicant problem has been treated extensively by McCluskey and Pyne in their two papers ([2], [3]).

Example D. Minimal including sums. Let f_1, \dots, f_n be a set of Boolean formulas, each of which is \cup -irreducible, i.e. is a product, and let g be a formula, (or function) such that g implies $f_1 \cup \dots \cup f_n$. Then $f_1 \cup \dots \cup f_n$ is called an including sum for g (the fact that $g \Rightarrow f$ is also sometimes expressed as " f includes g ;" hence the name "including sum"). It may also happen that g implies a "sub-sum" $f_{i_1} \cup \dots \cup f_{i_r}$ of $f_1 \cup \dots \cup f_n$. Such a sub-sum $f_{i_1} \cup \dots \cup f_{i_r}$ is said to be a minimal including sum for g (composed of one or more of the f_i) if a) g implies $f_{i_1} \cup \dots \cup f_{i_r}$, and b) g implies no other proper sub sum of $f_{i_1} \cup \dots \cup f_{i_r}$.

Denote by p_1, \dots, p_m the canonical states of g . Then each p_i implies $f_1 \cup \dots \cup f_n$. We define a matrix A as follows: $a_{ij} = 1$ if p_i implies f_j , and $a_{ij} = 0$ if p_i does not imply f_j . Then A is a matrix of 0's and 1's, and the minimal including sums for g correspond exactly to the irredundant covers of A .

If g is equivalent to the function 1, then $f_1 \cup \dots \cup f_n$ is equivalent to 1, also, and is called a "sum-to-one." In this case the problem is to find the minimal sums-to-one contained in $f_1 \cup \dots \cup f_n$.

The problem of finding the irredundant normal formulas representing a given function g is evidently a special case: for this we would take the f_j to be the prime implicants of g .

We may formulate the problem of finding the prime implicants of a given function F as a problem in minimal including sums. More exactly, we show how to find the prime implicates of the dual function, δF . The prime implicants of F are then the duals of the prime implicates of δF .

We take for \mathcal{C} the set of literals which are essential to F , and for R any set of terms whose union is a normal formula representing F . Then the set of literals is an including sum for F , and the minimal including sums are the prime implicates of F .

The theory of minimal including sums has been developed by Samson

and Calabi [5], and has been applied in [6] to the determination of higher order minimal formulas.

Example E. Externally stable sets. If G is a graph, a set T of vertices is called externally stable if each vertex of G not in T is joined by an edge of G to at least one element of T . A minimal externally stable set is one having a minimum number of vertices. The problem is : find the minimal externally stable sets of a given graph. To see that this is a special case of our general problem, we take $R = G = \{v_1, \dots, v_n\}$, where the v_i are the vertices of G , and we define a_{ij} to be 1 if v_i and v_j are joined by an edge of G , and $a_{ij} = 0$ if v_i and v_j are not joined by an edge of G . Then each row and column of $A = (a_{ij})$ corresponds to a vertex of G . If we define the cost of a set of columns to be the number of columns in the set, we obtain a cost function χ , and it is easy to see that the minimal externally stable sets of G are in one-to-one correspondence with the χ -minimal covers of A .

The algorithm given by Berge ([7]; pp.42-43) for finding a minimal externally stable set is essentially the branching method to be discussed below.

Example F. 1-widths of matrices. The concept of width of a matrix of zeros and ones plays an important role in many combinatorial problems of current interest ([8]). If A is a matrix of zeros and ones, the 1-width of A is by definition (transposed into our terminology) the smallest number of columns such that each row meets at least one of these columns. Thus the theory of 1-widths of matrices also falls within our general scheme.

The article by Edmonds referred to in Example A above shows that the more general concept of α -width (cf [8]) of matrices and 0's and 1's also can be formulated as a covering problem.

4. Matrix Formulation. The Naive Solution. Basically, there are three known methods for the solution of covering problems: the "naive" solution, the Boolean algebra, or "multiplying-out" method, and the branching technique.

We have omitted from consideration the application by McCluskey and Pyne [16] (see also Breuer, [9]) of linear programming techniques to the normal minimization problem. This technique may of course be applied in a similar fashion to find the \mathcal{V} -minimal covers of a general covering matrix, at least in the case that the cost of a cover is the sum of the costs of the columns which it contains. Certain serious difficulties (analyzed in [16]) are encountered in the use of the 'classical' linear programming techniques; further investigation devoted to the resolution of these difficulties would be desirable.

It will be convenient, for our subsequent exposition, to have the problem formulated in matrix terminology.

Let the elements of R be r_1, \dots, r_m and those of C , c_1, \dots, c_n . The matrix $A = (a_{ij})$ of the relation \mathcal{P} is by definition given by: $a_{ij} = 1$ if $c_j \in \mathcal{P}(r_i)$, and $a_{ij} = 0$ if $c_j \notin \mathcal{P}(r_i)$. We will denote the rows of A by r_1, \dots, r_m and the columns of A by c_1, \dots, c_n - the notation was originally chosen with this in mind. We say that row r_i meets column c_j if $a_{ij} = 1$, i.e. if $c_j \in \mathcal{P}(r_i)$. A cover of R , or as we shall say, of A , is a subset C' of the set of columns of A such that each row in R meets at least one column in C' . The translation of the notions of irredundant cover, \mathcal{V} -minimal and \leq -minimal covers into the terminology of rows and columns is simple.

Now it is well known that the Boolean algebra of subsets of a given set C is isomorphic; in a technical sense, to the Boolean algebra of n -tuples of 0's and 1's, where n is the number of elements in C . The explicit description of this isomorphism will yield us a way of transforming our covering problem into the problem of solving a certain set of Boolean equations.

Denote by $\mathcal{S}(C)$ the Boolean algebra of subsets of C , $C = \{c_1, \dots, c_n\}$ and by B_n the algebra of n -tuples of 0's and 1's. We define the map $e: \mathcal{S}(C) \rightarrow B_n$ by $S \rightarrow e_S = (y_1, \dots, y_n)$, where $y_j = 1$ if $c_j \in S$ and $y_j = 0$ if $c_j \notin S$. It is not at all difficult to see that the Boolean operations of union, intersection and complementation are preserved, and that, in the technical language, e is an isomorphism of the two Boolean algebras. If ϕ is the empty subset, e_ϕ is the n -tuple consisting entirely of zeroes; the subset C itself corresponds to the n -tuple, (1) , consisting entirely of 1's.

Consider the system of equations

$$(*) \quad \begin{cases} a_{11}x_1 \cup \dots \cup a_{1n}x_n = 1; \\ \vdots \\ a_{m1}x_1 \cup \dots \cup a_{mn}x_n = 1. \end{cases}$$

If C' is a cover of R , then the components y_1, \dots, y_n of the corresponding n -tuple $e_{C'} = (y_1, \dots, y_n)$ are a solution of the system (*). The reason for this is that if a row r_i meets $c_j \in C'$, then $a_{ij} = 1$, so that upon substitution of the y_j for the x_j in (*), the left side of the i^{th} equation becomes $a_{i1}y_1 \cup \dots \cup a_{in}y_n$ and since $a_{ij} = 1$, and $y_j = 1$ (c_j being an element of C'), we have that $a_{ij}y_j = 1$, and the left side is equal to 1. Thus (y_1, \dots, y_n) is a solution of the i^{th} equation of (*), and this is so for each i . Conversely, it is easy to see that if (y_1, \dots, y_n) is a solution to (*), the corresponding subset C' of C , for which $e_{C'} = (y_1, \dots, y_n)$, is a cover of R .

Our general problem (III) becomes then:

(III') Find the \leq -minimal solutions of the system (*).

It should be remarked that our assumed pre-order relation " \leq " has been transported bodily from the collection $\mathcal{S}(C)$ of subsets of C to the corresponding isomorphic algebra of n -tuples - we say that $e_{C''} \leq e_{C'}$, for two n -tuples C'' and C' , if $C'' \leq C'$.

The first method of solution of (III') is what can be called the "naive" method. It consists simply in examining, one after the other, the various n -tuples of 0's and 1's to determine the ones which are solutions of (*). From the list of solutions, the \leq -minimal covers can be located by inspection. We have already mentioned that this exhaustive method is not generally practical. A better, but still impractical, way of proceeding, would be to solve the system (*) by some algebraic means, analogous perhaps to successive elimination of variables in solving ordinary simultaneous systems with real coefficients. Techniques for solving linear Boolean equations are known; they do not seem as yet to have been considered in connection with covering problems. This seems to be a potentially useful direction for future work.

If just one \leq -minimal cover is wanted, the naive method may sometimes be useful. We have in mind the case that \leq is derivable from a cost function χ , with the cost of a cover being given as the sum of the costs of its columns. The procedure is to examine in succession sets of columns, starting with those of cardinality 1, then those with two columns, and so on. Each time that a cover is found, its cost is recorded, and compared with the costs of those already found. The procedure is carried out to the point at which it can be determined that the cost of any set of columns of larger cardinality would be larger than the minimum cost so far recorded. At this point, the recorded cover of smallest cost will be the desired minimal cover of A .

5. The Boolean Algebra, or Prime Implicant Method. This method is based on the following theorem, which shows that the irredundant covers of \mathcal{R} are in one-to-one correspondence with the prime implicants of a certain Boolean formula associated to the matrix A . Here, the a_{ij} are the entries of A .

Theorem: Define the Boolean formula, $F(A)$, by:

$$F(A) = (a_{11}c_1 \cup \dots \cup a_{1n}c_n) \dots (a_{m1}c_1 \cup \dots \cup a_{mn}c_n),$$

where we have used the letters c_1, \dots, c_n to denote both the columns of A , and Boolean literals. Then $C' = \{c_{i_1}, \dots, c_{i_r}\}$ is an irredundant cover of A if and only if the formula $F(C') = c_{i_1} \dots c_{i_r}$ is a prime implicant of $F(A)$.

Proof: We show that, in general, a subset $S = \{c_{i_1}, \dots, c_{i_r}\}$ of C is a cover of A if and only if the formula $F(S) = c_{i_1} \dots c_{i_r}$ is an implicant of $F(A)$ (i.e. $F(S) \Rightarrow F(A)$). Well, if $S = \{c_{i_1}, \dots, c_{i_r}\}$ is a cover of A , then $A(e_S) = (1)$, so that the system (*) is satisfied, with $e_S = \{y_1, \dots, y_n\}$:

$$\begin{cases} a_{11}y_1 \cup \dots \cup a_{1n}y_n = 1; \\ \vdots \\ a_{m1}y_1 \cup \dots \cup a_{mn}y_n = 1. \end{cases}$$

Thus $(a_{11}y_1 \cup \dots \cup a_{1n}y_n) \dots (a_{m1}y_1 \cup \dots \cup a_{mn}y_n) = 1;$

now the expression on the left here is that obtained by substituting y_j for c_j in $F(A)$, for $j=1, \dots, n$. Since $y_j = 1$ if and only if $c_j \in S$, it follows that the formula $F(S) = c_{i_1} \dots c_{i_r}$ takes the value one only upon this same substitution of the y_j for c_j . Thus the product $F(S) = c_{i_1} \dots c_{i_r}$ implies $F(A)$, and $F(S)$ is an implicant of $F(A)$. A reversal of the argument shows that conversely, if $F(S)$ is an implicant of $F(A)$ then S is a cover of A . Thus the problem of finding irredundant covers is transformed into the problem of finding the prime implicants of a certain formula given as the product of sums of literals, i.e. the problem of finding the star formula $F^*(A)$ of $F(A)$. The existing known methods for finding prime implicants can then be brought to bear. In particular, since $F(A)$ does not contain complements, $F^*(A)$ may be found by first multiplying-out $F(A)$ and then simplifying by deleting redundant terms and literals.

We remark that, as we have seen in the discussion of the examples, the problem of finding the prime implicants of a function can itself be formulated as a covering problem, so that any progress in finding better algorithms for computing \leq -minimal covers would lead to improved

methods for finding prime implicants.

The theorem, in the special case of irredundant normal formulas, is implicit in the work of Samson and Mueller [10]; the first explicit statement and proof seems to have been given by Petrick [11]. The presence-function defined by Gazalé [12] is also a function whose prime implicants correspond to irredundant normal forms (his derivation of it is somewhat different, however, and belongs more properly to the situation of minimal including sums discussed below in section 7).

6. Branching Methods. Consider the system (*) of equations for the covering problem:

$$(*) \quad \begin{cases} a_{11}x_1 \cup \dots \cup a_{1n}x_n = 1; \\ \vdots \\ a_{m1}x_1 \cup \dots \cup a_{mn}x_n = 1. \end{cases}$$

It is a trivial, but as this section will demonstrate, an extremely useful observation that an irredundant solution of (*) either will have $x_n = 1$, or will have $x_n = 0$. This suggests that we split the problem (*) up into two smaller ones, viz. first find those irredundant covers of A , if any, which do not contain column c_n , and then find those, if any, which do. In order to describe the procedure we need a bit more notation. We will denote the set of \leq -minimal covers of a matrix A by M_{c_1, \dots, c_n} , where c_1, \dots, c_n are the columns of A . If c_j is a particular column of A , we denote by $M_{c_1, \dots, c_n}^{\widehat{c}_j}$ the set of \leq -minimal covers of the matrix obtained from A by deleting not only column c_j but also by deleting in addition those rows of A which meet c_j . Thus, for example, if $A = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$, with columns $c_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ etc., then $M_{c_1, c_2, c_4}^{\widehat{c}_3}$ is the set of \leq -minimal covers of the matrix, $\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$.

Then our previous remark may be rendered symbolically as follows: the \leq -minimal covers of A which do not contain column c_n are exactly the covers in the set $M_{c_1, \dots, c_{n-1}}$; the \leq -minimal covers of A which do contain c_n are the covers in the set $M_{c_1, \dots, c_n}^{\widehat{c}_n}$. Then we may express the solution M_{c_1, \dots, c_n} to the covering problem in the

following form:

$$(**) M_{c_1 \dots c_n} = \min \left\{ M_{c_1, \dots, c_{n-1}} ; c_n \cdot M_{c_1, \dots, c_n}^{\hat{c}_n} \right\}.$$

Here by $c_n \cdot M_{c_1, \dots, c_n}^{\hat{c}_n}$ we mean the collection of sets $\{c_n\} \cup S'$ where S' ranges over the sets in $M_{c_1, \dots, c_n}^{\hat{c}_n}$. Here, also, by $\min \{X, Y\}$ we mean the collection of \leq -minimal sets in $X \cup Y$.

The formula (**) is intuitively obvious: a \leq -minimal cover of A either does not contain c_n , or else it is composed of c_n together with a \leq -minimal cover of those rows not already covered by c_n , i.e. those rows not meeting c_n .

The two sub-problems are each of a smaller order than A , the one having one column less, and the other having not only one column, but also at least one row less.

Further simplification may be had by iterating formula (**). We have for example, selecting column c_{n-1} as the next "pivot:"

$$M_{c_1 \dots c_n} = \min \left\{ \min \left\{ M_{c_1, \dots, c_{n-2}} ; c_{n-1} \cdot M_{c_1, \dots, c_{n-1}}^{\hat{c}_{n-1}} \right\} ; M_{c_1, \dots, c_n}^{\hat{c}_n} \right\}.$$

This may be written, using obvious properties of the $\min \{ ; \}$ operation,

$$(***) M_{c_1 \dots c_n} = \min \left\{ M_{c_1, \dots, c_{n-2}} ; M_{c_1, \dots, c_{n-1}}^{\hat{c}_{n-1}} ; M_{c_1, \dots, c_n}^{\hat{c}_n} \right\}.$$

It is necessary to point out that a sub-problem obtained by this type of deletion may have no solutions. We have not mentioned it before, but it is not difficult to see that a system of equations (*) will fail to have a solution if and only if some row of the matrix A consists entirely of zeros. For example, if in (***) c_n and c_{n-1} were the only columns meeting some row of A , then $M_{c_1, \dots, c_{n-2}}$ would be ϕ . In this case (***) would reduce to

$$M_{c_1 \dots c_n} = \min \left\{ c_{n-1} \cdot M_{c_1, \dots, c_{n-1}}^{\hat{c}_{n-1}} , c_n \cdot M_{c_1, \dots, c_n}^{\hat{c}_n} \right\}.$$

In practice we may use this iterative simplification by choosing for successive pivots all of the columns which meet a chosen row of A . We obtain then a determinate set of simpler problems, and we are sure that a \leq -minimal cover of the original matrix A must be a \leq -minimal cover of one at least of these sub-matrices.

Another practical method of procedure is the following. First choose an irredundant cover of A , and re-arrange the columns of A so that those in the cover are the first r columns c_1, \dots, c_r . Then $\{c_1, \dots, c_r\}$ is the only cover of the matrix $[c_1 \dots c_r]$, i.e. $M_{c_1 \dots c_r} = \{c_1, \dots, c_r\}$. Now consider the problem $M_{c_1 \dots c_r c_{r+1}}$. By (**) the \leq -minimal covers of this are:

$$M_{c_1 \dots c_r c_{r+1}} = \min \left\{ \{c_1, \dots, c_r\}, c_{r+1}, M_{c_1 \dots c_{r+1}} \right\}.$$

Having these, we repeat the procedure, adjoining the next column of A , using (**) again, and so on.

There are other variations which may be made in using the branching technique in specific problems. A good summary of various procedures can be found in the papers of McCluskey and Pyne ([1a]-[1c]). In section 2 above we have pointed out that Roth's extraction algorithm is basically the same as the branching technique. Menger's procedures [13] for finding minimal normal formulas seem, also, to consist among other simplifications, in an ingenious adaptation of the branching technique.

7. An application to minimal including sums. The Gazale-Rado' Technique.

Let f_1, \dots, f_n be a set of formulas and π_1, \dots, π_m a set of terms such that $\pi_1 \cup \dots \cup \pi_m \Rightarrow f_1 \cup \dots \cup f_n$. The problem of minimal including sums for $\pi_1 \cup \dots \cup \pi_m$ is as we have seen, to find those "sub-sums" of $f_1 \cup \dots \cup f_n$ which are also implied by each π_i , and which have the property that none of their proper sub-sums are implied by $\pi_1 \cup \dots \cup \pi_m$. The usual formulation of this as a covering problem uses the matrix A whose entry a_{ij} is 1 if $\pi_i \Rightarrow f_j$, and 0 otherwise. The following theorem shows that we may solve the problem in essentially the same

manner as our previous covering problems, except that now we admit the use of matrices whose entries are not necessarily 0 or 1, but may be any Boolean function. We need to define some terms occurring in the statement of the theorem.

Denote by $f_j(\pi_i)$ the function obtained from f_j by substituting 1, 0 (according as the corresponding literal is x or \bar{x}) for the variables x which occur in π_i , and leaving the rest of the variables in f_j untouched. If π_i is $x_1 \bar{x}_2$, for example, and f_j is $x_1 x_2 x_3 \cup x_2 x_4$, then $f_j(\pi_i)$ is the function $1 \cdot 0 \cdot x_3 \cup x_2 x_4 = x_2 x_4$.

We define the matrix B by stipulating that its entry b_{ij} shall be $f_j(\pi_i)$. We label the columns b_1, \dots, b_n and consider the b_j also as Boolean literals. If $S = \{b_{i_1}, \dots, b_{i_r}\}$ is any set of columns, we denote by $F(S)$ the product, $F(S) = b_{i_1} \dots b_{i_r}$ of the literals corresponding to these columns. If $S = f_{i_1} \cup \dots \cup f_{i_r}$ is a sub-sum of $f_1 \cup \dots \cup f_n$, we let this correspond to the set $S = \{b_{i_1}, \dots, b_{i_r}\}$ of columns. The formula $F(S)$ is then the product of the literals b_{i_1}, \dots, b_{i_r} corresponding to the summands of $f_{i_1} \cup \dots \cup f_{i_r}$.

Denote by S_{i_1}, \dots, S_{i_r} the minimal including sums-to-one in the i^{th} row of B ; we recall that this means the minimal including sums for the function 1, which in our case, since $\pi_1 \cup \dots \cup \pi_m \Rightarrow f_j$, is the union of the entries $f_j(\pi_i)$ in the i^{th} row of B . Finally, we denote by $F(B)$ the formula defined as follows:

$$F(B) = (F(S_{11}) \cup \dots \cup F(S_{1r_1})) \dots (F(S_{m1}) \cup \dots \cup F(S_{mr_m})).$$

The statement of the theorem is:

Theorem. The sub-sum $S = f_{i_1} \cup \dots \cup f_{i_r}$ is a minimal including sum for $\pi_1 \cup \dots \cup \pi_m$ if and only if $F(S)$ is a prime implicant of $F(B)$.

The proof follows the same pattern as that of our previous theorem concerning the relation between prime implicants and irredundant covers.

Thus, let $S = f_{i_1} \cup \dots \cup f_{i_r}$ be a sub-sum of $f_1 \cup \dots \cup f_n$ such that $F(S) \Rightarrow F(B)$. Then $F(S) \Rightarrow F(S_{i_1}) \cup \dots \cup F(S_{i_r})$ for each i , and since these formulas contain no complements, $F(S) \Rightarrow F(S_{i_1})$ and

some \mathcal{E} . Thus the columns corresponding to $S_{i\mathcal{E}}$ are among those corresponding to S . Thus we see that for each row, there is a minimal sum-to-one whose corresponding columns are a subset of the columns f_{i_1}, \dots, f_{i_r} . But this then leads immediately to the implication $\pi_1 \cup \dots \cup \pi_m \Rightarrow f_{i_1} \cup \dots \cup f_{i_r}$. Thus we have shown that each subsum of $f_1 \cup \dots \cup f_r$ such that $F(f_1, \dots, f_r) \Rightarrow F(B)$ is an including sum for $\pi_1 \cup \dots \cup \pi_m$. Conversely, a similar proof will show that if $S = f_{i_1} \cup \dots \cup f_{i_r}$ is an including sum for $\pi_1 \cup \dots \cup \pi_m$, then $F(S)$ is an implicant of $F(B)$. It follows that the minimal including sums correspond to the prime implicants, and vice versa.

We close with several remarks dealing with the application of this theorem.

The usual applications of the technique of minimal including sums, for example, finding irredundant normal formulas, start with the canonical sum of states representing the given function. In the prime implicant problem, likewise, the rows of the prime implicant table correspond to states or canonical terms. The preceding theorem shows that we may use any normal formula representation of the function to represent the rows of our matrix B . Thus, the summands, π_i , of the given normal formula would correspond to the rows of B , and the prime implicants of the function would correspond to the columns. Since, in general, we can obtain a normal formula which has fewer summands than states, we see that B has fewer rows than the usual covering matrix A would have. This means a saving in the amount of work necessary - in particular, the formula $F(B)$ will have fewer factors to be multiplied-out.

Off-setting this advantage, however, is the fact that this more general procedure requires the computation, for each row of B , of all of the minimal including sums of functions in that row. This may very well amount to more work, for some problems, than using the larger number of rows corresponding to the states, but together with entries of 0 and 1, only, in B .

There are ways of getting around the difficulty, but we have not yet investigated them to the extent sufficient to decide whether they are really worthwhile or not.

The entries $f_j(\pi_i)$ of the matrix B are what would have been called "ratios" by Gazalé, and would have been denoted by f_j/π_i ; see [12]. Gazalé however considered only terms for the f_j ; in this respect our procedure is an extension of his. Radó [14], in his treatment of the general "presence" function, gives a proof of our theorem in the special case that the $\{\pi_i\}$ are all of the canonical states for the function $f_1 \cup \dots \cup f_n$.

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