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OPTIMAL GROWTH IN A TWO-SECTOR MODEL OF CAPITAL ACCUMULATION

by

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1. Introduction

One of the basic problems in economic planning, in particular in
underdeveloped countries, is concerned with the rate at which society
should save out of current income to achieve a maximum growth. It is
closely related to the problem of how scarce resources at each moment
of time should be divided between consumers' goods industries and capi-
tal goods industries. In the present paper, we shall analyze the prob-
lem in the framework of the two-sector growth model as introduced by
Meade [3], Srinivasan [6], and Uzawa [8]. We shall abstract from the
complications which would arise by taking into account those factors
such as the changing technology and structure of demand, the role of
foreign trade (in particular, of capital imports), and tax policy that
are generally regarded as decisive in the determination of the course
of economic development. Instead, we shall focus our attention on
evaluating the impact of roundabout methods of production upon the wel-
fare of society, as expressed by a discounted sum of per capita consump-
tion. However, since our primary concern is with economic planning in
underdeveloped countries, we shall depart with respect to one important

point from the two-sector growth model as formulated in [3, 6, 8] which is, in general, concerned with an economy with fairly advanced technology and relatively abundant capital; namely, we shall postulate that a certain quantity of consumers' goods (per capita) is required to sustain a given rate of population growth. This restraint becomes ineffective for an economy with relatively abundant capital; however, for an economy with low capital-labor ratio and high rate of population growth, it results in the phenomenon frequently referred to as "the vicious circle of poverty"[2]. In the course of the discussion below on optimal growth, we shall briefly investigate the existence of such a vicious circle and its implications upon patterns of optimal growth.

Mathematically, our problem is that of finding a growth path over which the criterion function (i.e., the discounted sum of per capita consumption over the whole period) is maximized among all feasible growth paths. It is a problem in concave programming in linear spaces, to use the term of Hurwicz [2], and the techniques developed by him and others, particularly the extensions of the Kuhn-Tucker theorem, may be applicable. In the present case, however, it is possible to solve our problem without recourse to those advanced methods, and the optimal growth paths are instead characterized by a simple extension of the Euler equations in the classical calculus of variations. The mathematical structure of the auxiliary differential equations arising from the Euler equations differs markedly, according to whether consumers' goods

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[2]

    See, e.g., Nurkse [5, p. 4 f.] and Myrdal [4, p. 11 f.].

are more or less capital-intensive than capital goods, and the detailed structure of the optimal growth path differs in these two cases. Therefore, for the convenience of analysis, we shall first present the discussions for these two cases separately, and then the general case in which no restrictions are imposed on relative capital intensities will be briefly discussed. It will be generally shown that for any economy with relatively low capital-labor ratio, consumers' goods are produced in the amounts just necessary to satisfy the minimum requirements until a certain critical level is reached, and from then on the rate of production of consumers' goods is gradually increased toward a certain balanced rate. Our results are thus extensions of those obtained by Srinivasan [6] for the case in which the minimum wage rate is zero and for which consumers' goods are always more capital-intensive than capital goods.

2.   A Two-Sector Model of Capital Accumulation

To begin, let us describe the basic premises of our two-sector model in terms of a mathematical model $\underline{3/}$. We are concerned with an economy in which consumption goods and capital goods are composed of homogeneous quantities. Both goods are produced by combinations of two factors of production, labor and capital, but the possibility of joint products is excluded. The sole aim of the economy is to consume consumption goods, while capital goods are produced only to increase future production of consumption goods. Consumption goods may be

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$\underline{3/}$
    Such a model of the two-sector economy was first introduced in Meade [3]. The present formulation modifies slightly that introduced in Uzawa [8].

assumed instantaneously consumed and capital goods to depreciate at a certain rate, say μ , which is technologically given. It will be assumed that technological knowledge remains constant in the whole period in question, constant returns to scale and diminishing marginal rates of substitution between capital and labor prevail in each sector, and there exist no external (dis-) economies. The sector producing consumption goods will be referred to as the C-sector, while that producing capital goods as the I-sector.

To make the analysis simpler, it is assumed that the size of the working population and the rate at which it grows are exogenously given, and that labor is inelastically offered for employment at any moment of time. Let $L(t)$ denote the size of the working population at time t, then

$$(1) \qquad\qquad \dot{L}(t) / L(t) = n ,$$

where n stands for the rate of increase in labor forces.

It is furthermore assumed that the working population shares a stationary proportion to the total population and that no external (dis-) economies exist for consumption, so that the minimum amount of consumption goods per capita required to sustain the given labor growth n may be assumed determinate. Let w_{min} denote the minimum wage rate in terms of consumption goods corresponding to the labor growth n. [4]/

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[4]/

Several authors, in particular Buttrick [1] and Tsiang [7], have postulated certain relationships between the rate of labor growth and minimum wages, and have effectively analyzed the characteristics of various stages of economic growth.

In general, the minimum wage rate w_{min} is assumed positive.[5]

The aggregate quantity of capital $K(t)$ existing at any moment of time is determined by the accumulation of capital goods which have been produced in the past; namely, the rate of change in aggregate stock of capital at time t, $\dot{K}(t)$, is given by

$$(2) \qquad\qquad \dot{K}(t) = Y_I(t) - \mu\, K(t) \, ,$$

where $Y_I(t)$ stands for the rate at which new capital goods are produced at time t.

The rate $Y_I(t)$ at which new capital goods are produced, on the other hand, is determined by the quantities of capital and labor allocated to the I-sector, $K_I(t)$ and $L_I(t)$; namely

$$(3) \qquad\qquad Y_I(t) = F_I(K_I(t), L_I(t)) \, ,$$

where F_L is the production function which summarizes the production processes in the I-sector.

The rate of production of consumption goods at time t, $Y_C(t)$, is similarly determined by

$$(4) \qquad\qquad Y_C(t) = F_C(K_C(t), L_C(t)) \, ,$$

where $K_C(t)$ and $L_C(t)$, respectively, are the quantities of capital and labor, employed in the C-sector at time t.

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[5]
The case discussed by Srinivasan [6] and Uzawa [8] may be considered as a limiting case when the minimum wage rate tends to zero, which in the present paper will be discussed only to illustrate the techniques to be used for the general case.

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The quantities of capital and labor allocated to the two sectors should remain within the available quantities existing in the economy as a whole; i.e., [6]

(5) $$K_I(t) + K_C(t) = K(t) \ ,$$

(6) $$L_I(t) + L_C(t) = L(t) \ .$$

The conditions (5) and (6) in particular imply that both capital and labor may be transferred from one sector to another without any cost; our capital thus is malleable in Meade's terminology ([3], p. 45).

The quantity of consumption goods, on the other hand, must be sufficient to afford the minimum wage rate w_{min}; hence, we have inequality:

(7) $$Y_C(t) \geqq w_{min} \ L(t) \ .$$

The quantity of capital available to the economy at the beginning $(t = 0)$, $K(0)$, is given as one of the data, together with technological conditions and population growth. A path of consumption $\{Y_C(t); \ t \geqq 0\}$ is termed _feasible_ if it is possible to find allocations of capital and labor at each moment of time such that all the conditions (2-7) are satisfied.

In what follows, it will be assumed that production in each sector is subject to constant returns to scale, the marginal rate of substitution between capital and labor is smooth and diminishing, the marginal physical products of both factors are always positive, and both factors

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[6] In view of the assumptions made below, (8-10) both capital and labor are fully employed at any optimal growth path, and we may without loss of generality postulate the full employment of both capital and labor at any moment of time.

are indispensable. Let $k_j = K_j/L_j$ be the capital stock per unit of employment in the i-th sector $(i = C, I)$, and the function $f_j(k_j)$ be defined by:

$$f_j(k_j) = F_j(k_j, 1) \ , \quad i = C, I.$$

Then $f_j(k_j)$ is continuously twice differentiable and

$$(8) \qquad f_j(k_j) > 0 \ , \quad f_j'(k_j) > 0 \ , \quad f_j''(k_j) < 0 \ , \quad \text{for all } k_j > 0 \ ,$$

$$(9) \qquad\qquad f_j(0) = 0 \ , \quad f_j(\infty) = \infty \ ,$$

$$(10) \qquad\qquad f_j'(0) = \infty \ , \quad f_j'(\infty) = 0 \ .$$

3. Optimal Growth in the Two-Sector Model

Since there is only one consumption good in our two-sector economy, the social welfare may be determined, once we specify the rate of discount by which future consumption is weighed against present consumption. It will be assumed that the rate of discount is held at a fixed positive level δ. A feasible path will be termed _optimal_ (relative to the rate of discount δ) if it maximizes the discounted sum of per capita consumption

$$(11) \qquad\qquad \int_0^\infty \frac{Y_C(t)}{L(t)} \, e^{-\delta t} \, dt$$

among all feasible paths of consumption arising from the given capital stock $K(0)$ initially held in the economy.

In view of the assumptions (8-10), it is easily seen that the quantity (11) is finite for any feasible path, provided that

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(12) $$\lambda = n + \mu > 0 .$$

Let us first introduce auxiliary variables (Lagrange multipliers) $q(t)$, $p_I(t)$, $p_C(t)$, $r(t)$, $w(t)$, and $v(t)$, respectively, corresponding to the restraints (2), (3), (4), (5), (6), and (7), and consider the following quantity:

(13)
$$\int_0^\infty \{ Y_C(t) + p_C(t)(F_C(K_C(t),L_C(t)) - Y_C(t)) + p_I(t)(F_I(K_I(t),L_I(t)) - Y_I(t)) +$$

$$+ r(t)(K(t) - K_C(t) - K_I(t)) + w(t)(L(t) - L_C(t) - L_I(t)) +$$

$$+ q(t)(Y_I(t) - \mu K(t) - \dot{K}(t)) + v(t)(Y_C(t) - w_{min}L(t)) \} e^{-(n+\delta)t} \, dt,$$

where all variables are non-negative and $K(0)$ is a given quantity.

The expression (13) is concave in $Y_C(t)$, $Y_I(t)$, $K_C(t)$, $K_I(t)$, $K(t)$, $L_C(t)$ and $L_I(t)$. Suppose we have found a set of auxiliary variables $p_C(t)$, $p_I(t)$, $q(t)$, $v(t)$ for which the variables $Y_C(t)$, $Y_I(t)$, $K_C(t)$, $K_I(t)$, $K(t)$, $L_C(t)$, $L_I(t)$, maximizing the quantity (13) without any restraint, satisfy the feasibility conditions (2-7). Then the path of the corresponding $Y_C(t)$ is an optimal path. Our optimum problem thus is reduced to that of maximizing the quantity (13) for a given set of auxiliary variables. The latter is a concave problem in calculus of variations and may be obtained by solving the following Euler equations:

(14) $$v(t) \geqq 0 ,$$

with equality if $Y_C(t) > w_{min} L(t)$;

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$$(15) \qquad 1 + v(t) - p_C(t) = 0 \ ;$$

$$(16) \qquad q(t) - p_I(t) \leqq 0 \ ,$$

with equality if $Y_I(t) > 0$;

$$(17) \qquad p_C(t) \, \frac{\partial F_C(t)}{\partial K_C(t)} - r(t) \leqq 0 \ ,$$

with equality if $K_C(t) > 0$;

$$(18) \qquad p_I(t) \, \frac{\partial F_I(t)}{\partial K_I(t)} - r(t) \leqq 0 \ ,$$

with equality if $K_I(t) > 0$;

$$(19) \qquad p_C(t) \, \frac{\partial F_C(t)}{\partial L_C(t)} - w(t) \leqq 0 \ ,$$

with equality if $L_C(t) > 0$;

$$(20) \qquad p_I(t) \, \frac{\partial F_I(t)}{\partial L_I(t)} - w(t) \leqq 0 \ ,$$

with equality if $L_I(t) > 0$;

$$(21) \qquad r(t) - \mu \, q(t) = (n+\delta) \, q(t) - \dot{q}(t) \ ,$$

where $K(0)$ is a given quantity and all variables are non-negative and bounded.

4. <u>Reduction of Optimality Conditions</u>

In view of the constant-returns-to-scale assumption, it is possible to reduce the system of Euler equations (14-21) and the feasibility conditions (2-7) to those involving only per capita quantities.

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Let ω be an arbitrarily given wage-rentals ratio and define the optimum capital-labor ratio k_j in each sector by solving

$$(22) \qquad \omega = \frac{f_j(k_j)}{f'_j(k_j)} - k_j \quad (j = C, I) \, ,$$

in terms of k_j. By assumptions (8-10), such a capital-labor ratio k_j is uniquely determined for any wage-rentals ratio ω and it will be denoted by $k_j(\omega)$. From (22), we have

$$(23) \qquad \frac{dk_j(\omega)}{d\omega} = \frac{[f'_j(k_j(\omega))]^2}{-f_j(k_j(\omega)) \, f''_j(k_j(\omega))} > 0 \, .$$

We next introduce the supply price of capital, $p(\omega)$, in terms of consumption goods:

$$(24) \qquad p(\omega) = \frac{f'_C(k_C(\omega))}{f'_I(k_I(\omega))} \, .$$

The supply price thus defined corresponds to the price of capital (in terms of consumers' goods) which would just induce each entrepreneur in a competitive economy to produce an additional unit of new capital goods under the prevailing wage-rentals ratio ω. Logarithmically differentiate (24) and substitute (23) to get

$$(25) \qquad \frac{1}{p(\omega)} \frac{dp(\omega)}{d\omega} = \frac{1}{k_I(\omega) + \omega} - \frac{1}{k_C(\omega) + \omega} \, ,$$

which is positive or negative, according to whether consumption goods are more or less capital-intensive than capital goods.

Let us finally introduce:

$$y_j(t) = \frac{Y_j(t)}{L(t)} \quad ,$$

$$k_j(t) = \frac{K_j(t)}{L_j(t)} \quad ,$$

$$\ell_j(t) = \frac{L_j(t)}{L(t)} \quad , \qquad j = I, C$$

$$k(t) = \frac{K(t)}{L(t)} \quad .$$

A simple manipulation shows that the Euler equations (14-21) and the feasibility conditions (2-6) together are reduced to the following system:

(26) $\qquad y_C(t) = f_C(k_C(t))\ell_C \; , \quad y_I(t) = f_I(k_I(t))\ell_I \; ,$

(27) $\qquad k_C(t) \, \ell_C(t) + k_I(t) \, \ell_I(t) = k(t) \; ,$

(28) $\qquad \ell_C(t) + \ell_I(t) = 1 \; ,$

(29) $\qquad y_C(t) \geqq w_{min}$

(30) $\qquad p(t) \geqq q(t) \; , \quad \text{if} \;\; y_I(t) > 0 \; ,$

(31) $\qquad p(t) \leqq q(t) \; , \quad \text{if} \;\; y_C(t) > w_{min} \; ,$

(32) $\qquad \dot{k}(t) = y_I(t) - \lambda \, k(t) \; ,$

(33) $\qquad \dot{q}(t) = (\delta + \lambda) \, q(t) - r(t) \; ,$

where all variables are non-negative and bounded,

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$$k_C(t) = k_C(\omega(t)) \ , \quad k_I(t) = k_I(\omega(t)) \ ,$$

$$p(t) = p(\omega(t)) \ ,$$

$$r(t) = f_C'(k_C(t)) \ ,$$

$$\lambda = \mu + n > 0 \ ,$$

and $\qquad\qquad k(0) = K(0)/L(0)$ is given .

The auxiliary variable, $q(t)$, may be interpreted as the <u>demand price of capital</u> (in terms of consumers' goods) at time t. The relations (30) and (31) then simply mean that no capital goods are produced when the supply price of capital $p(t)$ exceeds the demand price of capital $q(t)$, while consumers' goods are produced just enough to meet the minimum requirement when the demand price $q(t)$ exceeds the supply price $p(t)$.

Our optimum problem now is reduced to solving the system of the optimality conditions (26-33). We shall first discuss the case in which consumption goods are always more capital-intensive than capital goods, and proceed to discuss the case in which consumption goods are always less capital-intensive then capital goods. Finally, the general case will be briefly discussed by using the results obtained for these two special cases.

5. <u>The Case when Consumption Goods are Always More Capital-Intensive Than Capital Goods</u>.

Let us first consider the case in which consumption goods are always more capital-intensive than capital goods; namely,

(34) $$k_C(\omega) > k_I(\omega) \ , \quad \text{for all} \ \ \omega > 0 \ .$$

To solve the system (26-33), it is found useful to investigate the structure of the differential equations which describe the behavior of capital-labor ratio k and wage-rentals ratio ω when capital goods are produced with positive quantities, and consumption goods exceed minimum requirements. In such a case, we have from (26), (27), (28), (30), and (31) that

(35) $$p(t) = q(t)$$

(36) $$y_C(t) = \frac{k(t)-k_I(t)}{k_C(t)-k_I(t)} \ f_C(k_C(t)), \ \ y_I(t) = \frac{k_C(t)-k(t)}{k_C(t)-k_I(t)} \ f_I(k_I(t)) \ .$$

The differential equations (32) and (33) are accordingly reduced to:

(37) $$\dot{k} = \frac{k_C(\omega)-k}{k_C(\omega)-k_I(\omega)} \ f_I(k_I(\omega))-\lambda \ k \ ,$$

(38) $$\frac{\overset{\circ}{p}(\omega)}{p(\omega)} = \lambda + \delta - f_I'(k_I(\omega)) \ ,$$

where for the sake of simplicity the variables are described without explicitly referring to the time variable t. The differential equations (37) and (38) will be referred to as the <u>auxiliary differential equations</u>.

In view of (25) and (34), the auxiliary differential equations (37) and (38) may be written as:

(39) $$\dot{k} = \{\frac{f_I[k_I(\omega)]}{k_C(\omega)-k_I(\omega)} + \lambda\}(\hat{k}(\omega)-k) \ ,$$

(40) $$\dot{\omega} = \frac{\lambda+\delta - f_I'(k_I(\omega))}{\frac{1}{k_I(\omega)+\omega} - \frac{1}{k_C(\omega)+\omega}} \ ,$$

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where

(41) $$\hat{k}(\omega) = \frac{f_I[k_I(\omega)]}{f_I[k_I(\omega)] + \lambda[k_C(\omega) - k_I(\omega)]} k_C(\omega) .$$

The quantity $\hat{k}(\omega)$ is always smaller than $k_C(\omega)$ and it is larger than $k_I(\omega)$ if and only if

$$\frac{f_I[k_I(\omega)]}{k_I(\omega)} > \lambda .$$

Since the average productivity of capital, $\dfrac{f_I(k_I)}{k_I}$, is a decreasing function of k_I and $k_I(\omega)$ is an increasing function of ω, we have

$$k_I(\omega) < \hat{k}(\omega) < k_C(\omega) \quad \text{if and only if} \quad \omega < \omega_\lambda ,$$

where ω_λ is defined by

(42) $$\frac{f_I(k_I(\omega_\lambda))}{k_I(\omega_\lambda)} = \lambda .$$

Let us now define the balanced wage-rentals ratio, ω^*, by

(43) $$f_I'(k_I(\omega^*)) = \lambda + \delta .$$

Define $k^* = \hat{k}(\omega)$, $k_I^* = k_I(\omega^*)$, $k_C^* = k_C(\omega^*)$. The determination of ω_λ and ω^* may be illustrated by Figure 1. The ratio ω_λ is always greater than the balanced ratio ω^*; in particular,

$$k_I^* < k^* < k_C^* .$$

The relationships between $k_C(\omega)$, $k_I(\omega)$, and $\hat{k}(\omega)$ are illustrated by Figure 2.
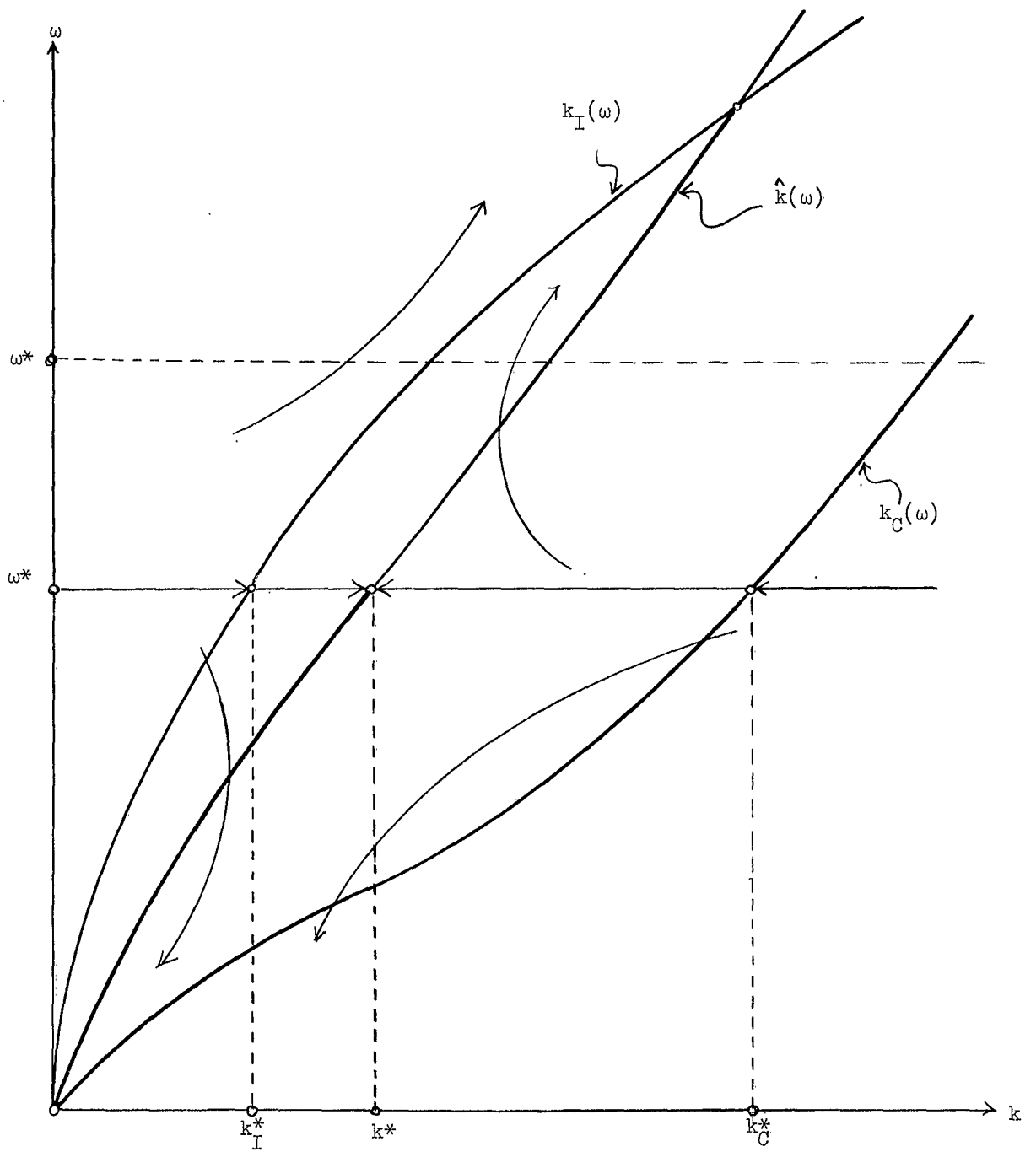
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Figure 1

$f_I(k_I)$

k_I

k^*

$\omega_{\lambda+\delta}^*$

ω_λ^*

ω_λ

λ

$\lambda+\delta$

λ

15

Fig. 2

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For any initial condition, the solution to the auxiliary differential equations (37) and (38) will be assumed to exist and to change continuously as the initial condition changes. The rate of change in k is positive, zero, or negative according to whether k is smaller than, equal to, or larger than $k(\omega)$, while the rate of change in ω is positive, zero, or negative according to whether ω is larger than, equal to, or smaller than the balanced ratio ω^*. Therefore, the structure of the solution paths to the auxiliary differential equations may be described by the arrow curves as illustrated in Figure 2.

The structure of the solution to the auxiliary differential equations described above will be used to solve the system (26-33) of the optimality conditions. To illustrate the method, let us first discuss the limiting case in which the minimum wage rate is zero:

$$w_{min} = 0 .$$

It is first noted that if the supply price of capital $p(t)$ is identical with the demand price $q(t)$ at any point on the $k_I(\omega)$ curve below the ω^*-line, and if the economy is specialized to the production of capital goods, then the demand price $q(t)$ satisfying the equation (33) falls while the supply price $p(t)$ rises; therefore, the economy continues the specialization in capital goods. This is easily seen from Figure 3, or it may be analytically shown by solving for the case in which the economy is specialized to capital goods.

It is similarly shown that if the supply price $p(t)$ is identical with the demand price $q(t)$ on the $k_C(\omega)$ curve above the ω^*-line, and if the economy is specialized in consumption goods, then the demand price rises while the supply price falls along the optimal path.

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These considerations lead us to the following solutions to the optimality conditions (26-33): (a) _If the initial capital-labor ratio_ $k(0)$ _is smaller than_ k_I^*, _then along the optimal path, the economy is specialized to the production of capital goods until the capital-labor ratio_ $k(t)$ _reaches the critical ratio_ k_I^*. _When the critical ratio_ k_I^* _is reached, both consumption goods and capital goods are produced, keeping the wage-rentals ratio at the level_ ω^*. _The optimal path then approaches asymptotically the balanced ratio_ k^* _along the_ ω^*-_line_. The precise analytical expressions may be given as follows:

Let the critical time t^* be defined by

$$(44) \qquad t^* = \int_{k(0)}^{k_I^*} \frac{dk}{f_I(k) - \lambda k} \ .$$

For $0 \leq t \leq t^*$: $k(t)$ and $\omega(t)$ are respectively obtained by solving

$$(45) \qquad \int_{k(0)}^{k(t)} \frac{dk}{f_I(k) - \lambda k} = t \ ,$$

$$(46) \qquad k(t) = k_I(\omega(t)) \ ,$$

and

$$(47) \qquad y_I(t) = f_I(k(t)) \ ,$$

$$(48) \qquad y_C(t) = 0 \ ,$$

$$(49) \qquad p(t) = p[\omega(t)] \ ,$$

$$(50) \quad q(t) = e^{-(\lambda+\delta)(t^*-t)} \left\{ \int_t^{t^*} f_C'[k_C(\omega(t))] e^{(\lambda+\delta)(t^*-\tau)} d\tau - p(t^*) \right\} .$$

For $t \geqq t^*$:

$$(51) \qquad k(t) = k^* - (k^*-k_I^*)e^{-\Theta(t-t^*)} ,$$

where

$$(52) \qquad \Theta = \frac{f_I[k_I(\omega^*)]}{k_C(\omega^*)-k_I(\omega^*)} + \lambda ,$$

$$(53) \qquad \omega(t) = \omega^* ,$$

$$(54) \qquad y_I(t) = \frac{k_C^*-k(t)}{k_C^*-k_I^*} \ f_I(k_I^*) ,$$

$$(55) \qquad y_C(t) = \frac{k(t)-k_I^*}{k_C^*-k_I^*} \ f_C(k_C^*) ,$$

$$(56) \qquad p(t) = q(t) = p(\omega^*) .$$

It is easily seen that $k(t)$ increasingly converges to k^*, $y_I(t)$ decreasingly converges to $y_I^* = \dfrac{k_C^*-k^*}{k_C^*-k_I^*} \ f_I(k_I^*)$, and $y_C(t)$ increasingly converges to $y_C^* = \dfrac{k^*-k_I^*}{k_C^*-k_I^*} \ f_C(k_C^*)$.

(b) If the initial capital-labor ratio $k(0)$ is larger than k_C^*, then along the optimal path the economy is specialized to the production of consumption goods until the capital-labor ratio $k(t)$ is reduced to the ratio k_C^*. When the capital-labor ratio k_C^* is reached, both consumption goods and capital goods are produced, keeping the wage-rentals ratio at the level ω^*. The optimal path then asymptotically approaches the balanced ratio k^*. The analytical expressions for this

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case may be given as follows:

The critical time t* is defined by

(57) $$t^* = \frac{1}{\lambda} \log \frac{k(0)}{k_C^*} \ .$$

For $0 \leqq t \leqq t^*$:

(58) $$k(t) = k(0)e^{-\lambda t} \ .$$

and $\omega(t)$ is obtained by solving

(59) $$k(t) = k_C(\omega(t)) \ ,$$

and

(60) $$y_I(t) = 0 \ .$$

(61) $$y_C(t) = f_I(k(t)) \ ,$$

and $p(t)$, $q(t)$ are given by (49) and (50).

For $t \geqq t^*$:

(52) $$k(t) = k^* + (k_C^* - k^*)e^{-\Theta(t - t^*)} \ ,$$

where the parameter Θ is defined by (52) , and $\omega(t)$, $y_I(t)$, $y_C(t)$, $p(t)$, and $q(t)$ are the same as those given by (53-56).

In this case, $k(t)$ decreasingly converges to k^*, $y_I(t)$ increasingly converges to y_I^*, and $y_C(t)$ decreasingly converges to y_C^*.

The optimal growth paths are indicated by the heavy arrow curves in Figure 3.

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Fig. 3

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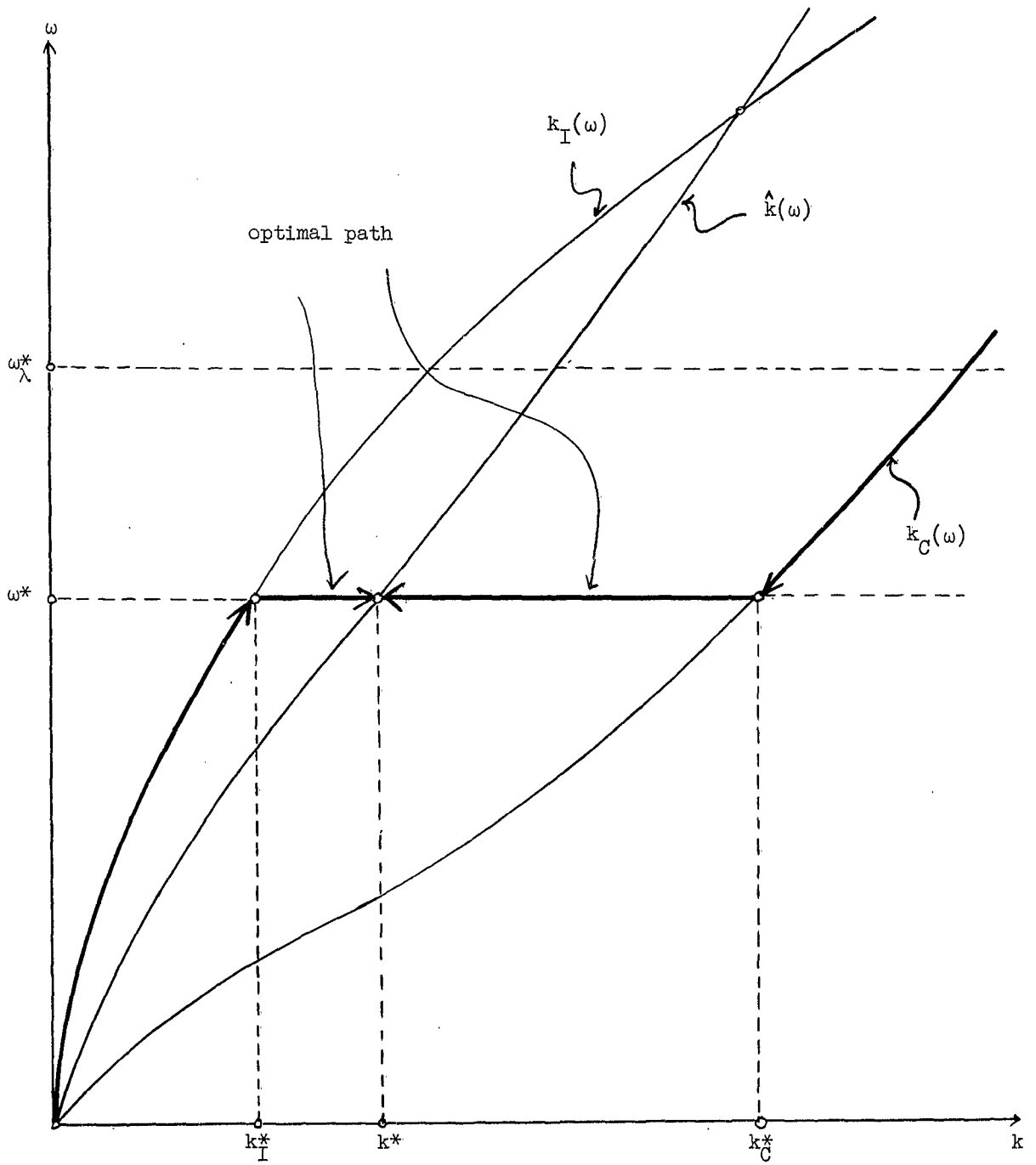
Let us now extend our analysis to the case in which the minimum
wage rate is positive:

$$w_{min} > 0 .$$

Then the per capita consumption $y_C(t)$ given by (36) is greater than
the minimum wage rate w_{min} if and only if

(63)
$$k(t) > k_{min} (\omega(t)) ,$$

where the function $k_{min}(\omega)$ is defined by

(64)
$$k_{min}(\omega) = k_I(\omega) + \frac{k_C(\omega) - k_I(\omega)}{f_C(k_C(\omega))} w_{min}$$

By assumption (34), $k_{min}(\omega)$ is always larger than $k_I(\omega)$, while
it is smaller than $k_C(\omega)$ if and only if $f_C[k_C(\omega)] > w_{min}$.

Let the capital-labor ratio ω_{min} be defined as the one satisfying

(65)
$$f_C[k_C(\omega_{min})] = w_{min} .$$

Then we have that

(66) $k_I(\omega) < k_{min}(\omega) < k_C(\omega)$ if and only if $\omega > \omega_{min}$.

On the other hand, $k_{min}(\omega) < \hat{k}(\omega)$ if and only if

(67)
$$\frac{f_I(k_I(\omega)) - \lambda k_I(\omega)}{f_I(k_I(\omega)) - \lambda k_I(\omega) + \lambda k_C(\omega)} f_I(k_I(\omega)) > \omega_{min} .$$

The left-hand side of the inequality (67) is an increasing function
of ω, provided $\omega < \omega_\lambda^*$, where ω_λ^* is the wage-rentals ratio for which
the optimum marginal productivity in the I-sector is λ; i.e.,

$$(68) \qquad f'_I(k_I(\omega^*_\lambda)) = \lambda \ .$$

Hence, by defining the wage-rentals ratio $\hat{\omega}$ by

$$(69) \qquad \frac{f_I(k_I(\hat{\omega})) - \lambda \ k_I(\hat{\omega})}{f_I(k_I(\hat{\omega})) - \lambda \ k_I(\hat{\omega}) + \lambda \ k_C(\hat{\omega})} \ f_I(k_I(\hat{\omega})) = \omega_{min} \ ,$$

we have that, for $\omega < \omega^*_\lambda$,

$$(70) \qquad k_{min}(\omega) < \hat{k}(\omega) \quad \text{if and only if} \quad \omega > \hat{\omega} \ .$$

The relationships of the $k_{min}(\omega)$ curve to the $k_C(\omega)$, $k_I(\omega)$, and $\hat{k}(\omega)$ curves may be typically illustrated in Figure 4.

Let us now denote the capital-labor ratios $\hat{k}$ and k_{min} corresponding to the wage-rentals ratios $\hat{\omega}$ and ω_{min}, respectively:

$$(71) \qquad \hat{k} = \hat{k}(\hat{\omega}) = k_{min}(\hat{\omega}) \ ,$$

$$(72) \qquad k_{min} = k_C(\omega_{min}) = k_{min}(\omega_{min}) \ .$$

It is easily established that if the initial capital-labor ratio $k(0)$ is smaller than $\hat{k}$, then the capital-labor ratio $k(t)$ is decreasing over any feasible path and eventually reaches the capital-labor ratio k_{min}. However, if the capital-labor ratio is smaller than the ratio k_{min}, the economy is not able to produce consumption goods to afford the minimum requirement even though all capital and labor are allocated to the C-sector. Therefore, if the capital-labor ratio is smaller than k_{min}, our model is inconsistent in the sense that no feasible solution satisfying the minimum wage requirement may be found. To explain fully the phenomena would require the introduction of a certain Malthusian function explicitly relating the population growth
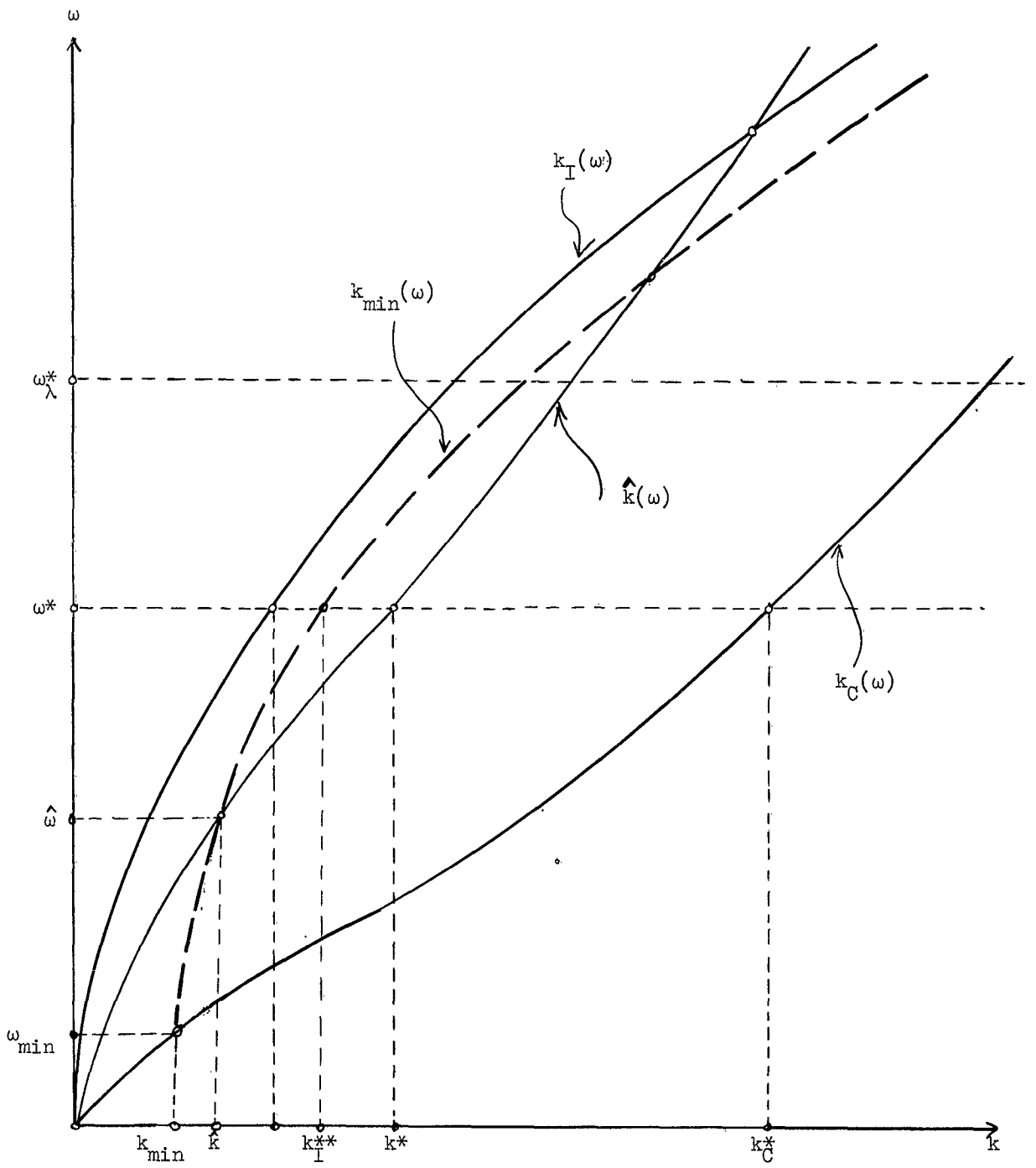
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Fig. 4

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to the minimum wage rate. In the present paper, however, we assume that the economy has somehow attained a capital-labor ratio larger than the critical capital-labor ratio $\hat{k}$.

The structure of the optimal growth is then analyzed, and is analogous to the previous case; namely, we have that: (a) <u>If the initial capital-labor ratio</u> $k(0)$ <u>is smaller than</u> k_I^* <u>but larger than the critical ratio</u> $\hat{k}$, <u>then along the optimal path the economy produces just enough consumption goods just to meet the minimum requirements, until the time when the capital-labor reaches the level</u> $k_{min}(\omega^*)$, <u>and from then on it proceeds to produce both capital goods and consumption goods for more than the minimum requirements.</u> <u>The optimal path increasingly approaches the balanced capital-labor ratio</u> k^*.

(b) <u>If the initial capital-labor ratio</u> $k(0)$ <u>is larger than</u> k_C^*, <u>then the optimal path proceeds exactly as in the case with zero minimum wage rate.</u>

The analytical expressions are similar to those for the previous case. In Figure 5, the structure of the optimal path is depicted with the heavy arrow curves.

6. <u>The Case when Capital Goods are always more Capital-Intensive than Consumption Goods.</u>

Let us next consider the case where <u>the capital good is always more capital-intensive than the consumption good;</u> namely

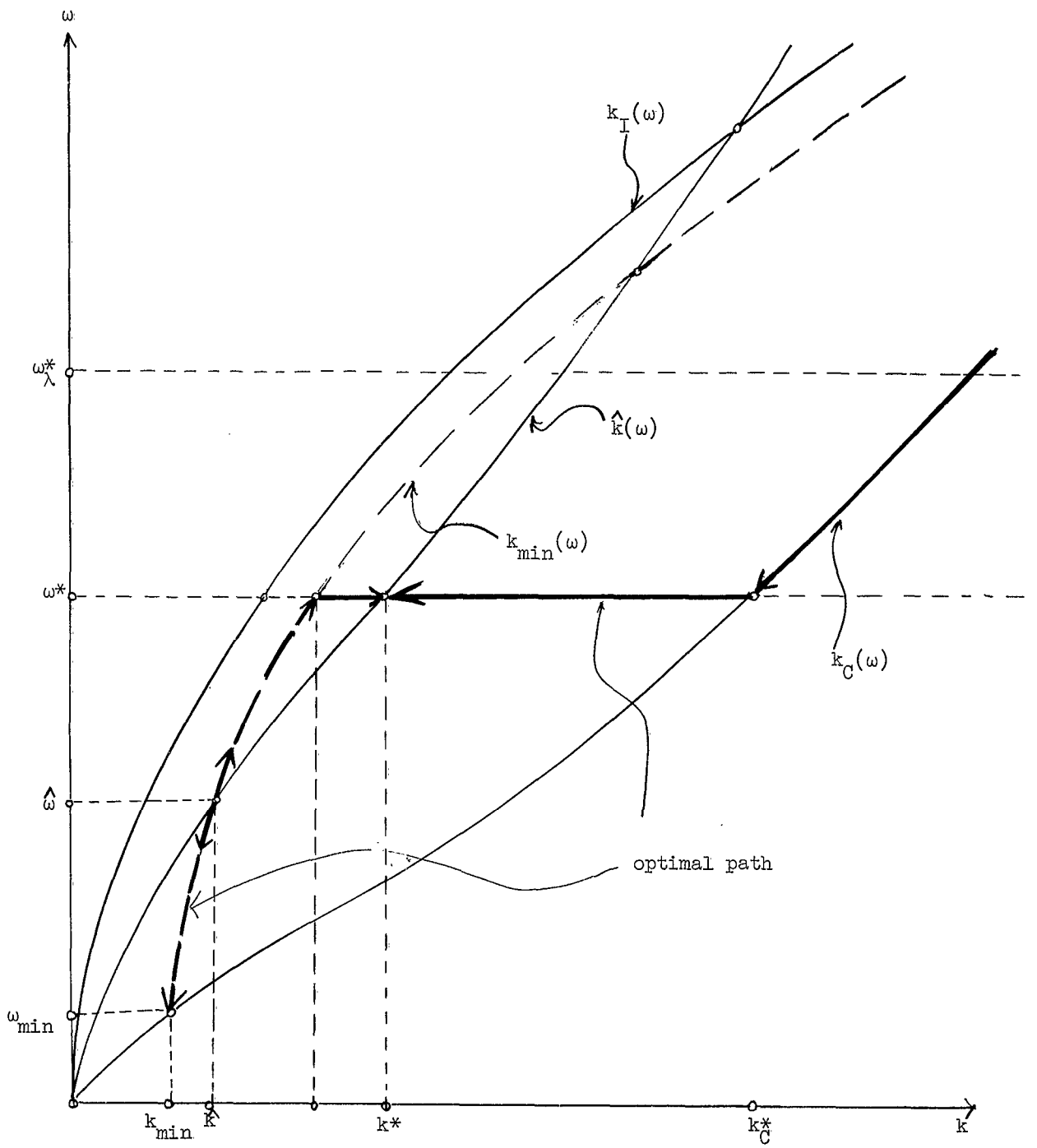$$(73) \qquad\qquad k_I(\omega) > k_C(\omega) , \quad \text{for all} \quad \omega > 0 .$$

Fig. 5

In this case, the auxiliary differential equations (37) and (38) may be rewritten as:

$$(74) \qquad \dot{k} = \{ \frac{f_I[k_I(\omega)]}{k_I(\omega) - k_C(\omega)} - \lambda](k - \hat{k}(\omega)) \ ,$$

$$(75) \qquad \dot{\omega} = \frac{f_I'[k_I(\omega)] - \lambda - \delta}{\frac{1}{k_C(\omega) + \omega} - \frac{1}{k_I(\omega) + \omega}}$$

where

$$(76) \qquad \hat{k}(\omega) = \frac{f_I[k_I(\omega)]}{f_I[k_I(\omega)] - \lambda[k_I(\omega) - k_C(\omega)]} \ k_C(\omega)$$

The quantity $\hat{k}(\omega)$ is always larger than $k_C(\omega)$ and smaller than $k_I(\omega)$ if and only if $\omega < \omega_\lambda$, where ω_λ is defined by (42).

The relationship among $k_C(\omega)$, $k_I(\omega)$, and $\hat{k}(\omega)$, are then illustrated by Figure 6.

For a wage-rentals ratio ω satisfying

$$\lambda < \frac{f_I[k_I(\omega)]}{k_I(\omega) - k_C(\omega)} \qquad ,$$

the rate of change in k has the same sign as $k - k(\omega)$; namely, k is increased or decreased according to whether k is larger or smaller than $\hat{k}(\omega)$. On the other hand, ω is increased or decreased according to whether ω is smaller or larger than ω^*.

The auxiliary differential equations are unstable in k. However, for any given wage-rentals ratio ω_o , it is possible to find a corresponding k_o such that the solution $(k(t), \omega(t))$ to the auxiliary

Fig. 6

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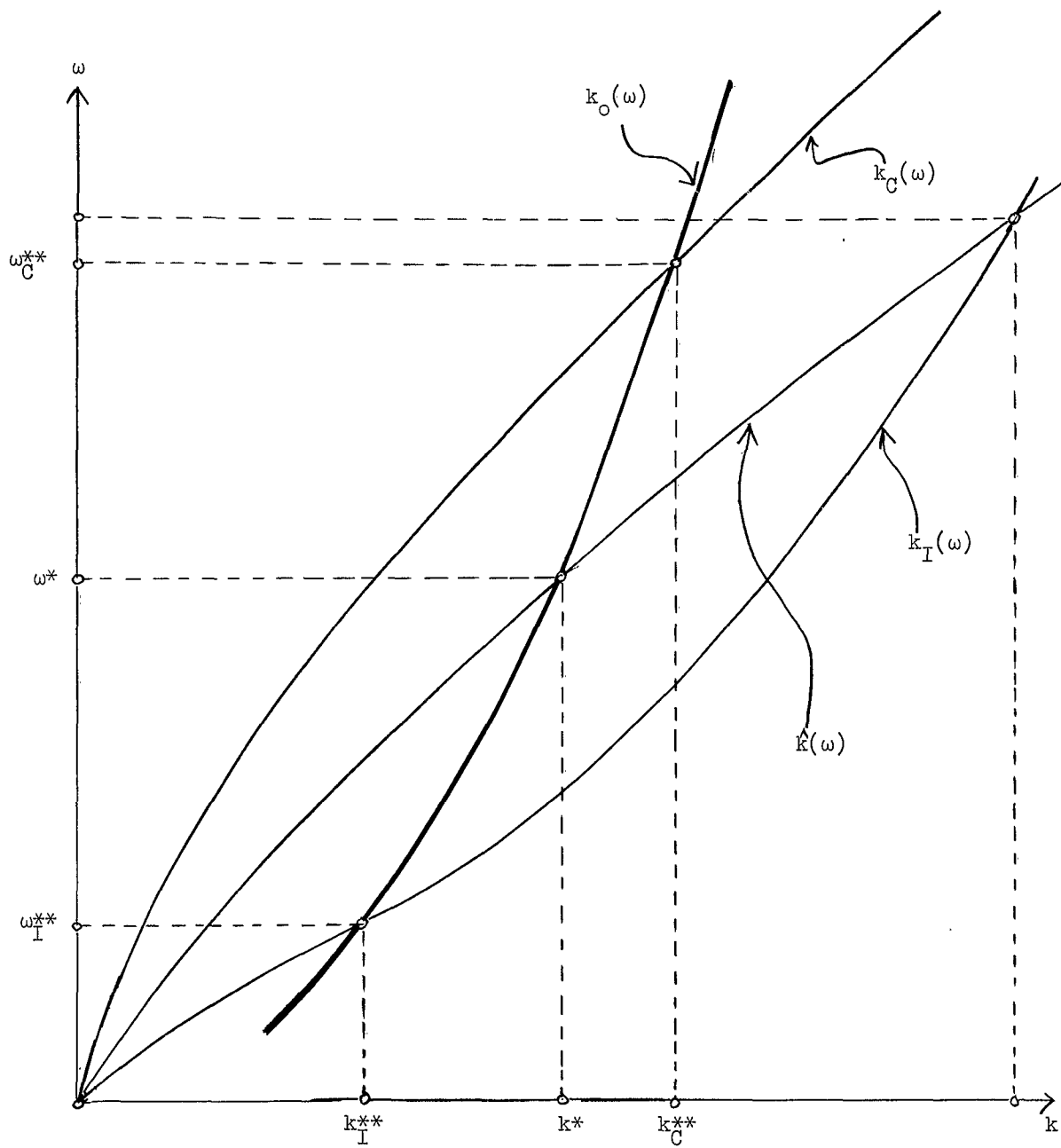
differential equations (74) and (75) with initial condition (k_o, ω_o) converges to (k^*, ω^*). The existence of such a k_o may be established simply by taking the supremum of the initial values of k for which the k-component of the solution to the auxiliary differential equation approaches zero. It is explicitly given by the following formula:

$$(77) \qquad k_o = \int_{\omega_o}^{\omega^*} e^{-A(\omega,\omega_o)} \alpha(\omega) \hat{k}(\omega) d\omega \ ,$$

where

$$(78) \qquad \alpha(\omega) = \frac{\dfrac{f_I[k_I(\omega)]}{k_I(\omega)-k_C(\omega)} - \lambda}{f_I'[k_I(\omega)]-\lambda-\delta} \ \left(\frac{1}{k_C(\omega)+\omega} - \frac{1}{k_I(\omega)+\omega} \right) \ ,$$

$$(79) \qquad A(\omega,\omega_o) = \int_{\omega_o}^{\omega} \alpha(\omega) \ d\omega \ .$$

In fact, the k-component of the solution $(k(t), \omega(t))$ to the auxiliary differential equations with initial condition (k_o, ω_o) is given by:

$$(80) \qquad k(t) = e^{A[\omega(t),\omega_o]} \left\{ k_o - \int_{\omega_o}^{\omega(t)} e^{-A(\omega,\omega_o)} \alpha(\omega) \hat{k}(\omega) d\omega \right\} \ .$$

Now we have that

$$\lim_{t\to\infty} \omega(t) = \omega^* \ , \quad \lim_{\omega\to\omega^*} A(\omega,\omega_o) = \infty \ ; \quad \text{hence,}$$

if $k(t)$ converges to k^*, as t tends to infinity, then

$$k_0 = \lim_{t \to \infty} \int_{\omega_0}^{\omega(t)} e^{-A(\omega,\omega_0)} \alpha(\omega) \, \hat{k}(\omega) \, d\omega$$

$$= \int_{\omega_0}^{\omega^*} e^{-A(\omega,\omega_0)} \alpha(\omega) \, \hat{k}(\omega) d\omega \; .$$

On the other hand, let k_0 be given by (77). Then, by using L'Hospital's rule, we have

$$\lim_{t \to \infty} k(t) = \lim_{\omega \to \omega^*} \frac{k_0 - \int_{\omega_0}^{\omega} e^{-A(\omega',\omega_0)} \alpha(\omega') \, \hat{k}(\omega') d\omega'}{e^{-A(\omega,\omega_0)}}$$

$$= \lim_{\omega \to \omega^*} \frac{-e^{-A(\omega,\omega_0)} \alpha(\omega) \, \hat{k}(\omega)}{-e^{-A(\omega,\omega_0)} \dfrac{\partial A(\omega,\omega_0)}{\partial \omega}}$$

$$= \lim_{\omega \to \omega^*} \hat{k}(\omega) = \hat{k}(\omega^*) \; .$$

It is easily seen that, for any positive ω_0, the integral in (77) exists and is positive; we may write

$$k_0 = k_0(\omega_0) \; .$$

The relationship of $k_0(\omega_0)$ to $k_C(\omega)$, $k_I(\omega)$, $\hat{k}(\omega)$ may be described by Figure 7.

The structure of optimal growth is analyzed by a method similar to one we have used in the previous case. Let us first discuss the limiting case in which the minimum wage rate is zero. Let the two critical points (k_I^*, ω_I^*) and $(k_C^{**}, \omega_C^{**})$ be defined as the intercepts of the $k_0(\omega)$

Fig. 7

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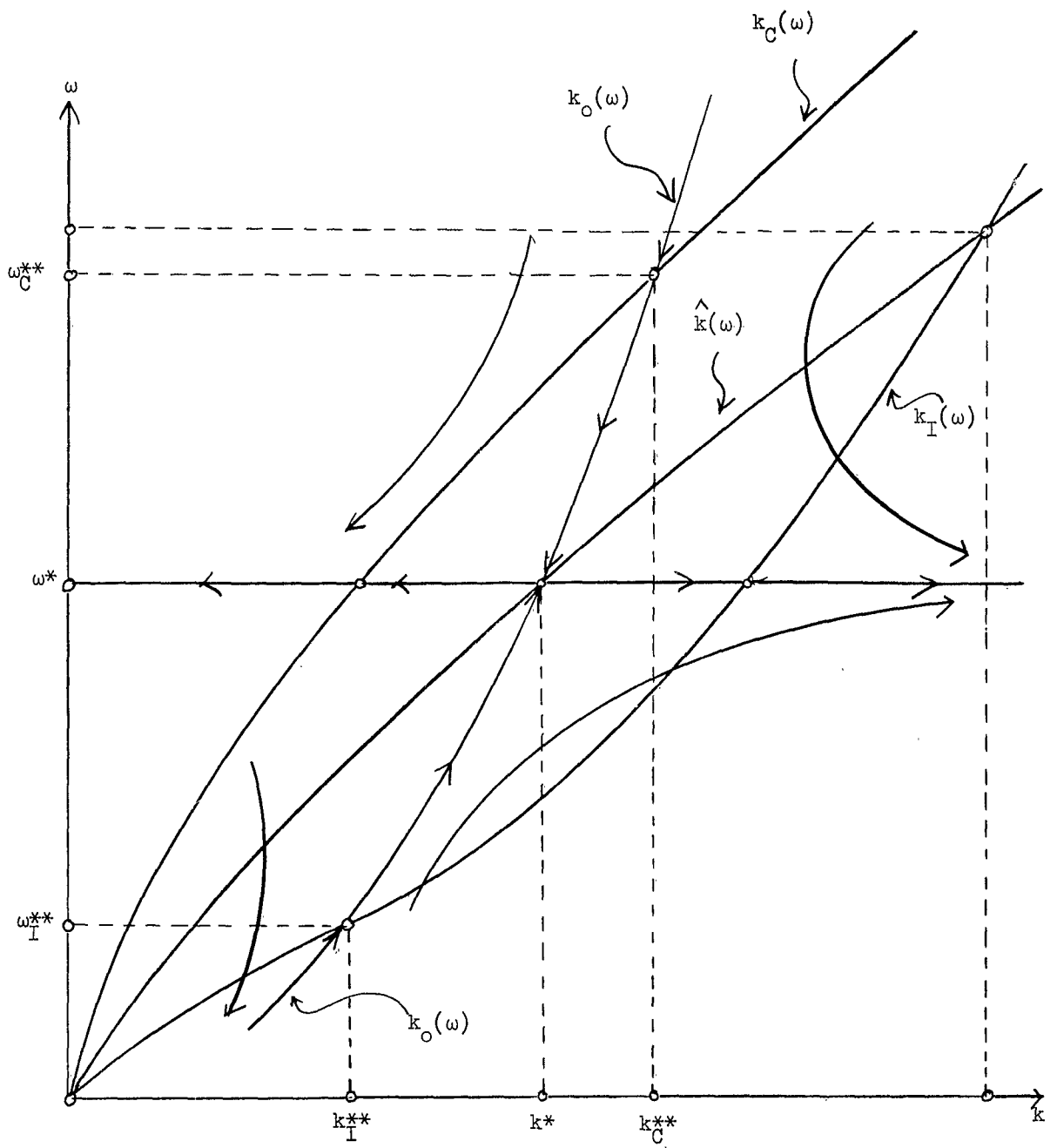
curve with the $k_I(\omega)$ curve and the $k_C(\omega)$ curve respectively:

$$k_I^{**} = k_o(\omega_I^{**}) = k_I(\omega_I^{**}) \ ,$$

$$k_C^{**} = k_o(\omega_C^{**}) = k_C(\omega_C^{**}) \ .$$

we have

(81) $$k_I^{**} < k^* < k_C^{**} \ .$$

It is noted that if the supply price of capital $p(t)$ is equal to the demand price of capital $q(t)$ on the $k_I(\omega)$ curve below the critical point (k_I^{**}, ω^{**}), and if the economy is specialized in capital goods, then the demand price of capital along the differential equation (33) falls while the supply price rises, and the economy continues the specialization in capital goods along the optimal path. Similarly, if the supply price of capital is equal to the demand price of capital on the $k_C(\omega)$ curve above the critical point (k_C^{**}, ω^{**}), and if the economy specializes in consumption goods, then the supply price rises while the demand price falls; thus the economy continues the specialization in consumption goods along the optimal path.

Therefore, the optimal paths are characterized by the following two propositions:

(a) If the initial capital-labor ratio $k(0)$ is smaller than the critical ratio k_I^{**}, then along the optimal path the economy is specialized to capital goods until the capital-labor ratio $k(t)$ reaches the critical ratio k_I^{**}. Once the critical level k_I^{**} is reached, the economy proceeds along the $k_o(\omega)$ curve toward the balanced state (k^*, ω^*).

(b) _If the initial capital-labor ratio_ $k(0)$ _is larger than the critical ratio_ k_I^{**} , _then along the optimal path the economy is specialized to capital goods until the capital-labor ratio_ $k(t)$ _is reduced to the critical ratio_ k_C^{**} , _and from then on it proceeds along the_ $k_o(\omega)$ _curve toward the balanced state_ (k^*, ω^*).

The analytical expressions for these two cases are similarly described in those given in (a) and (b) in the previous section.

In Figure 8, the optimal paths are indicated by the heavy arrow curves.

To discuss the general case in which the minimum wage rate is positive, let us introduce the function, $k_{max}(\omega)$, by:

$$(82) \qquad k_{max}(\omega) = k_I(\omega) - \frac{k_I(\omega) - k_C(\omega)}{f_C(k_C(\omega))} \; w_{min} \; .$$

It is then shown that the per capita consumption $y_C(t)$ exceeds the minimum wage rate w_{min} if and only if

$$(83) \qquad k(t) < k_{max}(\omega(t)) \; .$$

By assumption (73), $k_{max}(\omega)$ is always less than $k_I(\omega)$, while it is larger than $k_C(\omega)$ if and only if $f_C[k_C(\omega)] > w_{min}$; namely,

$$(84) \qquad k_{max}(\omega) > k_C(\omega) \; , \quad \text{if and only if} \quad \omega > \omega_{min} \; .$$

On the other hand, $k_{max}(\omega) > \hat{k}(\omega)$ if and only if the inequality (67) holds. Reasoning similar to that used in the previous section leads us again to the following conclusion that, for $\omega < \omega^*_\lambda$,

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Fig. 8

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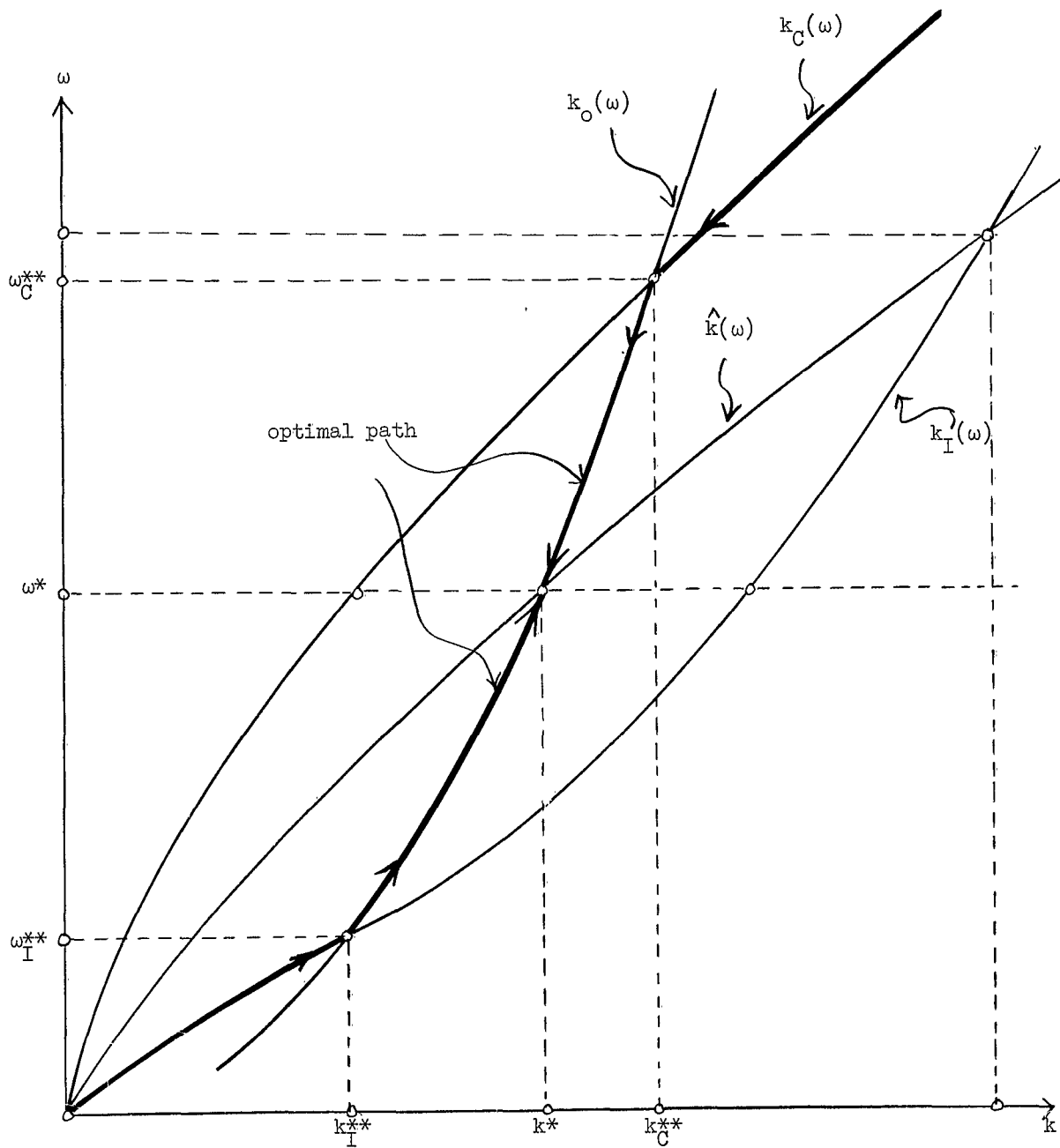
(85) $k_{max}(\omega) > \hat{k}(\omega)$ if and only if $\omega > \hat{\omega}$,

where $\hat{\omega}$ is defined by (69).

The relationship of the $k_{max}(\omega)$ curve to the $k_C(\omega)$, $k_I(\omega)$, and
$\hat{k}(\omega)$ curves is illustrated in Figure 9.

The critical capital-labor ratios $\hat{k}$ and k_{min} are defined by

(86) $\hat{k} = \hat{k}(\hat{\omega}) = k_{max}(\hat{\omega})$,

(87) $k_{min} = k_C(\omega_{min}) = k_{max}(\omega_{min})$.

The critical capital-labor ratio, $\hat{k}$, again represents the level
of the capital-labor ratio below which the economy suffers from the
steadily declining capital-labor ratio, whatever the allocation of the
scarce resources may be, while below the minimum ratio, k_{min}, the
economy cannot afford production of consumption goods with the minimum
requirement even though capital goods are not produced. When the initial
capital-labor ratio, $k(0)$, exceeds the critical ratio, k, the structure
of the optimal paths is characterized similarly to those described for
the previous cases. In Figure 10, the optimal paths are depicted by
the heavy arrow curves.

7.   Optimal Growth for the General Case.

The foregoing analysis may be, without modification, applied to
characterize the optimal paths for the general case in which the rela-
tive capital intensity may be reversed; namely, if consumption goods
are more capital-intensive than capital goods for the balanced wage-
rentals ratio ω^*, then the pattern described in Section 5 prevails
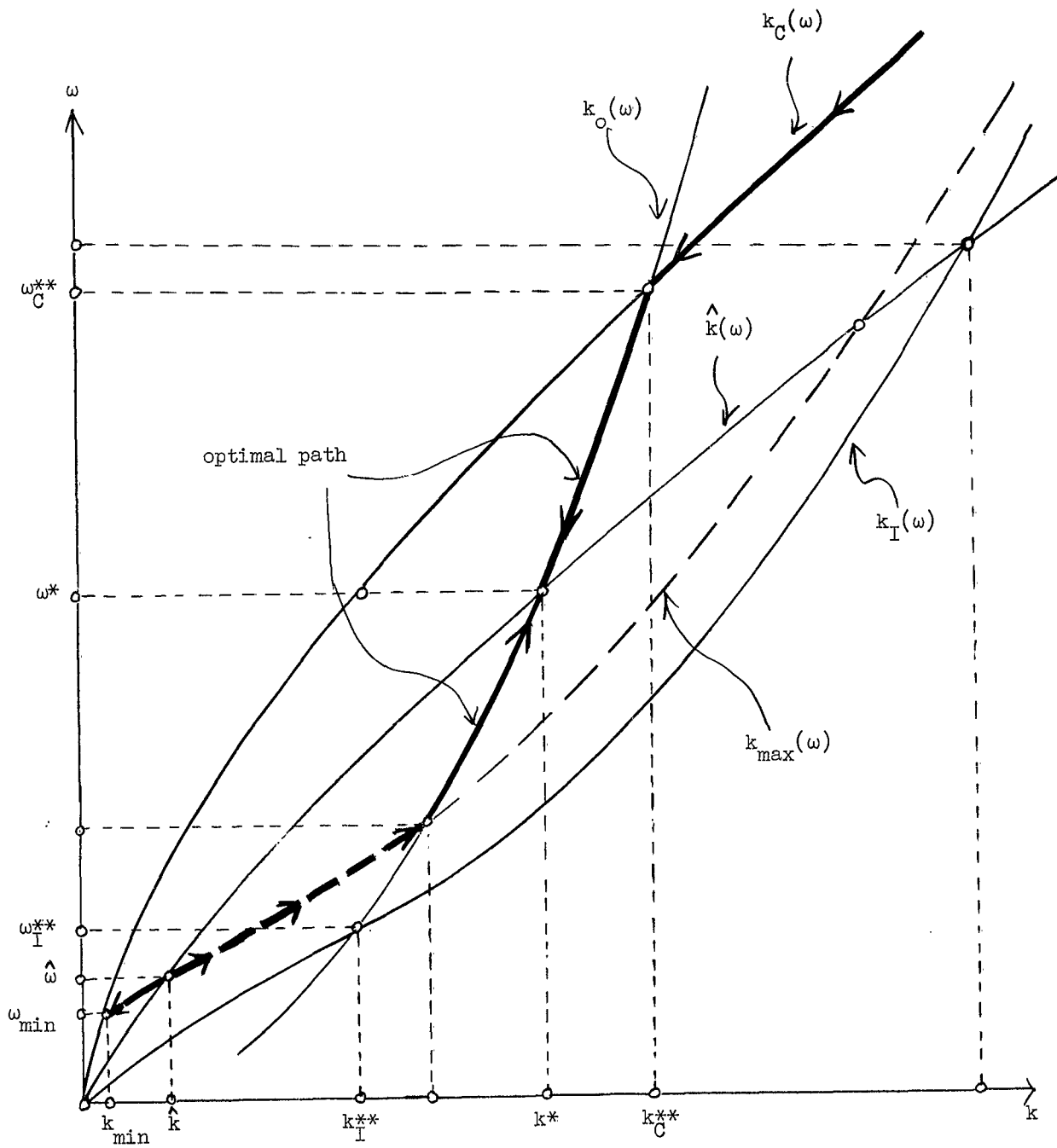
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Fig. 9

Fig. 10

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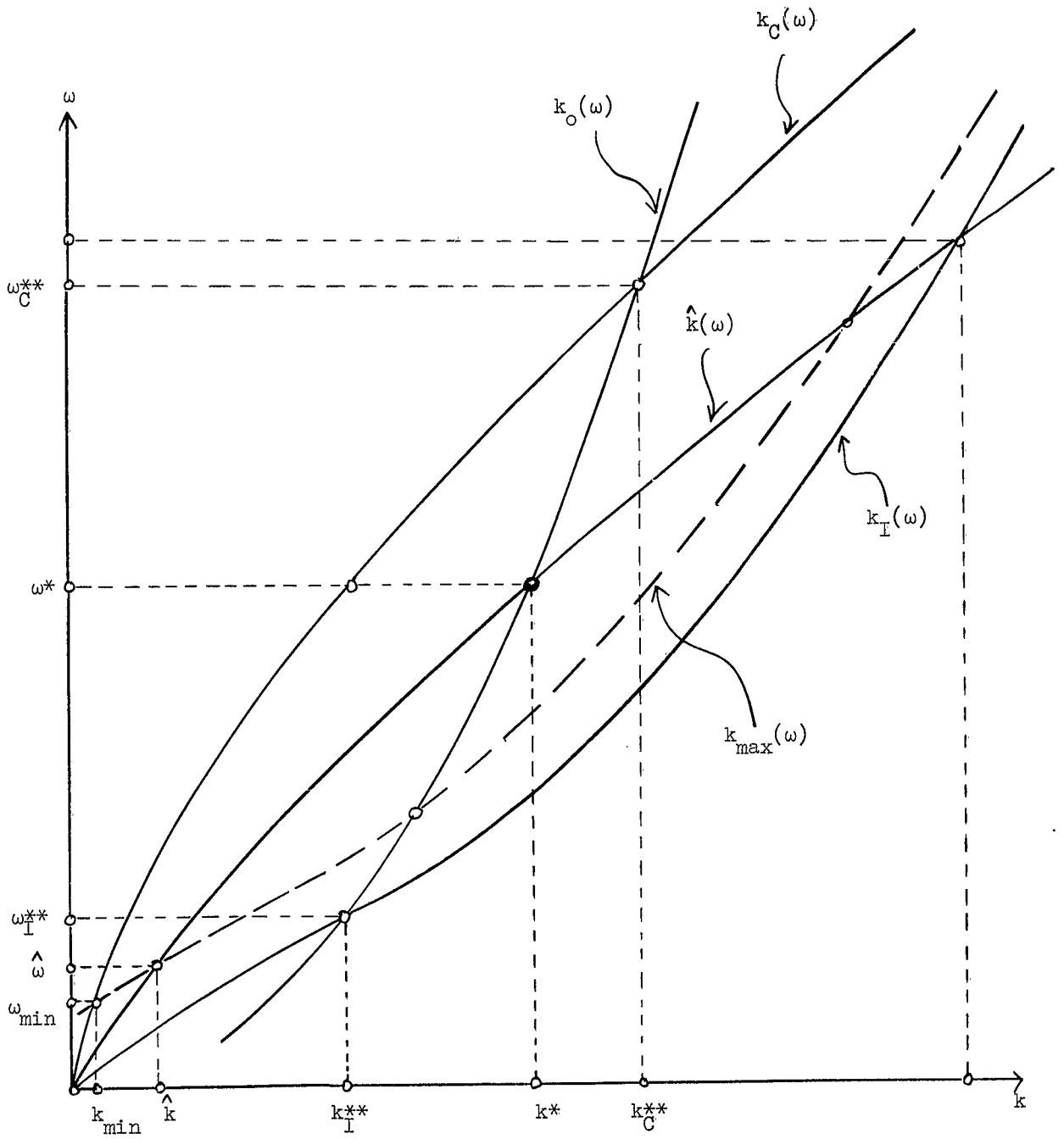
for the optimal paths, while the pattern in Section 6 is applied for the optimal paths in the economy in which consumption goods are less capital-intensive than capital goods for the balanced wage-rentals ratio ω^*. These two cases are illustrated in Figures 11 and 12 respectively.

The characterization we have obtained may be used to derive various conclusions concerning the optimal growth in the two-sector model of capital accumulation.

Let us first recall that, along the optimal path, the allocation of capital and labor between the two sectors and the quantities per capita of consumption goods and capital goods to be produced at any moment of time are determined by specifying the capital-labor ratio $k(t)$ and the wage-rentals ratio $\omega(t)$. Associated with the given discount rate δ, the balanced state (k^*, ω^*) is given by:

$$f_I'[k_I(\omega^*_{\lambda+\delta})] = \lambda + \delta$$

$$k^* = \hat{k}(\omega^*_{\lambda+\delta}) = \frac{f_I(k_I^*)}{f_I(k_I^*) + \lambda(k_C^*-k_I^*)} \ k_C^* \ ,$$

where
$$k_C^* = k_C(\omega^*) \ , \quad k_I^* = k_I(\omega^*)$$

The balanced quantities per capita of consumption goods and capital goods, y_C^* and y_I^*, are determined by:

$$y_C^* = \frac{k^*-k_I^*}{k_C^*-k_I^*} \ f_C(k_C^*) \ ,$$

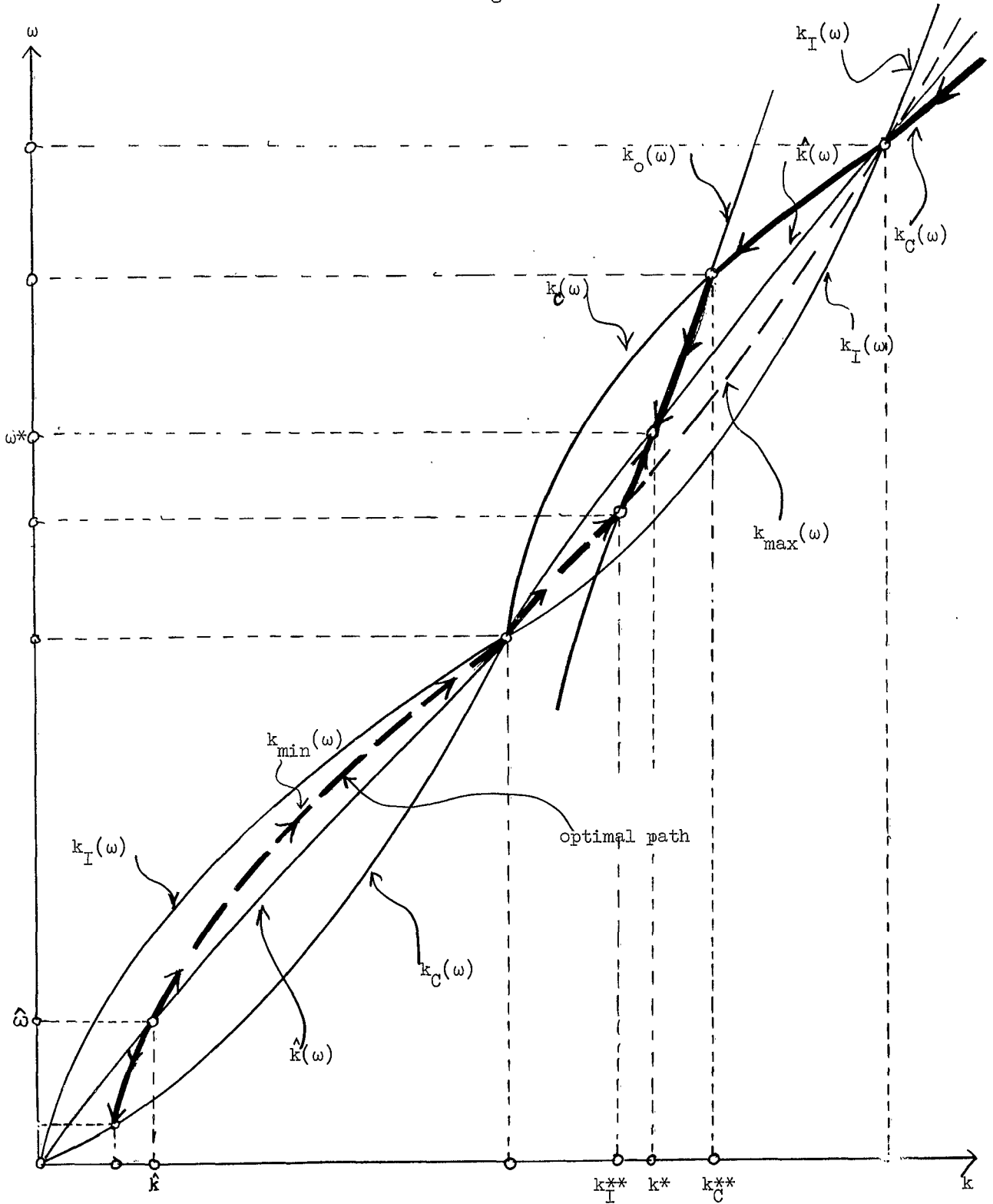$$y_I^* = \frac{k_C^*-k^*}{k_C^*-k_I^*} \ f_I(k_I^*) \ .$$

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Fig. 11

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Fig. 12

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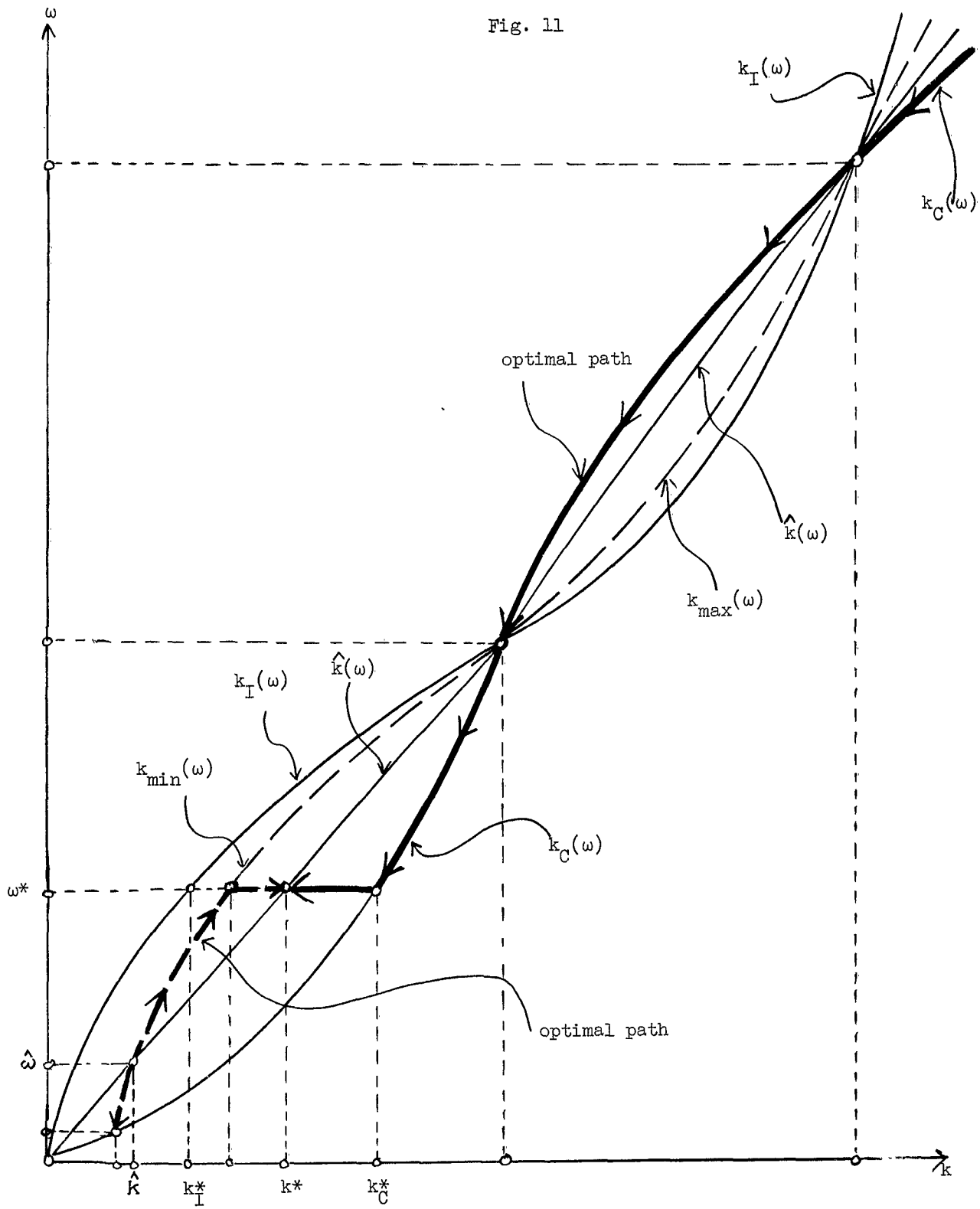
Along the optimal path, the capital-labor ratio $k(t)$, the wage-rentals ratio $\omega(t)$, the per capita consumption $y_C(t)$, and the per capita investment $y_I(t)$ all approach monotonically to the balanced capital-labor ratio k^*, wage-rentals ratio $\omega^*_{\lambda+\delta}$, per capita consumption y_C^*, and per capita investment y_I^*. The economy is specialized either in capital goods (consumption goods being produced only for minimum requirements) or in consumption goods until a certain critical level is reached for the capital-labor ratio.

The higher the discount rate δ, the lower the balanced capital-labor ratio k^* and the balanced wage-rentals ratio $\omega^*_{\lambda+\delta}$. As the discount rate δ tends to zero, the balanced wage-rentals ratio $\omega^*_{\lambda+\delta}$ approaches the level ω^*_λ defined by:

$$f'_I[k_I(\omega^*)] = \lambda .$$

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