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ON THE SOLUTIONS OF THE DIFFERENTIAL EQUATION  $y^{vi} = xy$ .

I. ANALYSIS

by

R. L. Duty and W. H. Reid

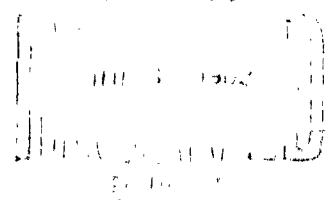
DIVISION OF APPLIED MATHEMATICS

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On the Solutions of the Differential Equation  $y^{v1} = xy$ .

I. Analysis\*

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Abstract

The differential equation  $y^{v1} = xy$  plays an important role in the asymptotic treatment of the stability of viscous flow between contra-rotating cylinders and, in one limiting case, solutions of this equation are required that remain bounded as  $x \rightarrow +\infty$ . A set of standard solutions have therefore been defined such that three of them, denoted by  $A_k(x)$ , are bounded as  $x \rightarrow +\infty$ , while the remaining three solutions, denoted by  $B_k(x)$ , are unbounded as  $x \rightarrow +\infty$ . The contour integral representations of these solutions are given, together with their power-series and asymptotic expansions. It is also shown that a slightly modified set of these solutions provide a "numerically satisfactory" set over the entire interval  $-\infty < x < +\infty$ .

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## 1. Introduction.

In a recent study of the stability of viscous flow between rotating cylinders (Duty & Reid 1961) it has been shown that, when the cylinders rotate in opposite directions, the solution can be expressed asymptotically in terms of the solutions of the comparison equation

$$y^{vi} = xy. \quad (1)$$

This equation clearly plays much the same role in the present theory as Airy's equation ( $y''=xy$ ) does in the asymptotic theory of second-order differential equations with a simple turning point and, once a suitable set of solutions have been defined and tabulated, the solution of the stability problem can be made explicit. For the present purposes it is sufficient to consider only real values of  $x$  though, more generally, if allowance were to be made for the possibility of overstable modes of instability, it would then be necessary to extend the analysis to complex values of  $x$ .

There are, of course, many different ways of defining a standard set of solutions of equation (1). Heading (1957), for example, has recently studied a class of differential equations which includes equation (1) as a particular case. The solutions defined by Heading, however, are not the most convenient ones for the present purposes since they are not all real for real values of  $x$ . We shall instead find it convenient to define a standard set of solutions in such a way that three of them remain bounded

as  $x \rightarrow +\infty$  (the remaining three being then necessarily unbounded as  $x \rightarrow +\infty$ ). The solutions defined in this way are thus analogous to the solutions of Airy's equations usually denoted by  $Ai(x)$  and  $Bi(x)$ .

The relevance of equation (1) to the problem of the stability of Couette flow was first recognized by Meksyn (1946, 1961) who attempted an asymptotic treatment of the problem on the hypothesis that the wave-number of the disturbance (rather than the Taylor number) is large. Although there are many superficial similarities between these two approaches, there still remain some important differences that have not been fully resolved. Furthermore, Meksyn's analysis of both the stability problem and the comparison equation was limited to the leading terms in the asymptotic expansions of the solutions and so gave no information on their behavior near the turning point.

## 2. Power-series solutions.

Six fundamental solutions of equation (1) can be found in the form of power-series by elementary methods (cf. Heading 1957). They can conveniently be written in the form

$$y_n(x) = x^{n-1} \sum_{s=0}^{\infty} a_s x^{7s} \quad (n=1,2,\dots,6) \quad (2)$$

where  $a_0 = 1$

and  $a_s = \frac{(n-1)!}{(n+7s-1)!} \prod_{p=0}^{s-1} (n+7p)$  for  $s \geq 1$ .

We now wish to determine those combinations of the power-series solutions that remain bounded as  $x \rightarrow +\infty$  and for this purpose we turn to a discussion of the solution of equation (1) by the method of contour integration.

### 3. Contour integral solutions.

The solution of equation (1) by means of a Laplace contour integral leads to the integral representation

$$y(x) = \int_C \exp(xt - t^7/7) dt, \quad (3)$$

where  $C$  is any open contour that starts in one and ends in another of the shaded sectors shown in figure 1. If we now let  $L_{rs}$  denote a contour starting in sector  $r$  and ending in sector  $s$  then, since the integrand of (3) has no singularity in the finite  $t$ -plane, all contours for given values of  $r$  and  $s$  are equivalent by Cauchy's theorem.

We shall be particularly concerned with contours  $L_{rs}$  for which  $s=r+1$  and  $r=1, 2, \dots, 7$  with  $L_{78} \equiv L_{71}$ . For such a contour we will denote the solution by

$$u_r(x) = \int_{L_{rs}} \exp(xt - t^7/7) dt \quad (s=r+1). \quad (4)$$

Note that in this way we have defined seven solutions. But again, by Cauchy's theorem, we have

$$\sum_{r=1}^7 u_r(x) = \oint \exp(xt - t^7/7) dt = 0.$$

We now wish to obtain the asymptotic behavior of the solutions (4) as  $x \rightarrow +\infty$  for we can then define a set of standard solutions that are real when  $x$  is real and that have the desired behavior as  $x \rightarrow +\infty$ .



4. The asymptotic expansions of the solutions as  $x \rightarrow +\infty$ .

In the application of the saddle-point method (or, more generally, the method of steepest descents) to the solutions (4), it is convenient to let  $t = x^{1/6}\tau$  so that

$$u_r(x) = x^{1/6} \int_{L_{rs}} \exp \{x^{7/6} w(\tau)\} d\tau \quad (s=r+1), \quad (5)$$

where 
$$w(\tau) = \tau - \tau^7/7. \quad (6)$$

The saddle-points, which are located at the roots of  $w'(\tau) = 0$ , therefore lie on the unit circle in the  $\tau$ -plane at the positions given by the six roots of  $+1$  (see table 1). If these saddle-points are denoted by  $\tau_1$  ( $i=0,1,2,\dots,5$ ) then we may note that

$$w(\tau_1) = \frac{6}{7} \tau_1 \quad \text{and} \quad w''(\tau_1) = -6\tau_1^*, \quad (7)$$

where  $\tau_1^*$  denotes the complex conjugate of  $\tau_1$ . If we also let

$$w = u+iv \quad \text{and} \quad \tau = \xi+i\eta, \quad (8)$$

$$\left. \begin{aligned} \text{then} \quad u(\xi, \eta) &= \xi - \frac{1}{7} \xi(\xi^6 - 21\xi^4\eta^2 + 35\xi^2\eta^4 - 7\eta^6) \\ \text{and} \quad v(\xi, \eta) &= \eta + \frac{1}{7} \eta(\eta^6 - 21\eta^4\xi^2 + 35\eta^2\xi^4 - 7\xi^6). \end{aligned} \right\} \quad (9)$$

The curves  $u(\xi, \eta) = \text{constant}$  form a family of contour lines in the  $\tau$ -plane and the curves  $v(\xi, \eta) = \text{constant}$  are their orthogonal trajectories (except at the saddle-points). In particular, the steepest paths through  $\tau_1$  are given by

$$v(\xi, \eta) = v(\xi_1, \eta_1). \quad (10)$$

The explicit determination of the contour lines and steepest paths that pass through the saddle-points is clearly a complicated matter and only a qualitative picture of them, adequate for the present purposes, is shown in figure 2.

When a path  $L_{rs}$  (with  $s$  not necessarily equal to  $r+1$ ) is deformed into a path of steepest descent,\* it may pass through one or more saddle-points. The dominant contribution to the solution associated with such a path then comes from the highest saddle-point on the path. Since the elevations of the saddle-points are determined by  $\text{re}\{w(\tau_i)\}$ , their relative heights can be ordered according to the following scheme:

$$\left. \begin{array}{l} \tau_0: +1 \\ \tau_1 \text{ and } \tau_5: +\frac{1}{2} \\ \tau_2 \text{ and } \tau_4: -\frac{1}{2} \\ \tau_3: -1 \end{array} \right\} \quad (11)$$

Consider now the solution  $u_1(x)$  associated with the path  $L_{12}$ . This path can be directly deformed into a steepest descents path passing through  $\tau_3$  and, by the saddle-point method, we then obtain

$$u_1(x) \sim 1(\pi/3)^{1/2} x^{-5/12} \exp(-\frac{6}{7} x^{7/6}). \quad (12)$$

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\* It may not, in fact, be so deformable and in such cases an equivalent path must be chosen that is deformable into a steepest descents path.

The descending series associated with this solution could be found, if necessary, by the method of steepest descents but this is somewhat difficult and not necessary for the present purposes. Although we are concerned here with real values of  $x$  only, it can be shown (cf. Heading 1957) that the expansion (12) remains valid (in the strict sense) in  $|\arg x| < 2\pi/7$ . In the Poincare sense, however, it is valid in the larger range  $|\arg x| < \pi$ .

The solutions  $u_2(x)$  and  $u_7(x)$  associated with the paths  $L_{23}$  and  $L_{71}$  respectively can be treated in a similar manner to yield

$$u_2(x) \sim + e^{+\pi i/3} (\pi/3)^{1/2} x^{-5/12} \exp(+\frac{6}{7} \tau_2 x^{7/6}) \quad (13)$$

$$\text{and } u_7(x) \sim - e^{-\pi i/3} (\pi/3)^{1/2} x^{-5/12} \exp(+\frac{6}{7} \tau_4 x^{7/6}). \quad (14)$$

Since  $\text{re}(\tau_2) = \text{re}(\tau_4) = -\frac{1}{2}$ , these two solutions are dominant with respect to  $u_1(x)$  but they are subdominant (though not maximally so) with respect to any solution whose path starts or ends in sectors 4, 5, or 6.

Thus, from these results, it is clear that any set of solutions that is bounded as  $x \rightarrow +\infty$  must be formed from linear combinations of  $u_1(x)$ ,  $u_2(x)$ , and  $u_7(x)$ . The further requirement that the solutions be real for real values of  $x$  then suggests the choice:

$$\left. \begin{aligned} A_1(x) &= \frac{1}{2\pi i} u_1, \\ A_2(x) &= \frac{1}{2\pi i} (u_2 + u_7), \\ \text{and } A_3(x) &= \frac{1}{2\pi} (u_2 - u_7). \end{aligned} \right\} \quad (15)$$

The leading terms in the asymptotic expansions of these solutions are then given by

$$\left. \begin{aligned} A_1(x) &\sim \frac{1}{2}(3\pi)^{-1/2}x^{-5/12} \exp\left(-\frac{6}{7}x^{7/6}\right), \\ A_2(x) &\sim (3\pi)^{-1/2}x^{-5/12} \exp\left(-\frac{3}{7}x^{7/6}\right)\sin\left(\frac{3\sqrt{3}}{7}x^{7/6} + \frac{\pi}{3}\right), \\ \text{and} \\ A_3(x) &\sim (3\pi)^{-1/2}x^{-5/12} \exp\left(-\frac{3}{7}x^{7/6}\right)\cos\left(\frac{3\sqrt{3}}{7}x^{7/6} + \frac{\pi}{3}\right). \end{aligned} \right\} (16)$$

Although the choice (15) would appear to be the most natural one in view of the symmetry that has been achieved, these solutions do not, unfortunately, form a "numerically satisfactory" set of solutions for large negative values of  $x$ . This matter will be examined more fully in the following section where advantage will be taken of the fact that an arbitrary multiple of  $A_1(x)$  can be added to  $A_2(x)$  or  $A_3(x)$  without changing the dominant series in their asymptotic expansions.

To complete the set of standard solutions we must now consider the solutions  $u_3, u_4, u_5$ , and  $u_6$ . Consider first the solution  $u_3$  associated with the path  $L_{34}$ . This path can be deformed into a steepest descents path but it is seen from figure 2 that it must then pass through two saddle-points with a right angle turn at the lower one. The contribution from the saddle-point at  $\tau_2$  is subdominant so that if we retain only the dominant term in the expansion arising from  $\tau_1$  then we have

$$u_3(x) \sim + e^{+\pi i/6} (\pi/3)^{1/2} x^{-5/12} \exp(+\frac{6}{7} \tau_1 x^{7/6}). \quad (17)$$

In an exactly similar way we obtain

$$u_6(x) \sim - e^{-\pi i/6} (\pi/3)^{1/2} x^{-5/12} \exp(+\frac{6}{7} \tau_5 x^{7/6}). \quad (18)$$

Consider next the solution  $u_4$  associated with the path  $L_{45}$ . This path is not directly deformable into a steepest descents path and we must therefore choose an equivalent path that is so deformable. A suitable equivalent path is seen from figure 2 to be  $L_{42} + L_{25}$ , each of which is deformable into a steepest descents path. Each of these paths must again pass through two saddle-points with a right angle turn at the lower one. If only the dominant term is retained then we have

$$u_4(x) \sim + (\pi/3)^{1/2} x^{-5/12} \exp(+\frac{6}{7} x^{7/6}). \quad (19)$$

In a similar way we obtain

$$u_5(x) \sim - (\pi/3)^{1/2} x^{-5/12} \exp(+\frac{6}{7} x^{7/6}). \quad (20)$$

From these results we can now define a set of standard solutions that are real and unbounded as  $x \rightarrow +\infty$ . The form of the asymptotic expansions (17) to (20) suggests the following choice:

$$\left. \begin{aligned} B_1(x) &= \frac{1}{2\pi} (u_4 - u_5), \\ B_2(x) &= \frac{1}{2\pi i} (u_3 + u_6), \\ \text{and } B_3(x) &= \frac{1}{2\pi} (u_3 - u_6). \end{aligned} \right\} \quad (21)$$

The leading terms in the asymptotic expansions of these solutions are then given by

$$\left. \begin{aligned} B_1(x) &\sim (3\pi)^{-1/2} x^{-5/12} \exp\left(+\frac{6}{7} x^{7/6}\right), \\ B_2(x) &\sim (3\pi)^{-1/2} x^{-5/12} \exp\left(+\frac{3}{7} x^{7/6}\right) \sin\left(\frac{3\sqrt{3}}{7} x^{7/6} + \frac{\pi}{6}\right), \\ \text{and} \\ B_3(x) &\sim (3\pi)^{-1/2} x^{-5/12} \exp\left(+\frac{3}{7} x^{7/6}\right) \cos\left(\frac{3\sqrt{3}}{7} x^{7/6} + \frac{\pi}{6}\right). \end{aligned} \right\} (22)$$

Having thus tentatively defined a set of standard solutions with the desired asymptotic behavior as  $x \rightarrow +\infty$ , we turn now to a study of their behavior as  $x \rightarrow -\infty$ .

5. The asymptotic expansions of the solutions as  $x \rightarrow -\infty$ .

In discussing the behavior of the solutions as  $x \rightarrow -\infty$  it is convenient to first replace  $x$  by  $-x$  in equation (4) so that

$$u_r(-x) = \int_{L_{rs}} \exp(-xt - t^7/7) dt \quad (s=r+1). \quad (23)$$

We again let  $t = x^{1/6}\tau$  so that

$$u_r(-x) = x^{1/6} \int_{L_{rs}} \exp\{x^{7/6}w(\tau)\} d\tau, \quad (24)$$

where  $w(\tau)$  now has the meaning

$$w(\tau) = -\tau - \tau^7/7. \quad (25)$$

The saddle-points therefore still lie on the unit circle in the  $\tau$ -plane but at the positions given by the six roots of  $-1$  (see table 2). The discussion of the level curves and steepest paths through these saddle points then proceeds as before and leads to the qualitative picture shown in figure 3. It is useful to note, however, that the relative heights of the saddle-points are now given by the following scheme:

$$\left. \begin{array}{l} \tau_2 \text{ and } \tau_3: +1 \\ \tau_1 \text{ and } \tau_4: 0 \\ \tau_0 \text{ and } \tau_5: -1 \end{array} \right\} \quad (26)$$

Consider now the solution  $A_1(-x)$  which is still associated with the path  $L_{12}$ . This path is no longer directly

deformable into a steepest descents path, and it is seen from figure 3 that we must therefore choose the equivalent path

$$L_{12} = L_{15} + L_{52}, \quad (27)$$

where the paths  $L_{15}$  and  $L_{52}$  are so deformable. They each pass through three saddle-points and make right angle turns at two of them. The contributions to the asymptotic expansion of  $u_1(x)$  from the saddle-points at  $\tau_2$  and  $\tau_3$  are dominant compared to the contributions from the other saddle-points.

Thus we have

$$A_1(-x) \sim (3\pi)^{-1/2} x^{-5/12} \exp\left(+\frac{3\sqrt{3}}{7} x^{7/6}\right) \sin\left(\frac{3}{7} x^{7/6} + \frac{\pi}{12}\right), \quad (28)$$

where we have omitted the neutral contributions from the saddle-points at  $\tau_1$  and  $\tau_4$  and the subdominant contributions from  $\tau_0$  and  $\tau_5$ .

Consider next the solution  $A_2(-x)$  associated with the paths  $L_{23}$  and  $L_{71}$ . These paths are directly deformable into steepest descents paths, each of which then passes through two saddle-points with right angle turns at one of them. The saddle-points at  $\tau_2$  and  $\tau_3$  therefore make the same contribution (except for a change of sign) to  $A_2(-x)$  as they do to  $A_1(-x)$  and thus we have

$$A_2(-x) \sim -A_1(-x). \quad (29)$$

This does not mean, of course, that  $A_1(-x)$  and  $A_2(-x)$  become



linearly dependent\* as  $x \rightarrow +\infty$ , but only that they differ by an amount that is exponentially small compared to either of them. To investigate this situation more precisely, consider their sum which can be written in the form

$$A_1(-x) + A_2(-x) = \frac{1}{2\pi i} \int_{L_7} \exp(-xt - t^7/7) dt. \quad (30)$$

The major contributions to (30) come from the saddle-points at  $\tau_1$  and  $\tau_4$ , the contributions from  $\tau_0$  and  $\tau_5$  being subdominant by comparison, and yield a neutral expansion, the leading term of which is

$$A_1(-x) + A_2(-x) \sim (3\pi)^{-1/2} x^{-5/12} \sin\left(\frac{6}{7} x^{7/6} + \frac{\pi}{4}\right). \quad (31)$$

This situation does not occur for  $A_3(-x)$ , since the direction of integration along one of the paths is reversed, and we have immediately the dominant term

$$A_3(-x) \sim (3\pi)^{-1/2} x^{-5/12} \exp\left(+\frac{3\sqrt{3}}{7} x^{7/6}\right) \cos\left(\frac{3}{7} x^{7/6} + \frac{\pi}{12}\right). \quad (32)$$

This solution therefore has the same exponential behavior as  $A_1(-x)$  but differs from it by  $\pi/2$  in phase.

Turning now to the B-type solutions we encounter no further difficulties. In the case of  $B_1(-x)$ , for example, we find that it is subdominant (since only the saddle-points at  $\tau_0$  and  $\tau_5$  are involved), with leading term

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\* Numerically, of course, they become indistinguishable, and this is confirmed by the existing tables (Hughes & Reid 1961). The set of bounded solutions  $A_k(-x)$  ( $k=1,2,3$ ) do not therefore form a "numerically satisfactory" set. A method for avoiding this difficulty is discussed in section 7.

$$B_1(-x) \sim (3\pi)^{-1/2} x^{-5/12} \exp(-\frac{3\sqrt{3}}{7} x^{7/6}) \cos(\frac{3}{7} x^{7/6} + \frac{5\pi}{12}). \quad (33)$$

Similarly  $B_2(-x)$  and  $B_3(-x)$  are neutral (because of the contributions from  $\tau_1$  and  $\tau_4$ ), with leading terms

$$\left. \begin{aligned} B_2(-x) &\sim - (3\pi)^{-1/2} x^{-5/12} \sin(\frac{6}{7} x^{7/6} + \frac{\pi}{4}) \\ \text{and } B_3(-x) &\sim + (3\pi)^{-1/2} x^{-5/12} \cos(\frac{6}{7} x^{7/6} + \frac{\pi}{4}). \end{aligned} \right\} \quad (34)$$

On comparing these results with equation (31) we see that

$$B_2(-x) \sim - A_1(-x) - A_2(-x) \quad (35)$$

or, more precisely, that

$$A_1(-x) + A_2(-x) + B_2(-x) = \frac{1}{2\pi i} \int_{L_{64}} \exp(-xt - t^7/7) dt \quad (36)$$

and this is asymptotic to the subdominant term

$$(3\pi)^{-1/2} x^{-5/12} \exp(-\frac{3\sqrt{3}}{7} x^{7/6}) \sin(\frac{3}{7} x^{7/6} + \frac{5\pi}{12}). \quad (37)$$

Thus, the combination of solutions (36) has the same exponential behavior as  $B_1(-x)$  but differs from it by  $\pi/2$  in phase.

6. The power-series for the standard solutions.

In order to obtain the power series representation for the standard solutions defined in section 4, we now consider a path  $L_{rs}$  (with  $s$  not necessarily equal to  $r+1$ ) to consist of two rays from the origin which bisect the sectors  $r$  and  $s$ . The sense of the path is from infinity to the origin along the ray that bisects sector  $r$  and from the origin to infinity along the ray that bisects sector  $s$ . If we now define the functions  $I_k(x)$  by the relation

$$I_k(x) = \frac{1}{2\pi i} \int_0^{\infty} \frac{\exp(2\pi i k/7)}{\exp(xt - t^7/7)} dt \quad (38)$$

( $k=0,1,2,\dots,6$ ) then the standard solutions can be expressed in the forms

$$\left. \begin{aligned} A_1 &= I_3 - I_4 \\ A_2 &= I_2 - I_3 + I_4 - I_5 \\ A_3 &= 1(I_2 - I_3 - I_4 + I_5) \end{aligned} \right\} \quad (39)$$

and

$$\left. \begin{aligned} B_1 &= 1(2I_0 - I_1 - I_6) \\ B_2 &= I_1 - I_2 + I_5 - I_6 \\ B_3 &= 1(I_1 - I_2 - I_5 + I_6). \end{aligned} \right\} \quad (40)$$

From the definition of  $I_k(x)$  it is evident that

$$I_k(x) = e^{2\pi i k/7} I_0(xe^{2\pi i k/7}), \quad (41)$$

and this relation can be used to express all of the functions

$I_k(x)$  (for  $k \neq 0$ ) appearing in equations (39) and (40) in terms of  $I_0(x)$  which can, in turn, be expressed as the infinite series

$$I_0(x) = \frac{1}{2\pi i} 7^{-6/7} \sum_{r=0}^{\infty} \left(\frac{r-6}{7}\right)! \frac{(7^{1/7}x)^r}{r!} . \quad (42)$$

By using these results we can now relate the standard solutions to the power-series  $y_n(x)$  defined in section 2 in the following manner:

$$\left. \begin{aligned} A_1(x) &= \sum_{n=1}^6 a_n \sin(6n\pi/7) y_n(x) \\ A_2(x) &= \sum_{n=1}^6 a_n \{ \sin(4n\pi/7) - \sin(6n\pi/7) \} y_n(x) \\ A_3(x) &= \sum_{n=1}^6 a_n \{ \cos(4n\pi/7) - \cos(6n\pi/7) \} y_n(x) \end{aligned} \right\} (43)$$

$$\text{and} \quad \left. \begin{aligned} B_1(x) &= \sum_{n=1}^6 a_n \{ 1 - \cos(2n\pi/7) \} y_n(x) \\ B_2(x) &= \sum_{n=1}^6 a_n \{ \sin(2n\pi/7) - \sin(4n\pi/7) \} y_n(x) \\ B_3(x) &= \sum_{n=1}^6 a_n \{ \cos(2n\pi/7) - \cos(4n\pi/7) \} y_n(x), \end{aligned} \right\} (44)$$

where the coefficients  $a_n$  are given by

$$a_n = \frac{7^{(n-7)/7}}{\pi(n-1)!} \left(\frac{n-7}{7}\right)! \quad (45)$$

It is also interesting to note that the particular combinations of these solutions considered in equations (30) and (36) are given by the series

$$\left. \begin{aligned} A_1(x)+A_2(x) &= \sum_{n=1}^6 a_n \sin(4n\pi/7)y_n(x) \\ \text{and } A_1(x)+A_2(x)+B_2(x) &= \sum_{n=1}^6 a_n \sin(2n\pi/7)y_n(x). \end{aligned} \right\} \quad (46)$$

The Wronskian of the solutions (43) and (44), which is, of course, a constant, is given by

$$W(A_1, A_2, A_3, B_1, B_2, B_3) = -1/\pi^3. \quad (47)$$

Some relations which are useful in checking the numerical values of the constants appearing in equations (43) and (44) are

$$\left. \begin{aligned} \prod_{n=1}^6 \left(\frac{n-7}{7}\right)! &= \frac{8\pi^3}{7}, \\ \sin \frac{2\pi}{7} \sin \frac{4\pi}{7} \sin \frac{6\pi}{7} &= \frac{\sqrt{7}}{8}, \\ \cos \frac{2\pi}{7} \cos \frac{4\pi}{7} \cos \frac{6\pi}{7} &= \frac{1}{8}, \\ \sin \frac{2\pi}{7} + \sin \frac{4\pi}{7} - \sin \frac{6\pi}{7} &= \frac{\sqrt{7}}{2}, \\ \cos \frac{2\pi}{7} + \cos \frac{4\pi}{7} + \cos \frac{6\pi}{7} &= -\frac{1}{2}. \end{aligned} \right\} \quad (48)$$

## 7. Discussion

The standard solutions  $A_k(x)$  and  $B_k(x)$  ( $k=1,2,3$ ) defined in section 3 were the ones tabulated by Hughes & Reid (1961) and used by Duty & Reid (1961) in their study of the stability of Couette flow. Although the results obtained would seem to be entirely satisfactory, it is now clear from the discussion given in section 5 (cf. equation (29) in particular) that the bounded solutions  $A_k(x)$  do not form a "numerically satisfactory" set of solutions for large negative values of  $x$ .

This situation can be improved, however, if we replace  $A_2(x)$  by

$$\bar{A}_2(x) = A_1(x) + A_2(x). \quad (49)$$

The dominant term in the asymptotic expansion of  $\bar{A}_2(x)$  as  $x \rightarrow +\infty$  is, of course, the same as for  $A_2(x)$  but, as  $x \rightarrow -\infty$ ,  $\bar{A}_2(x)$  is asymptotically neutral and, hence, the set of solutions  $A_1, \bar{A}_2, A_3$  do form a numerically satisfactory set. Furthermore, if a complete set of six solutions are required then  $B_2(x)$  is no longer satisfactory either (cf. equation (35)) and we should instead use

$$\bar{B}_2(x) = \bar{A}_1(x) + B_2(x). \quad (50)$$

The asymmetrical manner in which the solutions  $\bar{A}_2$  and  $\bar{B}_2$  have been defined here would appear to be unavoidable if we wish to have a set of six solutions that are numerically

satisfactory over the entire interval  $-\infty < x < +\infty$ .

Finally, it should be remarked that because of the oscillatory behavior of these solutions it would be advantageous to develop the equivalent representation of them in terms of phase and amplitude functions along the lines discussed by Miller (1946) for the Airy functions.

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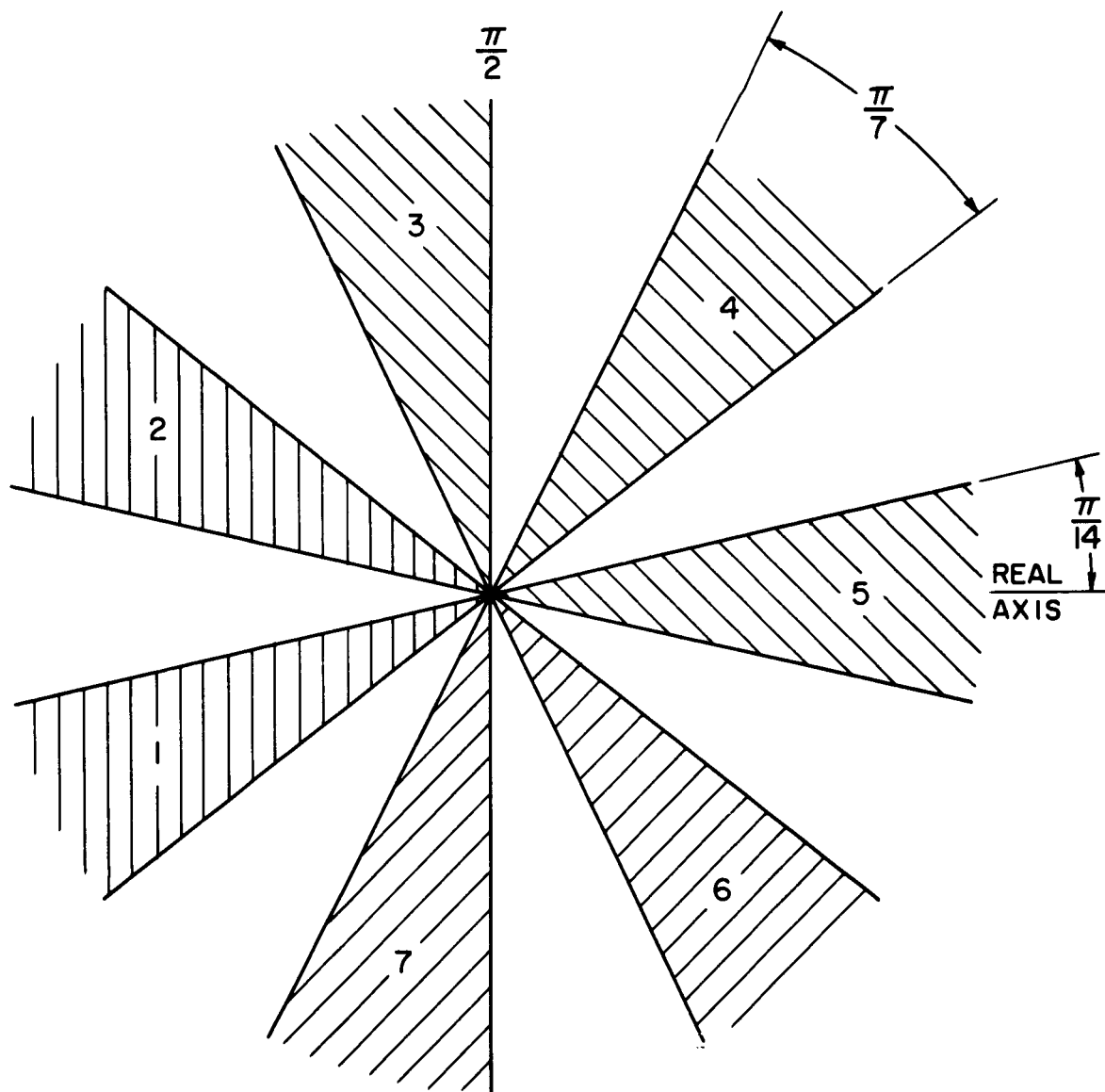


Table 1. The saddle-point configuration for  $x > 0$ .

| Saddle-point | Angular argument of the position of the saddle-point | Angle which the paths of steepest descent make with the positive real $\tau$ -axis |
|--------------|--|--|
| $\tau_0$     | 0  | 0 and $\pi$  |
| $\tau_1$     | $\pi/3$  | $\pi/6$ and $-5\pi/6$  |
| $\tau_2$     | $2\pi/3$   | $\pi/3$ and $-2\pi/3$  |
| $\tau_3$     | $\pi$  | $\pm\pi/2$   |
| $\tau_4$     | $-2\pi/3$  | $2\pi/3$ and $-\pi/3$  |
| $\tau_5$     | $-\pi/3$   | $5\pi/6$ and $-\pi/6$  |

Table 2. The saddle-point configuration for  $x < 0$ .

| Saddle-point | Angular argument of the position of the saddle-point | Angle which the paths of steepest descent make with the positive real $\tau$ -axis |
|--------------|--|--|
| $\tau_0$     | $\pi/6$  | $7\pi/12$ and $-5\pi/12$   |
| $\tau_1$     | $\pi/2$  | $3\pi/4$ and $-\pi/4$  |
| $\tau_2$     | $5\pi/6$   | $11\pi/12$ and $-\pi/12$   |
| $\tau_3$     | $-5\pi/6$  | $\pi/12$ and $-11\pi/12$   |
| $\tau_4$     | $-\pi/2$   | $\pi/4$ and $-3\pi/4$  |
| $\tau_5$     | $-\pi/6$   | $5\pi/12$ and $-7\pi/12$   |

FIGURE 1 The  $\tau$ -plane

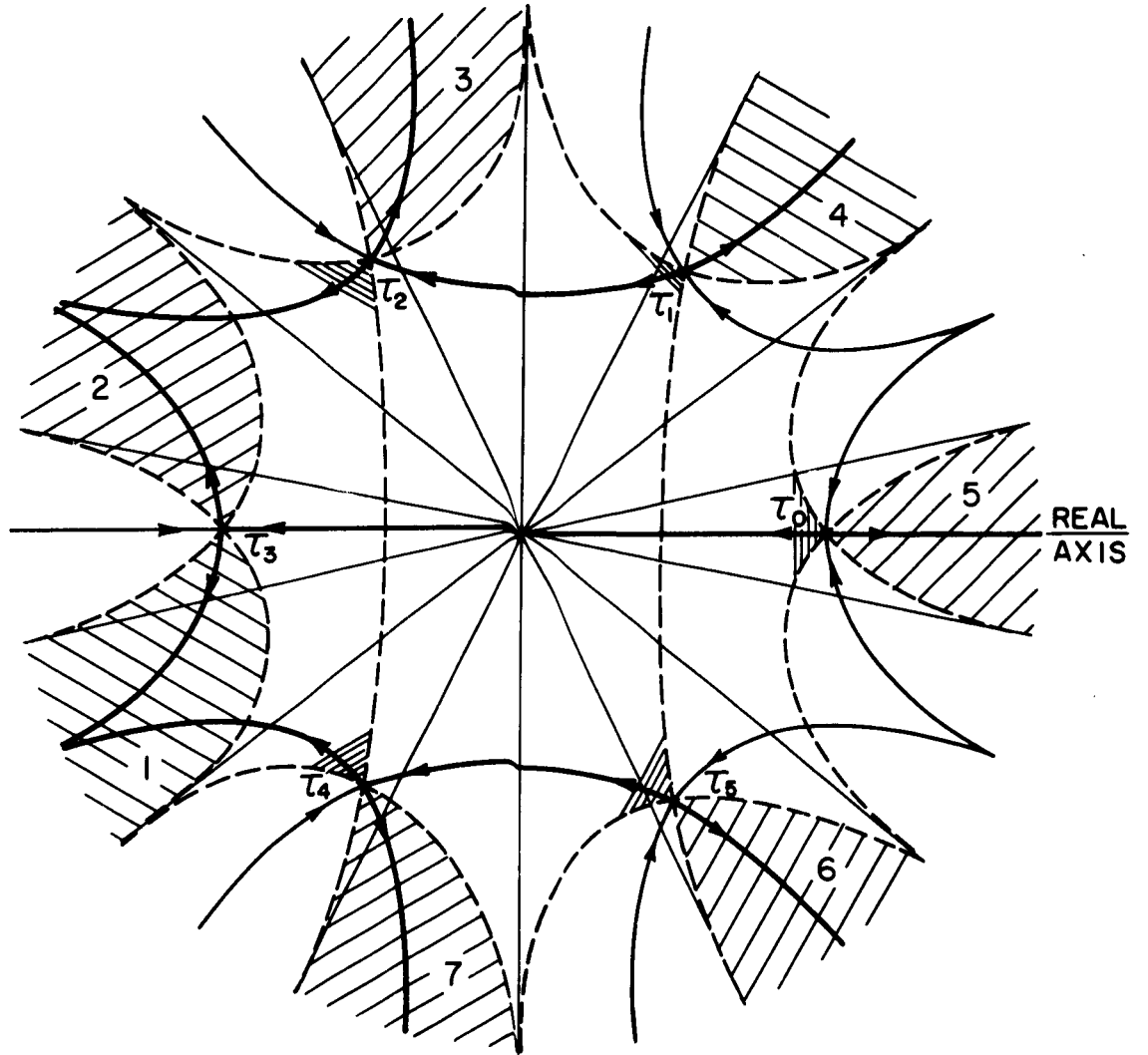


FIGURE 2 The saddle-point configuration for  $x > 0$

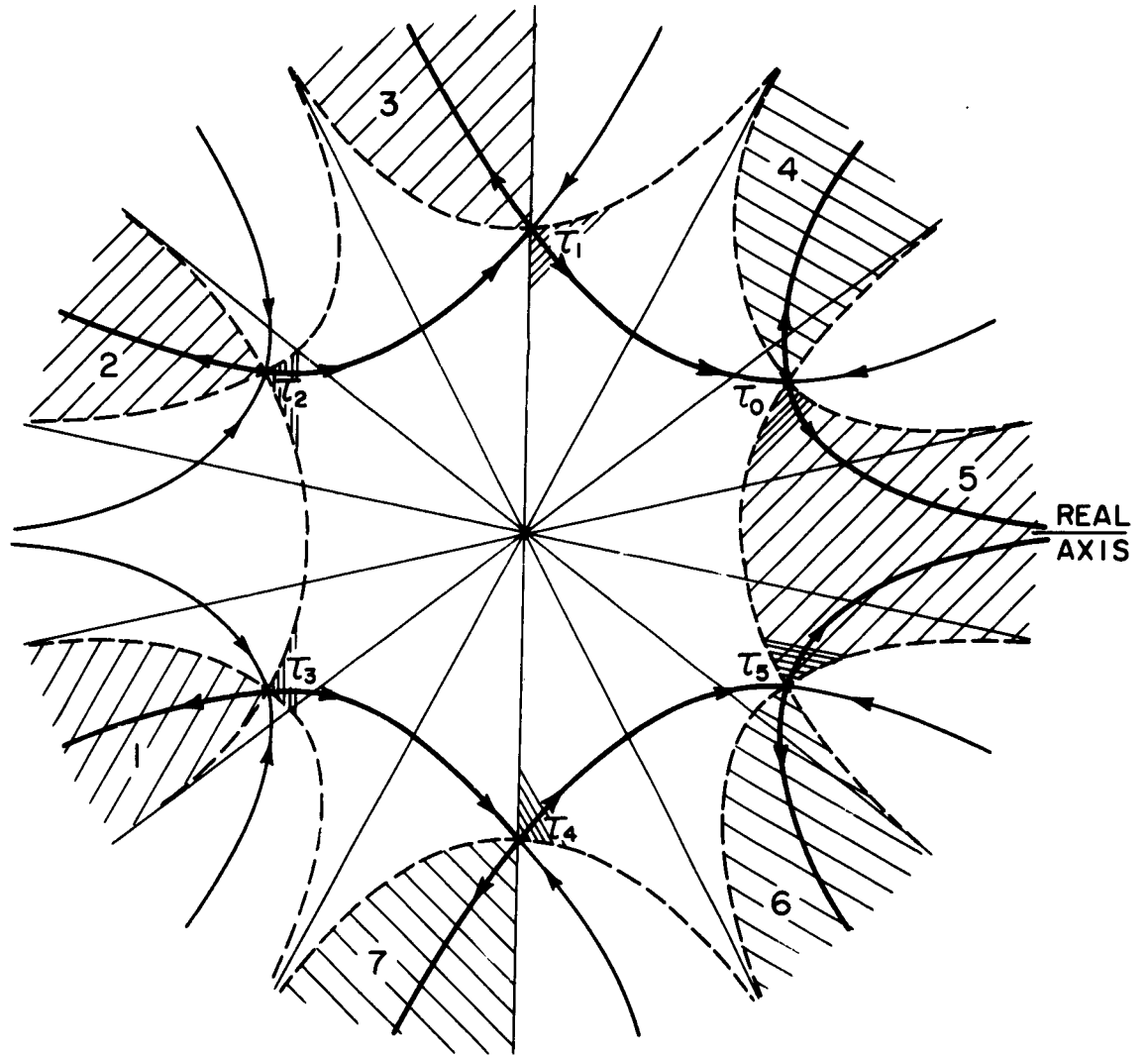


FIGURE 3 The saddle-point configuration for  $x < 0$