ERRATA in "Vibration in an Incompressible Fluid"
by L. Landweber

1. P. 10, line 4: A_{ii} instead of A_{ij}

2. P. 11, eq. (38): $\omega_n = \dfrac{K_n}{m + \frac{\pi\rho}{2n}}$ instead of $\dfrac{K_n}{m - \frac{\pi\rho}{2n}}$

3. P. 16, eq. (57): close parenthesis about $n + 1$; $(n + 1)$

4. P. 20, two equations following eq. (65):

   Change $-2\zeta\dot{P}_n(\zeta)$ to $+2\zeta\dot{P}_n(\zeta)$

   $-2\zeta\dot{Q}_n(\zeta)$ to $+2\zeta\dot{Q}_n(\zeta)$

5. P. 21, eq. (67): $(\zeta - 1)^r$ instead of $(x - 1)^r$

6. P. 22, two equations following eq. (71):

   Change $z = e^{-\xi}$ to $z = e^{\xi}$

   Change $F(\quad) = 1 + \frac{(\ )(\ )}{1\cdot(\ )(\ )} - \frac{(\ )(\ )(\ )(\ )}{1\cdot 2\cdot(\ )(\ )(\ )(\ )}{}^2 + \cdots$

   to

   $F(\quad) = 1 + \frac{(\ )(\ )}{1\cdot(\ )(\ )} + \frac{(\ )(\ )(\ )(\ )}{1\cdot 2\cdot(\ )(\ )(\ )(\ )}{}^2 + \cdots$

7. Pgs. 24 and 25, caption for tables:

   $Q_n^{\ 1}/\dot{Q}_n^{\ 1}$ instead of $Q_n^{\ 1}/Q_n^{\ 1}$

8. P. 27, eq. (75): $\ell_z = \dfrac{1}{\sqrt{1 + f_x^{\ 2} + f_z^{\ 2}}}$

9. P. 30, line following eq. (85): $y \gtrless 0$ instead of $y \lessgtr 0$

   eq. (86): $\doteq -\dfrac{V(x)}{2\pi} f_z(\ )$ instead of $\doteq \dfrac{V(x)}{2\pi} f_z(\ )$

   last line: $k(\xi, \zeta; \ x, z)$ instead of $k(\xi, \zeta, \ x, z)$

10. P. 33, equation following eq. (100): $R(x, x', z, z')$ instead of $R(x, x'; z, z')$

VIBRATION IN AN INCOMPRESSIBLE FLUID

Introduction

In the current procedure for calculating the natural frequencies of a vibrating ship the effect of the fluid is taken into account by increasing the mass at each section of the ship along its length by an added-mass. The latter is obtained by the so-called method of strip theory, first proposed independently by F. M. Lewis [1]* and J. L. Taylor [2], in which the added mass at a transverse section is first taken to be that for a two-dimensional form of that shape, and then the ordinates of this added-mass distribution curve are multiplied by a constant correction factor obtained from the exact potential-flow solution for a vibrating spheroid.

The aforementioned method has yielded predictions of vibration frequency in good agreement with experiment at the lower modes, but serious deviations observed at the higher ones raise the question whether more exact theories of both the elastic characteristics of the ship and the effect of the surrounding fluid might not be required. It has been reported [3] that an attempt is under way to improve upon the beam theory of ship vibration by taking into account the three-dimensional structure of the hull, which clearly becomes important at the higher modes of vibration. Complementary to this is the work reported here to improve upon strip theory by developing a unified hydroelastic theory of ship vibration.

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General Theory

First suppose that the body consists of n discrete masses connected by certain elastic restraints. Let q_i denote a set of generalized coordinates which describe the displacements, assumed small, of the system from equilibrium, and q_i the corresponding set of generalized velocities.

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*Numbers in [ ] indicate references at end of report.

Then the kinetic energy of the body, T_B, is expressible in the form

$$2T_B = \sum_{i,j} m_{ij} \dot{q}_i \dot{q}_j \quad , \quad i, j = 1, 2, \ldots N, \ N = 3n \tag{1}$$

in which the m_{ij} are generalized masses. In general the m_{ij} are functions of position, but for small oscillations about equilibrium they may be assumed to have the constant values corresponding to the equilibrium position, $q_i = 0$. Simultaneously the potential energy V of the system may be expressed in the form

$$2V = \sum_{i,j} k_{ij} q_i q_j \quad , \quad i, j = 1, 2, \ldots N \tag{2}$$

in which, for the same reason, the k_{ij} may be considered to be constants.

If the vibrating body is immersed in an incompressible fluid, the fluid will also be set into motion and have kinetic energy T_f which is expressible as the quadratic form [4]

$$2T_F = \sum_{i,j} a_{ij} \dot{q}_i \dot{q}_j \qquad i, j = 1, 2, \ldots N \tag{3}$$

in which the added masses a_{ij} are also constants. The determination of the a_{ij} is a hydrodynamic, potential-flow problem. This may be approached by introducing velocity potentials ϕ_i corresponding to unit magnitude of each generalized velocity q_i and formulating N Neumann problems for the ϕ_i. As is well known, the a_{ij} are expressible in terms of the ϕ_i by the relations [4]

$$a_{ij} = -\rho \iint \phi_i \frac{\partial \phi_j}{\partial n} dS \tag{4}$$

in which the integral extends over the wetted surfaces of the system. By formulating these Neumann problems as integral equations and replacing the integrals by quadrature formulas, the solution of the Neumann problems and the evaluation of the a_{ij} are reduced to linear algebra. This procedure becomes feasible if high speed computing equipment is available.

The dynamical equations of motion can now be obtained by employing Lagrange's formulation

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0 \tag{5}$$

in which $L = T_B + T_F - V$. Hence, from Eqs. (1), (2) and (3), we obtain

$$\sum_j \left[(m_{ij} + a_{ij})\ddot{q}_j + k_{ij} q_j\right] = 0 \qquad i,j = 1,2,\ldots N \tag{6}$$

a set of N linear, homogeneous, ordinary, differential equations. Such a system can be solved by assuming harmonic solutions with frequency ω of the form

$$q_i = \xi_i e^{\sqrt{-1}\,\omega t} \tag{7}$$

This, substituted into equation (6), yields the set of linear equations

$$\sum_j \left[\omega^2(m_{ij} + a_{ij}) - k_{ij}\right]\xi_j = 0 \tag{8}$$

which has solutions if and only if the determinant of the coefficients vanishes

$$\left|\,\omega^2(m_{ij} + a_{ij}) - k_{ij}\,\right| = 0 \tag{9}$$

The values of ω^2 for which equation (9) is satisfied are the eigenvalues of the matrix $\| k_{ij} \|$ with respect to $\| m_{ij} + a_{ij} \|$ [5]. In the present case, in which both matrices are positive definite, there are $N = 3n$ real eigenvalues and hence the same number of natural frequencies. When such a pair of matrices has been obtained, the determination of the eigenvalues is best performed with a high-speed computer for which appropriate programs are already available.

When the body is an elastic continuum, the expressions for the kinetic and potential energies will appear as integrals and the system may be considered as a limiting case of the one previously discussed, having an infinite number of degrees of freedom. A procedure for obtaining numerical solutions for the eigenvalue consists of approximating the system by one having a finite number of degrees of freedom which can then be treated by the method outlined above. This can be accomplished either by replacing the integrals by quadrature formulas and the space derivatives of the displacements (which occur in the expressions for the potential energy) by difference formulas, or by variationsl methods such as those of Rayleigh-Ritz and Galerkin.

In comparison with strip theory, the foregoing method introduces the surrounding fluid into the problem in a simple, unified manner, without assumptions, and takes into account quite naturally the hydrodynamic interference effects which are represented by the nondiagonal elements of the $\| a_{ij} \|$ added-mass matrix. In strip theory it is assumed that the nondiagonal elements are zero. When these elements are small in comparison with those of the principal diagonal, application of the method of matrix perturbation yields a procedure for correcting the frequencies calculated by strip theory.

These considerations are illustrated in the following by several cases, culminating with a formulation of the procedure for a ship undergoing a vertical-shear vibration.

A System of Three Degrees of Freedom

As an example of a system with a finite number of degrees of freedom, consider a massless string under tension τ, extending from $x = 0$ to $x = 4$, to which small spheres of radius a and mass m are attached at the points $x = 1, 2, 3$. Let y_1, y_2, y_3 denote small lateral displacements in the y-direction of the spheres at $x = 1, 2, 3$, respectively. If this system is displaced from equilibrium, and the effect of the surrounding fluid is neglected, its equations of motion are

$$\left.\begin{aligned} \ddot{y}_1 &= \alpha^2(-2y_1 + y_2) \\ \ddot{y}_2 &= \alpha^2(y_1 - 2y_2 + y_3) \\ \ddot{y}_3 &= \alpha^2(y_2 - 2y_3) \end{aligned}\right\} \qquad (10)$$

where $\alpha^2 = \tau/m$

If one put

$$y_j = \xi_j e^{i\omega t} \quad , \; j = 1,2,3 \tag{11}$$

equations (10) become

$$\left.\begin{aligned} &\xi_1(\omega^2-2\alpha^2)+\xi_2\alpha^2 = 0 \\ &\xi_1\alpha^2+\xi_2(\omega^2-2\alpha^2)+\xi_3\alpha^2 = 0 \\ &\xi_2\alpha^2+\xi_3(\omega^2-2\alpha^2) = 0 \end{aligned}\right\} \tag{12}$$

These yield the secular equation for

$$\begin{vmatrix} \omega^2-2\alpha^2 & \alpha^2 & 0 \\ \alpha^2 & \omega^2-2\alpha^2 & \alpha^2 \\ 0 & \alpha^2 & \omega^2-2\alpha^2 \end{vmatrix} = 0$$

the solutions of which are

$$\left.\begin{aligned} \omega_1 &= \alpha\sqrt{2-\sqrt{2}} \doteqdot 0.765\,\alpha \\ \omega_2 &= \alpha\sqrt{2} \doteqdot 1.414\,\alpha \\ \omega_3 &= \alpha\sqrt{2+\sqrt{2}} \doteqdot 1.846\,\alpha \end{aligned}\right\} \tag{13}$$

Next suppose that the spheres are immersed in a fluid of mass density ρ. The added mass of an isolated sphere of radius a is half the mass of the displaced fluid. Then, according to strip theory, one need only replace m by

$$M = m + \tfrac{1}{2} m_f \quad , \quad m_f = \tfrac{4}{3}\pi a^3 \rho \tag{14}$$

in (10), and hence, putting $\beta = \sqrt{\tau/m}$, we again obtain the expressions (13) for the frequencies, but with α replaced by β.

The velocity potential due to the motion of an isolated sphere with velocity U in an infinite fluid is that of a doublet of strength

$$\mu = \frac{1}{2} U a^3 \tag{15}$$

at the center of the sphere oriented in the direction of its motion. In the problem under consideration, however, we have three spheres moving in the y-direction with velocities $\dot{y}_1$, $\dot{y}_2$, and $\dot{y}_3$. Since the radius of each sphere is small in comparison with the distance between them, it is a good first approximation to assume that the doublet strength is unaltered by the presence of the other spheres. This approximation is equivalent to the strip-theory assumption.

In order to improve upon this first approximation it is necessary to take into account the image system within each sphere of the two external doublets, a second approximation which will suffice for the present purpose. The image system of a y-oriented doublet of strength μ at a distance c from the sphere is a similarly oriented doublet at the inverse point within the sphere of strength $\mu\, a^3/c^3$, and a line distribution of doublets, oriented in the negative y-direction, of total strength $-\frac{1}{2}\mu a^3/c^3$, giving a combined image strength of $\frac{1}{2}\mu\, a^3/c^3$. Thus, if $\dot{y}_2 = 1$, the doublet strength within the second sphere is $\mu_2 = \frac{1}{2} a^3$ and the sum of the doublet strengths of the image system of μ_2 within the first sphere is $\frac{1}{4} a^6$. According to the generalized Taylor added-mass theorem [6] this gives the induced added mass

$$A_{12} = \pi \rho a^6 = \frac{3}{4} a^3 m_F \tag{16}$$

In this manner the following set of values of the added masses are obtained

$$\left.\begin{aligned} A_{11} &= A_{22} = A_{33} = \tfrac{1}{2} m_F \\ A_{21} &= A_{12} = A_{23} = A_{32} = \tfrac{3}{4} a^3 m_F \\ A_{31} &= A_{13} = \tfrac{3}{32} a^3 m_F \end{aligned}\right\} \tag{17}$$

Hence the kinetic energy of the fluid is given by

$$2T_F = \frac{1}{2} m_F \left(\dot{y}_1^2 + \dot{y}_2^2 + \dot{y}_3^2 + 3a^3 \dot{y}_2 \dot{y}_3 + \frac{3}{8} a^3 \dot{y}_3 \dot{y}_1 + 3a^3 \dot{y}_1 \dot{y}_2\right) \tag{18}$$

Also we have for the kinetic energy of the spheres, T_s , and the potential energy of the system, V ,

$$2T_s = m(\dot{y}_1^2 + \dot{y}_2^2 + \dot{y}_3^2) \tag{19}$$

$$2V = 2\tau(y_1^2 + y_2^2 + y_3^2 - y_1 y_2 - y_2 y_3) \tag{20}$$

Hence, by (9), the secular equation for the frequencies becomes

$$\begin{vmatrix} \omega^2 - 2\beta^2 & \lambda\omega^2 + \beta^2 & \frac{1}{8}\lambda\omega^2 \\ \lambda\omega^2 + \beta^2 & \omega^2 - 2\beta^2 & \lambda\omega^2 + \beta^2 \\ \frac{1}{8}\lambda\omega^2 & \lambda\omega^2 + \beta^2 & \omega^2 - 2\beta^2 \end{vmatrix} = 0 \tag{21}$$

where $\lambda = \frac{3 m_F a^3}{4M}$

Since the terms in (21) contributed by the second approximation are small, the method of perturbations is appropriate for finding the new eigenvalues. For this purpose we first solve (12) (with α^2 replaced by β^2) for the three normalized eigenvectors ξ_{1j} , ξ_{2j} , ξ_{3j} , associated with ω_1 , ω_2 , ω_3 . This gives the orthogonal matrix

$$\xi_{ij} = \begin{pmatrix} \frac{1}{2} & \frac{1}{\sqrt{2}} & \frac{1}{2} \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{2} & -\frac{1}{\sqrt{2}} & \frac{1}{2} \end{pmatrix} \tag{22}$$

Applying the transformations

$$y_i = \xi_{ij} z_j \quad , \qquad \dot{y}_i = \xi_{ij} \dot{z}_j$$

we now obtain from (18), (19), and (20)

$$2T = 2T_s + 2T_F = M(\dot{z}_1^2 + \dot{z}_2^2 + \dot{z}_3^2)$$

$$+ \frac{\lambda M}{16}\left[(16\sqrt{2}+1)\dot{z}_1^2 - 2\dot{z}_2^2 - (16\sqrt{2}-1)\dot{z}_3^2 + 2\dot{z}_1\dot{z}_3\right] \tag{23}$$

$$2V = \tau\left[(2-\sqrt{2})\,\zeta_1^2 + 2\zeta_2^2 + (2+\sqrt{2})\,\zeta_3^2\right]$$

and the secular equation becomes

$$\begin{vmatrix} \omega^2[1+\frac{\lambda}{16}(16\sqrt{2}+1)-\beta^2(2-\sqrt{2}) & 0 & \frac{\lambda}{16}\omega^2 \\ 0 & \omega^2(1-\frac{\lambda}{8})-2\beta^2 & 0 \\ \frac{\lambda}{16}\omega^2 & 0 & \omega^2[1-\frac{\lambda}{16}(16\sqrt{2}-1)]-\beta^2(2+\sqrt{2}) \end{vmatrix} = 0 \qquad (24)$$

Neglecting terms in λ^2, one sees that the eigenvalues are now given by

$$\left.\begin{aligned} \omega_1^2 &= \frac{\beta^2(2-\sqrt{2})}{1+\frac{\lambda}{16}(16\sqrt{2}+1)} \\ \omega_2^2 &= \frac{2\beta^2}{1-\frac{\lambda}{8}} \\ \omega_3^2 &= \frac{\beta^2(2+\sqrt{2})}{1-\frac{\lambda}{16}(16\sqrt{2}-1)} \end{aligned}\right\} \qquad (25)$$

the numerators of which are the eigenvalues previously obtained from strip theory.

This simple example illustrates the general procedure for correcting the values of the frequencies obtained by strip theory for a system with a finite number of degrees of freedom.

Vibrating Plate

Next, as an example of a case with infinite degrees of freedom, consider a plate which bounds an infinite incompressible fluid on one side. Taking the $x - \zeta$ plane to be coincident with the plate in its equilibrium position, it will be supposed that the plate is undergoing a unidirectional oscillation with amplitude

$$\eta(x,t) = X(x)\cos\omega t \qquad (26)$$

in the y-direction at angular frequency ω, with nodes at the points $x = 0$, $\pm\pi, \pm 2\pi, \ldots$

The velocity potential for the motion of the fluid in the region $y > 0$ may be taken of the form

$$\Phi = \phi(x, y) \sin \omega t$$

with

$$\phi = \sum_{n=1}^{\infty} a_n e^{-ny} \sin nx \tag{27}$$

The boundary condition

$$\left(\frac{\partial \phi}{\partial y}\right)_{y=0} = -\sum_{n=1}^{\infty} n\, a_n \sin nx = -\omega X(x)$$

then gives

$$a_n = \frac{2\omega}{n\pi} \int_0^{\pi} X(\xi) \sin n\xi \, d\xi$$

It will suffice to obtain the kinetic energy T of the fluid bounded by the planes $x = 0$ and $x = \pi$ per unit width of the plate. This is given by

$$2T_F = -\rho \int_0^{\pi} \left(\phi \frac{\partial \phi}{\partial y}\right)_{y=0} dx = \rho\omega^2 \int_0^{\pi}\int_0^{\pi} A(x,\xi) X(x) X(\xi)\, dx\, d\xi \tag{28}$$

where

$$A(x,\xi) = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \sin nx \sin n\xi \tag{29}$$

The infinite series in (29) can be summed when $x \neq \xi$. We have [7]

$$\sum_{n=1}^{\infty} \frac{r^n}{n} \cos n\theta = -\frac{1}{2} \ln (1 - 2r \cos \theta + r^2)$$

Hence, writing (29) in the form

$$A(x,\xi) = \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \left[\cos n(x-\xi) - \cos n(x+\xi)\right] \tag{30}$$

we obtain

$$A(x,\xi) = \frac{1}{2\pi} \ln \frac{1 - \cos(x+\xi)}{1 - \cos(x-\xi)} = \frac{1}{\pi} \ln \frac{\sin \frac{1}{2}(x+\xi)}{\sin \frac{1}{2}(x-\xi)} \tag{31}$$

This shows that $A(x, \xi)$ has a logarithmic singularity at $x = \xi$. Hence, if the integral for the kinetic energy in (28) were replaced by a quadrature formula, the resulting matrix A_{ij} would have a strong principal diagonal. Since the elements A_{ij} of the principal diagonal would vary with i, as is seen from (31), these are not equivalent to a strip-theory added-mass approximation.

An exact treatment of the present case can be based on the maximum property of the eigenvalues. Put for the potential energy per unit width of the plate

$$2V = \int_0^\pi \int_0^\pi k(x,\xi)\, X(x)\, X(\xi)\, dx\, d\xi \cdot \cos^2 \omega t \tag{32}$$

and for its kinetic energy

$$2T_p = m\omega^2 \int_0^\pi \int_0^\pi \delta(x,\xi)\, dx\, d\xi \cdot \sin^2 \omega t \tag{33}$$

where $\delta(x, \xi)$ is the Dirac delta function and m is the mass per unit area of the plate. Hence, since the total energy is conserved, we must have

$$\int_0^\pi \int_0^\pi \left\{ \omega^2 \left[\rho A(x,\xi) + m\delta(x,\xi) \right] - k(x,\xi) \right\} X(x) X(\xi)\, dx\, d\xi = 0 \tag{34}$$

Assume $X(x) = \sum_{n=1}^{\infty} c_n \sin nx$. Then (34) becomes

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left\{ \omega^2 c_n^2 \left[\rho \frac{\pi}{2n} + m \right] - c_m c_n \int_0^\pi \int_0^\pi k(x,\xi) \sin mx \sin n\xi\, dx\, d\xi \right\} = 0 \tag{35}$$

The condition that ω is a maximum with respect to variations of c_1, c_2 ... yields

$$\omega^2 c_n \left[\rho \frac{\pi}{2n} + m \right] - \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} c_m \int_0^\pi \int_0^\pi k(x,\xi) \sin mx \sin n\xi\, dx\, d\xi = 0 \tag{36}$$

a set of linear equations for c_m, the determinant of which is the secular equation for obtaining the eigenvalues ω^2.

Assuming that the potential-energy integral is also diagonalized by the transformation $X(x) = \sum_{n=1}^{\infty} c_n \sin nx$, i.e.,

$$\int_0^{\pi}\int_0^{\pi} k(x,\xi)\, X(x)\, X(\xi)\, dx\, d\xi = \sum_{n=1}^{\infty} c_n^2 K_n$$

then we obtain instead of (36)

$$\sum_{n=1}^{\infty} c_n^2 \left[\omega^2\left(m + \frac{\pi\rho}{2n}\right) - K_n\right] = 0 \qquad (37)$$

Hence, setting the derivative with respect to c_n equal to zero, one sees that the condition for ω to be a maximum yields

$$\omega\left(m + \frac{\pi\rho}{2n}\right) = K_n$$

or

$$\omega_n = \frac{K_n}{m - \frac{\pi\rho}{2n}} \qquad (38)$$

Thus, for this case, the added mass is $\pi\rho/(2n)$, inversely proportional to the order of the mode.

The Vibrating String

Another illustration of a case with infinite degrees of freedom is furnished by the vibrating string. It will be supposed that the string of radius a is of infinite length but restricted to vibration with nodes at $x = 0, \pm\pi, \pm 2\pi, \ldots$ along its length. Let m denote the mass of the string per unit length and τ its tension, and suppose that the string is vibrating transversely in the y-direction with amplitude

$$\eta(x,t) = X(x) \cos \omega t \qquad (39)$$

at angular frequency ω .

The velocity potential may be assumed to be of the form [2]

$$\Phi = \phi(x, r, \theta) \sin \omega t$$

where (x, r, θ) are the cylindrical coordinates of a point, and $\phi(x, r, \theta)$ is a velocity potential expressible as the series

$$\phi(x,r,\theta) = \sum_{n=1}^{\infty} a_n K_1(nr) \sin nx \cos\theta \tag{40}$$

where $K_1(\lambda)$ denotes the modified Bessel function of the first order. From the boundary condition at the surface of the string

$$\left(\frac{\partial\phi}{\partial r}\right)_{r=a} = v(x)\cos\theta \quad , \quad v(x) = -\omega X(x)$$

we obtain

$$a_n = -\frac{2\omega}{\pi n K_1'(na)} \int_0^{\pi} X(\xi) \sin n\xi \, d\xi$$

where K_1' denotes the derivative of K_1 with respect to its argument, and hence (40) becomes

$$\phi = -\frac{2\omega}{\pi} \sum_{n=1}^{\infty} \frac{K_1(nr)}{n K_1'(na)} \cos\theta \int_0^{\pi} X(\xi) \sin nx \sin n\xi \, d\xi \tag{41}$$

The maximum value of the kinetic energy of the fluid bounded by the planes $x = 0$ and $x = \pi$ is then given by

$$2T_F = -\rho \int_S \phi \frac{\partial\phi}{\partial n} dS = \rho\omega \int_{x=0}^{\pi} \int_{\theta=0}^{2\pi} a\phi X(x) \, dx \, d\theta$$

evaluated on the surface $r = a$. Substituting from (41), we obtain

$$\left.\begin{aligned} 2T_F &= \rho\omega^2 \int_0^{\pi}\int_0^{\pi} A(x,\xi) X(x) X(\xi) \, dx \, d\xi \\ &\text{where} \\ A &= -2a^2 \sum_{n=1}^{\infty} G(na) \sin nx \sin n\xi \ , \quad G(\lambda) = -\frac{K_1(\lambda)}{\lambda K_1'(\lambda)} \end{aligned}\right\} \tag{42}$$

The modified Bessel function $K_1(\lambda)$ is defined by the series [8]

$$K_1(\lambda) = \frac{1}{\lambda} - \frac{1}{4} \sum_{n=0}^{\infty} \frac{\psi(n) + \psi(n+1) - 2\ln\frac{\lambda}{2}}{n!\,(n+1)!} \left(\frac{\lambda}{2}\right)^{2n} \tag{43}$$

where

$$\psi(0) = -C, \quad \psi(n) = -C + 1 + \frac{1}{2} + \cdots + \frac{1}{n}$$

and C is the Euler constant, $C = 0.5772 \ldots$ For small values of λ we obtain from (43) the asymptotic formula

$$G(\lambda) \approx 1 + \lambda^2 \ln \frac{\lambda}{2} + C\lambda^2, \quad \lambda \ll 1 \tag{44}$$

For large values of λ we have the asymptotic formula [10]

$$K_1(\lambda) \approx \sqrt{\frac{\pi}{2s}}\, e^{-u}\left(1 - \frac{1}{8s}\right)$$

where

$$s = \sqrt{\lambda^2+1}, \qquad u \approx s - \frac{1}{s}$$

We obtain then

$$K_1(\lambda) \approx \sqrt{\frac{\pi}{2\lambda}}\left(1 + \frac{3}{8\lambda}\right)e^{-\lambda}$$

and

$$K_1'(\lambda) \approx -\sqrt{\frac{\pi}{2\lambda}}\left(1 + \frac{7}{8\lambda}\right)e^{-\lambda}$$

and hence

$$G(\lambda) \approx \frac{1}{\lambda}\left(1 - \frac{1}{2\lambda}\right), \quad \lambda \gg 1 \tag{45}$$

For intermediate values of λ, $G(\lambda)$ may be computed from tabulated values of $K_n(\lambda)$ [8], using the identity

$$G(\lambda) = -\frac{K_1(\lambda)}{\lambda K_1'(\lambda)} = \frac{K_1(\lambda)}{K_1(\lambda) + \lambda K_0(\lambda)} \tag{46}$$

The maximum values of the kinetic and potential energies of a vibrating string are given by

$$\left.\begin{aligned} 2T_s &= m\omega^2 \int_0^\pi X(x)^2\, dx \\ 2V &= \tau \int_0^\pi \left(\frac{dX}{dx}\right)^2 dx \end{aligned}\right\} \tag{47}$$

Hence, by (42), the total kinetic energy of string and fluid is given by

$$2(T_s+T_F) = \omega^2\int_0^\pi\int_0^\pi [m\,\delta(x,\xi)+\rho A(x,\xi)]\,X(x)X(\xi)\,dx\,d\xi \qquad (48)$$

and the condition of constancy of the total energy gives

$$\omega^2\int_0^\pi\int_0^\pi [m\,\delta(x,\xi)+\rho A(x,\xi)]\,X(x)X(\xi)\,dx\,d\xi = \tau\int_0^\pi\left(\frac{dX}{dx}\right)^2 dx \qquad (49)$$

As in the previous case of the vibrating plate, exact solutions for the vibration frequencies can be obtained from the energy equation (49) by the Rayleigh - Ritz method. Assume

$$X(x) = \sum_{n=1}^{\infty} c_n \sin nx$$

Then (49) becomes

$$\omega^2\sum_{n=1}^{\infty} c_n^2\,[m-\pi a\rho G(na)] = \tau\sum_{n=1}^{\infty} n^2 c_n^2$$

The condition that the c_n be selected so that ω is a maximum or a minimum gives

$$\omega^2\,[m-\pi a\rho\, G(na)] = \tau n^2$$

and hence, denoting the frequency corresponding to n by ω_n,

$$\omega_n = n\sqrt{\frac{\tau}{m}}\,\left[1+\frac{1}{\sigma}G(na)\right]^{-\frac{1}{2}} \qquad (50)$$

where σ is the specific gravity of the string material.

According to strip theory, the frequency is given by

$$\omega_{ns} = n\sqrt{\frac{\tau}{m}}\,\left(1+\frac{1}{\sigma}\right)^{-\frac{1}{2}} \qquad (51)$$

Hence, from (50) and (51) we have

$$\frac{\omega_n}{\omega_{ns}} = \sqrt{\frac{\sigma+1}{\sigma+G(na)}} \doteq 1+\frac{1-G(na)}{2(\sigma+1)} \qquad (52)$$

For harmonics of moderate order the value of $G(\lambda)$ in (44) may be applied and (52) becomes

$$\frac{\omega_n}{\omega_{ns}} \doteqdot 1 + \frac{\lambda^2}{2(\sigma+1)}\left(\ln\frac{2}{\lambda} - C\right) \quad , \quad \lambda = na \tag{53}$$

For example, consider a steel string $(\sigma = 6.8)$ 2 ft long and 0.01 ft in diameter. Since the foregoing theory is based on a string length of π between nodes, the value of a in (52) will be taken to be

$$a = \frac{0.01\pi}{4} = 0.00785$$

This gives the following values of ω_n/ω_{ns} :

n	G(na)	ω_n/ω_{ns}
1	1	1
10	0.985	1.001
100	0.660	1.022

The example indicates that the strip-theory approximation would be an excellent one in the case of a vibrating string.

Vibrating Spheroid

In the cases of the infinite plate and the infinite string vibrating in a fluid, it has been possible to obtain exact solutions for the natural frequencies. Since the principal purpose of the present work is to illustrate an approximate numerical procedure for obtaining these frequencies, it would be of interest to compare the results of this numerical procedure, applied to either of the previous cases, with the known exact solution.

Although it is planned to undertake such a calculation in the continuation of this study, to date attention has been devoted to the development of procedures for obtaining the added-mass function for several cases of more practical interest than the preceding ones - viz., a circular bar of finite length, a spheroid, and a ship form. Of these, only in the case of the finite bar, discussed in Part 3 of this Final Report, has a complete application of the theory, taking into account the inertia and elastic properties of the bar, been consummated. The results obtained in the case of a vibrating spheroid will first be presented.

It has been shown [9] that the kinetic energy of a liquid, for a spheroid vibrating at its free surface, is expressible in the form

$$2T_F = -\frac{\rho k^3}{4\pi}(\zeta_0^2-1)\sum_{n=1}^{\infty}\sum_{s=1}^{\infty}\frac{Q_n^s(\zeta_0)}{\dot{Q}_n^s(\zeta_0)}\frac{(n-s)!}{(n+s)!}(2n+1)\left[\int_0^{2\pi}\int_{-1}^{1}F(\mu,\theta)P_n^s(\mu)\sin s\theta\, d\mu\, d\theta\right]^2 \tag{54}$$

where

$$F(\mu,\theta) = \mu u_e + (1-\mu^2)^{1/2}\zeta_0(\zeta_0^2-1)^{-1/2}(v_e\cos\theta + w_e\sin\theta)$$

Here (ζ, μ, θ) are the spheroidal coordinates of a point with focal distance $2k$. The surfaces $\zeta =$ const. are prolate spheroids, the value $\zeta = \zeta_0$ denoting the equilibrium surface of the vibrating spheroid. The polar and equatorial radii of the spheroid $\zeta = \zeta_0$ are

$$a = k\zeta_0 \qquad c = k(\zeta_0^2-1)^{1/2}$$

The functions P_n^s and Q_n^s are the associated Legendre functions of the first and second kind [10], and u_e, v_e, w_e are the components of the velocity vector $\bar{v}_e$, related to the velocity vector $\bar{V}_e$ of a point on the surface $\zeta = \zeta_0$ by

$$\bar{V}_e = \bar{v}_e(\zeta,\mu,\theta)\sin\omega t \tag{55}$$

Let us consider a pure shear motion with

$$u_e = v_e = 0 \quad , \quad w_e = w_e(\mu) \tag{56}$$

Then (54) becomes

$$\left.\begin{aligned} 2T_F &= \frac{\pi}{4}\rho k^3\zeta_0^2\int_{-1}^{1}\int_{-1}^{1}A(\mu,\nu)\,w_e(\mu)\,w_e(\nu)\,d\mu\, d\nu \\ &\text{where} \\ A(\mu,\nu) &= -\sqrt{(1-\mu^2)(1-\nu^2)}\sum_{n=1}^{\infty}\frac{Q_n^1(\zeta_0)}{\dot{Q}_n^1(\zeta_0)}\frac{2n+1}{n(n+1)}P_n^1(\mu)P_n^1(\nu)\end{aligned}\right\} \tag{57}$$

Let us write the kinetic and potential energies of the elastic spheroid in the form

$$2T_s = \int_{-1}^{1} \lambda_s(\mu)\, w_e(\mu)^2\, d\mu$$

$$2V = \int_{-1}^{1} k(\mu,\nu)\, X(\mu)\, X(\nu)\, d\mu\, d\nu$$

where $X(\mu)$ is the amplitude of the vibration, given by

$$w_e(\mu) = -\omega X(\mu)$$

From the conservation of the total energy we then have

$$\int_{-1}^{1}\int_{-1}^{1}\left\{\omega^2\left[\lambda_s(\mu)\,\delta(\mu,\nu)+\lambda_F A(\mu,\nu)\right]-k(\mu,\nu)\right\}X(\mu)X(\nu)\,d\mu\, d\nu = 0 \tag{58}$$

where $\lambda_F = \frac{\pi}{4}\rho k^3 \xi_0^2$

The Rayleigh-Ritz method will now be applied, assuming

$$X(\mu) = \sum_{r=1}^{\infty} c_r P_r(\mu)$$

Then (58) becomes

$$\omega^2 \sum_{r=1}^{\infty}\sum_{s=1}^{\infty}\left[c_r c_s \Lambda_{rs} - \lambda_F \sum_{n=1}^{\infty} \frac{Q_n}{Q_n'}\frac{2n+1}{n(n+1)} c_r c_s \int_{-1}^{1}\sqrt{1-\mu^2}\, P_n'(\mu)\, P_r(\mu)\, d\mu \cdot \int_{-1}^{1}\sqrt{1-\nu^2}\, P_n'(\nu)\, P_s(\nu)\, d\nu\right]$$

$$= \sum_{r=1}^{\infty}\sum_{s=1}^{\infty} K_{rs} c_r c_s \tag{59}$$

where

$$\Lambda_{rs} = \int_{-1}^{1} \lambda_s(\mu)\, P_r(\mu)\, P_s(\mu)\, d\mu$$

$$K_{rs} = \int_{-1}^{1}\int_{-1}^{1} k(\mu,\nu)\, P_r(\mu)\, P_s(\nu)\, d\mu\, d\nu$$

The integrals occurring in (59) can be simplified by means of the relations [8]

$$P_n'(\mu) = (1-\mu^2)^{1/2}\, \dot{P}_n(\mu)$$

$$(1-\mu^2)\,\dot{P}_n(\mu) = \frac{n(n+1)}{2n+1}\left(P_{n-1} - P_{n+1}\right)$$

Applying these yields

$$\int_{-1}^{1}\sqrt{1-\mu^2}\,P_n^1(\mu)\,P_r(\mu)\,d\mu = \int_{-1}^{1}(1-\mu^2)\,\dot{P}_n(\mu)\,P_r(\mu)\,d\mu$$

$$= \frac{n(n+1)}{2n+1}\int_{-1}^{1}(P_{n-1}-P_{n+1})\,P_r\,d\mu$$

Hence, summing with respect to r, we obtain

$$\sum_{r=1}^{\infty} c_r\int_{-1}^{1}\sqrt{1-\mu^2}\,P_n^1(\mu)P_r(\mu)\,d\mu = \frac{2n(n+1)}{2n+1}\left(\frac{c_{n-1}}{2n-1}-\frac{c_{n+1}}{2n+3}\right)$$

and the term containing the integrals in (59) reduces to

$$-4\lambda_F\sum_{n=1}^{\infty}\frac{Q_n^1}{\dot{Q}_n^1}\;\frac{n(n+1)}{2n+1}\left(\frac{c_{n-1}}{2n-1}-\frac{c_{n+1}}{2n+3}\right)^2 \qquad (60)$$

The condition that the eigenvalues ω^2 are stationary with respect to variations in the coefficients c_r, applied to (59) yields a set of homogeneous linear equations and the secular equation for determining the eigenvalues. The form of (60), which represents the contribution of the fluid inertia, indicates that there will be several nondiagonal elements due to the fluid in the secular equation. This will be studied in the continuation of the present work. In the remainder of this section the properties of the added-mass function will be developed.

Convergence of $A(\mu,\nu)$

For large values of n we have the asymptotic formulas [8]

$$\frac{Q_n^1}{\dot{Q}_n^1} \approx -\frac{2\sqrt{\zeta_0^2-1}}{2n+1}$$

$$P_n^1(\mu) \approx -\sqrt{\frac{n}{\pi\sin\theta}}\left(\cos\frac{2n+1}{2}\theta - \sin\frac{2n+1}{2}\theta\right)$$

$$P_n^1(\nu) \approx -\sqrt{\frac{n}{\pi\sin\theta'}}\left(\cos\frac{2n+1}{2}\theta' - \sin\frac{2n+1}{2}\theta'\right)$$

where $\mu = \cos\theta$, $\nu = \cos\theta'$. Hence the nth term of the series for $A(\mu,\nu)$ is, asymptotically,

$$\frac{2}{\pi(n+1)}\sqrt{\sin\theta\sin\theta'(\zeta_0^2-1)}\left[\cos\frac{2n+1}{2}(\theta-\theta') - \sin\frac{2n+1}{2}(\theta+\theta')\right]$$

Thus the series for $A(\mu,\nu)$ converges, except when $\mu = \nu$. Comparison with (30), the added-mass function for the infinite plate, indicates that the present function also has a logarithmic singularity at $\mu = \nu$ and consequently, if it were replaced by a quadrature formula, the resulting added-mass matrix A_{ij} would have a strong principal diagonal. This indicates that perturbation methods should be suitable for deriving corrected values of the frequencies from those obtained by strip theory.

Evaluation of $Q_n^1(\zeta)/\dot{Q}_n^1(\zeta)$

Tables of values of $Q_n^1(\zeta)$ and $\dot{Q}_n^1(\zeta)$, which are required for the numerical evaluation of the added-mass function (57), do not seem to be available for the values of ζ near unity associated with elongated spheroids. The functions $Q_n^1(\zeta)$ may be successively computed from the recurrence formula [8]

$$\left.\begin{aligned} n\, Q_{n+1}^1 &= (2n+1)\zeta Q_n^1 - (n+1) Q_{n-1}^1 \\ \text{where}\quad Q_1^1(\zeta) &= (\zeta^2-1)^{1/2}\left[\frac{1}{2}\ln\frac{\zeta+1}{\zeta-1} - \frac{\zeta}{\zeta^2-1}\right] \\ (\zeta^2-1)\, Q_2^1(\zeta) &= \frac{3}{2}(\zeta^2-1)\ln\frac{\zeta+1}{\zeta-1} + 2 - 3\zeta^2 \end{aligned}\right\} \qquad (61)$$

Also we have

$$(\zeta^2-1)\,\dot{Q}_n^1 = n\zeta Q_n^1 - (n+1) Q_{n-1}^1$$

and hence

$$\frac{Q_n^1}{\dot{Q}_n^1} = \frac{(\zeta^2-1)\, Q_n^1}{n\zeta Q_n^1 - (n+1) Q_{n-1}^1} \qquad (62)$$

An alternative form, which will be useful in deriving an asymptotic formula, is obtained by means of the relation [8]

$$Q_n^1(\zeta) = (\zeta^2-1)^{1/2}\dot{Q}_n(\zeta)$$

and

$$(\zeta^2-1)\,\dot{Q}_n(\zeta) = n(\zeta Q_n - Q_{n-1})$$

These yield the formula

$$\frac{Q_n^1}{\dot{Q}_n^1} = (\zeta^2-1)\,\frac{Q_n - Q_{n-1} + (\zeta-1)Q_n}{Q_{n-1} - Q_n + (\zeta-1)(Q_{n-1} + 2nQ_n) + (\zeta-1)^2 n Q_n} \tag{63}$$

From equations (58) and (59) values of $Q_n^1/\dot{Q}_n^1$ can be successively computed.

It is of interest to derive a series expansion of $Q_n(\zeta)$ about $\zeta = 1$ in order to obtain an asymptotic formula for $Q_n^1/\dot{Q}_n^1$ for values of ζ near unity. It will be shown that

$$Q_n(\zeta) = Q_0 - H_n^1 + \frac{n(n+1)}{2}(Q_0 - H_n^2)(\zeta-1) + \cdots + \frac{(n-r+1)\ldots(n+r)}{2^r(r!)^2}(Q_0 - H_n^r)(\zeta-1)^r + \cdots \tag{64}$$

where

$$Q_0(\zeta) = \frac{1}{2}\ln\frac{\zeta+1}{\zeta-1}\ , \qquad \zeta > 1$$

and

$$H_n^m = \sum_{r=m}^{n}\frac{1}{r}$$

It is interesting to observe the identity of the coefficients in (61) with those of the known expansion [10]

$$P_n(\zeta) = 1 + \frac{n(n+1)}{2}(\zeta-1) + \cdots + \frac{(n-r+1)\ldots(n+r)}{2^r(r!)^2}(\zeta-1)^r + \cdots \tag{65}$$

In proving (61) we will make use of the differential equation satisfied by the Legendre functions

$$\left.\begin{aligned}(\zeta^2-1)\,\ddot{P}_n(\zeta) - 2\zeta\dot{P}_n(\zeta) - n(n+1)P_n(\zeta) = 0\\ (\zeta^2-1)\,\ddot{Q}_n(\zeta) - 2\zeta\dot{Q}_n(\zeta) - n(n+1)Q_n(\zeta) = 0\end{aligned}\right\}$$

and the relation

$$Q_n = Q_0 P_n - W_{n-1}\ , \qquad W_{n-1} = \sum_{r=1}^{n}\frac{1}{r}P_{m-1}P_{n-r} \tag{66}$$

Let us also write $W_{n-1}(\zeta)$ as the Taylor series

$$W_{n-1}(\zeta) = \sum_{r=0}^{n-1} \frac{W_{n-1}^{(r)}(1)}{r!}(\chi-1)^r \tag{67}$$

Since $P_n(1) = 1$, we immediately have from (64)

$$W_{n-1}(1) = H_n' \tag{68}$$

From (66) and the above differential equation, we obtain

$$(\zeta^2-1)\ddot{W}_{n-1} + 2\zeta\dot{W}_{n-1} - n(n+1)W_{n-1} + \dot{P}_n = 0$$

which, by (65) and (67), can be expressed as the series in terms of $\lambda = \zeta-1$,

$$(\lambda+2)\sum_{r=2}^{n-1}\frac{W_{n-1}^{(r)}(1)}{(r-2)!}\lambda^{r-1} + 2(\lambda+1)\sum_{r=1}^{n-1}\frac{W_{n-1}^{(r)}(1)}{(r-1)!}\lambda^{r-1} - n(n+1)\sum_{r=0}^{n-1}\frac{W_{n-1}^{(r)}(1)}{r!}\lambda^r$$
$$+ 2\sum_{r=0}^{n}\frac{P_n^{(r)}(1)}{(r-1)!}\lambda^{r-1} = 0$$

Equating coefficients of λ^{r-1} now gives, after simplification,

$$W_{n-1}^{(r)}(1) = \frac{(n-r+1)(n+r)}{2r}W_{n-1}^{(r-1)}(1) - \frac{P_n^{(r)}(1)}{r}$$

or, from (65)

$$W_{n-1}^{(r)}(1) = \frac{(n-r+1)(n+r)}{2r}W_{n-1}^{(r-1)}(1) - \frac{(n-r+1)\cdots(n+r)}{r\cdot 2^r\cdot r!}$$

Thus, when $r = 1$, we have by (68)

$$\dot{W}_{n-1}(1) = \frac{n(n+1)}{2}\left[W_{n-1}(1) - 1\right] = \frac{n(n+1)}{2}H_n^2$$

and similarly, continuing successively with $r = 2, 3, \ldots$, we obtain

$$W_{n-1}^{(r)}(1) = \frac{(n-r+1)\cdots(n+r)}{2^r r!}H_n^{r+1} \tag{69}$$

It is now seen that (61) is derived by substituting the expansions (64) and (67) into (66).

The desired asymptotic formula for $Q_n^1/\dot{Q}_n^1$ is obtained by substituting (64) into (63) and neglecting powers of $(\zeta - 1)$ greater than the first. The result is

$$-\frac{1}{\zeta^2-1}\,\frac{Q_n^1}{\dot{Q}_n^1} \approx 1 - [2n(n+1)(Q_0 - H_n^1) + 1](\zeta - 1) \tag{70}$$

It does not appear to be useful to extend this asymptotic formula to higher powers of $(\zeta - 1)$ since, at the small values of n at which the formula gives sufficiently accurate results, better results are obtained from the recurrence relations (61) and (62). Equation (70) is valuable in that it displays the analytical behavior of $Q_n/\dot{Q}_n$ when n is small.

At large values of n the iteration procedure for computing $Q_n/\dot{Q}_n$ becomes inaccurate because of the accumulation of errors and an asymptotic formula suitable for large values of n is desirable. Available for this purpose is the expansion [11]

$$Q_n^m(\zeta_0) = e^{m\pi i}\,\frac{2^{m+n+1}(n+m)!}{1\cdot 3\cdots(2n+1)}\,\frac{z^{m-n}}{(z^2-1)^{m+1/2}}(\zeta_0^2-1)\,F\left(m+\tfrac{1}{2}, -m+\tfrac{1}{2}; n+\tfrac{3}{2}; \frac{1}{1-z^2}\right) \tag{71}$$

where

$$\zeta_0 = \cosh\xi\,, \qquad z = e^{-\xi}$$

and F is the hypergeometric series

$$F\left(m+\tfrac{1}{2}, -m+\tfrac{1}{2}; n+\tfrac{3}{2}; \frac{1}{1-z^2}\right) = 1 + \frac{(m+\frac{1}{2})(-m+\frac{1}{2})}{1\cdot(n+\frac{3}{2})(1-z^2)} - \frac{(m+\frac{1}{2})(m+\frac{3}{2})(-m+\frac{1}{2})(-m+\frac{3}{2})}{1\cdot 2\cdot(n+\frac{3}{2})(n+\frac{5}{2})(1-z^2)^2} + \cdots$$

The series (71) converges when $\zeta_0 > \frac{3}{4}\sqrt{2} = 1.061\ \ldots$ When $\zeta_0 < \frac{3}{4}\sqrt{2}$, the remainder after r terms of the series is less numerically than the $(r + 1)$th term; i.e., the series is asymptotic. The limiting value $\zeta_0 = \frac{3}{4}\sqrt{2}$ corresponds to a spheroid of length-diameter ratio $a/c = 3.0$, so that, for elongated spheroids, the asymptotic property of the series is very important.

For $m = 1$, the hypergeometric series becomes

$$F = 1 + \frac{3}{2(2n+3)}\,\frac{1}{(z^2-1)} - \frac{15}{8(2n+3)(2n+5)}\,\frac{1}{(z^2-1)^2} + \cdots$$

and hence, when $\zeta_0 < \frac{3}{4}\sqrt{2}$, the first two terms of F will give a good approximation provided the last term is very much less than unity. From the form of this last term it is seen that a nearly equivalent condition is

$$n + \frac{3}{2} \gg \frac{1}{\mathfrak{z}^2 - 1}$$

But from the relations

$$\zeta_0 = \frac{a}{k} = \frac{a}{\sqrt{a^2 - c^2}} = \frac{1}{2}\left(\mathfrak{z} + \frac{1}{\mathfrak{z}}\right)$$

we have

$$\frac{1}{\mathfrak{z}^2 - 1} = \frac{a}{2c} - \frac{1}{2}$$

Hence the above condition becomes

$$n \gg \frac{a}{2c} - 2 \tag{72}$$

For sufficiently large n we have then

$$\frac{Q'_{n-1}}{Q'_n} \approx \frac{n + \frac{1}{2}}{n+1} e^{\xi} \frac{1 + \frac{3}{2(2n+1)} \frac{1}{\mathfrak{z}^2 - 1}}{1 + \frac{3}{2(2n+3)} \frac{1}{\mathfrak{z}^2 - 1}} = \frac{2n+3}{2n+2} \frac{3\cosh\xi + (8n+1)\sinh\xi}{3\cosh\xi + (8n+9)\sinh\xi} e^{\xi}$$

and hence, from (60), we obtain after simplification

$$-\frac{Q'_n}{\dot{Q}'_n} \approx \frac{2(\zeta_0^2 - 1)}{(2n+1)\sqrt{\zeta_0^2 - 1} + \zeta_0 + \frac{6}{(8n+9)\sqrt{\zeta_0^2 - 1} + 3\zeta_0}} \tag{73}$$

Values of $Q_n^1/\dot{Q}_n^1$ calculated by the exact iteration formulas, (58) and (59), and by the two asymptotic formulas, (70) and (73), are compared in the following tables for spheroids of length-diameter ratios 7.1 and 71. In the former case it is seen that (70) agrees well with the values from the iteration formulas only up to $n = 3$; equation (73) becomes a good approximation for values of $n > 10$, and indeed yields values more reliable than those from the iteration formula, because of error accumulation, for $n > 40$. For the much more elongated body, $a/c = 71$, equation (70) yields acceptable results up to $n = 30$. At $n = 70$ the value from (73) is high by about 4 percent, the deviation from the value from the iteration formula reducing to about 2 percent at $n = 100$.

$$Q_n^{\ 1}/Q_n^{\ 1} ; \quad \zeta_0 = 1.01 , \quad a/c \approx 7.14$$

n	Eqs. (58, 59)	Eq. (70)	Eq. (73)
1	0.01860	0.01857	0.01584
2	0.01713	0.01712	0.01528
3	0.01572	0.01595	0.01446
4	0.01444	0.01533	0.01356
5	0.01330	0.01546	0.01268
6	0.01230	0.01649	0.01185
8	0.01065		0.01039
10	0.00935		0.00920
12	0.00832		0.00822
16	0.00679		0.00675
20	0.00573		0.00571
30	0.00410		0.00409
40	0.00322		0.00319
50			0.00261
60			0.00220
70			0.00191
80			0.00168

Q_n^1/Q_n^1 ; $\zeta_0 = 1.0001$, $a/c = 71$

n	Eqs. (58, 59)	Eq. (70)	Eq. (73)
1	0.0001997	0.0001997	0.0001388
2	0.0001992	0.0001992	0.0001399
3	0.0001985	0.0001985	0.0001414
4	0.0001977	0.0001977	0.0001427
5	0.0001969	0.0001969	0.0001439
6	0.0001958	0.0001958	0.0001449
8	0.0001936	0.0001936	0.0001466
10	0.0001912	0.0001911	0.0001478
12	0.0001886	0.0001885	0.0001486
16	0.0001830	0.0001829	0.0001492
20	0.0001773	0.0001772	0.0001485
30	0.0001630	0.0001644	0.0001437
40	0.0001497	0.0001956	0.0001365
50	0.0001377		0.0001286
60	0.0001272		0.0001206
70	0.0001179		0.0001131
80	0.0001097		0.0001070
90	0.0001024		0.0000997
100	0.0000960		0.0000941

One final interesting property of the added-mass function $A(\mu,\nu)$ in (57) will be derived. From the recurrence formula [8]

$$n\,P'_{n+1}(\mu) - (2n+1)\mu P'_n(\mu) + (n+1)P'_{n-1}(\mu) = 0$$

we have

$$\frac{2n+1}{n(n+1)}\,\mu\, P'_n(\mu) = \frac{P'_{n+1}(\mu)}{n+1} + \frac{P'_{n-1}(\mu)}{n}$$

$$\frac{2n+1}{n(n+1)}\,\nu\, P'_n(\nu) = \frac{P'_{n+1}(\nu)}{n+1} + \frac{P'_{n-1}(\nu)}{n}$$

Hence

$$\frac{2n+1}{n(n+1)}(\mu-\nu)P'_n(\mu)P'_n(\nu) = \frac{1}{n+1}\left[P'_{n+1}(\mu)P'_n(\nu) - P'_n(\mu)P'_{n+1}(\nu)\right] - \frac{1}{n}\left[P'_n(\mu)P'_{n-1}(\nu) - P'_{n-1}(\mu)P'_n(\nu)\right]$$

and consequently

$$\sum_{h=1}^{n}\frac{2h+1}{h(h+1)}(\mu-\nu)P'_h(\mu)P'_h(\nu) = \frac{1}{n+1}\left[P'_{n+1}(\mu)P'_n(\nu) - P'_n(\mu)P'_{n+1}(\nu)\right]$$
$$-\frac{1}{2}\left[P'_2(\mu)P'_1(\nu) - P'_1(\mu)P'_2(\nu)\right] + \frac{3}{2}(\mu-\nu)P'_1(\mu)P'_1(\nu)$$

But we have

$$P'_1(\mu) = (1-\mu^2)^{1/2}$$
$$P'_2(\mu) = 3\mu(1-\mu^2)^{1/2}$$

and then

$$\frac{1}{2}\left[P'_2(\mu)P'_1(\nu) - P'_1(\mu)P'_2(\nu)\right] = \frac{3}{2}(\mu-\nu)P'_1(\mu)P'_1(\nu)$$

Hence, finally

$$\sum_{h=1}^{n}\frac{2h+1}{h(h+1)}P'_h(\mu)P'_h(\nu) = \frac{P'_{n+1}(\mu)P'_n(\nu) - P'_n(\mu)P'_{n+1}(\nu)}{(n+1)(\mu-\nu)} \tag{74}$$

This gives the sum of the first n terms of the series for $A(\mu,\nu)$ when $\zeta_0 - 1 \ll 1$ and n is chosen sufficiently small so that $Q_n^1/\mathring{Q}_n^1 \approx 1$.

Kinetic Energy of a Vibrating Ship

The equation of the ship surface in its equilibrium position will be taken in the form

$$y = \pm f(x, z) \quad , \quad -1 \leqq x \leqq 1$$

where x , y , z are rectangular Cartesian coordinates with the x axis in the free surface, the z axis positive vertically upwards in the centerplane of symmetry of the hull, and the y axis completing the right-handed coordinate system. From (75) one obtains for the direction cosines ℓ_1 , ℓ_2 , ℓ_3 of the outward normal to the surface

$$\ell_1 = \frac{-f_x}{\sqrt{1+f_x^2+f_z^2}} \; , \quad \ell_2 = \pm \frac{1}{\sqrt{1+f_x^2+f_z^2}} \; , \quad \ell_3 = \frac{-f_z}{\sqrt{1+f_x^2+f_z^2}} \tag{75}$$

according as $y \gtrless 0$, where $f_x = \partial f/\partial x$, etc.

It will be assumed that the fluid is incompressible and inviscid, and that the flow is irrotational. These are reasonable assumptions for vibrations of very small amplitude. Then there exists a velocity potential which, for vibration in a particular mode at an angular frequency ω, may be written in the form

$$\Phi = \phi(x, y, z)\, e^{i\omega t}$$

The vibration will be assumed to be one of pure shear, expressible in the form

$$v(x,t) = V(x)\, e^{i\omega t}$$

It is required, then, to obtain a solution of Laplace's equation

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0$$

which on the surface (1) satisfies the boundary condition

$$\frac{\partial \phi}{\partial n} = \ell_3 V(x) \tag{76}$$

where n denotes distance in the direction of the outward normal, and on the free surface the condition

$$\phi = 0 \tag{77}$$

The amplitude of the kinetic energy of the fluid associated with the vibration is given by

$$2T = -\rho \int_S \phi \frac{\partial \phi}{\partial n} dS \tag{78}$$

the integration extending over the surface of the hull.

Two alternative methods for obtaining the kinetic energy of the fluid will now be formulated. The first method, in which it is assumed that the flow is generated by a distribution of sources on the centerplane, requires the solution of a Fredholm integral equation of the first kind; the second, in which a distribution of sources on the hull surface is sought, leads to one of the second kind. In principle, an exact solution for the latter equation exists and can be found by iteration. For the former, in general, only approximate solutions can be found, although for elongated bodies these approximations may be adequate for practical purposes. Furthermore, because of the avoidance of a mathematical singularity, the formulations of potential-flow problems by means of integral equations of the first kind have resulted in briefer numerical computing programs. There is a possibility, however, that a procedure for eliminating the singularity in the kernel of the integral equation of the second kind, which is presented in the following, may make the labor of solving this type integral equation more nearly comparable.

Distribution on Center Plane

The boundary condition $\phi = 0$ on the undisturbed position of the free surface, $y = 0$, can be satisfied at each instant by adding to the submerged portion of the hull its mirror image in the free surface and treating the double form as a single body, completely immersed in an infinite fluid, and undergoing vibration. In order to satisfy the boundary condition on the hull surface (76), assume that the flow is generated by a distribution of sources on the center plane of the hull and its image, of strength $m(\xi, \zeta)$, where ξ and ζ are used to denote values of x and z on the center plane, to distinguish from the coordinates of points on the hull surface.

In terms of this distribution the potential at an arbitrary point (x, y, z) of the fluid is given by

$$\phi = -\iint \frac{m(\xi, \zeta)}{R} d\xi d\zeta \tag{79}$$

where

$$R = \left[(x-\xi)^2+y^2+(z-\zeta)^2\right]^{1/2}$$

and the boundary condition (6) at a point of the surface S yields the Fredholm integral equation of the first kind

$$-\iint m(\xi,\zeta)\frac{\partial}{\partial n}\left(\frac{1}{R}\right)d\xi\, d\zeta = \ell_z V(x) \qquad (80)$$

In both (79) and (80) the integration extends over the surface of the center plane and its image in the free surface. The integral equation may also be expressed in the form

$$\left.\begin{aligned}&\iint m(\xi,\zeta)\, k(\xi,\zeta;x,z)\,d\xi\, d\zeta = V(x)\, f_z(x,z)\\ &\text{where}\\ &k(\xi,\zeta;x,z) = \frac{(x-\xi)f_x \mp f(x,z)+(z-\zeta)f_z}{[(x-\xi)^2+y^2+(z-\zeta)^2]^{3/2}}\end{aligned}\right\} \qquad (81)$$

If (81) were solved, its solution would be of the form

$$m(\xi,\zeta) = \iint h(\xi,\zeta;x,z)\, f_z(x,z)\, V(x)\, dx\, dz \qquad (82)$$

Through use of (76) and (79) one obtains for the kinetic energy of the fluid (78)

$$2T_F = \rho\iint A(x,x')\,V(x)\,V(x')\,dx\,dx'$$

where

$$A(x,x') = -\iiiint \frac{h(\xi,\zeta;x',z')}{[(x-\xi)^2+y^2+(z-\zeta)^2]^{1/2}}\, f_z f_{z'}\, d\xi\, d\zeta\, dz\, dz' \qquad (83)$$

Here the hull-surface elements dS occurring in (78) have been projected on the x-z plane, and only the projection from one side of the double hull form ($y \geqslant 0$) has been taken, since the energy calculated with the image included is twice the actual energy. The function $A(x,x')$ in (83) is the desired quantity from which an added-mass matrix can be obtained.

An explicit solution of (81) can be obtained for a very thin ship ($y \ll 1$). For this case the kernel of (80) behaves like a Dirac delta function and we obtain

$$-m(x,z) \iint \frac{\partial}{\partial n}\left(\frac{1}{R}\right) d\xi \, d\zeta \doteq \ell_z V(x) \qquad (84)$$

But by Gauss's flux theorem we have

$$-\left[\iint \frac{\partial}{\partial y}\left(\frac{1}{R}\right) d\xi \, d\zeta\right]_{y=0} = \pm 2\pi \qquad (85)$$

according as $y \lessgtr 0$, and, except near the borders of the center plane, the corresponding integrals containing derivatives with respect to y and z are zero. Hence, since

$$\left[\frac{\partial}{\partial n}\left(\frac{1}{R}\right)\right]_{y=0} = \ell_z \left[\frac{\partial}{\partial y}\left(\frac{1}{R}\right)\right]_{y=0}$$

we obtain from (84)

$$m(x,z) \doteq \frac{V(x)}{2\pi} f_z(x,z) \qquad (86)$$

Comparison of (86) with the form of the solution assumed in (82) shows that $h(\xi,\zeta;x,z)$ is a Dirac δ function which has the properties that it vanishes except when $\xi = x$, $\zeta = z$ and that

$$\iint h(\xi,\zeta;x,z)\, dx \, dz = -\frac{1}{2\pi} \qquad (87)$$

Hence we obtain for the added-mass function (83)

$$A(x,x') = \frac{1}{2\pi} \iint \frac{f_z f_{z'} \, dz \, dz'}{\left[(x-x')^2 + f^2(x,z) + (z-z')^2\right]^{1/2}} \qquad (88)$$

The equivalent matrix, obtained by replacing the double integral by a quadrature formula, would have strong diagonal elements $(x = x')$, although the non-diagonal elements would probably not be negligible.

Better approximations are obtained by solving (81) by iteration. Since $k(\xi,\zeta,x,z)$ peaks sharply in the neighborhood of $\xi = x$, $\zeta = z$, and is small elsewhere, it is seen that the value of the integral is affected only slightly by replacing $m(\xi, \zeta)$ by $m(x, z)$ in (81); i.e., the procedure which gives an exact solution when the kernel is a Dirac delta function will now be used to obtain a first approximation. Instead of (85) we now employ the actual value for the given ship form of the double integral of the kernel. This yields for the first approximation

$$\left.\begin{aligned} m_1(x,z) &= -\frac{H(x,z)}{2\pi} f_z(x,z)\, V(x) \\ \text{where} \qquad & \\ \frac{2\pi}{H(x,z)} &= -\iint k(\xi,\zeta;x,z)\, d\xi\, d\zeta \end{aligned}\right\} \tag{89}$$

and for the corresponding added-mass function

$$A_1(x,x') = \frac{1}{2\pi}\iint \frac{H(x',z')\, f_z\, f_{z'}\, dz\, dz'}{[(x-x')^2 + f^2(x,z) + (z-z')^2]^{1/2}} \tag{90}$$

In order to obtain a second approximation we write (81) in the form

$$\iint [m(\xi,\zeta) - m_1(\xi,\zeta)]\, k(\xi,\zeta;x,z)\, d\xi\, d\zeta$$
$$= V(x) f_z(x,z) - \iint m_1(\xi,\zeta)\, k(\xi,\zeta;x,z)\, d\xi\, d\zeta$$

Then, by the same reasoning as was used to derive $m_1(x, z)$, we obtain the second approximation

$$m_2(x,z) = m_1(x,z) - \frac{H(x,z)}{2\pi}\left[f_z V(x) - \iint m_1 k\, d\xi\, d\zeta\right] \tag{91}$$

or, from (89),

$$m_2(x,z) = 2\, m_1(x,z) - \iint \frac{H(x,z) H(\xi,\zeta)}{4\pi^2} f_\zeta(\xi,\zeta)\, k(\xi,\zeta;x,z)\, V(\xi)\, d\xi\, d\zeta \tag{92}$$

The resulting added-mass function is then, by (82), (83), and (90),

$$A_2(x,x') = 2A_1(x,x') + \iiiint \frac{H(\xi,\zeta)\, H(x',z')\, k(x',z';\xi,\zeta)\, f_z\, f_{z'}}{4\pi^2 [(x-\xi)^2 + f^2(x,z) + (z-\zeta)^2]^{1/2}}\, d\xi\, d\zeta\, dz\, dz' \tag{93}$$

This procedure could be continued to obtain additional successive approximations, but the second approximation will probably suffice.

Distribution on Hull Surface

It will now be assumed that the flow is generated by a distribution of sources on the surface of the hull and its image, of strength $m(\xi,\zeta)$. This gives for the potential at an arbitrary point (x, y, z) of the fluid

$$\left.\begin{aligned} \phi &= -\int_S \frac{m(\xi,\zeta)}{R}\,dS \\ \text{where} \qquad & \\ R &= \left[(x-\xi)^2+(y-\eta)^2+(z-\zeta)^2\right]^{1/2}, \quad \eta = f(\xi,\zeta) \end{aligned}\right\} \tag{94}$$

The boundary condition (6) at a point of the surface S now yields the Fredholm integral equation of the second kind

$$\left.\begin{aligned} 2\pi m(x,z) - \iint m(\xi,\zeta)\frac{\partial}{\partial n}\left(\frac{1}{R}\right)dS &= l_3 V(x) \\ \text{where} \qquad & \\ \frac{\partial}{\partial n}\left(\frac{1}{R}\right) &= -\frac{1}{R^3}\left[l_1(x-\xi)+l_2(y-\eta)+l_3(z-\zeta)\right] \end{aligned}\right\} \tag{95}$$

A procedure for eliminating or reducing the order of the singularity of the integrand in (95) when the points (x, y, z) and (ξ,η,ζ) coincide will now be indicated. By Gauss's flux theorem we have

$$\iint \frac{\partial}{\partial \nu}\left(\frac{1}{R}\right) dS = -2\pi \tag{96}$$

where ν denotes distance in the direction of the outward normal at a point (ξ,η,ζ) of the hull surface. Thus the integral equation (95) may be expressed in the form

$$4\pi m(x,z) - \iint\left[m(\xi,\zeta)\frac{\partial}{\partial n}\left(\frac{1}{R}\right) - m(x,z)\frac{\partial}{\partial \nu}\left(\frac{1}{R}\right)\right] dS = l_3 V(x) \tag{97}$$

which accomplishes the desired purpose.

An approximate solution of (32), obtained by neglecting the surface integral, is given by

$$m_1(x,\mathfrak{z}) = \frac{\ell_3 V(x)}{4\pi} \tag{98}$$

Successive approximations are then obtained from the iteration formula

$$4\pi\, m_{i+1} = \ell_3 V(x) + \iint \left[m_i(\xi,\zeta)\frac{\partial}{\partial n}\left(\frac{1}{R}\right) - m_i(x,\mathfrak{z})\frac{\partial}{\partial \nu}\left(\frac{1}{R}\right)\right] \frac{d\xi\, d\zeta}{\ell_2(\xi,\zeta)} \tag{99}$$

which yields a solution of the form

$$m(x,\mathfrak{z}) = \frac{1}{4\pi}\,\ell_3(x,\mathfrak{z})V(x) + \iint g(\xi,\zeta\,;x,\mathfrak{z})\,\frac{\ell_3(\xi,\zeta)}{\ell_2(\xi,\zeta)}\,V(\xi)\,d\xi\,d\zeta \tag{100}$$

Substituting (76) and (94) into (8), we obtain for the kinetic energy of the fluid

$$2T_F = 2\rho \iiiint \frac{m(x',\mathfrak{z}')\,\ell_3(x,\mathfrak{z})\,V(x)}{R(x,x';\mathfrak{z},\mathfrak{z}')\,\ell_2(x',\mathfrak{z}')\,\ell_2(x,\mathfrak{z})}\,dx\,dx'\,d\mathfrak{z}\,d\mathfrak{z}' ,$$

in which, by symmetry, only values of ℓ_2 corresponding to $y \gtrless 0$ need be taken, the resulting quadruple integral being multiplied by 2 instead of by 4, since the energy calculated with the image included is twice the actual energy. Substituting for $m(x,\mathfrak{z})$ from (100) gives then

$$2T_F = 2\rho \iiiint \frac{f_{\mathfrak{z}}(x,\mathfrak{z})\,V(x)}{R(x,x';\mathfrak{z},\mathfrak{z}')\,\ell_2(x',\mathfrak{z}')}\Big[\frac{\ell_3(x',\mathfrak{z}')}{4\pi}V(x') - \iint g(\xi,\zeta;x',\mathfrak{z}')\,f_{\zeta}(\xi,\zeta)\,V(\xi)\,d\xi\,d\zeta\Big]\,dx\,dx'\,d\mathfrak{z}\,d\mathfrak{z}'$$

which yields the added-mass function

$$A(x,x') = A_0(x,x') + 2\iiiint \frac{g(x',\mathfrak{z}';\xi,\zeta)\,f_{\mathfrak{z}}\,f_{\mathfrak{z}'}}{R(x,\xi,\mathfrak{z},\zeta)\,\ell_2(\xi,\zeta)}\,d\xi\,d\zeta\,d\mathfrak{z}\,d\mathfrak{z}' \tag{101}$$

where

$$A_0(x,x') = \iint \frac{f_{\mathfrak{z}}\,f_{\mathfrak{z}'}\,d\mathfrak{z}\,d\mathfrak{z}'}{2\pi R(x,x';\mathfrak{z},\mathfrak{z}')} \tag{102}$$

Summary and Recommendations

A dynamical theory of an elastic body vibrating in an incompressible, inviscid fluid ~~has been~~ WAS applied in five examples; ~~viz.,~~ a system of three degrees of freedom, an infinite plate, a taut string, a spheroid, and a ship. The case of a circular bar of finite length is separately considered in Part 3 of this Final Report. The first of these problems is solved by employing matri theory, the second and third by means of the method of Rayleigh-Ritz.

Since one of the purposes of the present work is to compare the Rayleigh-Ritz method with the proposed one of replacing surface integrals for the kinetic and potential energies by quadrature formulas so that matrix methods become applicable, it would be of interest to obtain solutions for the same problem by both methods. These can readily be performed for either the infinit plate or string. In the case of a vibrating spheroid, the mathematical properties required for application of either method have already been developed, and it remains to perform the numerical calculations for the eigenvalues. In the last case a way of applying the Rayleigh-Ritz method is not apparent, and only the formulation for the matrix method has been accomplished.

Various expressions for the added-mass function for a ship form, (88) (90), (93), (101), and (102), have been derived. It appears to be desirable to undertake a numerical study of these functions for a particular ship form, possibly one defined by simple mathematical formulas. The functions defined by (88), (90), and (102) are much simpler than those given by (93) and (101), although the latter two are more accurate and, in principle, that given by (101) is exact. A numerical study would indicate whether the simpler but less accurate forms were adequate for practical purposes.

It would also be of essential interest to compare the natural frequencies obtained with the added-mass matrices derived from the above added-mass functions with those computed by strip theory. This would show whether there is any practical advantage over strip theory of the present development of a more rational procedure for the inclusion of hydrodynamic inertia effects in vibration analyses.

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