THE IDENTIFICATION OF OVERDAMPED PROCESSES IN THE TIME-DOMAIN

by

Jerry M. Mendel

Research Report No. PIBMRI-1131-63

for

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Washington 25, D. C.

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April 19, 1963

MRI

POLYTECHNIC INSTITUTE OF BROOKLYN
MICROWAVE RESEARCH INSTITUTE

ELECTRICAL ENGINEERING DEPARTMENT

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Polytechnic Institute of Brooklyn
Microwave Research Institute
55 Johnson Street
Brooklyn 1, New York

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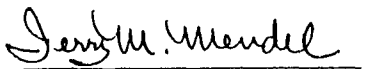

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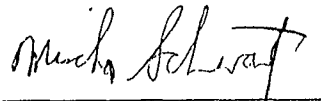

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Prepared for

The Air Force Office of Scientific Research
Washington 25, D. C.

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AN ABSTRACT

THE IDENTIFICATION OF OVERDAMPED PROCESSES IN THE TIME-DOMAIN

by

Jerry M. Mendel

Advisor: Dr. Ludwig Braun, Jr.

The problem of characterizing overdamped systems, from data in the time-domain, by means of exponential functions is studied. Particular attention is given to: (1) the choice of a suitable error criterion based upon performance measures the approximate system is to meet, (2) the choice of weighting functions and their effect on the approximations, and (3) the extension of the orthonormal exponential approximations of Kautz and Huggins to approximations of any asymptotic-order in the s-domain.

The spirit of the research carried out is in keeping with the philosophy that the approximate system will replace the actual system in practical analyses. For example, the actual system may be embedded in the forward path of a positional servomechanism. If a stability analysis of the closed-loop system is to be performed then it is desirable for the approximate plant, which will be used in the stability analysis, to indicate the correct stability behavior within an allowable margin of error. Since stability is intimately connected with high-frequency behavior it appears to be desirable to be able to match the high-frequency behavior of the actual and approximate systems. This naturally leads to the choice of exponential approximants of asymptotic-order, in the s-domain, greater than unity.

In each of the above areas of concentration, emphasis is placed on the effects that the asymptotic-order will have on the utility of the approximations in further anaylses.

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CHAPTER I. INTRODUCTION

I.1 Introductory Remarks

In both analysis and synthesis, it is necessary to be able to characterize a system by means of a mathematical model. In simple cases, this can be done by writing the differential equations describing the system and then solving them for the desired input-output ratio. In many cases, however, the system is so complex that the only feasible methods for obtaining a model are experimental in nature. These measurements can be made in either the frequency-domain or in the time-domain. Many problems, such as determining what measurements to make, whether to use periodic or aperiodic inputs, etc., exist in connection with the practical aspects of making these measurements. Assuming that the correct measurements have been made, it shall be our purpose to concentrate on the representation of these measurements by means of a suitable working model. This is the "identification problem".

Within the last decade a great deal of attention has been focussed upon the time-domain characterization of processes. Obviously, one important reason for such a characterization has to do with the economy of a single transient test as compared to the multitude of frequency tests necessary to obtain the same amount of information. This trend has been noticeable not only in the field of network synthesis, where it originated, but also in the areas of adaptive control, optimal control theory, electrocardiography, and speech-signal representation.

Most of the past research, in the area of identification of processes in the time-domain, has been concerned with achieving a close fit between the model and the data, the closeness of fit being measured by the integral-squared error criterion. The introduction of the orthonormal exponentials by Kautz[18] and Huggins[15, 17] aided the approximators immeasurably, for their work enabled one to approximate exponential responses by exponential signals. These approximations not only lend themselves quite nicely to further analysis involving transform techniques, but are also in a form amenable to network realization; hence, they are useful in both the analysis and synthesis problems.

Unfortunately, ensuring a close fit between the approximation (model) and the actual system does not necessarily lead to a close correspondence between the two with respect to other properties of the system. For example, approximating the plant f(t) in Figure I.1.1 may only be the first step in a detailed analysis of the closed-loop system in which this plant is embedded.

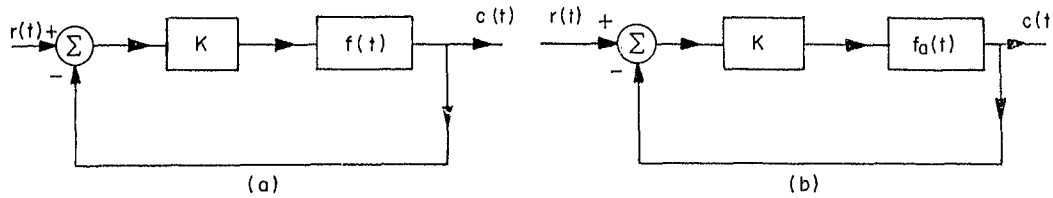


Figure I. 1. 1. A typical feedback system in which the plant f(t) in (a) is approximated by $f_a(t)$ in (b).

In particular, if one is interested in the stability of the closed-loop system, then there should be some correspondence between A. O. (F_a)* and A. O. (F), since the high-frequency behavior of the plant determines whether or not the closed-loop system becomes unstable as the loop gain increases. To illustrate the importance of this last sentence, consider a plant f(t) whose poles and zeros all lie in the left-half of the s-plane, and whose asymptotic-order is four. Not knowing the value of A. O. (F) ahead of time, assume that a model, $f_a(t)$, has been chosen to approximate the actual plant where A. O. $(F_a) = 2$. It is well known that for large enough values of the loop gain the actual closed-loop system does become unstable. The approximation, on the other hand, which may give a very good fit to f(t), indicates that the closed-loop system never becomes unstable (this last phrase tacitly assumes that $F_a(s)$ has no zeros in the right-half of the s-plane); thus, when one plans to utilize the approximation in future analyses, there appears to be more to the identification problem, in the time-domain, than merely achieving a close fit between the approximant and the true response. In control systems analyses, the asymptotic order of the plant' s model is of great importance, as has been mildlydemonstrated in the preceding sentences.

Rather than belabor the point at this time, the interested reader is referred to the introductory sections of Chapters 1-4 for more detailed introductory remarks. This chapter is concluded with a brief discussion of the actual problem studied and some of the assumptions made in the formulation of the problem.

---

* A. O. (F_a) is read "asymptotic-order of $F_a(s)$" and is defined by looking at the $\lim_{s\to\infty} F_a(s) = \frac{\gamma}{s^n}$ and letting A. O. $(F_a) = n$.

I. 2 Statement of the Problem

An identification of the system in Fig. I. 2. 1 is desired. The transfer function characterizing the system is to be of the following form:

$$T(s) = K_T \frac{s^v + a_{v-1} s^{v-1} + \ldots + a_1 s + a_o}{\prod_{\ell=0}^{N} (s + m_\ell)} \qquad (I.1)$$

Only the poles of T(s) must lie in the left-half of the s-plane. The zeros may be negative and real, positive and real, or complex. Since the impulse response of this

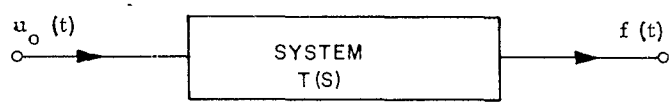


Figure I. 2.1. A description of the system which is to be identified.

system, quite obviously, is of the form $\sum_{k=0}^{N} A_k \epsilon^{-m_k t}$, where the m_k are real, it shall be approximated by a similar set of functions, that is

$$f(t) \approx f_a(t) = \sum_{k=0}^{n} C_k f_{a_k}(t) \qquad (I.2)$$

where the $f_{a_k}(t)$ are linear combinations of exponential functions. This study is limited to real exponents (overdamped processes) for two reasons. First, almost no work has been done in the area of approximating overdamped processes of high asymptotic-order, and secondly, the introduction of complex poles into Eq. (I. 1) vastly increases the analytical complexity of the problem.

It was felt that a concentrated study in one aspect of the more general identification problem might lead to results that could easily be extended to the more general case. It is also possible that by narrowing one's objectives he will discover theory that is particularly amenable to the case of real exponents. This may be completely overlooked in a more general study. As it turned out theory was developed that justified narrowing the scope of the problem. Actually, there are many systems that occur quite frequently in practice satisfying the requirements of Figure I. 2. 1, as stated in Eq. (I. 1).

It is assumed that the internal construction of the system in Figure I. 2. 1 is unknown and that only the external terminals are available for the application of test signals. In order to determine whether or not the system meets the requirements of Eq. (I. 1) one must be able to correlate the form of the time response y(t) with possible pole-zero locations in Y(s). Table I. 2. 1 presents a number of t-domain -- s-domain correlations which are useful in the analysis. It is based upon a set of ingenuous theorems proven by Brulé [7] in his Ph. D dissertation.

The problem before the analysist is one of choosing a proper set of $f_{a_k}(t)$ in Eq. (I. 2) and then determining the C_k based upon a minimization of some measure of the error between f(t) and $f_a(t)$. This error, e (t), is defined in Eq. (I. 3).

$$e(t) = f(t) - f_a(t) \tag{I. 3}$$

It is convenient to use the functional notation $F\left[\psi(e), w(t)\right]$ to represent a measure of the error that must be minimized, in order to find the C_k. The ψ (e) notation is short hand for "an operator ψ acting upon e(t)", and w(t) is known as a weighting function. w(t) can be chosen so as to weight the approximation heavily over certain intervals of time and lightly over others. $F\left[\psi(e), w(t)\right]$ is more commonly known as an error criterion. As an example, consider the ITE^2 * criterion defined in Eq. (I. 4).

$$ITE^2 = \int_0^\infty t\, e^2\, dt \tag{I. 4}$$

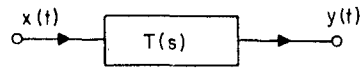
This equation can be put into the functional form $F\left[\psi(e), w(t)\right]$ by letting w(t) = t, $\psi(e) = e^2$, and $F = \int_0^\infty dt$. Note that the functional notation for the ITE^2 criterion is consistent with the mathematical definition of a functional, that is to say, it is a pure number. The weighting function, in this case w(t) = t, quite obviously emphasizes large values of t.

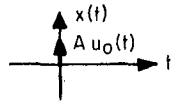
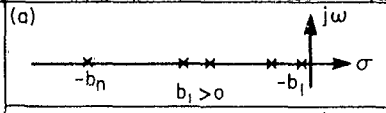
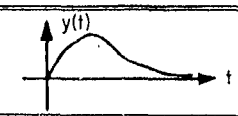
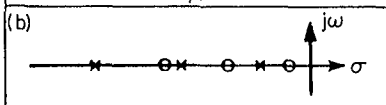
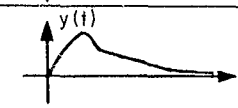
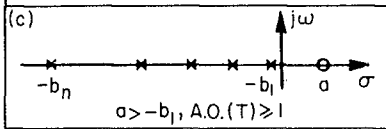
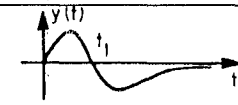
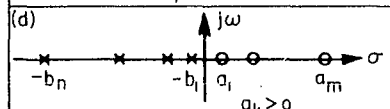
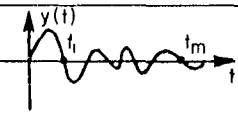
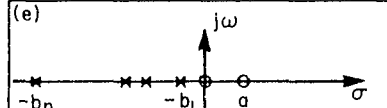
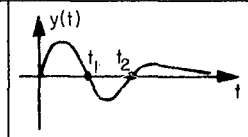
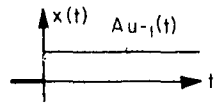
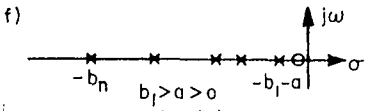
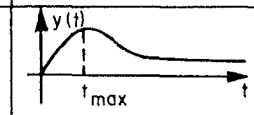
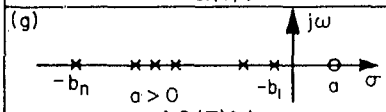
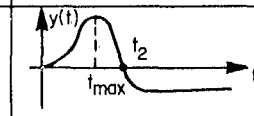
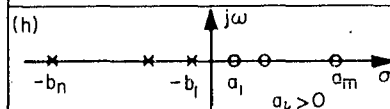
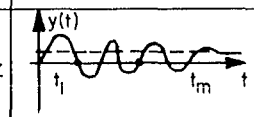
In the following four chapters, the choice of the approximants $f_{a_k}(t)$ [chapters 1, 3 and 4], a discussion of weighting functions [chapters 2 and 4] and their design [chapters 2 and 3], and the choice of error criteria leading to optimal approximations (chapter 4) are discussed in detail.

---

* ITE^2 is read, "Integral-Time-Squared-Error". Some authors prefer the abbreviation ITSE instead of ITE^2. Both abbreviations, however, are merely shorthand notations for the integral in Eq. (I. 4).

TABLE I.2.I SOME s-DOMAIN — t-DOMAIN CORRELATIONS.



x(t)	T(s)	y(t)	COMMENTS
	(a) 		1. y(t) > 0 FOR ALL t > 0 2. y(t) HAS ONLY ONE EXTREMUM
	(b) 		1. y(t) > 0 FOR ALL t > 0 2. T(s) = Z_{RC}(s)
	(c) 		1. y(t) = 0 EXACTLY ONCE FOR SOME t = t_1 = 0 2. y(t) HAS ONE MAX. AND MIN. ONLY IF A.O.(T) ≥ 2
	(d) 		1. y(t) HAS EXACTLY m INTERNAL ZEROS. 2. WHEN $\lvert a_k \rvert = \lvert b_\ell \rvert$ T(s) REPRESENTS AN ALL-PASS CONFIGURATION.
	(e) 		y(t) HAS EXACTLY 2 INTERNAL ZEROS
	(f) 		1. y(t) > 0 FOR ALL t > 0 2. y(t) HAS ONE EXTREMUM AT t = t_{max} > 0 3. t_{max} = t_1 IN (c).
	(g) 		1. y(t) = 0 EXACTLY ONCE FOR SOME TIME t = t_2 > 0 2. y(t) HAS ONE EXTREMUM AT t = t_{max} > 0 3. t_{max} = t_1 IN (c).
	(h) 		1. y(t) HAS EXACTLY m INTERNAL ZEROS. 2. WHEN $\lvert a_k \rvert = \lvert b_\ell \rvert$ T(s) REPRESENTS ALL-PASS CONFIGURATION

CHAPTER 1. UNIFORMLY-WEIGHTED, ORTHONORMAL, EXPONENTIAL APPROXIMANTS

In this chapter the integral-squared-error criterion is reviewed and a procedure for constructing sets of orthonormal exponential approximants is developed, in which any desired asymptotic-order may be designed into the Laplace transform of the approximants, ahead of time. These approximants are summarized by a recursion equation in the time-domain.

1.1 Introduction

The integral-squared-error criterion (IE^2) is perhaps the most widely known and used measure of performance for time-domain approximations. The term "time-domain" is used here to denote the interval of time from zero to infinity, that is $t \varepsilon (0, \infty)$.* A function $f(t)$ is said to be approximated in an "integral-squared sense" when it is approximated by the function $f_a(t) = \sum_{k=0}^{n} C_k \phi_k(t)$ in such a way that the IE^2 is a minimum. Given the definition of the IE^2 criterion,

$$IE^2 = \int_0^\infty w(t)\left[f(t) - f_a(t)\right]^2 dt \tag{1.1}$$

it is well known [18, 28] that the minimum IE^2 is

$$\min (IE^2) = \int_0^\infty w(t) f^2(t)\, dt - \sum_{k=0}^{n} C_k^2 \tag{1.2}$$

provided the elements $\phi_k(t)$ form an orthonormal set with respect to the weighting function $w(t)$ over the interval of time $(0, \infty)$. The condition for the orthonormality of the $\phi_k(t)$ is stated in Eq. (1.3).

$$\int_0^\infty w(t)\ \phi_i(t)\ \phi_j(t)\, dt = \delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases} \tag{1.3}$$

[Strictly speaking one should indicate that min (IE^2) is the minimum IE^2 with respect to the coefficients C_k in $f_a(t)$. It will be shown that, under certain conditions, the $\phi_k(t)$ can be formed from linear combinations of terms of the form $\epsilon^{-a_k t}$, where the exponents a_k are unknown. Liu[20] points out that the determination of the a_k from a minimization of the IE^2 with respect to the a_k involves lengthy and laborious computations, and what's more, the solution is not unique. It shall be assumed, throughout this entire dissertation, that the a_k have been specified a priori (see Appendix A

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* ε is a mathematical shorthand notation for, "is an element of".

for a discussion of a restriction pertinent to the choice of these exponents). The notation min (IE^2) will therefore always denote a minimization of the IE^2 with respect to the coefficients C_k .] As a result of the minimization the coefficients C_k are found to be

$$C_k = \int_0^\infty w(t)\, f(t)\, \phi_k(t)\, dt \qquad (1.4)$$

It is quite evident from this equation and Eq. (1.2) that the addition of an extra term $C_{n+1}\ \phi_{n+1}(t)$ to $f_a(t)$ in no way affects any of the previously calculated C_k's and will tend to decrease the min (IE^2) by $(C_{n+1})^2$. This represents the advantage of choosing an orthonormal set of functions $\phi_k(t)$ as compared to a non-orthonormal set of approximants.

Before proceeding to a discussion of how one constructs the orthonormal set of functions $\phi_k(t)$, it will be well to point out the importance of the infinite interval $(0, \infty)$. The following discussion is taken directly from Kautz' s[18] original research report, "It may be safely said that the number of practical problems in which precise transient behavior is desired over the entire interval $(0, \infty)$ is extremely small. Furthermore, for sufficiently large t, the response must drop off to zero either exponentially or as an exponentially damped sinusoid, so that no other type of behavior at infinity can be obtained, even if it were desired." (Kautz has tacitly assumed, in this last statement, that all recognizable components of the response being identified, such as a d-c component or a sinusoidal steady state component, have been removed by means of subtraction. The approximation is then made upon the remaining signal.) "One might very well reason, then, that methods for approximating a function should be developed over only a finite interval. That this conclusion is incorrect is brought out by the following three facts:

(1) If a finite interval is employed, almost nothing can be specified about the transient outside of this interval using existing approximation methods. It may behave very erratically, and as a rule it cannot be controlled.

(2) The use of an infinite interval does not imply that equal emphasis is given in the approximation to all sections of the time scale. In fact, normally very little weight is placed on reproducing the response for large values of t.

(3) It is difficult to carry out the analytical approximations over a finite interval."

The problem now is, "how does one construct the orthonormal set of functions $\phi_k(t)$?"

The answer to this question can be given by appealing to the Gram-Schmidt Orthonormalization Procedure[18, 28], which states that a unique set of orthonormal functions $\phi_k(t)$ can always be formed from a set of linearly-independent functions $g_\ell(t)$, in the following way:

$$\left.\begin{aligned} \phi_o(t) &= \lambda_o g_o(t) \\ \phi_1(t) &= \lambda_1\left[g_o(t) + \lambda_2 g_1(t)\right] \\ &\cdots\cdots\cdots\cdots\cdots \\ \phi_k(t) &= \lambda_m\left[g_o(t) + \lambda_i g_1(t) + \ldots + \lambda_j g_k(t)\right] \end{aligned}\right\} \quad (1.5)$$

The constants λ_ℓ are determined by substituting these equations into Eq. (1.3), which, depending upon the choice of the $g_\ell(t)$, usually represent a formidable set of simultaneous linear equations in the λ_ℓ, from which these constants must be solved for.

Since this discussion is being limited to the identification of the functions

$$f(t) = \sum_{k=0}^{N} A_k \epsilon^{-m_k t}, \quad m_k \text{ real} \quad (1.6)$$

it seems natural to choose the $g_\ell(t)$ as an exponential set of functions. If in particular

$$g_\ell(t) \; \epsilon \left\{\epsilon^{-a_\ell t}\right\} \quad \ell = 0, 1, \ldots \quad (1.7)$$

Kautz[18] has shown that the result of the Gram-Schmidt Procedure can be expressed very neatly in the frequency-domain, provided a constant weighting function, w (t), is used. This is the "uniformly-weighted" case. As a matter of fact, he has shown that the entire orthonormalization procedure can be carried out in the s-domain by means of Parseval's Equation* and Cauchy's Residue theorem, the results being

$$\left.\begin{aligned} \Phi_o(s) &= \sqrt{2a_o}\;\; \frac{1}{s + a_o} \\ \Phi_1(s) &= \sqrt{2a_1}\;\; \frac{(s - a_o)}{(s + a_o)(s + a_1)} \\ &\cdots\cdots\cdots\cdots \\ \Phi_k(s) &= \sqrt{2a_k}\;\; \frac{(s - a_o)\ldots(s - a_{k-1})}{(s + a_o)(s + a_1)\ldots(s + a_k)} \end{aligned}\right\} \quad (1.8)$$

for w (t) = 1.

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*Parseval's Equation[22] states that

$$\int_o^\infty h_1(t)\, h_2(t)\, dt = \frac{1}{2\pi j} \int_{c-j\infty}^{c+j\infty} H_1(s)\, H_2(-s)\, ds$$

Braun[4, 22] has extended these results to an exponential weighting function, ϵ^{-2bt}. In this case

$$\left.\begin{aligned}
\Phi_o(s) &= \sqrt{2(b + a_o)}\ \frac{1}{(s + a_o)} \\
\Phi_1(s) &= \sqrt{2(b + a_1)}\ \frac{(s - 2b - a_o)}{(s + a_o)(s + a_1)} \\
&\cdots\cdots\cdots \\
\Phi_k(s) &= \sqrt{2(b + a_k)}\ \frac{(s - 2b - a_o)\ldots(s - 2b - a_{k-1})}{(s + a_o)(s + a_1)\ldots(s + a_k)}
\end{aligned}\right\} \quad (1.9)$$

A discussion of choosing one form for the weighting function over another will be put off at this time until Chapter 2. In that chapter a great deal will be said about weighting functions, in general, including the two cases of a constant and an exponential.

It is obvious from Eqs. (1.8) and (1.9) and the fact $f_a(t) = \sum_{k=0}^{n} C_k \phi_k(t)$ that $F_a(s)$ will <u>always</u> have an asymptotic-order equal to unity, since A.O. (F_a) equals A.O. (Φ_k), and A.O. (Φ_k) is equal to unity. It is also interesting to note that if $\Phi_k(s)$ is considered to be the transfer function of a linear filter, then $\phi_k(t)$, which is the impulse response of the filter, will be oscillatory (Table I.2.1), and in fact will contain exactly k internal crossings of the t-axis, provided $k \geq 1$.

Practically speaking, if the function f(t), which is to be approximated by $f_a(t)$, has a non-zero initial value (at $t = 0^+$) then Kautz's (Braun's) results represent the natural choice for the $\phi_k(t)$. This does not mean that the one term approximation $C_o \phi_o(t)$, or the two term approximation $C_o\phi_o(t) + C_1\phi_1(t)$, or even the ten term approximation $C_o\phi_o(t) + \ldots + C_9\phi_9(t)$, where the $\phi_k(t)$ are found from Eq. (1.8) [or Eq. (1.9)], will match the initial value of f(t) exactly. What is meant, is that in the limit, as more and more terms are added to the approximation, the initial value of the approximation will coincide with the non-zero initial value of f(t). This is due primarily to the facts that A.O. (F_a) = A.O. (F), and that both of these are equal to unity.

On the other hand, if f(t) has a zero intial value, and (r - 1) zero initial derivatives at $t = 0^+$, where r must somehow be determined from the given f(t) data, it would make much more sense to choose a set of $\phi_k(t)$ having a similar behavior at $t = 0^+$. This means choosing a set of $\phi_k(t)$ such that A.O. $(\Phi_k) = r + 1$.

[This last result was found by applying the Initial Value theorem, from Laplace Transform Theory, r times to a function of asymptotic-order r + 1, and observing that the initial value and exactly (r - 1) initial derivatives of that function vanish.] This argument can be amplified by considering $f_a(t)$ to be the identification of a plant f(t), such as the one shown in the forward path of the simple positional servomechanism, Fig. 1.1.1. The absolute stability of this closed-loop system [using the approximate plant $f_a(t)$] will depend upon the asymptotic-order we have chosen for $f_a(t)$, as we have already indicated on page 2. It is well known that if the A. O. (F) $\geq$ 3 the servomechanism will always become unstable for sufficiently large values of loop gain. The use of an approximation having an asymptotic-order equal to unity, on the

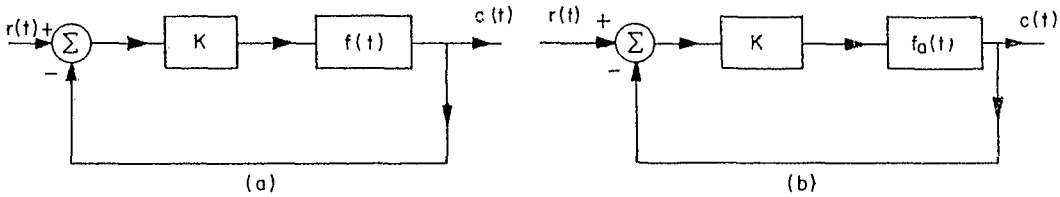

Figure 1.1.1. A typical feedback system in which the plant f(t) in (a) is approximated by $f_a(t)$ in (b).

other hand, implies that the system is absolutely stable; that is, no matter how large one makes the loop gain, the closed-loop system is stable.

It seems very desirable, therefore, to be able to extend Kautz's and Braun's results to cases where A. O. $(\Phi_k) > 1$. It also would be particularly advantageous to the user to have such sets summarizable either in the s-domain, by a recursion equation similar to Eqs. (1.8) and (1.9), or in the time-domain, whichever may be more appropriate.

In quest of such relationships, Braun[5] considered the general set of $\phi_k(t)$ where A. O. $(\Phi_k) = 2$, and attempted to perform the orthonormalization (with respect to a unity weighting function) in the s-domain by assuming

$$\left.\begin{aligned} \Phi_o(s) &= \frac{E_o}{(s+\alpha_o)(s+\alpha_1)} \\ \Phi_1(s) &= \frac{E_1(s+\beta_o)}{(s+\alpha_o)(s+\alpha_1)(s+\alpha_2)} \\ &\cdots\cdots\cdots\cdots \\ \Phi_k(s) &= \frac{E_k}{E_{k-1}}\,\Phi_{k-1}(s)\,\frac{(s+\beta_{k-1})}{(s+\alpha_k)} \end{aligned}\right\} \qquad (1.10)$$

Substituting these equations into the s-domain version of Eq. (1.3),

$$\frac{1}{2\pi j} \int_{\sigma - j\infty}^{\sigma + j\infty} \Phi_i(s)\, \Phi_j(-s)\, ds = \delta_{ij} \qquad (1.11)$$

and applying Cauchy's Residue Theorem, by closing the contour in either the right-half or the left-half of the s-plane, [due to the analyticity of $\Phi_i(s)$ and $\Phi_j(-s)$ in the right-half and left-half of the s-plane, respectively] one is confronted with a very formidable set of equations which must be solved for the zeros β_k and the constants E_k* in terms of the poles α_k. After obtaining a few of the zeros, β_k, and constants E_k, Braun concludes that it is difficult to derive a general recursion formula for a set of $\phi_k(t)$ in the s-domain in terms of the α_k, when the A.O. $(\Phi_k) = 2$. This does not mean that such sets are impossible to construct. On the contrary, it implies that one should choose the α_k first and then orthonormalize the elements $\phi_k(t)$. Naturally, this means that, each time the set of α_k's is changed, the entire orthonormalization procedure must be repeated. In general, these conclusions are valid and apply to the construction of any set of $\phi_k(t)$ when A.O. $(\Phi_k) > 1$.

The remaining sections of this chapter are devoted to the case of "equally-spaced" poles, where the term "equally-spaced" implies that $\alpha_k - \alpha_{k-1}$ = constant, in Eqs. (1.8) and (1.9)**. It will be shown that, in this case, the elements $\phi_k(t)$ can be found from a very general recursion equation in the time-domain. What is more, it will be proven that the generality of this equation enables one to use it to construct sets of orthonormalized exponentials whose Laplace Transform $\Phi_k(s)$ can be of <u>any asymptotic-order</u>, whatsoever. Rather than present the theory, theorems and proofs in their entire generality all of the important results shall first be developed for a set of $\phi_k(t)$ orthonormalized with respect to a unity weighting function. A discussion of a weighting function which is much more general than the type Braun has used (ϵ^{-2bt}), but one which reduces to ϵ^{-2bt} under special circumstances, is the subject of Chapter 2. The design of this weighting function will occupy a large portion of Chapter 2 since its design is necessary for the synthesis procedure developed in Chapter 3 for obtaining a set of functions, $\phi_k(t)$, which are orthonormal with respect to this more general weighting function. Here, again, it will be shown that the asymptotic-order of the $\Phi_k(s)$ can be chosen with complete freedom.

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* The zeros β_k are the orthogonalization constants while the constants E_k ensure the normality of the $\phi_k(t)$.

**More will be said about this type of pole spacing in Sections 1.3 and B.5.

To set the stage for Section 1.3, in which the recursion equation for the elements $\phi_k(t)$ is derived, one must first discuss a set or orthonormal polynomials known as the Jacobi Polynomials, since the recursion equation will be expressed in terms of these polynomials.*

* Actually, Armstrong[2,3] as early as 1957 indicated a correspondence between the orthonormal exponential set and the classical Jacobi Polynomials for very special sets of pole locations (see Section 3.1); however, he never considered the more general nature of this correspondence or the possibility of extending Kautz' s work to the case of a more general weighting function.

1.2 Jacobi Polynomials

The Jacobi Polynomials $\mathcal{J}_m(a, c; x)$ are m'th degree polynomials in x which are defined[21] by the following series:

$$\mathcal{J}_m(a, c; x) = \sum_{\ell=0}^{m} (-1)^{\ell} \binom{m}{\ell} \frac{\Gamma(a+m+\ell)\,\Gamma(c)}{\Gamma(a+m)\,\Gamma(c+\ell)}\, x^{\ell} \qquad (1.12)$$

$$(-c \neq 0, 1, 2, \ldots, m-1)$$

They are orthonormal with respect to the weighting function $x^{c-1}(1-x)^{a-c}$ over the interval (0,1),* that is

$$\int_0^1 x^{c-1}(1-x)^{a-c}\mathcal{J}_m(a, c; x)\mathcal{J}_n(a, c; x)\,dx = K_m \delta_{mn} \qquad (1.13)$$

provided that

$$c > 0, \text{ and } a > c-1 \qquad (1.14)$$

K_m is the orthonormality constant and is shorthand for

$$K_m = \frac{m!\left[\Gamma(c)\right]^2 \Gamma(m+a-c+1)}{(a+2m)\,\Gamma(a+m)\,\Gamma(c+m)} \qquad (1.15)$$

In Table 1.2.1 one can see the relationship between the Jacobi Polynomials and some of the more classical polynomials of mathematical physics. Actually, the Legendre polynomials are merely special cases of the more general Gegenbauer Polynomials, which are in themselves a special case of the very general Jacobi Polynomials. If one considers the two constants a and c, in $\mathcal{J}_m(a, c; x)$, as representing two degrees of freedom, then the Gegenbauer Polynomials are a set of polynomials having one degree of freedom (ν) while the Legendre and Tschebyscheff polynomials have zero degrees of freedom in the sense that they are defined by assigning numerical values to both a and c .

The purpose of relating the Jacobi Polynomials to the polynomials listed

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* Szego[28] defines the Jacobi Polynomials orthonormal over the interval (-1,1); however, to differentiate between this set and the one defined above, he uses the symbol $P_m^{(\alpha, \beta)}(z)$. A linear transformation exists between the variables z and x, which enables one to use either notation. Specifically, $P_m^{(a, c)}(2x-1) = \mathcal{J}_m(a, c; x)$.

in Table 1.2.1 will become clear in Section 1.4, where it will be shown that many of the familiar orthogonal polynomial approximations of a function can be rephrased in terms of the orthonormal exponential approximation.

Table 1.2.1. Relation Between Jacobi Polynomials and More Familiar Polynomials

Polynomial*	Relationship to Jacobi Polynomials
Gegenbauer,** $C_m^{\nu}(x)$	$C_m^{\nu}(x) = (-1)^m \frac{(2\nu)_m}{m!} \mathcal{J}_m(2\nu, \nu+1/2; \frac{1+x}{2})$
Legendre, $P_m(x)$	$P_m(x) = \mathcal{J}_m(1, 1; \frac{1-x}{2}) = C_m^{1/2}(x)$
Tschebyscheff,*** $T_m(x)$	$T_m(x) = \mathcal{J}_m(0, 1/2; \frac{1-x}{2})$

Recurrence formulae, integral relationships, and other facts about Jacobi Polynomials may be found in references 10, 21 and 28. Page 300 of reference 9 is very useful for the evaluation of the Gamma functions appearing in Eqs. (1.12) and (1.15). Of course, when the argument of the Gamma function is an integer, say ν, then $\Gamma(\nu) = (\nu-1)!$[27].

One is now in a position to derive the recursion equation for the elements $\phi_k(t)$ of the set of orthonormalized exponentials in terms of the classical Jacobi Polynomials.

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* For the standard definitions of these polynomials see Magnus and Oberhettinger[21].

** $(2\nu)_m = 2\nu(2\nu+1)(2\nu+2)\ldots(2\nu+m-1)$; $(2\nu)_0 = 1$

*** $T_m(x)$ is a Tschebyscheff Polynomial of the first kind to be distinguished from the Tschebyscheff Polynomials of the second kind, $U_m(x)$ (Ref. 21, pg. 78).

1.3 $\phi_k(t)$: A Recurrence Equation in the Time-Domain

By transforming the domain of orthonormality for the Jacobi Polynomials $\mathcal{J}_m(a, c; x)$ from (0, 1) in the x-domain to (0, ∞) in the t-domain, letting

$$x = \epsilon^{-pt} \tag{1.16}$$

where p is some positive constant, Eq. (1.13) becomes

$$\int_0^\infty \epsilon^{-cpt}(1-\epsilon^{-pt})^{a-c}\mathcal{J}_m(a, c; \epsilon^{-pt})\mathcal{J}_n(a, c; \epsilon^{-pt})dt = \frac{K_m}{p}\delta_{mn} \tag{1.17}$$

If the form of this equation is compared with the form of Eq. (1.3)[with w(t) = 1], it is clear that

$$\phi_k(t) = (-1)^k\sqrt{\frac{p}{K_k}}\,\epsilon^{-\frac{cp}{2}t}(1-\epsilon^{-pt})^{\frac{a-c}{2}}\mathcal{J}_k(a, c; \epsilon^{-pt}) \tag{1.18}$$

$$k = 0, 1, 2, \ldots$$

which is a recursive equation for the elements $\phi_k(t)$ in the time-domain.

Starting with the recursion equation relating $\mathcal{J}_{k+1}(a, c; x)$, $\mathcal{J}_k(a, c; x)$ and $\mathcal{J}_{k-1}(a, c; x)$ given in Section 10.8 of Ref. (10) one is able to derive a similar equation for the elements $\phi_{k+1}(t)$, $\phi_k(t)$, and $\phi_{k-1}(t)$.

$$2(k+1)(k+a+c+1)(2k+a+c)\sqrt{K_{k+1}}\,\phi_{k+1}(t) =$$

$$-(2k+a+c+1)\left[(2k+a+c)(2k+a+c+2)(2\epsilon^{-pt}-1)+a^2-c^2\right]\times$$

$$\sqrt{K_k}\,\phi_k(t) - 2(k+a)(k+c)(2k+a+c+2)\sqrt{K_{k-1}}\,\phi_{k-1}(t) \tag{1.18a}$$

This equation might prove to be useful in a computer simulation of the set $\phi_k(t)$. If, on the other hand, the (k + 1)th element $\phi_{k+1}(t)$ is to be found via manual calculations then it is usually easier to find it directly from Eq. (1.18), merely replacing k by k + 1 in that equation.

One obvious advantage of determining the elements $\phi_k(t)$ from Eq. (1.18)

---

* Strictly speaking, when one compares Eq. (1.13) with Eq. (1.17) he can only derive a proportionality relationship between the $\phi_k(t)$ and the terms in Eq. (1.17). The constant $(-1)^k$ is the proportionality factor.

is that it has completely removed the difficulties inherent in the orthonormalization of the $\phi_k(t)$. Essentially, this was accomplished by introducing the well-defined Jacobi Polynomials in whose definition, Eq. (1.12), the orthonormalization has already been carried out. One is, therefore, able to express an unknown set of orthonormal functions in terms of a well-known set of orthonormal functions, and, in so doing he has eliminated a lot of the algebraic tedium that is usually a part of an orthonormalization procedure.

In the following section some of the properties of the $\phi_k(t)$, in Eq. (1.18), will be studied in order to throw more light on the significance of these functions.

1.4 The Nature of the $\phi_k(t)$.

This section is begun with a theorem which states the most significant property of the functions $\phi_k(t)$.*

Theorem 1. The asymptotic-order of $\Phi_k(s)$, the Laplace Transform of $\phi_k(t)$, is $\frac{a-c}{2} + 1$ for all k, provided that $\frac{a-c}{2}$ is either a positive integer or zero.**

Proof:

Consider $\left[\phi_k(t)\right]_N = (-1)^k \sqrt{\frac{K_k}{p}}\ \phi_k(t)$ [the subscript N denotes a normalized version of $\phi_k(t)$] as the product of two functions h(t) and g(t) where $h(t) = (1 - \epsilon^{-pt})^{\frac{a-c}{2}}$ and $g(t) = \epsilon^{\frac{-cp}{2}t}\, \mathcal{J}_k(a, c; \epsilon^{-pt})$.

The l' th derivative of $\left[\phi_k(t)\right]_N$ with respect to t, $\left[\phi_k^{(\ell)}(t)\right]_N$, can then be expressed in terms of h(t), g(t), and their derivatives by means of the Liebnitz differentiation formula,

$$\left[\phi_k^{(\ell)}(t)\right]_N = \sum_{r=0}^{\ell} \frac{\ell!}{r!\,(\ell - r)!}\, h^{(\ell-r)}(t)\, g^{(r)}(t) \tag{1.19}$$

from which one sees that

$$\left.\begin{aligned}
\left[\phi_k^{(1)}(t)\right]_N &= h^{(1)}(t)\, g(t) + h(t)\, g^{(1)}(t) \\
\left[\phi_k^{(2)}(t)\right]_N &= h^{(2)}(t)\, g(t) + 2h^{(1)}(t)\, g^{(1)}(t) + h(t)\, g^{(2)}(t) \\
&\vdots \qquad\qquad\qquad\qquad \vdots \\
\left[\phi_k^{(\frac{a-c}{2}-1)}(t)\right]_N &= h^{(\frac{a-c}{2}-1)}(t)\, g(t) + \ldots + h(t)\, g^{(\frac{a-c}{2}-1)}(t) \\
\left[\phi_k^{(\frac{a-c}{2})}(t)\right]_N &= h^{(\frac{a-c}{2})}(t)\, g(t) + \ldots + h(t)\, g^{(\frac{a-c}{2})}(t) \\
&\vdots \qquad\qquad\qquad\qquad \vdots
\end{aligned}\right\} \tag{1.20}$$

provided that $\frac{a-c}{2}$ is a positive integer.

---

* Henceforth, when the functions $\phi_k(t)$ are referred to, they are taken to be synonomous with the set defined in Eq. (1.18), unless otherwise stated.

** Choosing integer or zero values for $\frac{a-c}{2}$ ensures one of a finite series in ϵ^{-pt} for $\phi_k(t)$.

Substituting the expressions for h(t) and g(t) into these equations and setting $t = 0^+$ one can easily show that

$$\left.\begin{aligned} \left[\phi_k^{(1)}\ (0^+)\right]_N &= 0 \\ \left[\phi_k^{(2)}\ (0^+)\right]_N &= 0 \\ &\vdots \\ \left[\phi_k^{(\frac{a-c}{2}-1)}\ (0^+)\right]_N &= 0 \\ \left[\phi_k^{(\frac{a-c}{2})}\ (0^+)\right]_N &\neq 0 \end{aligned}\right\} \qquad (1.21)$$

It is also true that $\left[\phi_k\ (0^+)\right]_N = 0$. Next, one appeals to the Initial Value Theorem from Laplace Transform Theory which states that

$$\left.\begin{aligned} \left[\phi_k\ (0^+)\right]_N &= \lim_{s\to\infty} s\ \Phi_{N_k}(s)\ ^* \\ \left[\phi_k^{(1)}\ (0^+)\right]_N &= \lim_{s\to\infty} s\left\{s\,\Phi_{N_k}(s) - \left[\phi_k\ (0^+)\right]_N\right\}\ ^{**} \\ &\vdots \\ \left[\phi_k^{(\frac{a-c}{2}-1)}\ (0^+)\right]_N &= \lim_{s\to\infty} s\left\{s^{\frac{a-c}{2}-1}\,\Phi_{N_k}(s) - s^{\frac{a-c}{2}}\left[\phi_k\ (0^+)\right]_N - \cdots \right. \\ &\qquad \left. \cdots - \left[\phi_k^{(\frac{a-c}{2}-2)}(0^+)\right]_N\right\} \end{aligned}\right\} \qquad (1.22)$$

---

* $\Phi_{N_k}(s)$ is the Laplace Transform of the k' th normalized element $\left[\phi_k(t)\right]_N$.

** Due to the continuity of the $\left[\phi_k(t)\right]_N$ and their $\frac{a-c}{2} - 2$ derivatives at the origin, the 0^- notation in all of the derivative transforms has been replaced by the 0^+ notation.

Substituting Eqs. (1.21) into these equations one can show that, at most

$$\lim_{s \to \infty} s^{\frac{a-c}{2}} \Phi_{N_k}(s) = 0 \qquad (1.23)$$

which implies that the degree of the denominator of $\Phi_{N_k}(s)$ must be exactly $\frac{a-c}{2} + 1$ larger than the degree of its numerator. It has, therefore, been proven that, for positive integral values of $\frac{a-c}{2}$, A. O. $\left(\Phi_{N_k}\right) = \frac{a-c}{2} + 1$. It is quite obvious that for $\frac{a-c}{2} = 1, 2, \ldots, \ell$, A. O. $\left(\Phi_{N_k}\right) = 2, 3, \ldots \ell + 1$. To complete the picture let $\frac{a-c}{2} = 0$. From the asymptotic-order equation, one sees that this is the unity asymptotic-order case. That this is correct can easily be seen by setting $\frac{a-c}{2} = 0$ in the formula for the $\left[\phi_k(t)\right]_N$.

In order to obtain a simple visual interpretation of this theorem, the results were plotted in Figure 1.3.1. From this figure one sees that after the asymptotic-order is specified he is still free to choose either a or c. Note also that the results of this theorem in no way violate the earlier constraints on a and c [Eq. (1.14)] which are shown by dashed lines on this figure.

The true significance of our recursive equation for the $\phi_k(t)$ is now clear. By properly choosing a and c one can generate sets of orthonormal exponentials whose Laplace Transforms are of any asymptotic-order whatsoever, without having to go through the usual orthonormalization procedures.

Unfortunately, however, there is a severe restriction on the pole locations of the $\Phi_k(s)$ - - - they can only be equally spaced. This will be proven in the following theorem.

Theorem 2. The poles of $\Phi_k(s)$ are located along the negative-real axis in the s-domain at $-(q + \frac{c}{2})p$, where $0 \leq q \leq$ A. O. $(\Phi_k) + k - 1$ and is integral. These poles are equally spaced, p units apart.

Before the proof of this theorem is presented, it will be instructive to illustrate the application of the theorem by means of an example.

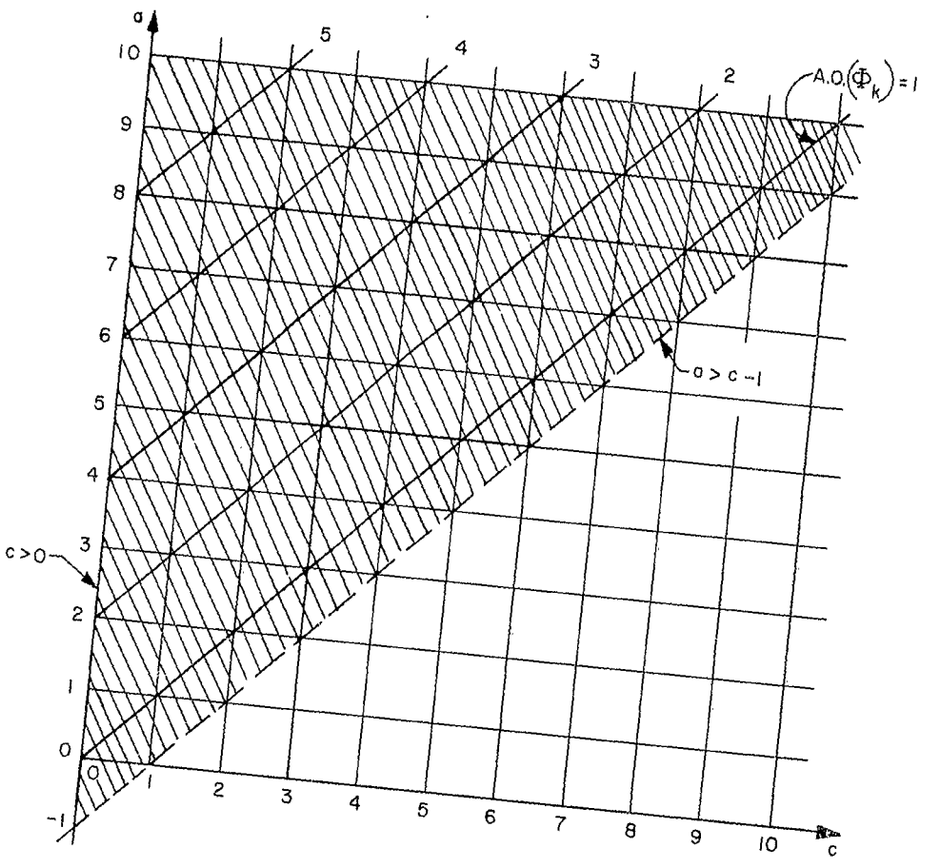

Figure 1.3.1. A summary of Theorem 1 indicating the valid regions (shaded) from which a and c may be chosen.

Example

For A. O. $(\Phi_k) = 3$ the poles of $\Phi_k(s)$ are located at $-(q + \frac{c}{2})p$, where $0 \leq q \leq k + 2$; thus, the poles of $\Phi_o(s)$ are located at $-\frac{c}{2}p$, $-(1 + \frac{c}{2})p$, and $-(2 + \frac{c}{2})p$ while the poles of $\Phi_1(s)$ are located at $-\frac{c}{2}p$, $-(1 + \frac{c}{2})p$, $-(2 + \frac{c}{2})p$, and $-(3 + \frac{c}{2})p$.

Proof:

Writing $\phi_k(t)$ as

$$\phi_k(t) = (-1)^k \sqrt{\frac{p}{K_k}}\, \epsilon^{-\frac{cp}{2}t}\, h_k(t) \tag{1.24}$$

where

$$h_k(t) = (1 - \epsilon^{-pt})^{\frac{a-c}{2}}\, \mathcal{J}_k(a, c; \epsilon^{-pt}) \tag{1.25}$$

it is quite simple to show by expanding $h_k(t)$ that it contains the terms $1, \epsilon^{-pt}, \epsilon^{-2pt}, \ldots, \epsilon^{-kpt}, \ldots, \epsilon^{-\left(\frac{a-c}{2} + k\right)pt}$. This means that $H_k(s)$ will have all of its poles situated along the negative-real axis (including the origin) of the s-plane at $0, -p, -2p, \ldots, -kp, \ldots\ldots, -(k + \frac{a-c}{2})p$; or alternatively, one can state that the poles of $H_k(s)$ are located at $-(qp)$ where $0 \leq q \leq k + \frac{a-c}{2}$ and is integral.

Taking the Laplace Transform of Eq. (1.24), one is able to conclude, in a rather straight forward manner, that the poles of $\Phi_k(s)$ are located along the negative-real axis of the s-plane at $-(q + \frac{c}{2})p$ where $0 \leq q \leq k + \frac{a-c}{2}$ and is integral; but by Theorem 1, the range of values q may take can be written as $0 \leq q \leq$ A. O. $(\Phi_k) + k - 1$. That the pole spacing is p is obvious from the location of the poles of $H_k(s)$ which have been enumerated in detail.

If one does not wish to use a set of evenly-spaced poles as the basis for his approximation then, of course, the orthonormalization procedure must be carried out anew every time a different set of poles is used. It has been shown that this becomes more tedious to accomplish the larger A. O. (Φ_k) becomes. The only exception to these last statements occurs when A. O. $(\Phi_k) = 1$, for in this case one can fall back upon Kautz' s recursive equation, (1.8), which is in a form that is independent of the spacings of the a_k.

Generally speaking, one has a great deal of latitude in the choice of the a_k,

the only restriction having to do with the completeness* of the set $\phi_k(t)$. The work of Reiss[24], Leight[19], and Brown[6] seems to indicate, however, that a geometric pole spacing may be the optimum one to choose. Unfortunately, their conclusions are based upon the identification of such a simple plant (the impulse response of the plant is ϵ^{-mt}) that one questions the extension of their results to more complicated situations. At the present time, no other literature is available on the topic of optimum pole locations (as far as the author knows) for the approximations of processes having more than a single pole; thus, it is not possible to arrive at a final conclusion about the optimum locations of the a_k. If one should decide to choose an equally spaced set of a_k as the poles of his approximation, then, of course, the theory and methods developed in this chapter and the following two chapters would be directly applicable. If, on the other hand, a set of non-equally spaced a_k were chosen, then, based upon the asymptotic-order of the approximation and the location of the a_k, the approximants $\phi_k(t)$ could be constructed via Eq. (1.3). Naturally, this procedure would have to be repeated in its entirety for every different set of a_k.

Let us remark, at this point, that two ways for constructing the functions $\phi_k(t)$ whose asymptotic-order is unity and whose poles are evenly spaced have been presented - - - Kautz' s equation in the s-domain and the recurrence equation, (1.18), in the time-domain. It would be reassuring for us to know that both results are equivalent.

Before stating and proving that this is so, the following notation, which will be used in the statement and proof of the theorem below, is introduced: $\phi_k^J(t)$ and $\phi_k^K(t)$ correspond to the orthonormal exponential approximants given in Equations (1.18) [Jacobi Polynomial representation (J)] and (1.8) [Kautz' s representation (K)], respectively. $\Phi_k^J(s)$ is the Laplace transform of $\phi_k^J(t)$ and $\Phi_k^K(s)$ is the Laplace transform of $\phi_k^K(t)$

Theorem 3. (Uniqueness Theorem)

$$\left[\phi_k^J(t) \Big|_{a=c} \right] \Longleftrightarrow \left[\Phi_k^K(s) \Big|_{a_k = (k + \frac{c}{2})p} \right] \qquad (1.26)$$

* See Appendix A for a discussion of "completeness" and its affects on choosing the sets of a_k.

provided $\phi_k^K(t)\Big|_{a_k = (k + \frac{c}{2})p}$ does not contain any null functions. *

The constraint $a = c$ on the $\phi_k^J(t)$ set is a direct consequence of Theorem 1, while constraining the poles of the $\Phi_k^K(s)$ set to be equally spaced at $-(k + \frac{c}{2})p$ is a direct consequence of theorem 2.

The method used in proving the theorem is a simple one; first, it shall be shown that $\Phi_k^J(s)\Big|_{a = c} = \Phi_k^K(s)\Big|_{a_k = (k + \frac{c}{2})p}$; the converse will then follow.

Proof:

For $a = c$, Eq. (1.18) becomes

$$\phi_k^J(t)\Big|_{a = c} = (-1)^k \sqrt{\frac{p}{K_k}}\, \epsilon^{-\frac{cp}{2}t}\, \mathcal{F}_k(c, c; \epsilon^{-pt}) \tag{1.27}$$

where, from Eqs. (1.12) and (1.15)

$$\mathcal{F}_k(c, c; \epsilon^{-pt}) = \sum_{\ell = 0}^{k} (-1)^\ell \binom{k}{\ell} \frac{\Gamma(c + k + \ell)\,\Gamma(c)}{\Gamma(c + \ell)\,\Gamma(c + k)}\, \epsilon^{-\ell pt} \tag{1.28}$$

and

$$\sqrt{K_k} = \frac{k!\,\Gamma(c)}{\Gamma(c+k)\sqrt{c + 2k}} \tag{1.29}$$

Substituting these last two equations into Eq. (1.27) and taking its Laplace transform one finds that

$$\Phi_k^J(s)\Big|_{a = c} = \sum_{\ell = 0}^{k} (-1)^{k+\ell} \frac{\sqrt{p(c+2k)}\,\Gamma(c+k+\ell)}{\ell!\,(k-\ell)!\,(c+\ell)} \times \frac{1}{\left[s + (\ell + \frac{c}{2})p\right]} \tag{1.30}$$

One must now demonstrate the equivalence of Eqs. (1.8) and (1.30). This will be done by means of an inductive type argument. In particular, when $k = r$

$$\Phi_r^K(s) = \sqrt{2a_r}\, \frac{(s - a_o)\ldots(s - a_{r-1})}{(s + a_o)(s + a_1)\ldots(s + a_r)} \tag{1.31}$$

and

$$\Phi_r^J(s)\Big|_{a = c} = \sum_{\ell = 0}^{r} (-1)^{r+\ell} \frac{\sqrt{p(c + 2r)}\,\Gamma(c + r + \ell)}{\ell!\,(r - \ell)!\,\Gamma(c + \ell)} \times \frac{1}{\left[s + (\ell + \frac{c}{2})p\right]} \tag{1.32}$$

* A "null function" is a function whose area, in the range from 0^- to ∞, is zero.

Equation (1. 31) can now be developed into the partial fraction expansion

$$\Phi_r^K(s) = \frac{\gamma_o}{s+a_o} + \frac{\gamma_1}{s+a_1} + \cdots + \frac{\gamma_{r-1}}{s+a_{r-1}} + \frac{\gamma_r}{s+a_r} \qquad (1.33)$$

This equation shall be compared with Eq. (1. 32) verifying the equality of the coefficients γ_o, γ_{r-1}, and γ_r and their respective counterparts in Eq. (1. 32) under the condition imposed by the theorem, that $a_r = (r + \frac{c}{2})p$. If this is so then the $\Phi_r^J(s)\Big|_{a=c}$ are merely the partial fraction expansions of the $\Phi_r^K(s)\Big|_{a_r = (r+\frac{c}{2})p}$

1. γ_o. (the residue of the zeroth pole)

From Eqs. (1. 31) and (1. 33) it can be shown easily that

$$\gamma_o = (-1)^r\sqrt{2a_r}\ \frac{(2a_o)(a_o+a_1)\cdots(a_o+a_{r-1})}{(a_1-a_o)(a_2-a_o)\cdots(a_r-a_o)} \qquad (1.34)$$

which reduces to

$$\gamma_o = (-1)^r\sqrt{2p(r+\tfrac{c}{2})}\ \frac{\Gamma(c+r)}{r!\,\Gamma(c)} \qquad (1.35)$$

when $a_r = (r+\frac{c}{2})p$ is substituted into Eq. (1. 34).

To find the corresponding value of γ_o from Eq. (1. 32) r is set equal to zero. Doing this and comparing the coefficient of the $\frac{1}{s+\frac{c}{2}p}$ term to Eq. (1. 35) one concludes that they are the same.

2. γ_{r-1}. [the residue of the (r - 1)th term]

Proceeding in exactly the same fashion as in the preceding case, it is noted that the residue γ_{r-1} in Eq. (1. 33) is

$$\gamma_{r-1} = (-1)^r\sqrt{2a_r}\ \frac{(a_o+a_{r-1})(a_1+a_{r-1})\cdots(2a_{r-1})}{(a_o-a_{r-1})(a_1-a_{r-1})\cdots(a_{r-2}-a_{r-1})(a_r-a_{r-1})} \qquad (1.36)$$

which reduces to

$$\gamma_{r-1} = -\sqrt{2p(r+\tfrac{c}{2})}\ \frac{\Gamma(c+2r-1)}{(r-1)!\,\Gamma(c+r-1)} \qquad (1.37)^*$$

* Great care must be taken when factoring (-1) out of the terms in the denominator of Eq. (1. 36)after a_r has been replaced by $(r+\frac{c}{2})p$. In this case $(-1)^{r-1}$ was factored out.

This corresponds exactly with the residue of the pole at $s = -a_{r-1} = -(r - 1 + \frac{c}{2})p$ in Eq. (1.32), which can be found merely by substituting $k = r - 1$ into that equation.

3. γ_r. (the residue of the rth term).

Finally, it can be shown that the residue γ_r in Eq. (1.33) becomes, after replacing a_r by $(r + \frac{c}{2})p$,

$$\gamma_r = \sqrt{2p\,(\ell + \frac{c}{2})}\quad \frac{\Gamma(c + 2r)}{r!\,\Gamma(c+r)} \tag{1.38}$$

which, once again, can be shown to correspond exactly with the residue of the pole at $s = -a_r = -(r + \frac{c}{2})p$ in Eq. (1.32)

The first part of the theorem has now been proven, that is, it has been demonstrated that

$$\phi_k^{J}(t)\Big|_{a = c} \Longrightarrow \Phi_k^{K}(s)\Big|_{a_k = (k + \frac{c}{2})p} \tag{1.39}$$

since it has just been shown that

$$\Phi_k^{J}(s)\Big|_{a = c} = \Phi_k^{K}(s)\Big|_{a_k = (k + \frac{c}{2})p} \tag{1.40}$$

That

$$\Phi_k^{K}(s)\Big|_{a_k = (k + \frac{c}{2})p} \Longrightarrow \phi_k^{J}(t)\Big|_{a = c} \tag{1.41}$$

follows at once from Eq. (1.40) and the uniqueness theorem from Laplace Transform Theory[27] which states: "Two functions of time having the same Laplace Transform are equal almost everywhere in the range from $t = 0^-$ to $t = \infty$ differing at most by a null function". Since null functions have been excluded from $\phi_k^{K}(t)\Big|_{a_k = (k + \frac{c}{2})p}$ one can be assured that Eq. (1.41) is true. This completes the proof.

The significance of this theorem should not be underestimated. If, for example, the inverse were true then the statements made on page 8 (in connection with the Gram-Schmidt Procedure), regarding the uniqueness of an orthonormal set would be violated.

There, it was pointed out that the uniqueness of an orthonormal set $\phi_k(t)$, formed from a collection of functions $g_\ell(t)$ depends solely upon the linear independence of the elements $g_\ell(t)$. Since the set of elements in both the Kautz expression, Eq. (1.8), and Eq. (1.18) are identically the same (under the conditions of this theorem), that is, $g_\ell(t) \; \epsilon \left\{ \epsilon^{-(\ell + \frac{c}{2})pt} \right\}$, the sets must be identical. It has just been proven that they are indeed equal.

This section is concluded with an example illustrating the results of Theorems 1 and 2.

Example: Special Sets of Functions $\phi_k(t)$.

It has been shown in Section 1.2 that the Gegenbauer and Legendre polynomials are special cases of the more general Jacobi Polynomials (Table 1.2.1). In this example the relationships existing between the orthonormal exponential approximants $\phi_k(t)$ and the Gegenbauer and Legendre Polynomials are summarized. This summary serves to relate the Gegenbauer and Legendre Polynomial approximations of a function, f (t), to the exponential approximations of f (t). Table 1.4.1 presents the results and is self-explanatory. Figure 1.4.1 compares the asymptotic-order behavior of the Gegenbauer and Legendre sets with the asymptotic-order behavior of the Jacobi set. Essentially, it is a reproduction of Fig. 1.3.1 upon which the asymptotic-order relationships for the Gegenbauer and Legendre sets have been superimposed. Note that the asymptotic order of the Legendre set is fixed at unity, while for the Gegenbauer set it may assume any integral value; however, once the asymptotic order of the Gegenbauer set is specified, a and c are both fixed. This is in contrast to the Jacobi set where one is still free to choose either a or c.

It would be desirable now to extend all of the results presented in this chapter to the case of a weighting function which is much more general than the uniform weighting function used in this chapter. The next chapter discusses just such a weighting function. In chapter 3 the results of this chapter are extended with the aid of the results presented in chapter 2.

Table 1.4.1. Special Sets of the Functions $\phi_k(t)$.

$\phi_k(t)$ set	$\phi_k(t)$, $k = 0, 1, 2, \ldots$	a	c	A. O. (Φ_k)
Gegenbauer *	$\frac{k!}{(2\nu)_k}\sqrt{\frac{p}{K_k}}\,\epsilon^{-\left(\nu \mp \frac{1}{2}\right)\frac{pt}{2}}\,(1-\epsilon^{-pt})^{\frac{\nu-\frac{1}{2}}{2}}\,C_k^{\nu}(2\epsilon^{-pt}-1)$ $\nu = \frac{1}{2}, \frac{5}{2}, \frac{9}{2}, \ldots$	2ν	$\nu + \frac{1}{2}$	$\frac{\nu}{2} + \frac{3}{4}$
Legendre	$(-1)^k\sqrt{p(1+2k)}\;\epsilon^{-\frac{p}{2}t}\;P_k(1-2\epsilon^{-pt})$	1	1	1

$\phi_k(t)$ set	Pole Locations, a_k	Location of a_o	Comments
Gegenbauer	$(j + \frac{\nu}{2} + \frac{1}{4})p$ $0 \le j \le \frac{\nu}{2} - \frac{1}{4} + k$	$(\frac{\nu}{2} + \frac{1}{4})p$	When $\nu = \frac{1}{2}$ the Gegenbauer set reduces to the Legendre set.
Legendre	$(j + \frac{1}{2})p$ $0 \le j \le k$	$\frac{1}{2}p$	

* Refer to Table 1.2.1 for the relation between the Jacobi Polynomials and this set of polynomials.

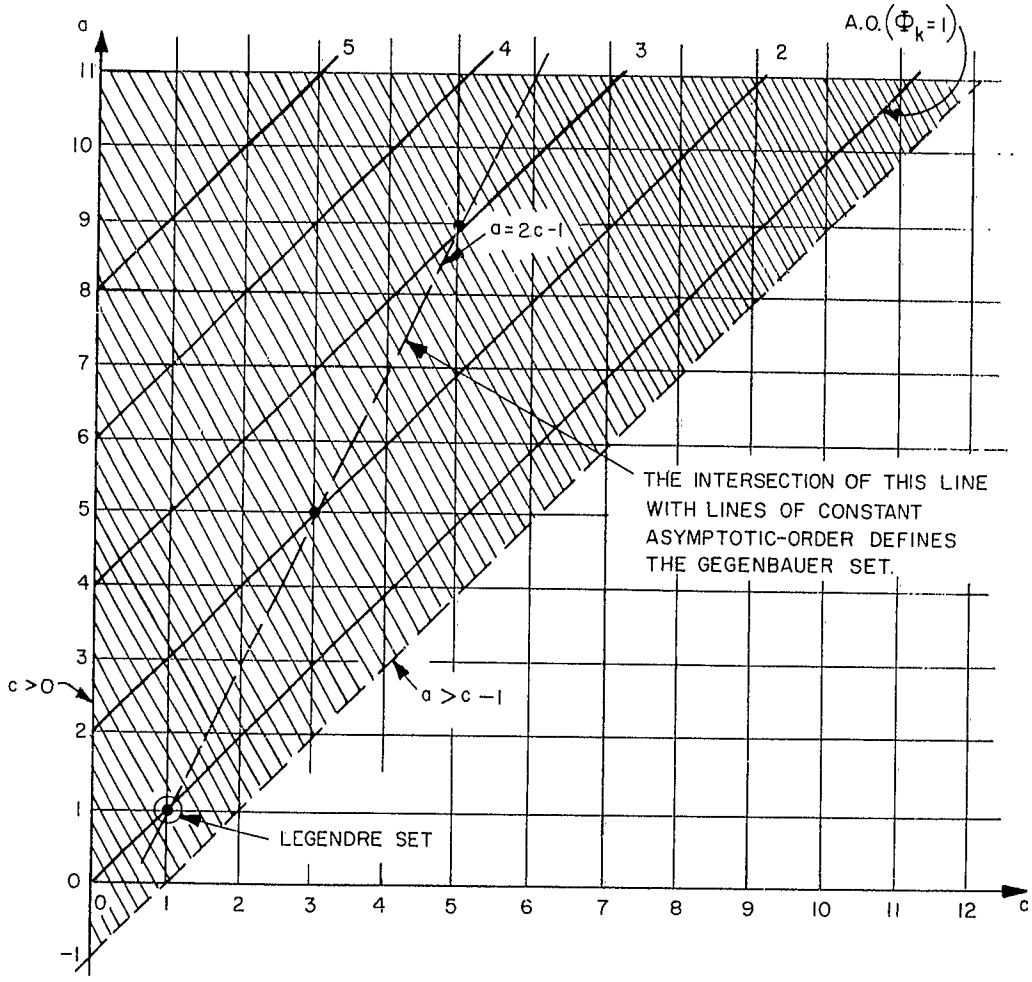

Figure 1.4.1. Summary of Table 3, indicating the locus of A.O. (Φ_k) for the Gegenbauer and Legendre Sets as compared to the more general results of Theorem 1.

CHAPTER 2. GENERALIZED WEIGHTING FUNCTIONS AND THEIR DESIGN

2.1 Why a Weighting Function?

The unweighted * integral-squared error criterion emphasizes large errors heavily and de-emphasizes small errors, for all time greater than zero. This is due to the squaring operation. If one assumes for the moment that w(t) in Eq. (1.1) (repeated below) is unity, as was done throughout Chapter 1,

$$IE^2 = \int_0^\infty w(t) \left[f(t) - f_a(t) \right]^2 dt \qquad (1.1)$$

he is in effect stating that $f_a(t)$ is to approximate $f(t)$ closely over the entire range of time $(0, \infty)$. To require this may, in many cases be unnecessary, impractical, and even undesirable. For example, suppose one needs only to obtain a close fit over a finite range of time. Two cases of this are shown in Fig. 2.1.1. The impulse response in Fig. 2.1.1a is typical of systems having a very high asymptotic-order. Since much of the important information is concentrated in the interval (t_1, t_2) it seems wise to emphasize this region in the IE^2 criterion. The only way one can introduce a time dependence effect into the IE^2 criterion is to choose a weighting function dependent upon time. Choosing the weighting function shown in Fig. 2.1.1c gives the desired effect; for, the product of this weighting function and the error function $f_1(t) - f_{a_1}(t)$ [in Eq. (1.1)] tends to emphasize the error in the interval (t_1, t_2) and to de-emphasize it outside of the interval.

The system characterized by the impulse response in Fig. 2.1.1b contains a delay; that is, for an input applied at $t = 0$ there is no output until time T. To emphasize the fact that the response is actually zero in the time interval $(0, T)$ one might choose a weighting function of the form shown in (d) of Fig. 2.1.1. Once again, it is to be noted that in both (a) and (b) of Fig. 2.1.1 the emphasis and de-emphasis of the error function has been accomplished by means of a time-dependent weighting function, and not by changing the limits of integration in Eq. (1.1).**

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* By convention, the term "unweighted" refers to the case where w(t) is independent of time. It is a bit misleading, however, since the prefix un - would seem to imply zero weighting. Perhaps "uniformly - weighted" would be a more appropriate choice of terminology.

** The emphasis of the time interval (t_1, t_2) in Fig. 2.1.1(a) and the de-emphasis of the interval $(0, T)$ in Fig. 2.1.1(b) could also have been achieved using rectangular-pulse weighting functions. A rectangular weighting function, unfortunately, introduces considerable analytical difficulty. This is further discussed in Section 2.4.

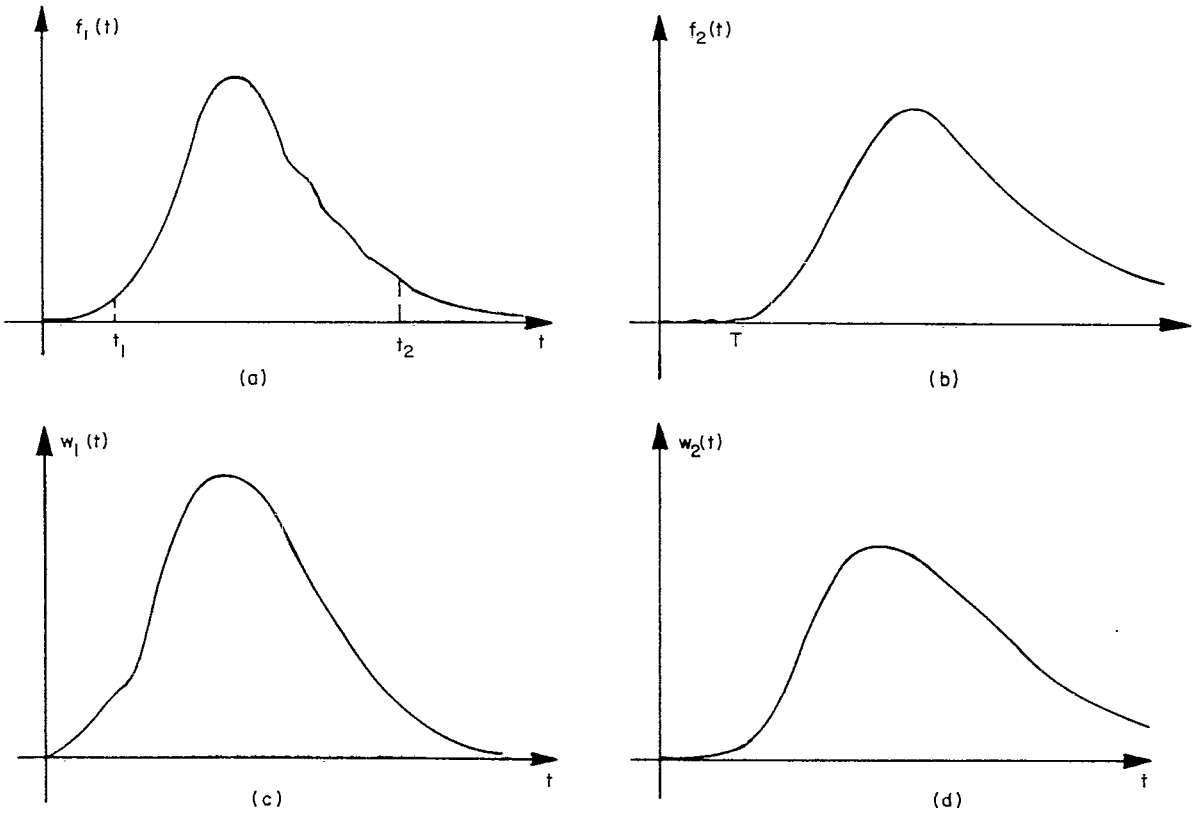

Figure 2.1.1. Impulse responses and weighting functions useful in the identification of the impulse responses. (a), (b) impulse responses; (c), (d) weighting functions.

So far, it has been indicated how one might choose a weighting function from the viewpoint of obtaining a close fit between $f_a(t)$ and $f(t)$. In many cases, one is also interested in using the identification, $f_a(t)$, in a frequency-domain analysis. If, in particular, high-frequency effects are important, then one must weight the error function quite heavily for small values of t. This can be dedeuced from the Initial-Value Theorem:

$$f_a(0^+) = \lim_{s \to \infty} s F_a(s) \tag{2.1}$$

On the other hand, if it is the low-frequency effects which are of primary concern, then, from the Final-Value Theorem

$$f_a(\infty) = \lim_{s \to 0} s F_a(s) \tag{2.2}$$

one concludes that a weighting function emphasizing large values of t should be chosen.

Heavy weighting of small values of time is also very important in the identification used in adaptive control systems[4, 22, 32] where one is primarily concerned with characterizing a plant as quickly as possible and over a short interval of time (0, T). This has been done in the past by choosing an exponential weighting function. Increasing the amplitude and decreasing the time constant of the exponential function provides the desired emphasis in the interval (0, T)*.

Weighting functions emphasizing large values or small values of t will be discussed at greater length in Chapter 4. In the remaining sections of this chapter weighting functions of the form shown in Fig. 2.0.1 c and d shall be studied in greater detail.

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* For a discussion of other weighting functions that might prove to yield far superior results in the Adaptive Identification problem see Chapter 5.

2.2 w(t) and its Properties

In this section some properties of the weighting function in Eq. (2.3) shall be investigated.

$$w(t) = \epsilon^{-k_o k_1 t} (1 - \epsilon^{-k_1 t})^{k_2} \qquad (2.3)^*$$

$$k_o, k_1, k_2 \geq 0$$

Typical portraits of w(t) appear in Fig. 2.2.1, from which it is evident that, when k_2 is zero, w(t) reduces to an exponential weight function, while, when both k_2 and k_o are zero, w(t) is constant. The general form of w(t) appears in (d) of Fig. 2.2.1. It has a single maximum at time $t_{max.}$, where

$$t_{max.} = \frac{1}{k_1} \ln\left(1 + \frac{k_2}{k_o}\right) \qquad (2.4)$$

Depending upon the choice of values for k_o, k_1, and k_2, w(t) may be made very narrow or very wide. A measure of the width of w(t) may be formulated in terms of the distance between the two inflection points of w(t). If t_1 and t_2 are the times of the first and second inflection points respectively, (see Fig. 2.2.2) then ΔT, the distance between these inflection points is

$$\Delta T = t_2 - t_1 \qquad (2.5)$$

t_1 and t_2 are found to be the two solutions of the equation

$$\frac{d^2 w(t)}{dt^2} = 0 \qquad (2.6)$$

and are

$$t_{1,2} = -\frac{1}{k_1} \ln\left[\frac{2k_o^2 + 2k_o k_2 + k_2 \pm \sqrt{k_2(4k_o^2 + 4k_o k_2 + k_2)}}{2(k_o + k_2)^2}\right] \qquad (2.7)^{**}$$

---

* It is tacitly assumed throughout the remainder of the dissertation that w(t) is amplitude normalized, that is, its amplitude is multiplied by unity. Multiplying w(t) by a constant does not change any of the results presented in this chapter.

** To calculate t_1 (t_2) use the positive (negative) square root in the numerator of this equation.

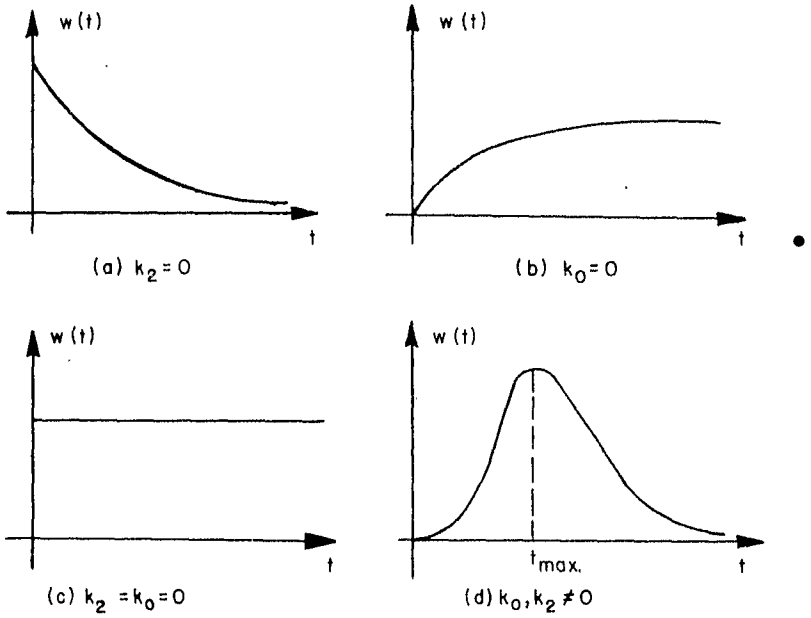

Figure 2.2.1 Some typical portraits of w(t).

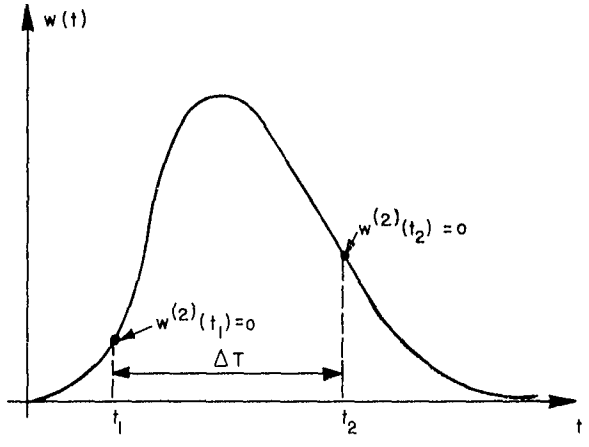

Figure 2.2.2. Relationship of ΔT to w(t).

whereupon ΔT becomes

$$\Delta T = \frac{1}{k_1} \ln \left[\frac{2k_o^2 + 2k_o k_2 + k_2 + \sqrt{k_2 (4k_o^2 + 4k_o k_2 + k_2)}}{2k_o^2 + 2k_o k_2 + k_2 - \sqrt{k_2 (4k_o^2 + 4k_o k_2 + k_2)}} \right] \qquad (2.8)^*$$

$$k_2 \neq 0$$

This equation and Eq. (2.4), for $t_{max.}$, will be very useful in the design procedure developed in the following section for selecting the constants k_o, k_1, and k_2.

* When $k_2 = 0$, ΔT has no meaning since $w(t) = \epsilon^{-k_o k_1 t}$ has no inflection points. Under this condition $\Delta T = 0$ in Eq. (2.8) which is meaningless.

2.3 The Design of w(t)

In this section, a method for determining the constants k_o, k_1, and k_2 in Eq. (2.3), from a given set of specifications on w(t), will be discussed. It is by no means the only method which may be devised, but, rather, is one which sheds a great deal of light onto the effects which these constants have on the shape of w(t). An alternate procedure which is more germane to the theory presented in Chapter 3 will be discussed in that chapter; however, most of the results of this section will be directly applicable to that method also.

Three constants are to be determined; therefore, three independent specifications on w(t) must be given. The three conditions listed below are only one set out of many possible sets; however, they are a particularly advantageous set in that they enable one to readily control the shape of w(t).

1. The initial behavior of w(t) may be accounted for by pre-specifying k_2; since, quite obviously, from the form of w(t), the number of derivatives of w(t) which are zero at t = 0 depends solely upon its value. k_2 also controls the width, ΔT, of w(t) as shall be shown. The pre-specified value of k_2 shall be designated as k_{2_s} *, where in general $k_{2_s} \geq 0$ **.

2. After a region of maximum emphasis of f(t) has been decided upon, $t_{max.}$ is chosen so as to coincide with the center of this region. By locating $t_{max.}$ in this manner, it is possible to design w(t) in such a manner that it emphasizes the desired region of maximum emphasis of f(t). The pre-specified time $t_{max.}$ shall be designated t_s.

3. It shall be desirable to have the amplitude of w(t) become approximately zero within some pre-specified time interval, say $4\tau_s$. Naturally, $4\tau_s$ must be chosen such that $4\tau_s > t_s$. This condition prevents one from attempting to design a weighting function where, for example, $\tau = \frac{1}{10}$ second and $t_{max.} \approx 1$ second.

These 3 conditions can be expressed in terms of k_o, k_1, and k_2 as follows:

---

* The subscript s stands for "specified a priori".

** Choosing integer values for k_{2_s} usually results in numerical simplifications and also enables one to correlate the initial behavior of w(t) in the time-domain to the asymptotic-order of W(s) in the s-domain.

1. $k_2 = k_{2_s}$ (2.9)

2. $t_{max.} = t_s = \frac{1}{k_1} \ln \left(1 + \frac{k_{2_s}}{k_o} \right)$ (2.10)

3. $\tau_s = \frac{1}{k_o k_1}$ (2.11)

Procedure for Designing w(t).

1. Values for $t_{max.}$, k_2 and τ are chosen based upon the specifications for w(t).

2. By trial and error, k_1 is found from the equation

$$k_1 = \frac{\epsilon^{k_1 t_s} - 1}{k_{2_s} \tau_s} \quad (2.12)$$

3. Finally, k_o is determined from the result of step 2 and Eq. (2.11), that is

$$k_o = \frac{1}{k_1 \tau_s} \quad (2.13)$$

That the actual process of determining k_1 from Eq. (2.12), by trial and error, is a rapidly converging one can best be illustrated by means of an example.

Example 1.

The impulse response of the overdamped process shown in Fig. 2.3.1 is to be

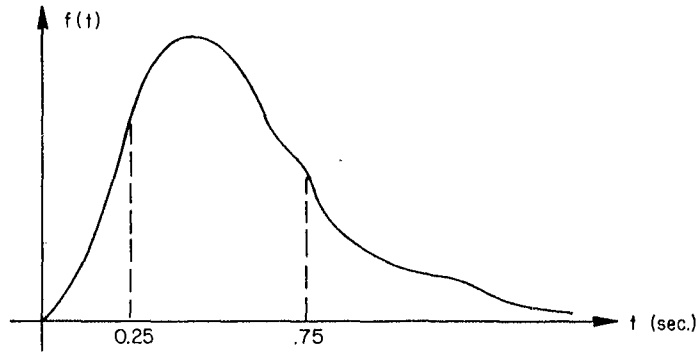

Figure 2.3.1. Impulse response of an overdamped process.

approximated in an integral-squared sense, such that the approximation is particularly accurate for $0.25 \leq t \leq 0.75$ sec. As the first step in the approximation procedure (to be outlined in Chapter 3) a weighting function must be designed.

1.* The center of the region of maximum emphasis of f(t) occurs at t = 0.50 sec; thus, based upon the discussion on page 35 t_s is chosen to be 1/2 sec. At the moment, there is no basis for the choice of one value of k_2 over another, however, such a basis will be established shortly. Here k_2 is arbitrarily chosen to be two, i.e., $k_{2_s} = 2$. In order to make sure that w(t) emphasizes the entire region $0.25 \leq t \leq 0.75$ heavily, as desired, τ_s is chosen to be 0.3 second.

2. $$k_1 = \frac{\epsilon^{\frac{1}{2}k_1} - 1}{0.6} \qquad (2.12a)$$

Table 2.3.1. Calculation of k_1

Assumed Value of k_1	$\frac{\epsilon^{\frac{1}{2}k_1} - 1}{0.6}$
1.0	1.085
0.9	0.950
0.8	0.816
0.7	0.700

3. With $k_1 = 0.7$ and $\tau_s = 0.3$, k_o, in Eq. (2.13), is found to be 4.76. Summarizing, one can easily verify that the weighting function w(t) is

$$w(t) = \epsilon^{-3.33t}\left(1 - \epsilon^{-0.7t}\right)^2 \qquad (2.14)$$

* These numbers correspond to the numerical designations assigned to the steps in the design procedure on page 36.

One now may pose the following question: "Having designed a weighting function via the trial and error procedure outlined above, is there any shortcut available for designing other weighting functions based upon the information available from the design of the first one?". The following theorem presents just such a shortcut.

Before stating and proving the theorem some new nomenclature will be introduced. Associated with a weighting function $w^1(t)$, where the unity superscript indicates that this function was actually designed by means of the trial and error procedure, are the following constants: the pre-specified $k^1_{2_s}$, t^1_s and τ^1_s and the calculated k^1_o and k^1_1. Similarily, associated with the j' th weighting function $w^j(t)$ are the constants k^j_2, $t^j_{max.}$, τ^j, k^j_o and k^j_1.

Theorem 4. Given $k^j_{2_s}$, $t^j_s = t^1_s$ and $k^j_1 = k^1_1$, then

$$k^j_o = \left(\frac{k^j_{2_s}}{k^1_{2_s}} \right) k^1_o \tag{2.15}$$

and

$$\tau^j = \left(\frac{k^1_{2_s}}{k^j_{2_s}} \right) \tau^1_s \tag{2.16}$$*

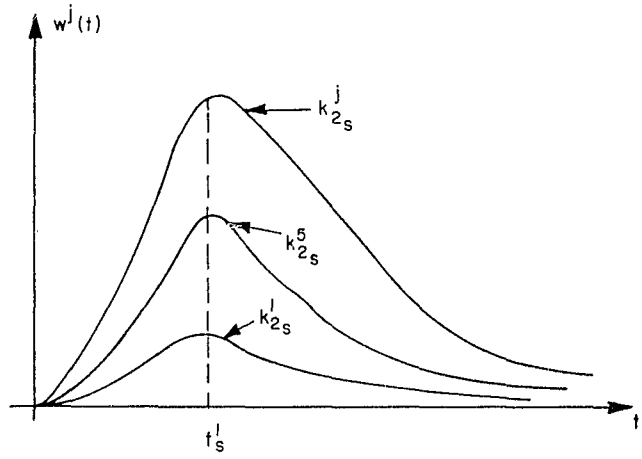

Figure 2.3.2. Illustration of Theorem 4.

* This theorem can also be used out of the context of the design procedure presented in this section. Such an application is discussed in Chapter 3, Section 3.2.

A graphical interpretation of this theorem is given in Fig. 2.3.2. One sees, from this figure, that the theorem enables him to construct a family of weighting functions, each having its maximum at the same value of time, and, each having a different value of k_{2_s}. It was pointed out, in Example 1 of this section, that, while there exists a practical basis for the pre-specification of $t_{max.}$, there does not yet exist a practical basis for the way in which to pre-specify k_2. In Example 1 a weighting function was designed for a specific value of $t_{max.}$ and an arbitrarily chosen value of k_2. Theorem 4 now enables one to rapidly determine the effect of an increase or decrease in k_2 on the shape of w(t), for a specific value of $t_{max.}$.

The seemingly stringent requirement that all of the k_1^j's be identical for all of the $w^j(t)$'s actually is a direct consequence of the general theory presented in Chapter 3. There it will be shown that this constant equals the pole spacing of the elements $\Phi_k(s)$ and that this spacing is usually fixed a priori; hence, for a given pole spacing, a family of weighting functions may be constructed, but they must all have the property $k_1^j = k_1^1$.

The proof of theorem 4 follows.

Proof:

From Eq. (2.10) and the given information one finds

$$k_o^1 = \frac{k_{2_s}^1}{\epsilon^{k_1^1 t_s^1} - 1} \tag{2.17}$$

and

$$k_o^j = \frac{k_{2_s}^j}{\epsilon^{k_1^1 t_s^1} - 1} \tag{2.18}$$

After k_o^j is divided by k_o^1 one is able to conclude that

$$\frac{k_o^j}{k_o^1} = \frac{k_{2_s}^j}{k_{2_s}^1} \tag{2.19}$$

which proves the first part of the theorem. If one proceeds to substitute

$$\tau^j = \frac{1}{k_1^1 k_o^j} \tag{2.20}$$

and

$$\tau^i = \frac{1}{k_1^1 \, k_o^1} \tag{2.21}$$

into Eq. (2.19) one obtains the expression

$$\tau^j = \left(\frac{k_{2_s}^1}{k_{2_s}^j} \right) \tau^1 \tag{2.22}$$

which verifies the second part of the theorem.

Before illustrating the use of this theorem by means of an example, it will be useful to try and gain a bit more physical insight into how the choice of the constants $k_{2_s}^j$ affects $w^j(t)$. It has already been pointed out (page 35) that k_2 controls the initial and final behaviors of the weighting function $w(t)$. It shall presently be shown that, under the conditions of theorem 4, k_2 also controls the width of $w(t)$, as measured by ΔT in Eq. (2.8).

Corollary 1. Under the conditions of Theorem 4 and by the definition of ΔT in Eq. (2.8)

$$\lim_{k_{2_s}^j \to \infty} \Delta T^j = 0 \tag{2.23}$$

where ΔT^j is the width of the j' th weighting function.

Proof:

The proof follows directly from Eq. (2.8) and the statement of Theorem 4. Utilizing this information one can show that

$$\Delta T^j = \frac{1}{k_1^1} \ln \left(\frac{2C' k_{2_s}^j + 1 + \sqrt{4C' k_{2_s}^j + 1}}{2C' k_{2_s}^j + 1 - \sqrt{C' k_{2_s}^j + 1}} \right) \tag{2.24}$$

where

$$C' = \left(\frac{k_o^1}{k_{2_s}^1} \right) \left(\frac{k_o^1}{k_{2_s}^1} + 1 \right) \tag{2.25}$$

Dividing numerator and denominator of the logarithmic argument in Eq. (2.24) by $k_{2_s}^j$ and letting $k_{2_s}^j$ become very large it is easy to show, by means of limit arguments, that

$$\lim_{k_{2_s}^j \to \infty} \Delta T^j = 0 \tag{2.26}$$

In a similar fashion one may also show that ΔT^j is a monotonically decreasing function of $k_{2_s}^j$; that is to say, increasing $k_{2_s}^j$ always narrows $w^j(t)$ while decreasing it widens $w^j(t)$. This is borne out in the following example.

Example 2.

In this example the design of weighting functions based upon the specifications given in Example 1 is continued. Here, as in Example 1, weighting functions are designed such that $t_{max.} = 0.50$ second. The effect of choosing values for k_2 other than two, in the design of these weighting functions via Theorem 4, is studied in detail.

Table 2.3.2 summarizes three designs and their properties. $w^1(t)$ is the weighting function designed in Example 1. $w^2(t)$ and $w^3(t)$, on the other hand, were obtained directly from the entries in the first row of Table 2.3.2 and Eqs. (2.15) and (2.16) of Theorem 4. The properties of $w^2(t)$ and $w^3(t)$ are listed in the second and third rows of Table 2.3.2 respectively. Figure 2.3.3 presents a graphical summary of the results. For clarity, $w^1(t)$, $w^2(t)$, and $w^3(t)$ have been scaled up by factors of 100, 20, and 2000 respectively. Note that $w^2(t)$ is the widest function and that this corresponds to the unity $k_{2_s}^j$ case; thus, the example substantiates the theory. If the width of the weighting function is used as a measure of the time of maximum emphasis of $w^j(t)$, then, based upon the specification stated in Example 1, that f(t) is to be approximated most accurately for $0.25 \leq t \leq 0.75$ sec. (see Fig. 2.3.1), and the widths of the three weighting functions listed in Table 2.3.2, one concludes that $w^3(t)$ is the most satisfactory design, of the three weighting functions listed.

Table 2.3.2. Weighting Functions Designed With The Aid of Theorem 4.

j	$k_{2_s}^j$	k_1^j	k_o^j	τ^j	ΔT^j	$w^j(t)$
1	2	0.70	4.76	0.30	0.708	$\epsilon^{-3.33t}(1-\epsilon^{-0.7t})^2$
j	$k_{2_s}^j$	$k_1^j = k_1^1$	$k_o^j = \frac{k_{2_s}^j}{k_{2_s}^1} k_o^1$	τ^j	ΔT^j	$w^j(t)$
2	1	0.70	2.38	0.60	0.998	$\epsilon^{-1.67t}(1-\epsilon^{-0.7t})$
3	3	0.70	7.14	0.20	0.578	$\epsilon^{-5t}(1-\epsilon^{-0.7t})^3$

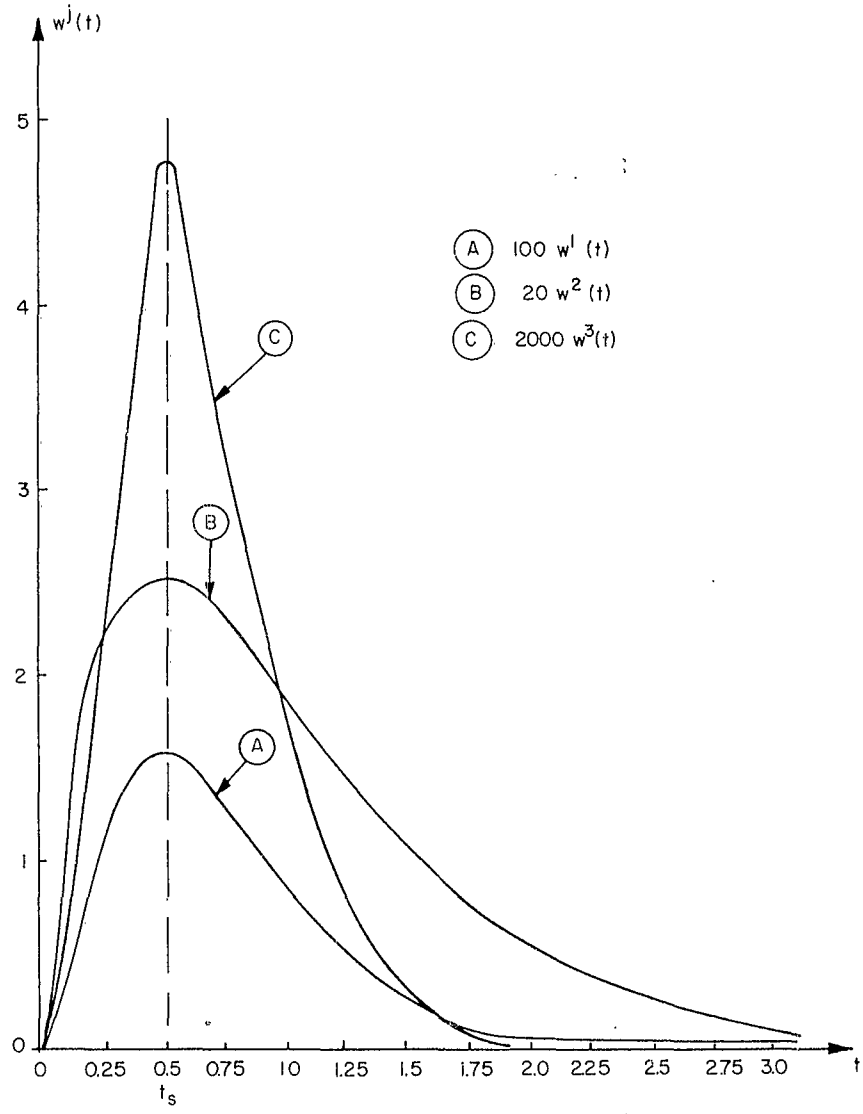

Figure 2. 3. 3 A comparison of the three weighting functions listed in Table 2. 3. 2.

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2.4 Why the Weighting Function $w(t) = \epsilon^{-k_o k_1 t}\left(1 - \epsilon^{-k_1 t}\right)^{k_2}$?

Thus far, no mention has been made of the reasons for choosing the particular weighting function

$$w(t) = \epsilon^{-k_o k_1 t}\left(1 - \epsilon^{-k_1 t}\right)^{k_2} \tag{2.3}$$

or for spending so much time discussing its properties, and a procedure for designing it based upon these properties. What makes this weighting function so useful?

It has already been indicated, in Section 2.1, that, many times one is merely concerned with the problem of fitting or approximating a function closely over a closed time interval, (t_1, t_2). If this be the case, then the most desirable form to choose for w(t) is the rectangular gating function in Figure 2.4.1a. This is equivalent to changing the domain of integration in Eqs. (1.1), (1.3) and (1.4), from the interval $(0, \infty)$ to the interval (t_1, t_2). Changing these limits results in very tedious computations. It is also interesting to note that Kautz's results and all of the results presented in Chapter 1 are useless in this case.

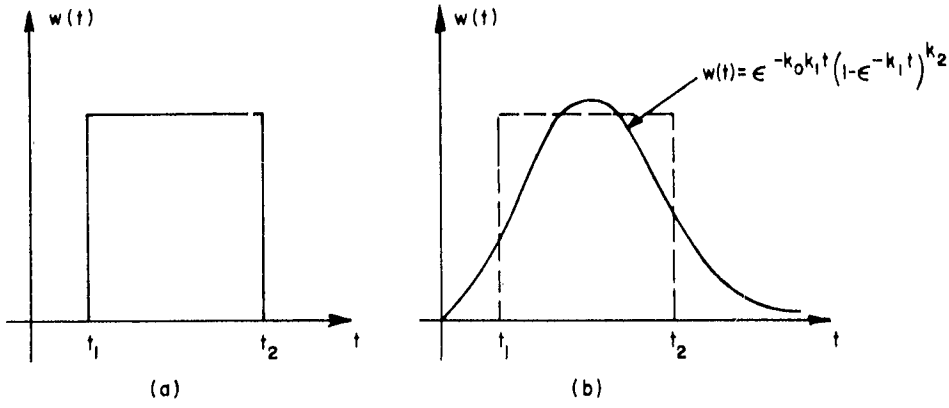

Figure 2.4.1. Weighting functions emphasizing the interval of time (t_1, t_2). (a) Gating function, (b) $w(t) = \epsilon^{-k_o k_1 t}\left(1 - \epsilon^{-k_1 t}\right)^{k_2}$ approximating the gating function.

One must therefore conclude that the analytical difficulties encountered when gating-type weighting functions are used precludes their usefulness. This does not mean that the idea of a gating function is incorrect; it suggests, rather, that one might choose a weighting function approximating the gating function in shape - - - one which is more manageable in the Eqs. (1.1), (1.3), and (1.4) than the gating function is. The weighting function in Eq. (2.3) is just such a function. That this is so is

amplified in Fig. 2.4.1b where the gating function $\left[u_{-1}(t - t_1) - u_{-1}(t - t_2)\right]$ is approximated by $w(t) = \epsilon^{-k_o k_1 t}\left(1 - \epsilon^{-k_1 t}\right)^{k_2}$.

A second reason for choosing w(t) in Eq. (2.3) is that this is the natural choice to make if the results of Chapter 1 [where w(t) was unity] are to be extended to a more general weighting function. This is perhaps the more important of the two reasons as will be seen in the next chapter.

CHAPTER 3. THE ORTHONORMAL EXPONENTIAL SET FOR EXPONENTIAL WEIGHTING FUNCTIONS

3.1 Introduction

The purpose of this Chapter is to extend the results of Chapter 1 to the cases of exponential weighting functions. This is accomplished with the aid of the theory presented in Chapter 2.

Recall that the elements, $\phi_k(t)$, of an orthonormal set satisfy the relationship

$$\int_0^\infty w(t)\ \phi_i(t)\ \phi_j(t)\,dt = \delta_{ij} \tag{3.1}$$

and, in particular, recall that if the $\phi_k(t)$ are chosen from a set of exponential functions, wherein the poles of the $\Phi_k(s)$ are evenly spaced along the negative - real axis of the s-plane, Eq. (3.1) can be written in terms of the classical Jacobi Polynomials, $\mathcal{J}_m(a, c; x)$, given by

$$\int_0^\infty \epsilon^{-cpt}(1-\epsilon^{-pt})^{a-c}\,\mathcal{J}_i(a, c; \epsilon^{-pt})\,\mathcal{J}_j(a, c; \epsilon^{-pt})\,dt = \frac{K_j}{p}\,\delta_{ij} \tag{3.2}$$

When $w(t) = 1$ in Eq. (3.1) it is shown in Chapter 1, that

$$\phi_k(t) = (-1)^k\sqrt{\frac{p}{K_k}}\ \epsilon^{-\frac{cp}{2}t}(1-\epsilon^{-pt})^{\frac{a-c}{2}}\,\mathcal{J}_k(a, c; \epsilon^{-pt}) \tag{3.3}$$

This was obtained by comparing Eqs. (3.1) and (3.2) after the weighting function $\epsilon^{-cpt}(1-\epsilon^{-pt})^{(a-c)}$, in Eq. (3.2), had been split into two equal portions $\left[\epsilon^{-\frac{cp}{2}t}(1-\epsilon^{-pt})^{\frac{(a-c)}{2}}\right]$; the two equal portions of the weighting function $\epsilon^{-cpt}(1-\epsilon^{-pt})^{(a-c)}$ were first associated with the factors $\mathcal{J}_i(a, c; \epsilon^{-pt})$ and $\mathcal{J}_j(a, c; \epsilon^{-pt})$ in Eq. (3.2), before the comparison of Eqs. (3.1) and (3.2) was made.

If, instead of splitting the entire weighting function in Eq. (3.2) into two equal parts, a portion of the weighting function, say $\epsilon^{-\lambda_1 cpt}(1-\epsilon^{-pt})^{\lambda_2(a-c)}$, is first removed then, upon comparison of Eqs. (3.1) and (3.2), it is possible to conclude that

$$\phi_k(t) = (-1)^k\sqrt{\frac{p}{K_k}}\ \epsilon^{-(1-\lambda_1)\frac{cp}{2}t}\left(1-\epsilon^{-pt}\right)^{(1-\lambda_2)\left(\frac{a-c}{2}\right)}\mathcal{J}_k(a, c; \epsilon^{-pt}) \tag{3.4}$$

$$k = 0, 1, 2, \ldots$$

where $$0 \leq \lambda_1, \lambda_2 \leq 1 \tag{3.5}$$

and

$$w(t) = \epsilon^{-\lambda_1 cpt} (1 - \epsilon^{-pt})^{\lambda_2 (a-c)} \tag{3.6}$$

Equation (3.4) is an equation for the elements $\phi_k(t)$ of an exponential set which are orthonormal with respect to the weighting function in Eq. (3.6). When both λ_1 and λ_2 are equal to zero, Eq. (3.4) reduces to Eq. (3.3). For all values of λ_1 and λ_2, as defined in Eq. (3.5), w(t), in Eq. (3.6), corresponds exactly to the weighting function

$$\epsilon^{-k_o k_1 t} \left(1 - \epsilon^{-k_1 t}\right)^{k_2}$$

discussed in the previous chapter, provided

$$\left.\begin{aligned} k_o &= \lambda_1 c \\ k_1 &= p \\ k_2 &= \lambda_2 (a-c) \end{aligned}\right\} \tag{3.7}$$

Thus, all of the results pertaining to the properties, and to the design of $\epsilon^{-k_o k_1 t} \left(1 - \epsilon^{-k_1 t}\right)^{k_2}$, apply directly to w(t) in Eq. (3.6) provided the transformations in Eq. (3.7) are employed.

Equations (3.4) and (3.6) form the basis for the remaining sections of this chapter. First the properties of the ϕ_k (t) shall be reviewed in the light of the more general weighting function, Eq. (3.6). Then a design procedure for uniquely determining the constants λ_1, λ_2, c, a, and p in Eqs. (3.4) and (3.6) shall be developed. Following the steps of this procedure it will be straight forward to construct the set of exponentials, in Eq. (3.4), orthonormal with respect to the weighting function in Eq. (3.6).

3.2 The Nature of the $\phi_k(t)$.

The format of this section is similar to that of Section 1.3, wherein the asymptotic-order of the $\Phi_k(s)$, the pole spacing of the $\Phi_k(s)$, and the conditions under which Eq. (1.18) reduced to Kautz' s results, Eq. (1.8), were investigated. There the results were stated and proved in the form of three theorems. Here the corresponding results for the more general set of elements in Eq. (3.4) shall merely be stated. The proofs of these statements are identical to those of their special counterparts in Section 1.3, except for some symbolic changes due to the inclusion of the more general weighting function, Eq. (3.6).

Theorem 1G.* The asymptotic-order of $\Phi_k(s)$, the Laplace Transform of $\phi_k(t)$, is $(1 - \lambda_2)(\frac{a-c}{2}) + 1$ for all k, provided that $(1 - \lambda_2)(\frac{a-c}{2})$ is a positive integer or zero.

Theorem 2G. The poles of the $\Phi_k(s)$ are located along the negative-real axis in the s-domain at $-\left[q + \frac{c}{2}(1 - \lambda_1)\right]p$ where $0 \leq q \leq$ A. O. $(\Phi_k) + k - 1$, and is integral. These poles are spaced p units apart.**

Table 3.2.1 summarizes the results of these theorems. The weighting functions listed in the first column correspond to the four functions sketched in Fig. 2.2.1.

The results of Theorem 1G can also be summarized as in Fig. 1.4.1 if a and c are replaced by $a' = a(1 - \lambda_2)$ and $c' = c(1 - \lambda_2)$ respectively in that figure.***

Theorem 3G. (Uniqueness Theorem)

$$\left[\phi_k^J(t)\Big|_{\substack{a = c \\ w(t) = \epsilon^{-\lambda_1 cpt}}}\right] \Longleftrightarrow \left[\Phi_k^B(s)\Big|_{\substack{a_k = \left[k + \frac{c}{2}(1-\lambda_1)\right]p \\ w(t) = \epsilon^{-\lambda_1 cpt}}}\right] \quad (3.8)^{****}$$

to within a set of null functions.

---

* The "G" notation indicates that this is the general form of the corresponding theorem presented in Section 1.3.

** See Appendix B for a more extensive discussion of the pole spacing question.

*** The inequality $a > c - 1$ becomes $a' > c' - (1 - \lambda_2)$ which actually is a family of constraints dependent upon λ_2. When $\lambda_2 = 0$, $a' > c' - 1$ and for all $0 < \lambda_2 < 1$ this line moves to the left approaching the line $a' = c'$ as λ_2 approaches unity.

**** The J superscript in $\phi_k^J(t)$ refers to the Jacobi Polynomial equation for $\phi_k(t)$, Eq. (3.4). The B superscript in $\Phi_k^B(s)$, on the other hand, refers to the set of $\Phi_k(s)$ in Eq. (1.9) developed by Braun[4, 22].

Table 3.2.1. Summary of Theorems 1G and 2G.

λ_1	λ_2	$\dot{w}(t)$	A.O. (Φ_k)	Pole Locations of $\Phi_k(s)$
0	0	1	$\frac{a-c}{2} + 1$	$-(q + \frac{c}{2})p$ *
$\lambda_1 \neq 0$	0	$\epsilon^{-\lambda_1 cpt}$	$\frac{a-c}{2} + 1$	$-\left[q + \frac{c}{2}(1-\lambda_1)\right]p$
0	$\lambda_2 \neq 0$	$\left(1-\epsilon^{-pt}\right)^{\lambda_2 (a-c)}$	$(1-\lambda_2)\left(\frac{a-c}{2}\right)+1$	$-\left(q + \frac{c}{2}\right)p$
$\lambda_1 \neq 0$	$\lambda_2 \neq 0$	$\epsilon^{-\lambda_1 cpt}\left(1-\epsilon^{-pt}\right)^{\lambda_2 (a-c)}$	$(1-\lambda_2)\left(\frac{a-c}{2}\right)+1$	$-\left[q + \frac{c}{2}(1-\lambda_1)\right]p$

In effect, this theorem states that Eq. (1.9), obtained by Braun for the weighting function ϵ^{-2bt}, is identical to the results given by Eq. (3.4), provided, of course, it is understood that this is only true for the set with unity asymptotic-order and evenly-spaced poles - - where $b = \frac{\lambda_1 cp}{2}$.

* $0 \leq q \leq$ A.O. $(\Phi_k) + k - 1$, and is integral

Armstrong's Example.

In a Letter to the Editor[2] appearing in the I. R. E. Transactions on Circuit Theory, Armstrong compares the integrals

$$\int_0^\infty \epsilon^{-3t}(1-\epsilon^{-t})\mathcal{F}_m(4,3;\epsilon^{-t})\,\mathcal{F}_n(4,3;\epsilon^{-t})\,dt = \frac{n!\left[\Gamma(3)\right]^2\Gamma(n+2)}{2(2+n)\,\Gamma(n+4)\,\Gamma(n+3)}\,\delta_{mn} \tag{3.9}$$

and

$$\int_0^\infty \epsilon^{-t}(1-\epsilon^{-t})\,\phi_n(t)\,\phi_m(t)\,dt = \delta_{mn} \tag{3.10}$$

from which he concludes that a suitable representation for the elements $\phi_k(t)$ is

$$\phi_k(t) = (-1)^k\left[\frac{(k+1)(k+2)^3(k+3)}{2}\right]^{1/2}\epsilon^{-t}\,\mathcal{F}_k\left(4,3;\epsilon^{-t}\right) \tag{3.11}$$

It is easy to show from Eqs. (3.1), (3.4), (3.6), (3.10), and (3.11) that this example is the special case

$$a = 4,\ c = 3,\ p = 1,\quad \lambda_2 = 1,\ \text{and}\ \lambda_1 = \frac{1}{3}\ . \tag{3.12}$$

It is interesting to note that by choosing $\lambda_2 = 1$ Armstrong has constructed a set of orthonormalized exponentials where A. O. $(\Phi_k) = 1$. This set, however, is orthonormal with respect to the weighting function $\epsilon^{-t}(1-\epsilon^{-t})$ which is a more complicated weighting function than the exponential weight Braun[4, 22] used. Armstrong's weighting function emphasizes a region of time in the vicinity of its maximum, which occurs at $t_{max.} = 0.693$ second.

From Table 3.2.1 it is seen that the poles of Armstrong's set occur at $-(q+1)$, $0 \le q \le k$, the first pole, a_o, being at -1.

3.3. A Design Procedure for Constructing the $\phi_k(t)$.

For convenience, Eq. (3.4), for the elements $\phi_k(t)$, is repeated below.

$$\phi_k(t) = (-1)^k \sqrt{\frac{p}{K_p}}\, \epsilon^{-(1-\lambda_1)\frac{cp}{2}t} (1 - \epsilon^{-pt})^{(1-\lambda_2)(\frac{a-c}{2})} \mathcal{J}_k(a, c; \epsilon^{-pt}) \quad (3.4)$$

In order to carry out the numerical computations indicated in Eq. (3.4) one must somehow choose values for the constants a, c, p, λ_1 and λ_2 . In this section, one procedure for determining these constants, from a set of specifications on both w(t), in Eq. (3.6), and $\phi_k(t)$, in Eq. (3.4), is developed.

(a) First-pole location.

From the pole-location theorem, Theorem 2G, it can be shown that the first pole, a_o, of $\Phi_k(s)$ will lie along the negative-real axis of the s-plane at

$$a_o = \frac{c}{2}(1 - \lambda_1)p \quad (3.13)$$

It should be possible to roughly locate this pole from an inspection of the portrait of f (t). As an example, suppose that f (t) approaches zero amplitude in about 8 seconds. It is safe to assume, then, that if a single term in f (t) contributed the 8 second response it would be of the form $\epsilon^{-1/2t}$. Based upon this assumption one might choose $a_o = \frac{1}{2}$.

(b) Specification of p.

It has been shown in the pole-location theorem, Theorem 2G, that the pole-spacing of the poles of $\Phi_k(s)$ exactly equals p. p must be chosen a priori based upon a consideration of the accuracy required of the approximation $f_a(t)$. It can be shown that choosing one value of p over another affects the rate of convergence of $f_a(t)$ as measured by the IE^2 numeric. For example, it may take 3 terms in $f_a(t)$ $\left[f_a(t) = C_o \phi_o(t) + C_1 \phi_1(t) + C_2 \phi_2(t)\right]$ with $p = p_1$, before the integral-squared error numeric is less than or equal to some pre-accepted accuracy level, while it may only require 2 terms in $f_a(t)$ with $p = p_2$, before the integral-squared error numeric is less than or equal to the same pre-accepted accuracy level. It has been found that choosing the value of p such that the poles of $\Phi_k(s)$, and consequently the poles of $F_a(s)$, lie in the vicinity of the poles of F(s) and surround the poles of F(s) leads to approximations that converge very rapidly. Naturally, this choice of p depends upon the analysts understanding of the dynamics of the system being studied.

(c) Specification of A. O. (F_a).

It has been shown that specifying A. O. (F_a) is equivalent to specifying A. O. (Φ_k), and that

$$\text{A. O. } (\Phi_k) = (1 - \lambda_2)\left(\frac{a-c}{2}\right) + 1 \tag{3.14}$$

In general, A. O. (F_a) must be stated a priori. For systems of low asymptotic-order this is not too difficult to do. One need only look at the portrait of f(t) in order to determine whether or not f(o) and/or $f^{(1)}$ (o) are zero. On the other hand, it is usually impossible to determine directly from f(t) the asymptotic-order of systems of high asymptotic-order. In these cases, one may have to resort to a rather detailed preliminary analysis of the system involving, perhaps, high-frequency testing, since the asymptotic-order is related to the high-frequency behavior of a system.

The determination of the asymptotic-order is probably the most difficult problem that the analyst faces. Much more work needs to be done in the areas of experimental and analytical methods useful in the practical determination of the asymptotic-order of a system.

(d) Design of the weighting function.

The constants $\lambda_1 c$ and $\lambda_2 (a-c)$ in Eq. (3.6), which is repeated below for convenience, must be determined.

$$w(t) = \epsilon^{-\lambda_1 cpt}\left(1 - \epsilon^{-pt}\right)^{\lambda_2 (a-c)} \tag{3.6}$$

It has already been pointed out that for

$$\left.\begin{aligned} \lambda_1 c &= k_o \\ p &= k_1 \\ \text{and} \quad \lambda_2 (a-c) &= k_2 \end{aligned}\right\} \tag{3.7}$$

the design of Eq. (3.6) is identical to the design of

$$w(t) = \epsilon^{-k_o k_1 t}\left(1 - \epsilon^{-k_1 t}\right)^{k_2} \tag{3.15}$$

which was studied in Chapter 2. The design procedure developed in that chapter must be modified in this case to accommodate the requirement that p be specified a priori

[step (b) on Page 51]. The procedure is outlined below. First the specifications on w(t) are stated:

1. k_1 is specified at the outset as the spacing of the poles of $\Phi_k(s)$.

2. $t_{max.}$, in Eq. (2.4), is chosen so that w(t) emphasizes a region of time necessary to effect the maximum emphasis of f(t), over that same region of time. It is designated t_s.

3. w(t) must approach zero in $4\tau_s$ seconds (see page 35) for a more complete discussion of the choice of τ_s).

Procedure for designing w(t).

Based upon the above specifications on w(t), calculate

1. $k_1 = p$ (3.17)

2. $k_o = \frac{1}{p\tau_s}$ (3.18)

3. $k_2 = \frac{\epsilon^{pt_s} - 1}{p\tau_s}$ (3.19)

4. Evaluate ΔT from Eq. (2.8) in order to determine w(t)'s width. If, for example, ΔT which is repeated below for convenience,

$$\Delta T = \frac{1}{k_1} \ln \left[\frac{2k_o^2 + 2k_o k_2 + k_2 + \sqrt{k_2(4k_o^2 + 4k_o k_2 + k_2)}}{2k_o^2 + 2k_o k_2 + k_2 - \sqrt{k_2(4k_o^2 + 4k_o k_2 + k_2)}} \right] \tag{2.8}$$

is too small, decrease k_2, and, with k_1 and $t_{max.}$ fixed, re-evaluate k_o and τ using Theorem 4.

Designing w(t) by this method one determines the constants k_o and k_2, as desired. Substituting the results of steps (a) through (d) into Eqs. (3.7), (3.13), and (3.14) one is able to determine the four constants c, λ_1, a, and λ_2 uniquely.

$$c = k_o + \frac{2a_o}{p} \tag{3.20}$$

$$\lambda_1 = \frac{1}{1 + \frac{2a_o}{pk_o}} \tag{3.21}$$

$$a = k_o + k_2 + \frac{2a_o}{p} + 2\left[\text{A. O. } (\Phi_k) - 1\right] \tag{3.22}$$

$$\lambda_2 = \frac{1}{1 + \frac{2}{k_2}\left[\text{A. O. } (\Phi_k) - 1\right]} \tag{3.23}$$

3.4 Summary.

It has been the purpose of this chapter to present a general method for constructing sets of orthonormalized exponentials so that these sets may then be used as the approximants in the identification problem. By introducing the classical Jacobi Polynomials one has been led to a recursion equation in the time-domain for determining the elements of the exponential set. In particular, these elements have the properties that,

1. they are orthonormal with respect to the weighting function $\epsilon^{-k_o k_1 t} \left(1 - \epsilon^{-k_1 t}\right)^{k_2}$,
2. their Laplace Transforms may be of any desired asymptotic-order, and
3. their poles lie along the negative-real axis of the s-plane and are equally spaced.

CHAPTER 4. THE CHOICE OF A SUITABLE ERROR CRITERION

4.1 Introduction

Thus far, the discussions in Chapters 1 and 3 have considered the approximation of a function, f(t), in an integral-squared sense. It was pointed out in those chapters that the addition of more and more terms to the approximation $f_a(t) = \sum_{k=0}^{m} C_k \phi_k(t)$ resulted in an approximation that fitted f(t) closer and closer.

It has been mentioned that, in many circumstances, a close fit between $f_a(t)$ and f(t) is not the only measure of performance that must be satisfied by the approximation. In Fig. 4.1.1 $f_a(t)$ represents the identification of a plant, f(t), located in the forward path of a positional servomechanism. The characterization of the plant by $f_a(t)$ is merely the first step in a study of the closed-loop system. Ideally, it would be desirable for the results of such a study to agree in some way with the actual results

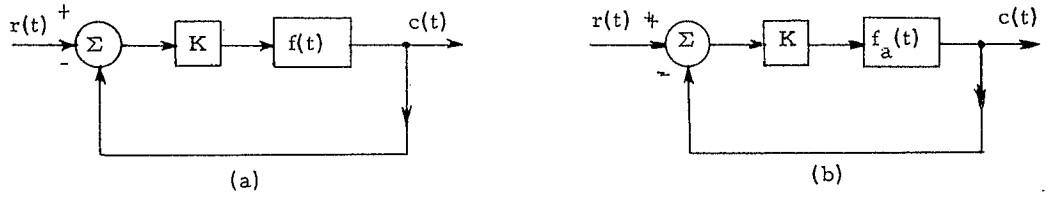

Figure 4.1.1. A single-loop feedback system in which f(t), in (a), represents the plants actual characteristics while $f_a(t)$, in (b), approximates these characteristics.

that would have been obtained had f(t) been known and used in the analyses.

One may, for example, be interested in performing a stability analysis on the closed-loop system. Two types of problems occur in this case. The first can be called the absolute-stability problem and has to do with whether or not the system ever becomes unstable and for what values of gain, K, it does so. Here it must be assumed that K takes on all values. Practically speaking, if the actual plant, f(t), causes the closed-loop system to become unstable for gain K_1 it would seem desirable to have the approximate plant indicate a similar behavior, within a certain allowable margin of error.

The second type of problem can be called the relative-stability problem. Here one is interested in the behavior of the closed-loop response as K varies over some

known finite range of values. Is c(t) always overdamped or underdamped for the range of K of interest, or is it overdamped for some values of K and underdamped for others? These are some of the questions that a relative-stability analysis answers. If, for example, the dynamics of the actual plant cause c(t) to be underdamped for $K_1 \leq K \leq K_2$ then it would be desirable to have the approximate plant indicate a similar behavior over this range of K, again within an allowable margin of error.

Some of the techniques applicable to both types of stability analysis utilize the frequency response of the plant (Nyquist-polar and Bode-logarithmic plots, for example). $F(j\omega)$ can be found analytically if f(t) is known (which, of course, is not the case here, since f (t) is being approximated), or experimentally, if the plant is available for frequency-response measurements. In any event, it also appears necessary to have some correspondence between both the magnitude and the phase of $F_a(j\omega)$ and of $F(j\omega)$ over some range of ω if $F_a(j\omega)$ is to predict the stability of the closed-loop system correctly.

Figure 4.1.2 summarizes some of the performance criteria used by the author for choosing the functions that best approximate the plant f(t). These are by no means the only measures of performance that can be chosen and, in some circumstances, they may not be appropo at all. Wescott[31] summarizes the difficulties of choosing a performance measure when he states: "The ultimate decision of what constitutes good performance is based upon human judgement or even personal opinion . . . . the end result is in the nature of a hit or a miss . . . It is not possible in a general way to legislate for all cases, and the field must be narrowed."

There is no reason to assume that the IE^2 criterion should lead to approximations which simultaneously minimize the IE^2 and the performance criteria of Fig. 4.1.2.

The purpose of this chapter is to present the details and the results of a study carried out to determine which error criteria (many of which are well established in the field of Automatic Control) out of a selected group of eleven lead to approximations which best approximate f(t) - - best in the sense that these approximations satisfy one or more of the performance measures listed in Fig. 4.1.2. Also of concern will be the effects on the performance of the $f_a(t)$, as measured by these same performance criteria, of choosing approximations of incorrect asymptotic-order, as compared to approximations having the correct asymptotic-order.

The eleven error criteria employed in this study were subdivided into three families of error criteria: the IT^nE, IT^nE^2 and AE - IT^nAE families, where

The best $f_a(t)$ chosen on the basis of a comparison of the closeness of fit between the approximations and f(t).

The best $f_a(t)$ chosen on the basis of a comparison of the closeness of fit between the $|F_a(j\omega)|$ and the $|F(j\omega)|$.

The best $f_a(t)$ chosen on the basis of a comparison of the closeness of fit between the $\angle F_a(j\omega)$ and the $\angle F(j\omega)$.

The best $f_a(t)$ chosen on the basis of a comparison of the similarity between the relative-stability of the closed-loop systems in Fig. 4.1.1 a and b, for a range of loop gain, K, greater than zero.

The best $f_a(t)$ chosen on the basis of a comparison of the similarity between the absolute-stability of the closed-loop systems in Fig. 4.1.1 a and b, for a range of loop gain, K, greater than zero.

The best $f_a(t)$ chosen on the basis of a comparison of the similarity between the absolute-stability of the closed-loop systems in Fig. 4.1.1 a and b, for a range of loop gain, K, less than zero.

Figure 4.1.2. Performance measures used in determining optimum approximations.

$$IT^nE = \int_0^\infty t^n (f - f_a)\, dt \tag{4.1}$$

$$IT^n E^2 = \int_0^\infty t^n (f - f_a)^2\, dt \tag{4.2}$$

$$AE = \max_t |f - f_a| \tag{4.3}^*$$

and

$$IT^n AE = \int_0^\infty t^n |f - f_a|\, dt \tag{4.4}$$

* The maximum is taken with respect to time as indicated by the t under max.

Of particular interest to this study were the IE, ITE, IT^2E, $IT^{-1/2}E$, IE^2, ITE^2, $IT^{-1/2}E^2$, AE, IAE, ITAE, and $IT^{-1/2}AE$ criteria. This choice of error criteria was motivated in part by the works of Graham and Lathrop[12], Wescott[31], Handa[14], Newton et al[23], and Walkovitch et al[30] who investigated various means for designing the system in Fig. 4.1.1 a. Their designs were based upon minimizing such measures of the error e(t) = r(t) - c(t) as the IE, ITE, IT^2E, IE^2, ITE^2, AE, IAE, and ITAE criteria. It is interesting to note their common results in the light of what has been said on page 57 about the IE^2 criterion and the choice of the best $f_a(t)$. These results are stated below.

(1) While the IE^2 criterion is analytically easier to handle it does not lead to the most selective design* or to the synthesis of a system having the best closed-loop performance (in terms of rise-time, bandwidth, etc.), and

(2) The optimal designs occur when the measure of e(t) is heavily weighted for large time thus placing emphasis on "late" errors.

It is hoped that our study will lead to a similar set of conclusions. In particular, answers will be sought for the following 14 questions.

1. In each family of error criteria, which error criterion leads to approximations of best fit between $f_a(t)$ and $f(t)$?

2. In each family of error criteria, which error criterion leads to approximations of best fit between $|F_a(j\omega)|$ and $|F(j\omega)|$?

3. In each family of error criteria, which error criterion leads to approximations of best fit between $\angle F_a(j\omega)$ and $\angle F(j\omega)$?

4. In each family of error criteria, which error criterion leads to approximations having the closest relative-stability ($K \geq 0$) correspondence between $F_a(s)$ and $F(s)$?

5. In each family of error criteria, which error criterion leads to approximations having the closest absolute - stability ($K \geq 0$) correspondence between $F_a(s)$ and $F(s)$?

6. In each family of error criteria, which error criterion leads to optimum approximations - - optimum in the sense of one or more of the performance measures listed in Fig. 4.1.2.?

7. From the eleven error criteria, which error criterion leads to approximations of best fit between $f_a(t)$ and $f(t)$?

8. From the eleven error criteria, which error criterion leads to approximations of best fit between $|F_a(j\omega)|$ and $|F(j\omega)|$?

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* By "selective" is meant that a minimum value of the measure of e(t) is sharp as system parameters are varied.

9. From the eleven error criteria, which error criterion leads to approximations of best fit between $\angle F_a(j\omega)$ and $\angle F(j\omega)$?

10. From the eleven error criteria, which error criterion leads to approximations having the closest relative - stability ($K \geq 0$) correspondence between $F_a(s)$ and $F(s)$?

11. From the eleven error criteria, which error criterion leads to approximations having the closest absolute - stability ($K \geq 0$) correspondence between $F_a(s)$ and $F(s)$?

12. From the eleven error criteria, which error criterion leads to optimum approximations - - optimum in the sense of one or more of the performance measures listed in Fig. 4.1.2. ?

13. Do the error criteria in the answers to the twelve preceding questions have anything in common, such as the same weighting function?

14. Does choosing an approximation of incorrect asymptotic-order have a noticeable affect on the relative - and absolute - stability of the closed-loop system?

The answers to these 14 questions will be considered in Section 4.4 and will be based upon the analyses discussed in Section 4.3.

Before the eleven error criteria are discussed in greater detail let us reemphasize the type of functions under study. Here, as in the preceding chapters, all conclusions will be based upon approximations of the function

$$f(t) = \sum_{k=0}^{N} A_k \epsilon^{-m_k t}, \quad m_k \text{ real} \tag{4.5}$$

by

$$f_a(t) = \sum_{k=0}^{m} C_k \phi_k(t) \tag{4.6}$$

where the $\phi_k(t)$ are linear combinations of exponential functions. In order to utilize the theory developed in the preceding chapters, for the IE^2 criterion, the poles of the $\Phi_k(s)$ are chosen from an equally-spaced set. In this case, the $\phi_k(t)$ become

$$\phi_k(t) = \sum_{\nu=0}^{k} B_\nu \epsilon^{-\beta t} \left[\sum_{\ell=0}^{\sigma-1} (-1)^\ell \binom{\sigma-1}{\ell} \epsilon^{-(\nu+\ell)pt} \right] \tag{4.7}$$

where $\epsilon^{-\beta t}$ is a factor of $\phi_k(t)$, $-\beta$ being the first pole of $\Phi_k(s)$,

p is the pole-spacing of the $\Phi_k(s)$ and subsequently it is also the pole-spacing of $F_a(s)$, and

σ is A.O. (Φ_k) which is the same as A.O. (F_a).

If the $\phi_k(t)$ are orthonormalized, as must be the case in the IT^nE^2 family, the constants B_ν will be found using the orthonormalization procedure. If, on the other hand, no such orthonormalization is utilized, as in the approximations involving the IT^nE and $AE - IT^nAE$ families, the B_ν will eventually be absorbed into the coefficients, C_k, of Eq. (4.6); hence, in these cases, all of the B_ν, in Eq. (4.7) will be set equal to unity, in order to simplify the algebra in the calculation of the C_k.

In any event, the coefficients C_k in Eq. (4.6) are found by minimizing one of the error criteria in Eqs. (4.1), (4.2), (4.3), or (4.4). These error criteria and the calculation of the C_k are discussed in greater detail in the following section.

4.2 The Error Criteria

In this section the eleven error criteria are phrased in terms of the functional notation used in Chapter I in order to obtain a graphical account of the effects the various weighting functions have on these criteria.

(a) IT^nE^2 Family

Equation (4.2) can be expressed in functional notation by letting $t^n (f - f_a)^2 = \left[\psi(e),\ t^n\right]$, where $e = f - f_a$ and $\psi(e) = e^2$.

$$IT^nE^2 = \int_o^\infty \left[\psi(e),\ t^n\right] dt = F\left[e^2,\ t^n\right] \tag{4.8}$$

Here $F \equiv \int_o^\infty dt$.

The second column of Fig. 4.2.1 summarizes the effects of the three weighting functions 1, t, and $t^{-1/2}$ on e^2 (t) while the last column illustrates exactly what is meant by the IE^2, ITE^2, and $IT^{-1/2}E^2$ measures. Figure 4.2.2 details the projective geometric techniques used for obtaining the entries in Fig. 4.2.1.

One of the major problems facing the analyst when using the IT^nE^2 criteria in signal analysis is the determination of the constants B_ν in Eq. (4.7) via an orthonormalization procedure. This problem has already been resolved for n = 0 (IE^2) in the first three chapters of the dissertation. Unfortunately, there is no simple way of extending these results to other values of n. For values of n not equal to zero one can always fall back upon Eq. (1.3), which is repeated below as Eq. (4.9),

$$\int_o^\infty t^n \phi_i(t)\ \phi_j(t)\, dt = \delta_{ij} \tag{4.9}$$

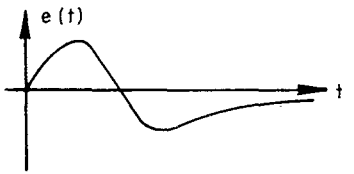
and carry out the orthonormalization of the $\phi_k(t)$ in Eq. (4.7) by <u>first</u> specifying β and the pole spacing p. This procedure must be carried out anew every time one chooses to use a different set of poles in $\Phi_k(s)$, for doing this, naturally, implies a change in β and/or p in Eq. (4.7).

(b) IT^nE Family

Equation (4.1) can be rewritten in terms of functional notation as

$$IT^nE = \int_o^\infty \left[\psi(e),\ t^n\right] dt = F\left[e,\ t^n\right] \tag{4.10}$$

$$IT^n E^2 = \int_0^\infty \left[\psi(e), t^n\right] dt, \quad \psi(e) = e^2$$



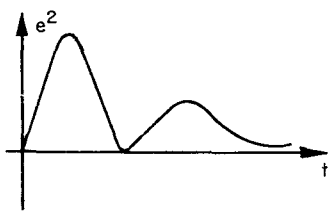
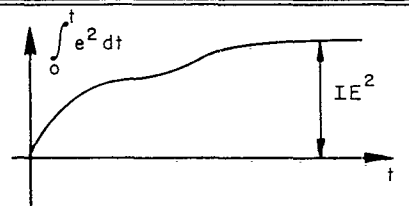
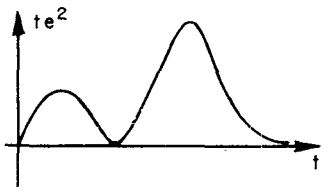
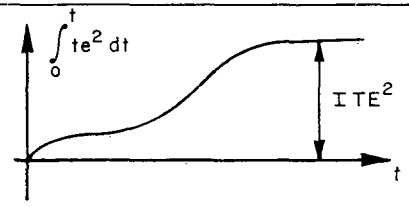
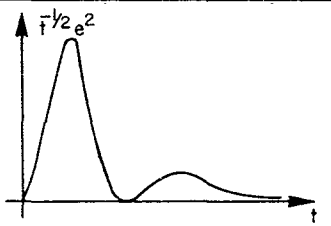
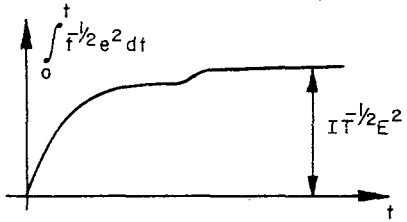
$\left[\psi(e), t^n\right]$	$\left[\psi(e), t^n\right]$ versus t	$IT^n E^2$	$\int_0^t \left[\psi(e), t^n\right] dt$ versus t
e^2 (t) (n=0)		IE^2	
te^2(t) (n=1)		ITE^2	
$t^{-1/2}e^2$(t) $(n=-\frac{1}{2})$		$IT^{-1/2}E^2$	

Figure 4.2.1 Effect of weighting functions on $IT^n E^2$ criteria.

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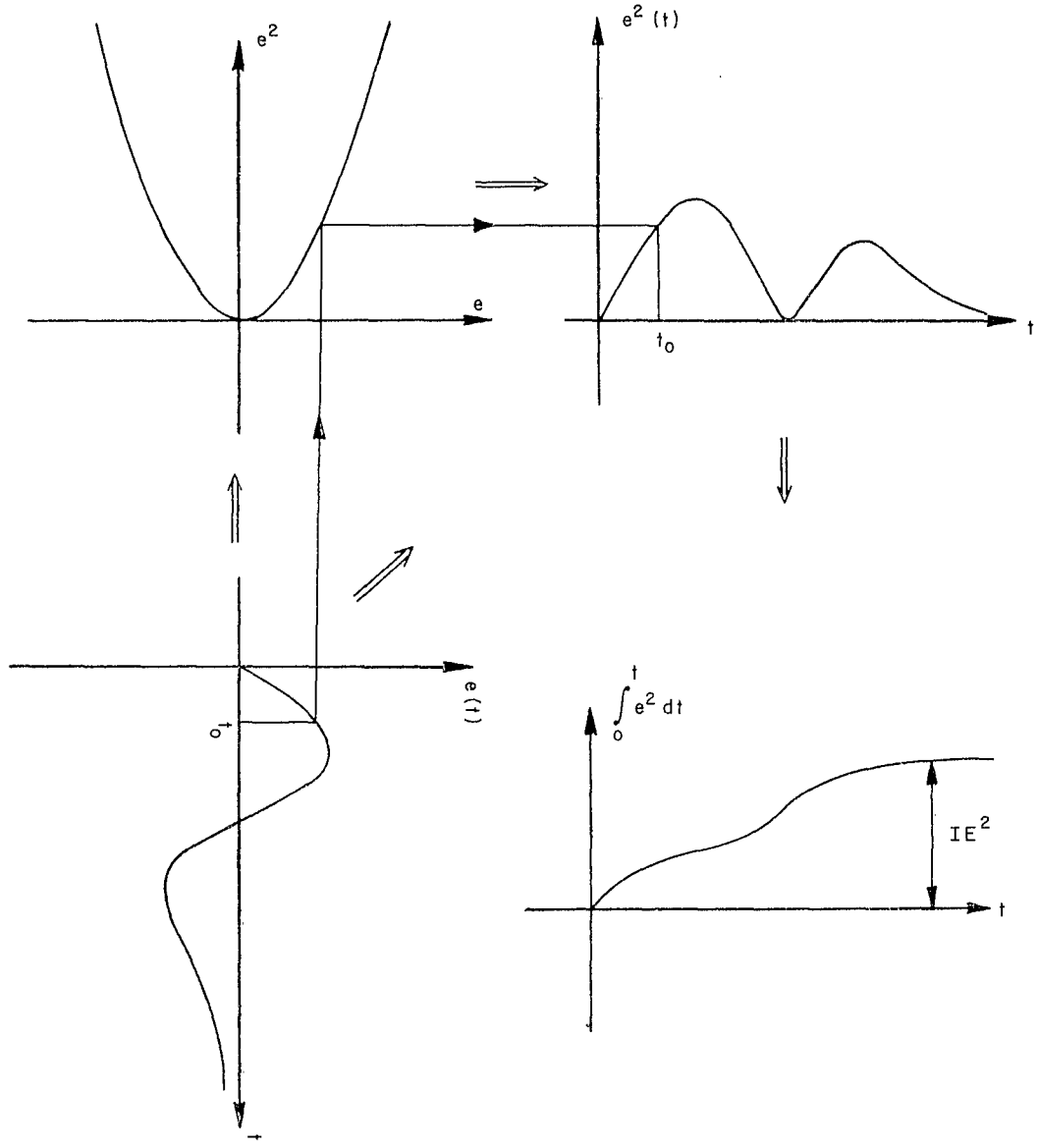

Figure 4.2.2. Construction for obtaining e^2 (t) vs. t and $\int_0^t e^2\,dt$ vs t.

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where $e = f - f_a$, $\psi(e) = e$, and $F \equiv \int_o^\infty dt$. Figure 4.2.3 summarizes the IT^nE family for $n = 0, 1, 2,$ and $-1/2$.

Substituting Eqs. (4.6) and (4.7) into Eq. (4.10), it is straightforward to show that the minimization of Eq. (4.10) results in a single equation in the $(m + 1)$ unknowns, $C_o, C_1, \ldots, C_m$. This condition occurs because the minimum value of Eq. (4.10) can actually be chosen to be zero; hence, one needs m additional equations if one is to obtain a unique solution for the $(m + 1)$ C_k. These might be given in the form of constraints between $f_a(t)$ and $f(t)$, and the derivatives of $f_a(t)$ and $f(t)$, at various values of time.

It is important to remember that the minimization of an IT^nE criterion is not sufficient for determining the constants in the approximation $f_a(t) = \sum_{k=0}^{m} C_k \phi_k(t)$ of $f(t)$.

(c) AE - IT^nAE Family

With $\left[\psi(e), t^n\right] = t^n |f - f_a|$, $e = f - f_a$, and $\psi(e) = |e|$ Eqs. (4.3) and (4.4) can be written as

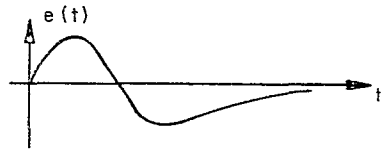
$$AE = \max_t \left[\psi(e)\right] \tag{4.11}$$

$$IT^nAE = \int_o^\infty \left[\psi(e), t^n\right] dt = F\left[|e|, t^n\right] \tag{4.12}$$

The dependence of these criteria on $|e|$ and t^n is illustrated in Figure 4.2.4 for $n = 0, 1,$ and $-1/2$.

The minimization of Eqs. (4.11) and (4.12) to find the $(m + 1)$ constants $C_o, C_1, \ldots, C_m$ in the approximation Eq. (4.6) can not be done analytically. This is due to the fact that one has chosen to work with a function of the absolute value of the error e(t) which is non-analytic. To be able to carry out the integration in Eq. (4.12) by hand, one would have to know the internal zero crossings of e(t) along the t-axis. Solving for these zeros is usually an extremely difficult task. The determination of the $(m + 1)$ coefficients from the minimization of Eq. (4.11) is an even more difficult task than the determination of the $(m + 1)$ constants in Eq. (4.12). This is due to the additional maximization of $|e|$ with respect to t, in Eq. (4.11). A trial and error (scanning) procedure for the minimization of the AE and IT^nAE criteria on a digital computer is discussed in Appendix C. There, the problems of finding the coefficients in Eqs. (4.11) and (4.12) reduce to the development of optimal computer programs and scanning techniques.

$$IT^n E = \int_0^\infty \left[\psi(e), t^n\right] dt, \psi(e) = e$$



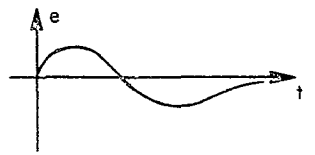
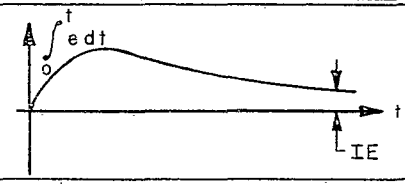
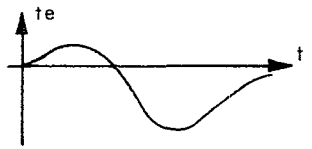
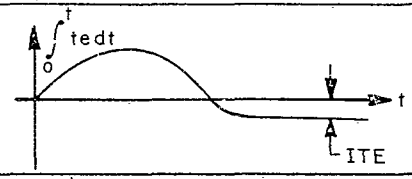
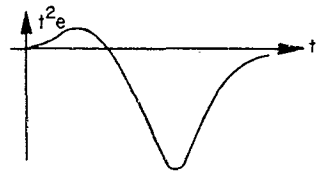
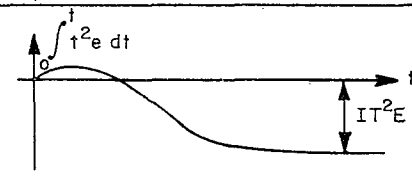
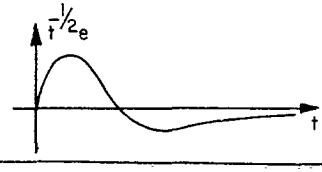
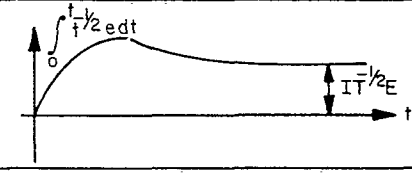
$\left[\psi(e), t^n\right]$	$\left[\psi(e), t^n\right]$ versus t	$IT^n E$	$\int_0^t \left[\psi(e), t^n\right] dt$ versus t
e (t) (n = 0)		IE	
te (t) (n = 1)		ITE	
$t^2 e(t)$ (n = 2)		$IT^2 E$	
$t^{-1/2} e(t)$ $n = -\frac{1}{2}$		$IT^{-1/2} E$	

Figure 4.2.3. Effect of weighting functions on $IT^n E$ criteria.

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$$AE = \max_{t} \left[\psi(e)\right]$$

$$IT^n AE = \int_0^{\infty} \left[\psi(e), t^n\right] dt, \quad \psi(e) = |e|$$

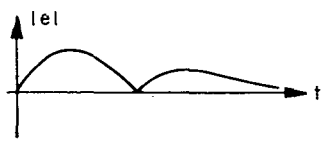
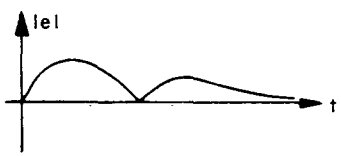
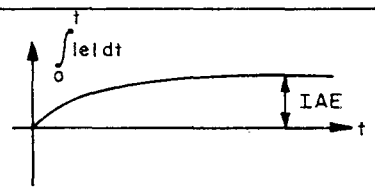
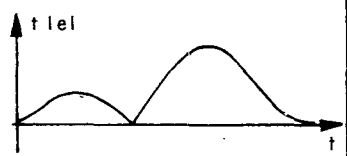
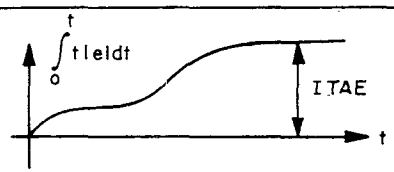
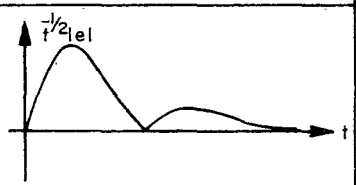
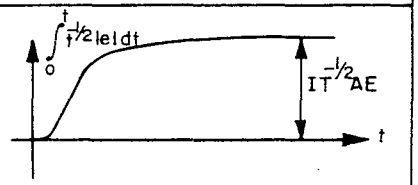
$\left[\psi(e), t^n\right]$	$\left[\psi(e), t^n\right]$ versus t	ERROR CRITERION	ERROR CRITERION versus TIME
$\lvert e \rvert$ $(n=0)$		AE	
$\lvert e \rvert$ $(n=0)$		IAE	
$t\lvert e \rvert$ $(n=1)$		ITAE	
$t^{-1/2}\lvert e \rvert$ $\left(n=-\frac{1}{2}\right)$		$IT^{-1/2}AE$	

Figure 4.2.4. Effect of weighting functions on AE - $IT^n AE$ criteria.

M.R.I.-19133

Based upon the discussions in this section and in Section 4.1, one is now in a position to set up a problem which, hopefully, will provide the answers to the questions posed at the end of Section 4.1. This is done in the next section.

4.3. The Error Problem

In order to answer the questions posed at the end of Section 4.1, the functions f(t) listed in the first column of Table 4.3.1 are approximated by the functions in the last column of that table. Figure 4.3.1 summarizes the three major steps taken in the analysis. These steps are:

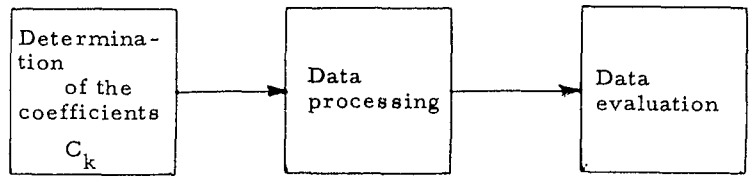


Figure 4.3.1. Summary of the three major steps taken in the error problem analysis.

(a) Determination of the coefficients C_k.

Corresponding to each entry in Table 4.3.1 eleven approximations were determined, one for each of the eleven error criteria. The calculation of the C_k, which has already been discussed in some detail in Section 4.2 is summarized in the flow chart of Figure 4.3.2. A summary of the numerical values for the C_k, calculated as indicated on Fig. 4.3.2, corresponding to the approximations listed in Table 4.3.1, is presented in Appendix D. Two problems worthy of further discussion arose during the calculation of the C_k.

The first of these has to do with the problem of roundoff effects due to keeping only a finite number of decimal places in a computation. It was found that these effects are particularly severe for the calculations of the C_k within the IT^nE^2 family. A desk calculator was used to evaluate these constants and the results were rounded off to four places during the calculations, which was well above the accuracy requirements of other parts of the problem. Four places, although quite sufficient for the C_k, is not nearly good enough for the calculations of the IT^nE^2 numeric in the three-term approximations. In many cases negative values are obtained for these errors! The inadmissability of such results is obvious from the definition of the IT^nE^2 criteria in Eq. (4.2).

Hamming[13] discusses a number of theories that have been developed by numerical analysts dealing with estimates of computational accuracies before a computation is begun. The reader will find many enlightening discussions of (to quote Hamming[13]) "the round-off noise problem" in the textbook written by Hamming.

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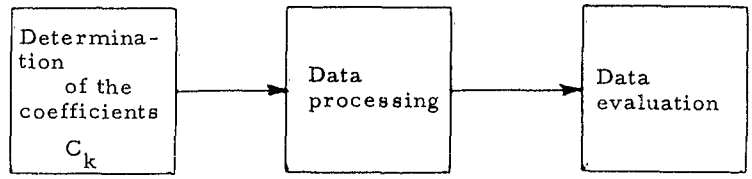

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Table 4.3.1. Summary of the functions used to approximate $\epsilon^{-2t} - \epsilon^{-4t}$ and $\epsilon^{-2t} + \epsilon^{-4t}$.*

f(t)	A.O. (F_a)	$f_a(t)$
$\epsilon^{-2t} - \epsilon^{-4t}$	1	$C_o \epsilon^{-t}$
		$C_o \epsilon^{-3t}$
		$C_o \epsilon^{-5t}$
		$C_o \epsilon^{-t} + C_1 \epsilon^{-3t}$
		$C_o \epsilon^{-t} + C_1 \epsilon^{-5t}$
		$C_o \epsilon^{-3t} + C_1 \epsilon^{-5t}$
		$C_o \epsilon^{-t} + C_1 \epsilon^{-3t} + C_2 \epsilon^{-5t}$
$\epsilon^{-2t} - \epsilon^{-4t}$	2	$C_o (\epsilon^{-t} - \epsilon^{-3t})$
		$C_o (\epsilon^{-t} - \epsilon^{-5t})$
		$C_o (\epsilon^{-3t} - \epsilon^{-5t})$
		$C_o (\epsilon^{-t} - \epsilon^{-3t}) + C_1 (\epsilon^{-3t} - \epsilon^{-5t})$
		$C_o (\epsilon^{-t} - \epsilon^{-3t}) + C_1 (\epsilon^{-3t} - \epsilon^{-5t}) + C_2 (\epsilon^{-5t} - \epsilon^{-7t})$
$\epsilon^{-2t} - \epsilon^{-4t}$	3	$C_o (\epsilon^{-t} - 2\epsilon^{-3t} + \epsilon^{-5t})$
		$C_o(\epsilon^{-t} - 2\epsilon^{-3t} + \epsilon^{-5t}) + C_1(\epsilon^{-3t} - 2\epsilon^{-5t} + \epsilon^{-7t})$
		$C_o(\epsilon^{-t} - 2\epsilon^{-3t} + \epsilon^{-5t}) + C_1(\epsilon^{-3t} - 2\epsilon^{-5t} + \epsilon^{-7t}) + C_2 (\epsilon^{-5t} - 2\epsilon^{-7t} + \epsilon^{-9t})$
$\epsilon^{-2t} + \epsilon^{-4t}$	1	$C_o \epsilon^{-t}$
		$C_o \epsilon^{-3t}$
		$C_o \epsilon^{-5t}$
		$C_o \epsilon^{-t} + C_1 \epsilon^{-3t}$
		$C_o \epsilon^{-t} + C_1 \epsilon^{-5t}$
		$C_o \epsilon^{-3t} + C_1 \epsilon^{-5t}$
		$C_o \epsilon^{-t} + C_1 \epsilon^{-3t} + C_2 \epsilon^{-5t}$

* A graphical portrait of $\epsilon^{-2t} + \epsilon^{-4t}$ immediately reveals that A.O. (F) = 1; thus, the approximations of $\epsilon^{-2t} + \epsilon^{-4t}$ are chosen such that A.O. (F_a) = 1. The approximations of $\epsilon^{-2t} + \epsilon^{-4t}$ serve primarily to check the validity of the conclusions drawn from the approximations of $\epsilon^{-2t} - \epsilon^{-4t}$.

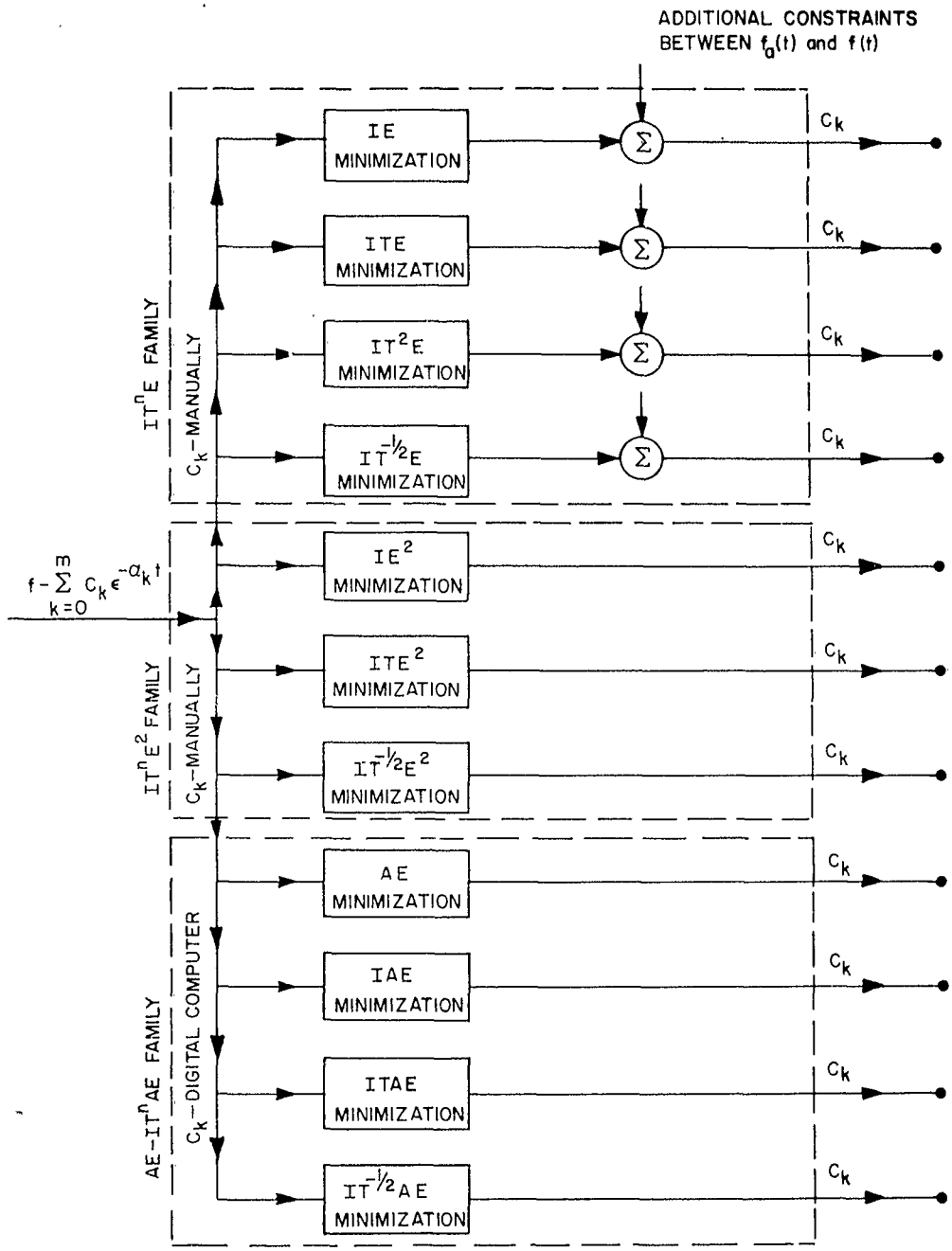

Figure 4.3.2. Flow chart indicating the determination of the coefficients C_k in the approximation $f_a(t) = \sum_{k=0}^{m} C_k \phi_k(t)$, where $\phi_k(t) = \epsilon^{-\alpha_k t}$.

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The second problem has to do with the limitations of the IBM 650 computer which was the digital computer used to evaluate the constants, C_k, within the AE-IT^{n}AE family of error criteria. It is not feasible to perform a three-dimensional scan on the 650 due to the prohibitive length of time such a scan requires. The machine is not fast enough for such a program; thus, most of the conclusions in this chapter are based upon the one and the two term approximations.

(b) Data Processing

In order to determine which error criteria led to the best approximations as discussed in Section 4.1, the eleven approximations corresponding to each entry in Table 4.3.1 were processed in the manner indicated on the flow chart in Figure 4.3.3. A brief explanation of the calculations listed on Fig. 4.3.3. that are not self-explanatory follows.

i) Magnitude-and Phase-Error Calculations:

The magnitude and phase angle of $F(j\omega)$ and $F_a(j\omega)$ were compared at seven discrete values of ω (ω = 0, 1, 2, 3, 4, 5, and 10 rad./sec.) by forming the magnitude and phase error functions, $E_M(\omega)$ and $E_\phi(\omega)$ respectively. These calculations were carried out on the IBM 650 computer. The seven values of ω (ω = 0, 1, 2, 3, 4, 5, and 10 rad./sec.) were chosen so that a comparison of the low-midband -, and high-frequency ["high" in relation to the location of the two poles of $\mathcal{L}\left\{f(t) = \epsilon^{-2t} - \epsilon^{-4t}\right\}$, at s = -2 and s = -4, and to the location of the two poles and single zero of $\mathcal{L}\left\{f(t) = \epsilon^{-2t} + \epsilon^{-4t}\right\}$, at s = -2, -4 and s = -3 respectively] behavior of the magnitude and phase angle of $F(j\omega)$ and $F_a(j\omega)$ could be made.

ii) Absolute-Stability Analysis:

Root-locus techniques were used to determine whether or not the system in Fig. 4.1.1, which is repeated below as Fig. 4.3.4, became unstable when the actual

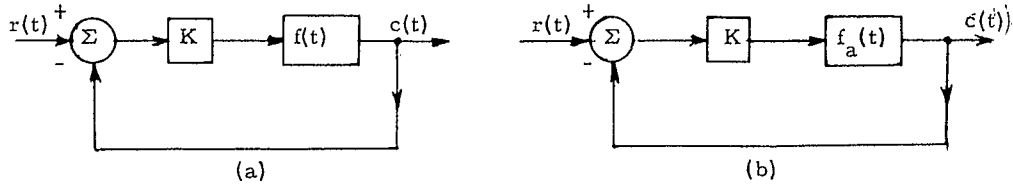

Figure 4.3.4. A single-loop feedback system in which f(t), in (a), represents the plants actual characteristics while $f_a(t)$, in (b), approximates these characteristics.

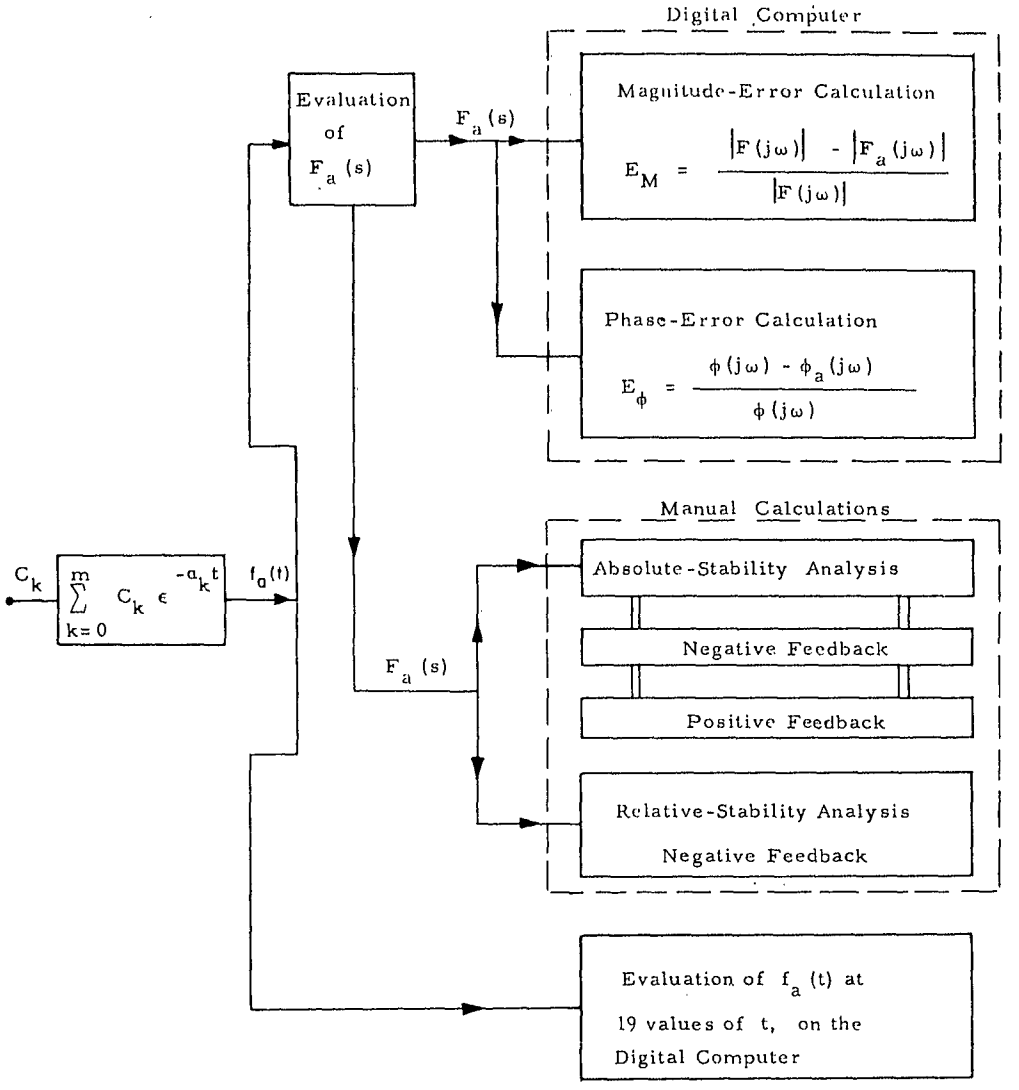

Figure 4.3.3. Flow chart indicating the processing of the approximations $f_a(t) = \sum_{k=0}^{m} C_k \epsilon^{-a_k t}$.

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plant was replaced by each one of the eleven approximations to the plant. If the system became unstable the gain at which instability occurred was also recorded. This was done for both positive and negative values of gain K. Figure 4.3.5 summarizes the stability behavior of the system containing the actual plants under study; that is, for $f(t) = \epsilon^{-2t} - \epsilon^{-4t}$, and for $f(t) = \epsilon^{-2t} + \epsilon^{-4t}$.

iii) Relative-Stability Analysis:

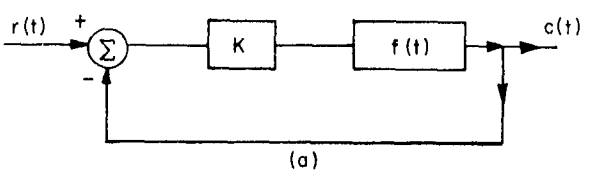
By using root-locus techniques it was possible to determine the loop-gain K required for the closed-loop response, c(t) [see Fig. 4.3.4,] to behave in some prescribed manner, e.g. $\zeta = 0.707$. This was done for each one of the eleven one -, two -, and three - term approximations listed in Appendix D, and was also done for the actual plant, f(t), in order that the loop gain necessary for the actual system to achieve a prescribed value of damping (ζ) could be compared to the loop gain necessary for the approximate system to achieve the same value of ζ. The purpose of such analyses was to determine which error criteria led to approximations whose root loci correspond closely to the loci of the actual plant over a reasonable range of gain K. Stated another way, the purpose of the analyses was to determine which error criteria led to approximations whose closed-loop response corresponded closely to the closed-loop response of the actual plant over a reasonable range of K.

(c) Data Evaluation

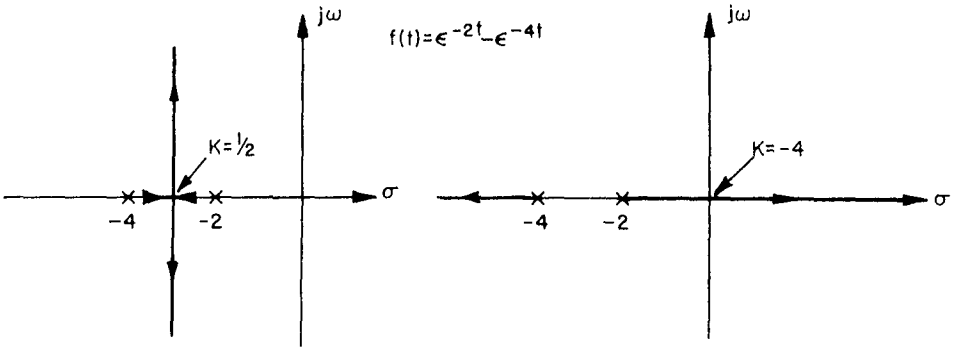
The processed approximations were evaluated as indicated in Fig. 4.3.6 in order to determine which error criterion or which groups of criteria, yield approximations satisfying the performance measures listed therein. This was done in two ways.

First, the eleven approximations were split up into three families corresponding to the three families of error criteria from which they had been derived. The data was then evaluated by comparing only the approximations from within a family of error criteria. This was done in order to determine which error criterion (or criteria), from within a family of error criteria, leads (lead) to optimum approximations - - optimum in the sense that the $f_a(t)$ satisfy any one or more of the performance measures listed in Fig. 4.3.6.

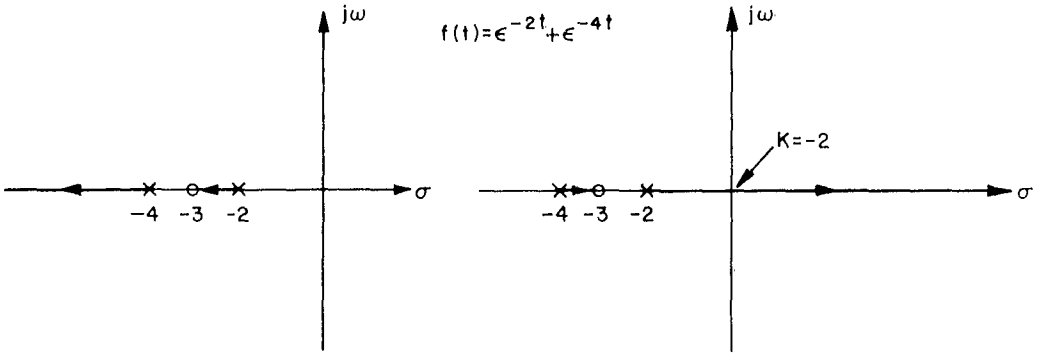
As an example, consider the determination of the error criterion (or criteria) in the IT^nE family which leads (lead) to approximations having the "best fit between $f_a(t)$ and $f(t)$". For each of the one-and two-term approximations listed in Table 4.3.1 four approximations were calculated, one corresponding to each of the four error criteria in the IT^nE family. These approximations are summarized in Appendix D.



(a)



(b) $K \geq 0$ (c) $K \leq 0$



(d) $K \geq 0$ (e) $K \leq 0$

Figure 4.3.5. Stability analysis of the closed-loop system in (a) for $f(t) = \epsilon^{-2t} - \epsilon^{-4t}$, (b) $K \geq 0$ and (c) $K \leq 0$ and for $f(t) = \epsilon^{-2t} + \epsilon^{-4t}$, (d) $K \geq 0$ and (e) $K \leq 0$.

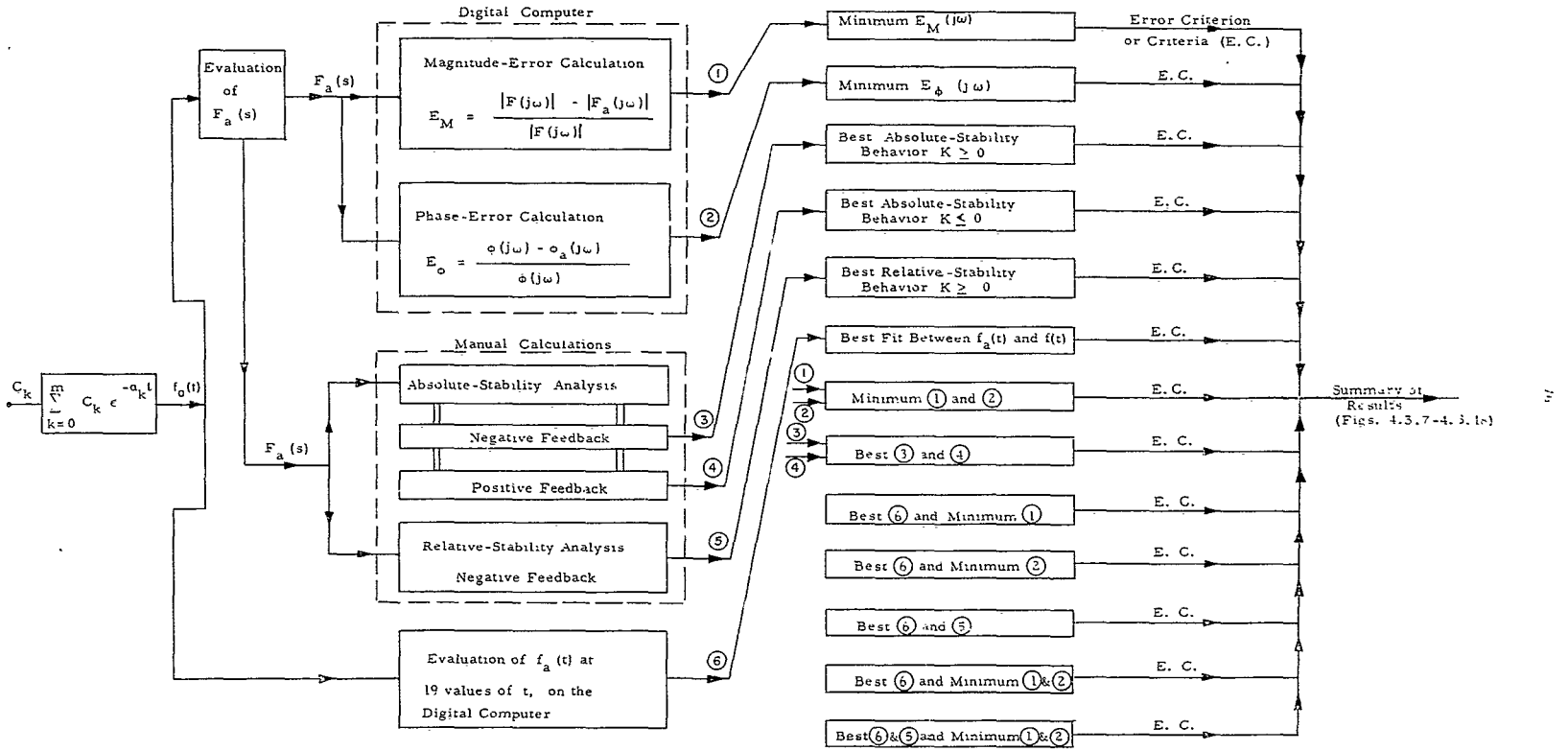


Figure 4.3.6. Flow chart indicating the processing of the approximations $f_a(t) = \sum_{k=0}^{m} C_k \phi_k(t)$ and the evaluation of the processed approximations.

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Each set of four approximations was then compared to f(t) at the 19 values of t at which these functions had been evaluated. Based upon this comparison the function (or functions) that best approximated f(t), in the sense of the best fit between $f_a(t)$ and f(t), was (were) found. Corresponding to this function (or these functions) was one of the four error criteria (or more than one) within the IT^nE family. The result of interest is the frequency (number of times) with which an error criterion within the IT^nE family leads to an optimum $f_a(t)$ - - optimum in the sense of the closest fit between $f_a(t)$ and f(t) .

The frequency with which an error criterion leads to optimum approximations -- optimum in the sense of any one or more of the performance measures listed in Fig. 4.3.6, versus error criteria, is tabulated in Appendix E. Tables E.1, E.2, E.3, and E.4 summarize the frequency with which an error criterion leads to optimum approximations when the sample-space is restricted to the error criteria from within a family of error criteria. The frequency, that a particular error criterion produces the closest fit between $f_a(t)$ and f(t), that a particular error criterion produces the closest fit between $|F_a(j\omega)|$ and $|F(j\omega)|$, that a particular error criterion produces the closest fit between $\angle F_a(j\omega)$ and $\angle F(j\omega)$, that a particular error criterion produces the closest relative-stability correspondence between $F_a(s)$ and F(s), and, that a particular error criterion produces the closest absolute-stability correspondence between $F_a(s)$ and F(s), is plotted versus error criteria in Figures 4.3.7a, 4.3.8a, 4.3.9a, 4.3.10a, and 4.3.11a respectively. These plots were obtained by adding the tabulated data in Tables E.1, E.2, E.3, and E.4 for each one of the above performance measures and for each error criterion; these sums were then plotted versus their respective error criterion (for a particular performance measure).

Returning to the example of the determination of the error criterion (or criteria) in the IT^nE family which leads (lead) to approximations having the "best fit between $f_a(t)$ and f(t)", one is able to conclude, upon inspection of Fig. 4.3.7a, that the $IT^{-1/2}E$ criterion leads to more approximations having the closest fit between $f_a(t)$ and f(t) than do the IE, ITE, and IT^2E criteria.

A summary of the conclusions which one is able to draw from Figures 4.3.7a, 4.3.8a, 4.3.9a, 4.3.10a, and 4.3.11a is presented in the next section, Section 4.4.

Figures 4.3.12a, 4.3.13a, 4.3.14a, and 4.3.15a present an alternate display of the results tabulated in Tables E.1, E.2, E.3, and E.4 respectively. These figures were obtained by first summing the numbers in each error criterion column, in each one of the tables, E.1, E.2, E.3, and E.4 respectively, and by then plotting each one of these sums versus its respective error criterion. The sum of these numbers

corresponds to the frequency with which an error criterion leads to an optimum $f_a(t)$ -- optimum in the sense of any one or more of the performance measures listed in Fig. 4.3.6. Figure 4.3.16 is an additive summary of Figs. 4.3.12a, 4.3.13a, 4.3.14a, and 4.3.15a. The important conclusions drawn from all of these figures are discussed in the next section.

In the second method of data evaluation, the data was evaluated by comparing the approximations from all of the error criteria. This was done in order to determine which error criterion (or criteria) leads (lead) to optimum approximations - - optimum in the sense that the $f_a(t)$ satisfy any one or more of the performance measures listed in Fig. 4.3.6.

As an example, consider the determination of the error criterion (criteria) which leads (lead) to approximations having the "best fit between $f_a(t)$ and $f(t)$" . For each of the one - and two-term approximations listed in Table 4.3.1 eleven approximations were calculated, one corresponding to each of the eleven selected error criteria (IE, ITE, IT^2E, $IT^{-1/2}E$, IE^2, ITE^2, $IT^{-1/2}E^2$, AE, IAE, ITAE, $IT^{-1/2}AE$). These approximations are summarized in Appendix D. Each set of eleven approximations was then compared to $f(t)$ at the 19 values of t at which these functions had been evaluated. Based upon this comparison the function (or functions) that best approximated $f(t)$, in the sense of the best fit between $f_a(t)$ and $f(t)$, was (were) found. Corresponding to this function (or these functions) was one of the eleven error criteria (or more than one). The result of interest is the frequency (number of times) with which an error criterion leads to an optimum $f_a(t)$ - - optimum in the sense of the closest fit between $f_a(t)$ and $f(t)$.

Tables E.5, E.6, E.7, and E.8 summarize the frequency with which an error criterion leads to optimum approximations when the sample-space includes all eleven error criteria. The frequency, that a particular error criterion produces the closest fit between $f_a(t)$ and $f(t)$, that a particular error criterion produces the closest fit between $|F_a(j\omega)|$ and $|F(j\omega)|$, that a particular error criterion produces the closest fit between $\angle F_a(j\omega)$ and $\angle F(j\omega)$, that a particular error criterion produces the closest relative - stability correspondence between $F_a(s)$ and $F(s)$, and, that a particular error criterion produces the closest absolute - stability correspondence between $F_a(s)$ and $F(s)$, is plotted versus error criteria in Figures 4.3.7b, 4.3.8b, 4.3.9b, 4.3.10b, and 4.3.11b respectively. These plots were obtained by adding the tabulated data in Tables E.5, E.6, E.7, and E.8 for each one of the above performance measures and for each error criterion; these sums were then plotted versus their respective error criterion (for a particular performance measure).

Returning to the example of the determination of the error criterion (or criteria) which leads (lead) to approximations having the "best fit between $f_a(t)$ and $f(t)$" , one is able to conclude, upon inspection of Fig. 4.3.7b, that the $IT^{-1/2}$ AE and $IT^{-1/2}E$ criteria lead to more approximations having the closest fit between $f_a(t)$ and $f(t)$ than do the other nine error criteria.

A summary of the conclusions which one is able to draw from Figures 4.3.7b, 4.3.8b, 4.3.9b, 4.3.10b, and 4.3.11b is presented in the next section, Section 4.4.

Figures 4.3.12b, 4.3.13b, 4.3.14b, and 4.3.15b present an alternate display of the results tabulated in Tables E.5, E.6, E.7, and E.8 respectively. These figures were obtained by first summing the numbers in each error criterion column, in each one of the tables, E.5, E.6, E.7, and E.8 respectively, and by then plotting each one of these sums versus its respective error criterion. The sum of these numbers corresponds to the frequency with which an error criterion leads to an optimum $f_a(t)$ - - optimum in the sense of any one or more of the performance measures listed in Fig. 4.3.6. Fig. 4.3.17 is an additive summary of Figs. 4.3.12b, 4.3.13b, 4.3.14b, and 4.3.15b. The important conclusions drawn from all of these figures are discussed in the next section.

While it was not possible to calculate the three-term approximations for the AE, IAE, ITAE, and $IT^{-1/2}$ AE criteria, due to the limitations of the IBM 650 comupter, it was possible to calculate the three-term approximations, listed in Table 4.3.1, for the other seven error criteria. The results of these calculations can be found in Appendix D. These approximations were processed (see Fig. 4.3.3) and the resulting data was then evaluated by using the second method for data evaluation, described on page 78. Naturally, the sample - space, in this case, consists only of seven error criteria: IE, ITE, IT^2E, $IT^{-1/2}E$, IE^2, ITE^2, and $IT^{-1/2}E^2$ criteria. Table E.9 summarizes the frequency with which an error criterion leads to optimum three-term approximations when the sample-space includes these seven error criteria. Figure 4.3.18 presents a graphical summary of Table E.9. It was obtained by first summing the numbers in each error criterion column in Table E.9, and by then plotting each one of these sums versus its respective error criterion. This sum corresponds to the frequency with which an error criterion leads to optimum three-term $f_a(t)$ - - optimum in the sense of any one or more of the performance measures listed in Fig. 4.3.6.

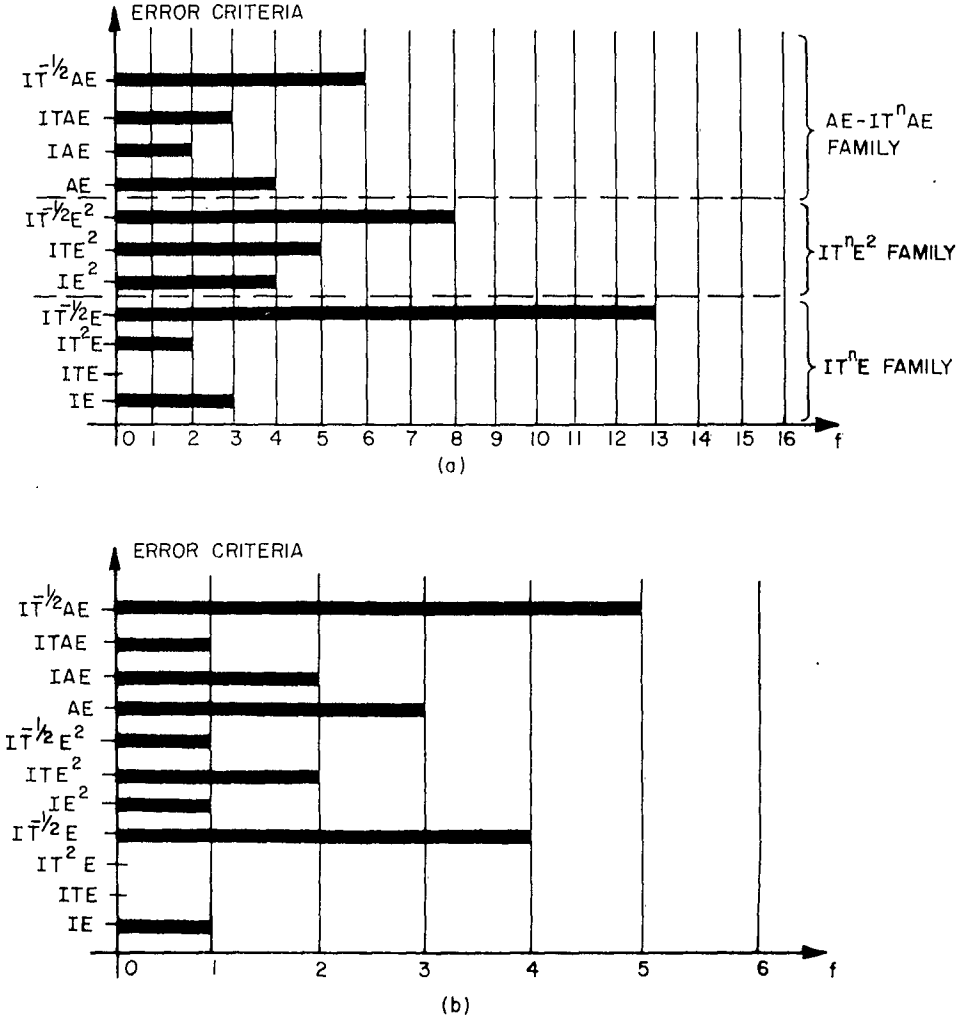

Figure 4.3.7. Frequency (number of times), that a particular error criterion produced the closest fit between $f_a(t)$ and f(t), versus error criteria. (a) Sample-space restricted to a family of error criteria (indicated by dashed lines); (b) sample-space included all eleven error criteria.

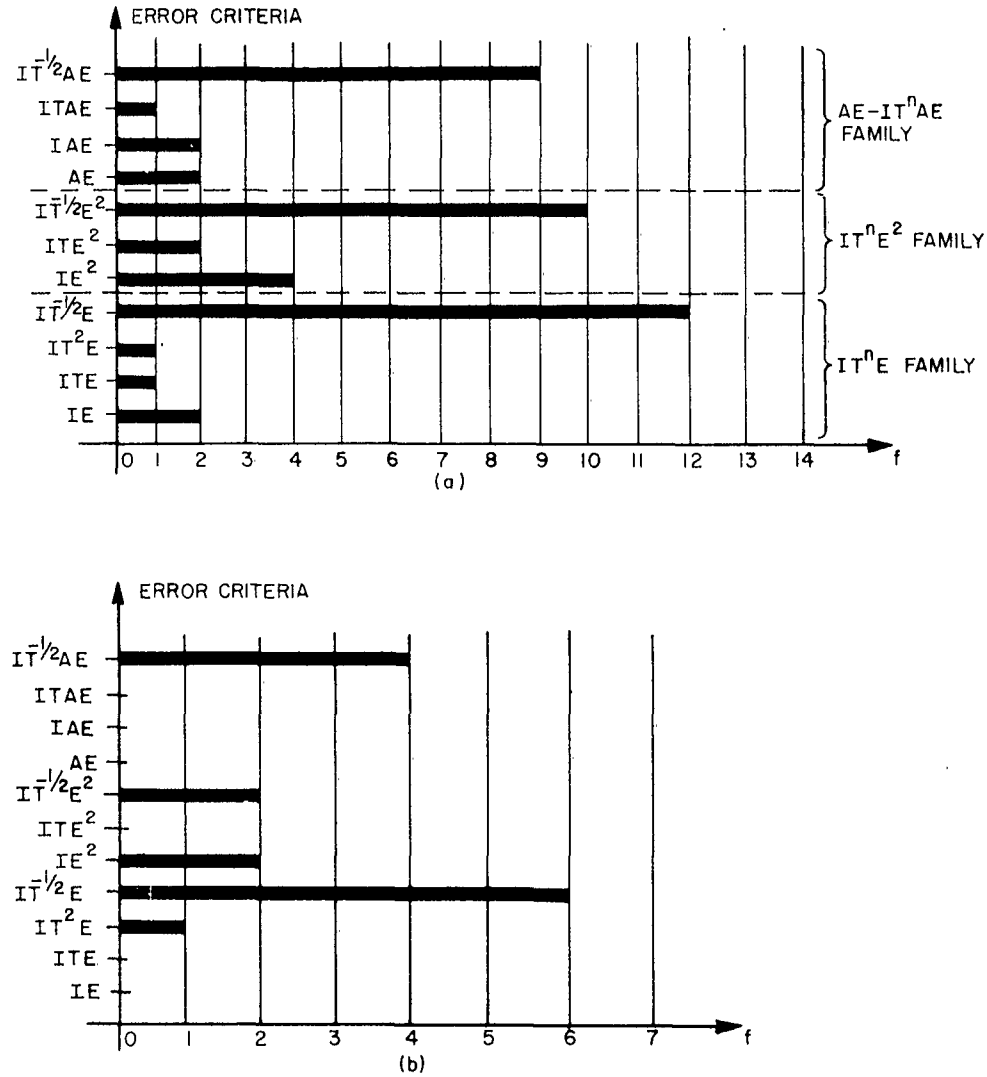

Figure 4.3.8. Frequency (number of times), that a particular error criterion produced the closest fit between $|F_a(j\omega)|$ and $|F(j\omega)|$, versus error criteria. (a) Sample-space restricted to a family of error criteria (indicated by dashed lines); (b) sample-space included all eleven error criteria.

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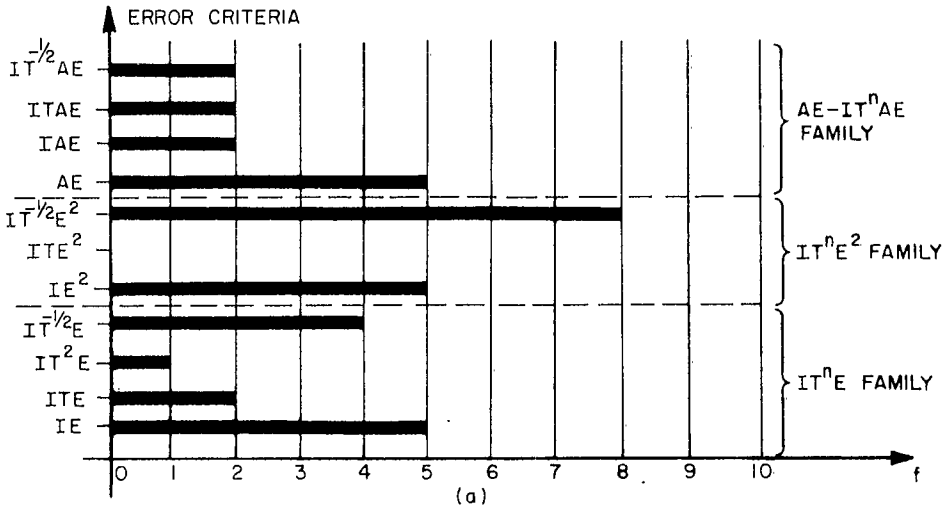

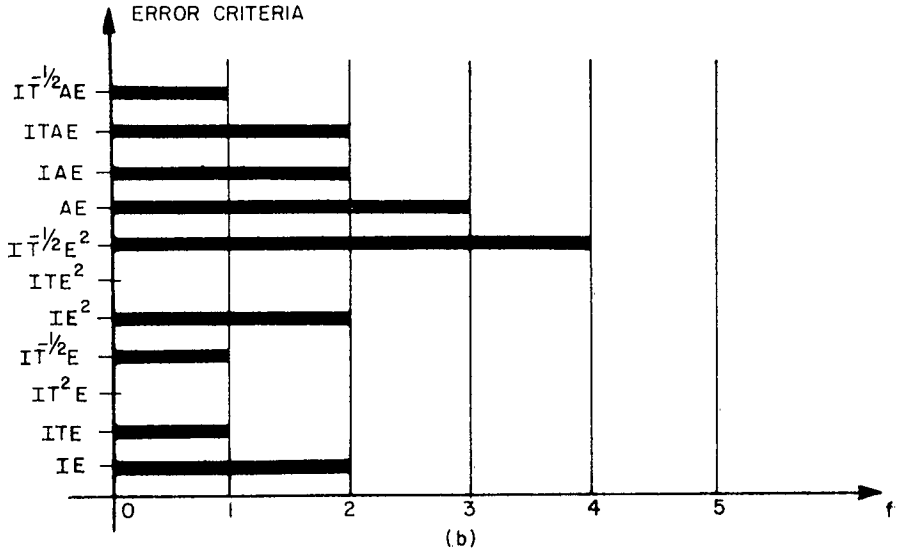

Figure 4.3.9. Frequency (number of times), that a particular error criterion produced the closest fit between $\angle F_a(j\omega)$ and $\angle F(j\omega)$, versus error criteria. (a) Sample-space restricted to a family of error criteria, (indicated by dashed lines); (b) sample-space included all eleven error criteria.

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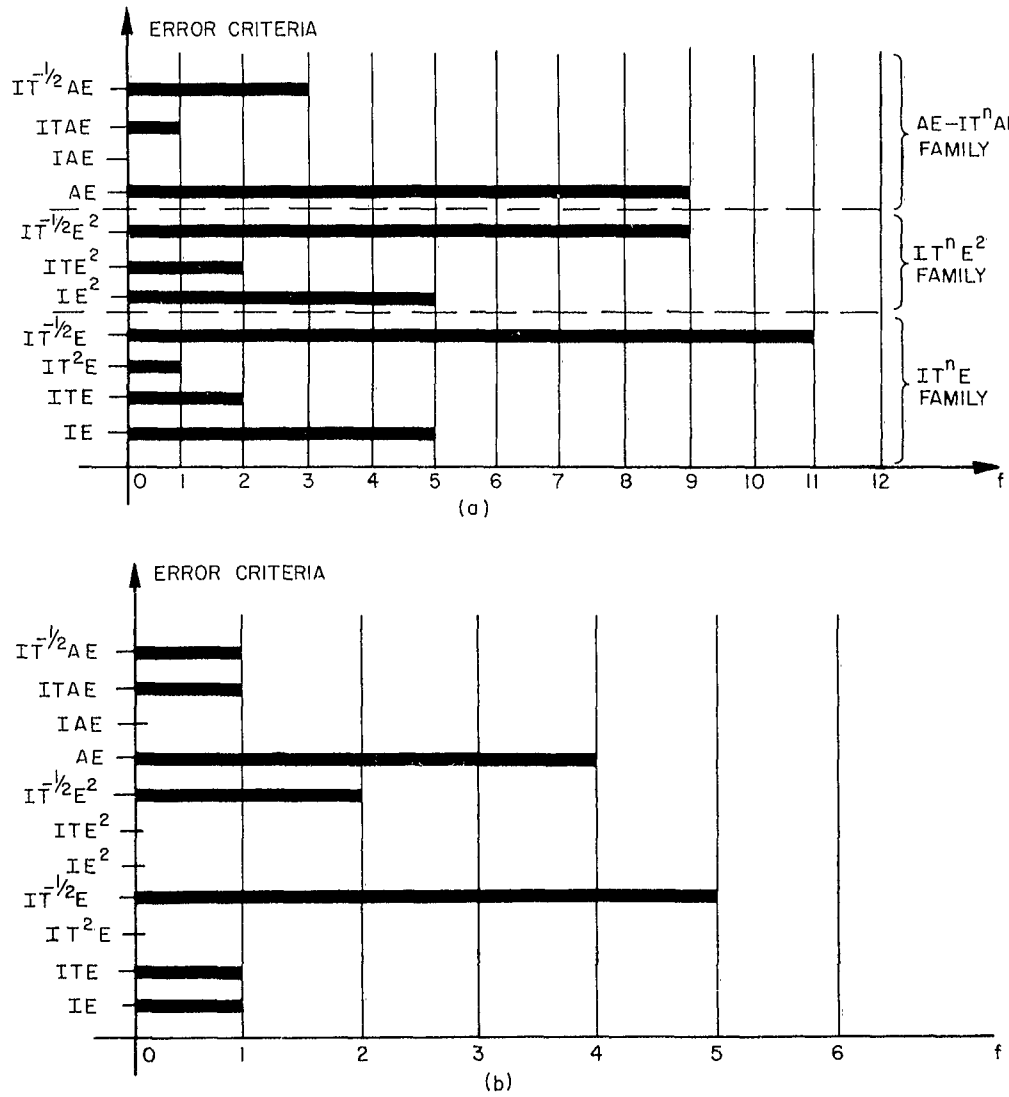

Figure 4.3.10. Frequency (number of times), that a particular error criterion produced the closest relative-stability correspondence between $F_a(s)$ and $F(s)$, versus error criteria. (a) Sample-space restricted to a family of error criteria (indicated by dashed lines); (b) sample-space included all eleven error criteria.

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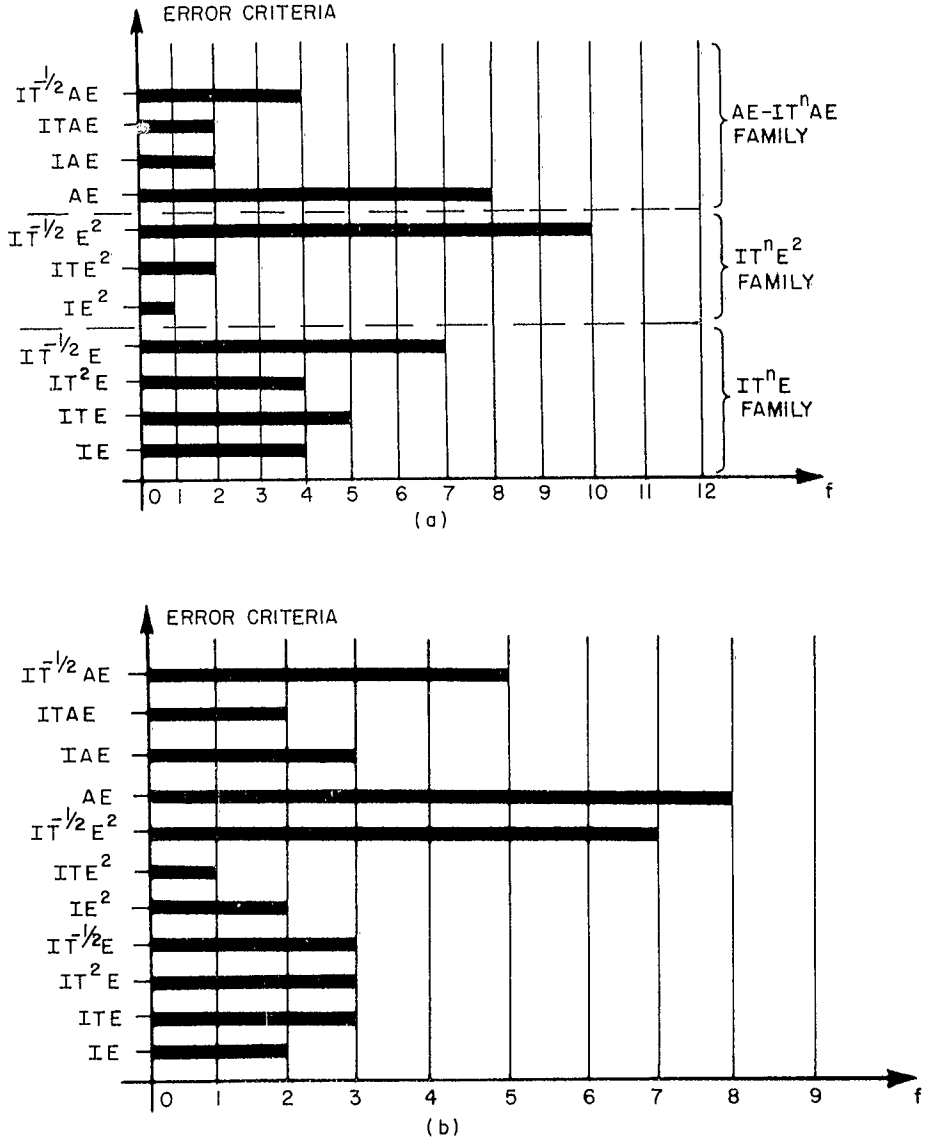

Figure 4.3.11. Frequency (number of times), that a particular error criterion produced the closest absolute-stability correspondence ($K \geq 0$) between $F_a(s)$ and $F(s)$, versus error criteria. (a) Sample-space restricted to a family of error criteria (indicated by dashed lines); (b) sample-space included all eleven error criteria.

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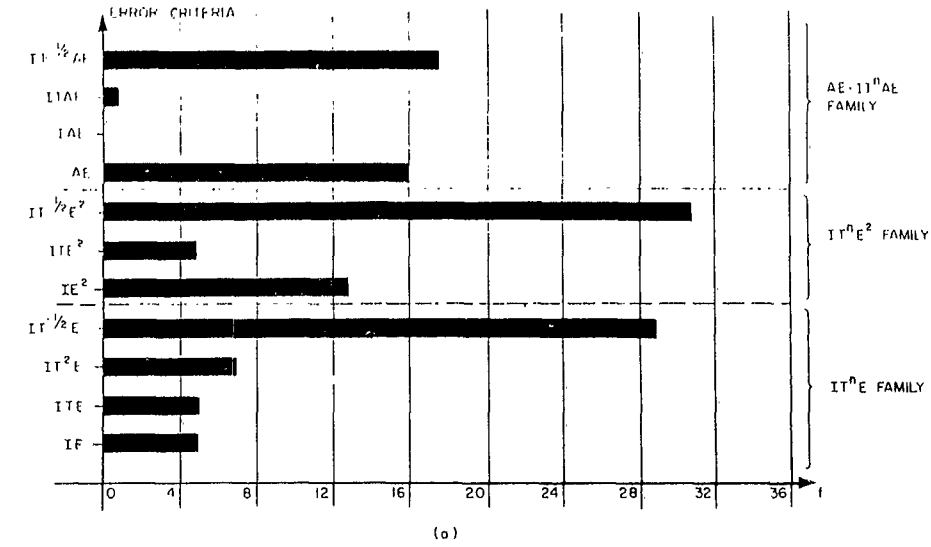


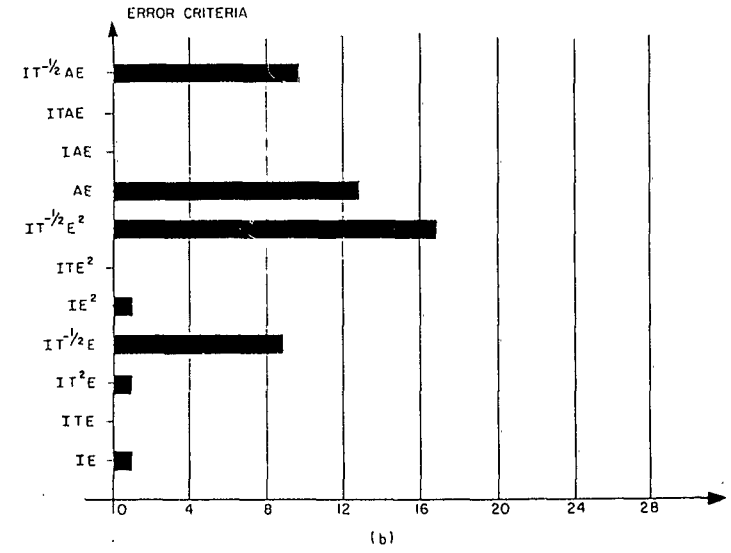


Figure 4.3.12. Frequency (number of times), that a particular error criterion led to an optimum $f_a(t)$ $[A.O.(F_a) = 1]$ - - optimum in the sense that the $f_a(t)$ satisfied any one or more of the performance criteria listed on Fig. 4.3.6, versus error criteria. Here $f(t) = \epsilon^{-2t} - \epsilon^{-4t}$. (a) Sample-space restricted to a family of error criteria (indicated by dashed lines); (b) sample-space included all eleven error criteria.

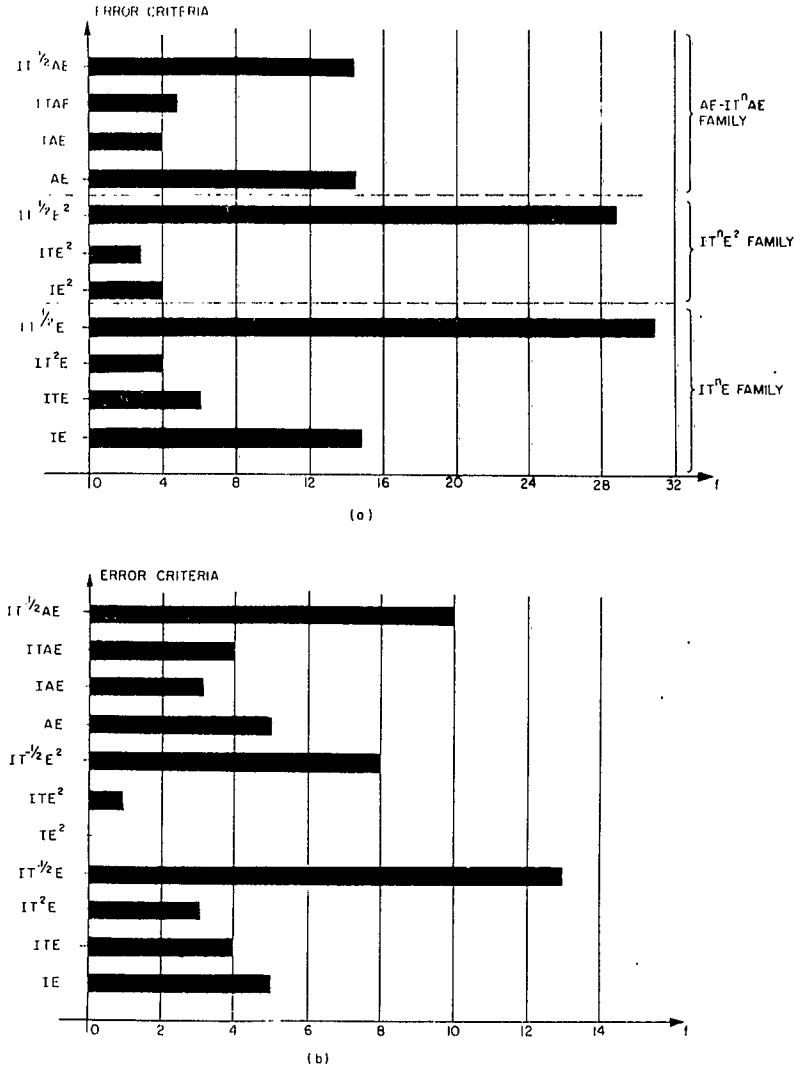
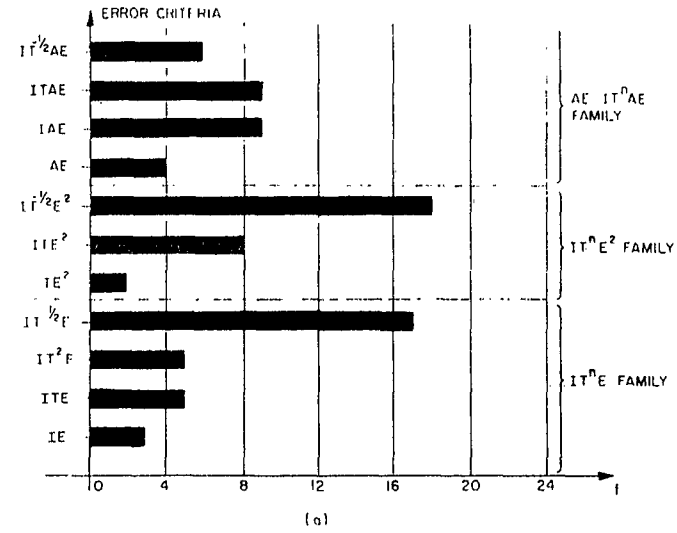


Figure 4.3.13. Frequency (number of times), that a particular error criterion led to an optimum $f_a(t)$ $\left[\text{A. O. } (F_a) = 2\right]$ - - optimum in the sense that the $f_a(t)$ satisfied any one or more of the performance criteria listed on Fig. 4.3.6, versus error criteria. Here $f(t) = \epsilon^{-2t} - \epsilon^{-4t}$. (a) Sample-space restricted to a family of error criteria (indicated by dashed lines); (b) sample-space included all eleven error criteria.



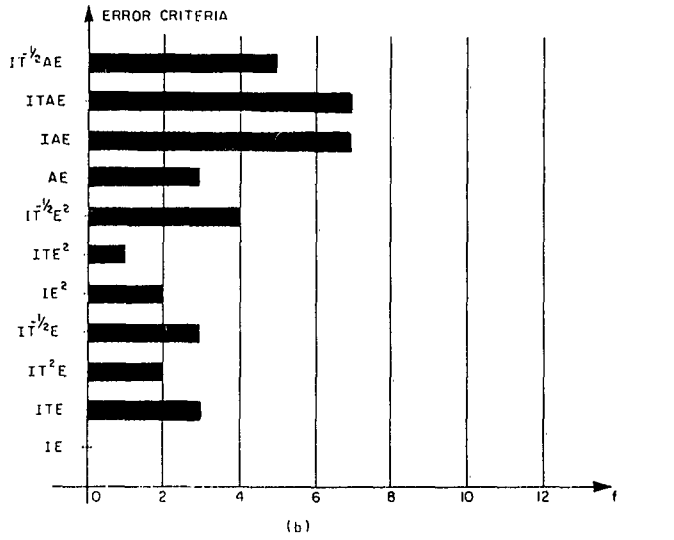


Figure 4.3.14. Frequency (number of times), that a particular error criterion led to an optimum $f_a(t)$ $\left[\text{A. O. } (F_a) = 3\right]$ - - optimum in the sense that the $f_a(t)$ satisfied any one or more of the performance criteria listed on Fig. 4.3.6, versus error criteria. Here $f(t) = \epsilon^{-2t} - \epsilon^{-4t}$. (a) Sample-space restricted to a family of error criteria (indicated by dashed lines); (b) sample-space included all eleven error criteria.

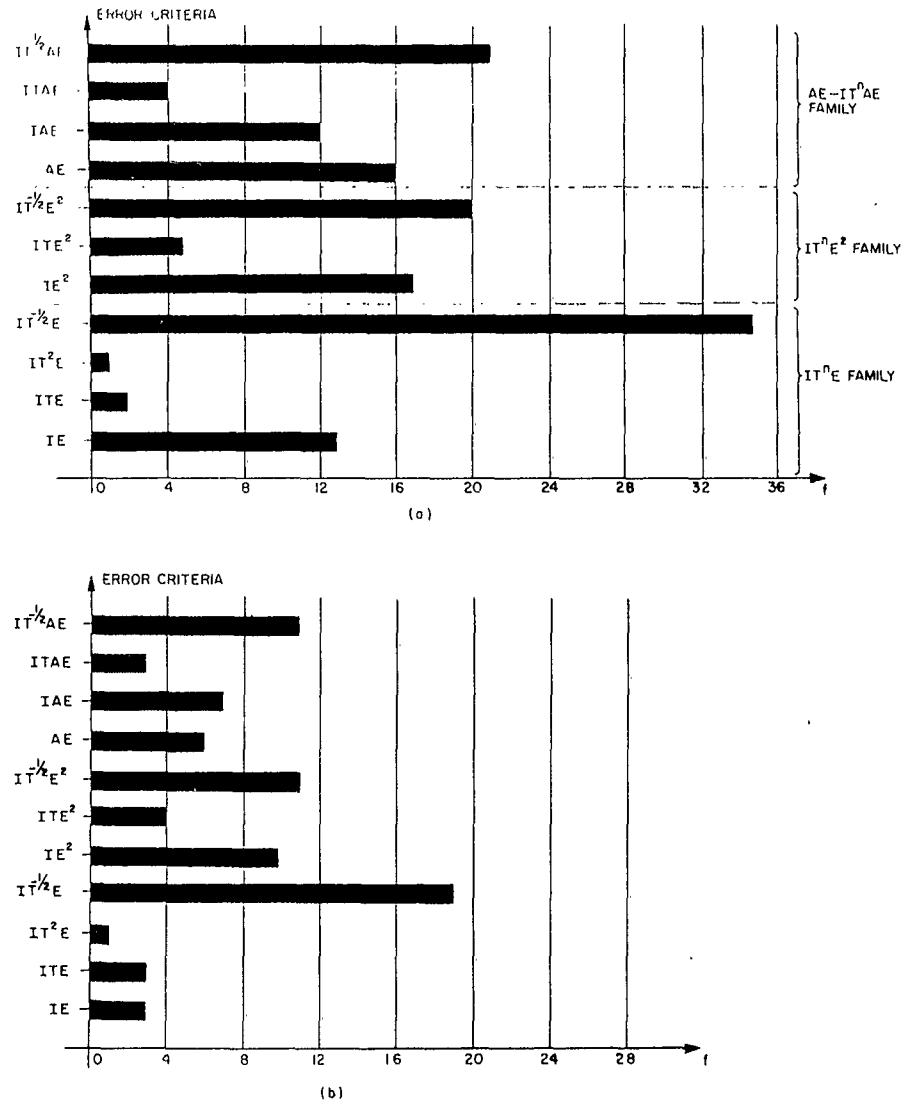


Figure 4.3.15. Frequency (number of times), that a particular error criterion led to an optimum $f_a(t)$ $\left[\text{A.O. } (F_a) = 1\right]$ - - optimum in the sense that the $f_a(t)$ satisfied any oneor more of the performance criteria listed on Fig. 4.3.6, versus error criteria. Here f(t) $= \epsilon^{-2t} + \epsilon^{-4t}$ . . (a) Sample-space restricted to a family of error criteria (indicated by dashed lines); (b) sample-space included all eleven error criteria.

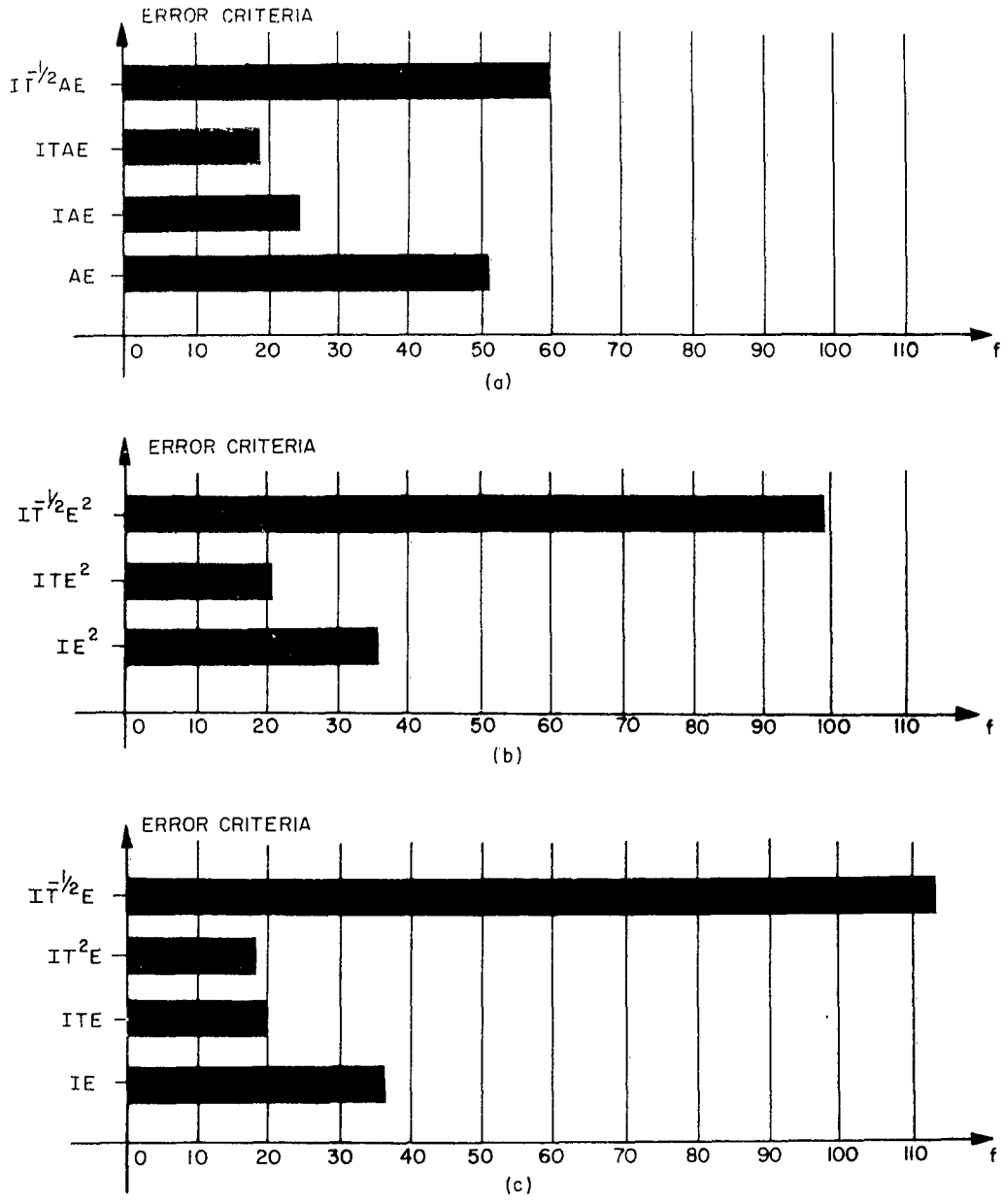


Figure 4.3.16. Additive summary of Figures 4.3.12a, 4.3.13a, 4.3.14a, and 4.3.15a. (a) AE-IT^nAE family, (b) IT^nE^2 family, (c) IT^nE family.

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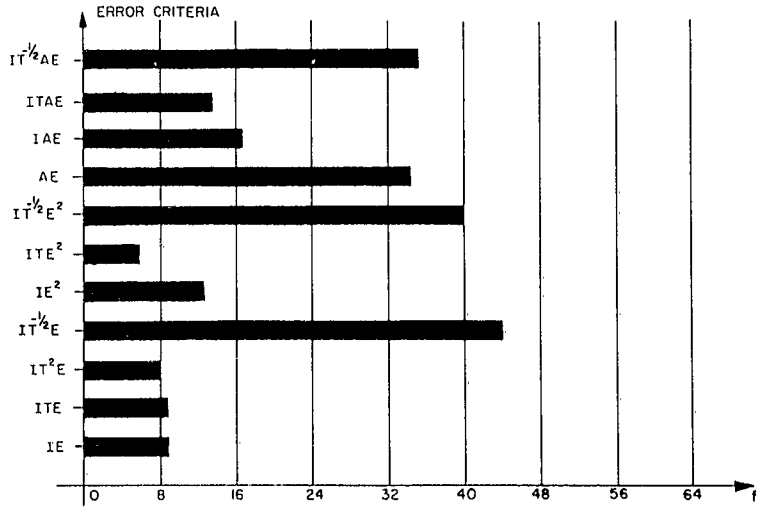

Figure 4.3.17. Additive summary of Figures 4.3.12b, 4.3.13b, 4.3.14b, and 4.3.15b.

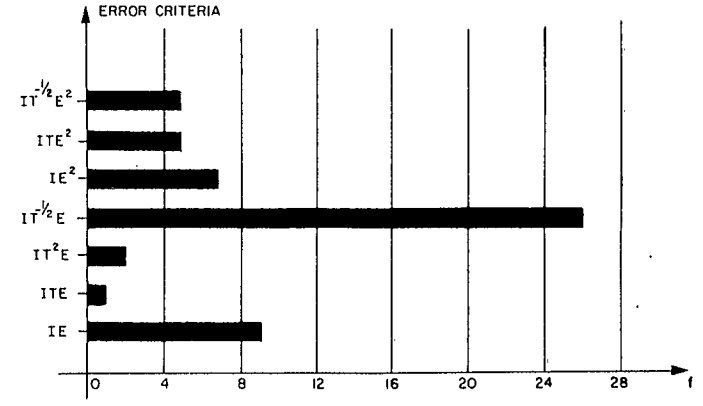

Figure 4.3.18. Frequency (number of times), that a particular error criterion led to an optimum three term $f_a(t)$ - - optimum in the sense that the $f_a(t)$ satisfied any one or more of the performance criteria listed on Fig. 4.3.6, versus error criteria. Here $f(t) = \epsilon^{-2t} - \epsilon^{-4t}$ [A.O. $(F_a) = 1, 2,$ and 3] and $f(t) = \epsilon^{-2t} + \epsilon^{-4t}$ [A.O. $(F_a) = 1$]. Sample-space restricted to seven error criteria.

4.4 Results

The results of this study are presented below as answers to the fourteen questions posed at the end of Section 4.1. They were obtained from the evaluated data, most of which is presented in tabular form in Appendix E and in graphical form in Figs. 4.3.7 - 4.3.18.

A. Approximations of best fit between $f_a(t)$ and $f(t)$: Within each family of error criteria, weighting small-time heavily leads to approximations of best fit to f(t) [Fig. 4.3.7a]. More specifically, within the IT^nE family of error criteria, the $IT^{-1/2}E$ criterion leads to approximations of best fit to f(t); within the IT^nE^2 family of error criteria, the $IT^{-1/2}E^2$ criterion leads to approximations of best fit to f(t), and; within the AE - IT^nAE family of error criteria, the $IT^{-1/2}AE$ criterion leads to approximations of best fit to f(t).

From the eleven selected error criteria, the $IT^{-1/2}AE$ and the $IT^{-1/2}E$ criteria lead to approximations of best fit to f(t) [Fig. 4.3.7b].

B. Approximations of best fit between $|F_a(j\omega)|$ and $|F(j\omega)|$: Within each family of error criteria, weighting small-time heavily leads to approximations having the best fit between $|F_a(j\omega)|$ and $|F(j\omega)|$ (Fig. 4.3.8a). More specifically, within the IT^nE family of error criteria, the $IT^{-1/2}E$ criterion leads to approximations of best fit between $|F_a(j\omega)|$ and $|F(j\omega)|$; within the IT^nE^2 family of error criteria, the $IT^{-1/2}E^2$ criterion leads to approximations of best fit between $|F_a(j\omega)|$ and $|F(j\omega)|$, and; within the AE-IT^nAE family of error criteria, the $IT^{-1/2}AE$ criterion leads to approximations of best fit between $|F_a(j\omega)|$ and $|F(j\omega)|$.

From the eleven selected error criteria, the $IT^{-1/2}E$ criterion leads to approximations of best fit between $|F_a(j\omega)|$ and $|F(j\omega)|$ (Fig. 4.3.8b).

C. Approximations of best fit between $\angle F_a(j\omega)$ and $\angle F(j\omega)$: Within the IT^nE, IT^nE^2, and AE - IT^nAE families of error criteria, the IE and $IT^{-1/2}E$, $IT^{-1/2}E^2$, and AE criteria, respectively, lead to approximations of best fit between $\angle F_a(j\omega)$ and $\angle F(j\omega)$ [Fig. 4.3.9a].

From the eleven selected error criteria, the AE and the $IT^{-1/2}E^2$ criteria lead to approximations of best fit between $\angle F_a(j\omega)$ and $\angle F(j\omega)$ [Fig. 4.3.9b].

D. Approximations having the closest relative-stability correspondence between $F_a(s)$ and $F(s)$: Within the IT^nE and IT^nE^2 families of error criteria, weighting small-time heavily leads to approximations having the closest relative-stability correspondence between $F_a(s)$ and $F(s)$. More specifically, within the IT^nE and IT^nE^2 families of error criteria, the $IT^{-1/2}E$ and $IT^{-1/2}E^2$ criterion, respectively, lead to approximations having the closest relative-stability correspondence between $F_a(s)$ and $F(s)$. Within the AE-IT^nAE family of error criteria, the AE criterion leads to optimum approximations - - optimum in the sense of the closest relative-stability correspondence between $F_a(s)$ and $F(s)$ [Fig. 4.3.10a].

From the eleven selected error criteria, the AE and $IT^{-1/2}E$ criteria lead to approximations having the closest relative-stability correspondence between $F_a(s)$ and $F(s)$ [Fig. 4.3.10b].

E. Approximations having the closest absolute-stability ($K \geq 0$) correspondence between $F_a(s)$ and $F(s)$: Within the IT^nE and IT^nE^2 families of error criteria, weighting small-time heavily leads to approximations having the closest absolute-stability correspondence between $F_a(s)$ and $F(s)$. More specifically, within the IT^nE and IT^nE^2 families of error criteria, the $IT^{-1/2}E$ and $IT^{-1/2}E^2$ criterion, respectively, lead to approximations having the closest absolute-stability correspondence between $F_a(s)$ and $F(s)$. Within the AE - IT^nAE family of error criteria, the AE criterion leads to optimum approximations - - optimum in the sense of the closest absolute-stability correspondence between $F_a(s)$ and $F(s)$ [Fig. 4.3.11a].

From the eleven selected error criteria, the AE and $IT^{-1/2}E^2$ criteria lead to approximations having the closest absolute stability correspondence between $F_a(s)$ and $F(s)$ [Fig. 4.3.11b].

F. Approximations which are optimum in the sense of one or more of the performance measures listed in Fig. 4.1.2: Within the IT^nE and IT^nE^2 families of error criteria, the $IT^{-1/2}E$ and $IT^{-1/2}E^2$ criterion, respectively, lead to approximations which are optimum in the sense of one or more of the performance measures listed in Fig. 4.1.2 (Fig. 4.3.16). It would not be fair to make a similar statement for the $IT^{-1/2}AE$ criterion without first explaining the extraordinarily strong showing made by the AE criterion.

The reason, for the very strong showing made by the AE criterion, is evident as a comparison of Eqs. (4.10) and (4.11) reveals. Consider, for the moment, a two-term approximation in which the coefficients C_o and C_1 have been found by a minimization of either the AE in Eq. (4.10) or one of the criteria in the IT^nAE family in Eq. (4.11).

The minimization of the AE actually involves a three-dimensional computer scan of the C_o - C_1 - t space while the minimization of the $IT^{-1/2}AE$, for example, merely involves a two-dimensional scan of the C_o - C_1 space*. It would appear from this and the results of Fig. 4.3.16a that the extra dimension involved in the minimization of the AE criterion is sufficient to enhance the selectivity of this criterion; however, the $IT^{-1/2}AE$ criterion still leads to 15% more of the "optimum approximations", on the average, then the AE criterion does, as is borne out on (a) of Fig. 4.3.16.

From the eleven selected error criteria the $IT^{-1/2}E$, $IT^{-1/2}E^2$, AE, and $IT^{-1/2}AE$ criteria lead to approximations which are optimum in the sense of one or more of the performance measures listed in Fig. 4.3.6. While a selectivity of error criteria exists on the "family of error criteria" level, this selectivity no longer seems to exist when the sample-space is extended to all eleven error criteria. Based upon the summary in Fig. 4.3.18, for the three term approximations, and the results presented in A through E, on the preceding pages, one is led to the conclusion that the $IT^{-1/2}E$ and the AE criterion seem to be the most selective error criteria; they lead to approximations that are optimum with respect to more of the performance measures listed in Fig. 4.3.6 then do the other nine error criteria.

G. Common properties of the error criteria in A - F, on the preceding pages:

The results in A-F, on the preceding 2 pages clearly indicate that it is, in most cases, the same weighting function that these error criteria have in common. The important fact, however, is not that the common weight is $\frac{1}{\sqrt{t}}$ but rather that emphasizing small-time heavily leads to optimal approximations. This seems plausible, since many of the performance measures in Fig. 4.3.6 involve the concept of stability which, as has been pointed out in Section 2.1, is intimately connected with high-frequency behavior or, by the Initial Value Theorem, with small-time behavior.

H. Approximations of incorrect asymptotic-order: From the large amount of processed and evaluated data involved in this study it was possible to conclude that, on the average, the numerical values of the error between the actual loop gain necessary to achieve a specified c(t) and the loop gain required by the approximate plant to achieve the same closed-loop response were smaller for the approximations of correct asymptotic-order than they were for those of incorrect asymptotic-order.

As an example, consider the plant $f(t) = \epsilon^{-2t} - \epsilon^{-4t}$ and the loop gain required to cause the closed-loop response c(t), in Fig. 4.3.4, to oscillate with a damping

* For an extensive discussion of multi-dimensional computer scanning, see Appendix C.

ratio $\xi = 0.707$. The loop gain required of the output of the system for it to oscillate with $\xi = .707$, when the actual plant is embedded in the forward path in Fig. 4.3.4a, is $K = 5$. The loop gain closest to the actual gain of $K = 5$, when A.O. $(F_a) = 2$ in the system of Fig. 4.3.4b, is produced by the ITE^2 approximation listed in Table D.11, and is $K = 4.63$. The loop gains closest to the actual gain, when A.O. $(F_a) = 1$ and 3, on the other hand, are produced by the $IT^{-1/2}E$ and $IT^{-1/2}E^2$ approximations listed in Tables D.4 and D.14 respectively and are $K = 2.6$ and $K = 2.88$. In this case the superiority of the approximation of correct asymptotic-order as compared to those of incorrect asymptotic-order is quite obvious.

CHAPTER 5. CONCLUDING THOUGHTS AND RECOMMENDATIONS

The primary purpose of the research described in the preceding chapters was to investigate some of the considerations which influence the choice of method for the identification of signals in the time-domain. Particular attention has been given to the asymptotic-order of the approximation, weighting functions, and the choice of a suitable error criterion.

It has been shown in Chapters 1-3, that the sets of orthonormalized exponential approximants developed by Huggins[17], Kautz[18], and Braun[4], whose asymptotic-order, in the s-domain, is unity, are not always applicable to the "identification" of a process which is embedded in the forward path of a positional servomechanism. Stability considerations, in many instances, necessitate the use of approximants of asymptotic-order greater than unity; such approximants are developed in the first three chapters.

It has also been shown that weighting small-time heavily leads to approximations (of a process) which satisfy many of the performance criteria necessary for not only a close correlation between the approximations and the process in the time-domain but also for a close correlation between the approximations and the process in the s-domain, the latter correlation being very important when the identification of the process is merely the first step in an analysis of a closed-loop servo in which the process is embedded.

Rather than formally summarize all of the results of the preceding four chapters, some of the problems which remain to be investigated, many of which are natural extensions of these results, shall briefly be indicated.

1. The importance of matching the asymptotic-order of the process being identified has been demonstrated throughout Chapters 1-4. The determination of the asymptotic-order of a process from practical, experimental, time-domain measurements made upon the process, under the assumption that the process may be represented by a lumped-parameter model, remains one of the most challenging problems facing the analyst.

2. The approximants developed in Chapters 1 and 3 are useful in the identification of overdamped processes. The extension of these results to the underdamped, or complex-pole case, if at all possible, would be of interest, since many systems are indeed characterized by underdamped impulse responses. At the same time, the feasibility of using complex-pole approximants should be investigated; they generally lead to much greater computational difficulties, as compared to the real-pole approximations.

3. The matrix weighting function developed in Appendix B, in connection with the pole-spacing question, while attractive on the one hand, in that it enables one to obtain unevenly-spaced poles in $F_a(s)$, involves an excessive amount of computational effort, on the other hand. Might this difficulty be overcome by adapting the matrix weighting function to a computer solution of the problem?

4. Weighting small-time heavily is important in the solution of the identification problem in adaptive systems. The possibility of extending the discussion of the weighting function $\epsilon^{-k_o k_1 t}\left(1 - \epsilon^{-k_1 t}\right)^{k_2}$ to negative values of k_2 would provide one with a basis for extending the results of Chapters 1 and 3 to orthonormal exponential approximants - - approximants which are weighted most heavily for very small time -- of any asymptotic-order. These approximations would then be consistent with the requirements demanded in the solution of the identification problem in adaptive control systems.

5. Based upon the "small-time" weighting results of Chapter 4, one must ask whether or not it is possible to develop analytical solutions of the identification problem which are based upon error criteria incorporating weights of the form $t^{-\frac{1}{n}}$, $n = 2, 3, 4, \ldots$. If this could be done then it might be possible to substantiate (or disprove) theoretically the results of that chapter.

6. The results of Chapter 4 were based upon a set of performance measures suited to single-loop feedback control systems. Given a particular application (network synthesis, stability analysis, adaptive control, etc.), the determination of what constitutes an optimum set of performance measures which the approximation must satisfy in order for it to "best" represent the time signal remains to be answered. These sets of performance measures will most likely be different for each application.

7. Based upon the results of item 6 above, one could ask what the optimum weighting function to be used in each application should be.

8. The determination of the coefficients C_k in the approximation $f_a(t) = \sum_{k=0}^{m} C_k \phi_k(t)$, using an error criterion from within the AE - IT^nAE family of error criteria, involves multi-dimensional computer scanning techniques, as discussed in Appendix C. Bellman's Dynamic Programming[34] might provide a useful alternate approach to the problem of the determination of the C_k, in contrast to the more direct approach described in Appendix C. The dynamic programming approach would lead to a complete set of one, two, and three term approximations naturally, due to the very nature of the dynamic programming approach.

9. Reiss[24] and Leight[19] have investigated the optimum pole locations for the poles of $F_a(s)$ when $f(t) = \epsilon^{-mt}$. The choice of these optimally located poles depends upon a priori knowledge of a range over which the pole of F(s) is known to vary, and upon a minimization of min (IE^2) with respect to the poles a_0, a_1, ... , of $F_a(s)$. An extension of their results to more complicated processes would be invaluable. Since it has been shown that weighting small-time heavily leads to approximations which "best" represent the actual process, for the positional servomechanism application, it would be very valuable, to the control analyst, to know the optimal pole locations of the poles of $F_a(s)$, where, in this case, these locations would be determined in part by minimizing an error criterion, weighted heavily for small time, with respect to the poles of $F_a(s)$.

The solutions to these problems should throw considerably more light upon the overall identification problem.

It is hoped that if anything at all has been achieved in this dissertation, it has been to make the reader aware of the distinction that exists between the "approximation in the time-domain" problem and the "identification in the time-domain" problem. In an area such as Numerical Analysis the analyst is often interested in "approximating" a signal. He does this by choosing an approximation, $f_a(t)$, such that the fit between $f_a(t)$ and f(t) in the time-domain is close. The systems engineer, on the other hand, "identifies" a process. He does this by choosing an approximation, $f_a(t)$, such that there not only is a close correspondence between $f_a(t)$ and f(t) in the time-domain, but such that there is also a correspondence between $f_a(t)$ and f(t) in the s-domain.

APPENDIX A. COMPLETENESS AND THE CHOICE OF THE a_k

Definition. When $\lim_{n\to\infty} (\min IE^2)^* = 0$ for any square integrable $\phi(t)$, one says that the set $\{\phi_\ell(t)\}_{\ell=0}^{\infty}$ is closed[20] in the space of all square integrable functions over the interval $t \in (0,\infty)$. The notation $L_2(0,\infty)$ is shorthand for, "the space of all square integrable functions over the interval $t \in (0,\infty)$".

Definition. A set $\{\phi_\ell(t)\}_{\ell=0}^{\infty}$ is complete[20] in $L_2(0,\infty)$ if no non-zero element exists which is orthogonal to every other element of that set.

Theorem A.1** Any arbitrary orthogonal system $\{\phi_\ell(t)\}_{\ell=0}^{\infty}$ in $L_2(0,\infty)$ is capable of being completed to an orthogonal system, which is complete in $L_2(0,\infty)$[1].

Theorem A.2 A system of functions of $L_2(0,\infty)$ is closed if and only if it is complete[1].

Alexits[1] summarizes these concepts in two profound sentences.

> The completeness of an orthonormal system is of great consequence for the theory of convergence of orthonormal expansions. For instance, assuming the expansion of a given $L_2(0,\infty)$ - integrable function f(t) in the functions of an appropriate orthonormal system $\{\phi_\ell(t)\}_{\ell=0}^{\infty}$ to be convergent, it is ensured only by the completeness of this system that the sum of the expansion $\sum_{\ell=0}^{n} C_\ell \phi_\ell(t)$ should represent f(t) almost everywhere.

If the $\phi_\ell(t)$ are formed from an exponential set of functions then alternate statements comparable to the two definitions and two theorems above can be made. Szaz's Theorem summarizes these statements.

Theorem A.3 (Szaz's Theorem[20]) If $\phi_\ell(t)$ is a linear combination of the elements from a set of exponential functions $\{\epsilon^{-a_k t}\}_{k=0}^{\ell}$, Re $a_k > 0$,*** such that A.O. $(\Phi_\ell) = 1$, then the set $\{\phi_\ell(t)\}_{\ell=0}^{\infty}$ is closed in $L_2(0,\infty)$ if and only if the series $\sum_{m=0}^{\infty} \frac{\text{Re } a_{in}}{1+|a_m|^2}$ diverges.

---

* $\min IE^2 = \int_0^\infty w(t) f^2(t)\, dt - \sum_{k=0}^{n} C_k^2$

** The theorems in this section will be stated without proofs. Their proofs can be found in the references.

*** Generally speaking, the notation a_k implies that a is a function of k. As an example a_k could be chosen to be $k + 2$. In this case $a_0 = 2$, $a_1 = 3$, ... , $a_\ell = \ell + 2$.

Thus one sees that he is not completely at liberty in choosing the sets of a_k. The set must satisfy the above theorem. Liu[20] has extended this result to the case where A. O. $(\Phi_\ell) \geq 2$.

Theorem A. 4 (Generalized Szaz Theorem) If $\phi_\ell(t)$ is a linear combination of the elements from a set of exponential functions $\left\{\epsilon^{-a_k t}\right\}_{k=0}^{\ell}$, Re $a_k > 0$, such that A. O. $(\Phi_\ell) \geq 2$, then the set $\left\{\phi_\ell(t)\right\}_{\ell=0}^{\infty}$ is closed in $L_2(0,\infty)$ if and only if the set $\left\{\phi_\ell(t)\right\}_{\ell=0}^{\infty}$ formed from the same exponential set, $\left\{\epsilon^{-a_k t}\right\}_{k=0}^{\ell}$, such that A. O. $(\Phi_\ell) = 1$, is closed in $L_2(0,\infty)$.

This means that the same sets of a_k can be used to generate the orthonormal approximants $\Phi_\ell(s)$ of any asymptotic-order.

By applying the integral test to the infinite series in Szaz' s Theorem one arrives at a more manageable test for determining whether or not the series in Theorem A. 3 diverges.

Corollary A. 1 If $\lim_{m\to\infty} \ln\left(\frac{a_m^2 + 1}{a_o^2 + 1}\right)^{1/2} \longrightarrow \infty$ (a_m real and positive) then $\sum_{m=0}^{\infty} \frac{a_m}{1+a_m^2}$ diverges, and the set $\left\{\phi_\ell(t)\right\}_{\ell=0}^{\infty}$ is complete[5].

Some complete and non-complete sets are listed below. In each case the validity of these results can be checked by applying Corollary A. 1 to the set of a_k.

a. Complete sets

1. $a_k = a_o (1 + bk)^{\nu}$, $\nu = 1, 2, 3, \ldots$
2. $a_k = a_o b^k$
3. $a_k = a_o (1 + bk)^k$
4. $a_k = a_o (1 + bk)^{-k}$
5. $a_k = \ln (\epsilon^{a_o} + bk)$

} $k = 0, 1, 2, \ldots$, and b is an arbitrary positive constant.

b. Non-complete sets

1. $a_k = a_o (1 + bk)^{-\nu}$, $\nu = 1, 2, 3, \ldots$
2. $a_k = a_o b^{-k}$

} $k = 0, 1, 2, \ldots$, and b is an arbitrary positive constant.

APPENDIX B. MATRIX WEIGHTING FUNCTIONS AND SPACINGS OF POLES OF $F_a(s)$

In this appendix the IE^2 criterion is generalized to a matrix criterion by weighting each element, $\phi_k(t)$, differently in Eq. (1.3). The individual weighting of the $\phi_k(t)$ is achieved by introducing the concept of a matrix weighting function. The approximation $\sum_{k=0}^{n} C_k \phi_k(t)$ is then reinterpreted in the light of the matrix IE^2 criterion and the matrix weighting function. The theory is then applied to the exponential approximation where, in particular, the spacing of the poles of $F_a(s)$ is investigated. It is shown that the introduction of a matrix weighting function allows one to achieve pole-spacings in $F_a(s)$ other than the equally-spaced type of pole-spacing which the approximations discussed in Chapters 1 and 3 are restricted to have.

B.1 The Matrix Weighting Function Defined.

$\overline{\overline{W}}(t)$ * is defined as an $(n+1) \times (n+1)$ symmetrical matrix weighting function

$$\overline{\overline{W}}(t) = \begin{bmatrix} w_{00} & w_{01} \cdots w_{0j} \cdots w_{0n} \\ w_{10} & w_{11} \cdots w_{1j} \cdots w_{1n} \\ \vdots & \vdots \quad \vdots \quad \vdots \\ w_{i0} & w_{i1} \cdots w_{ij} \cdots w_{in} \\ \vdots & \vdots \quad \vdots \quad \vdots \\ w_{n0} & w_{n1} \cdots w_{nj} \cdots w_{nn} \end{bmatrix} \qquad \text{(B.1)}^{**}$$

$$w_{ij} = w_{ji} \qquad \text{(B.2)}$$

whereupon Eq. (1.3) can be written as

$$\int_0^\infty \bar{\phi}_i \, \overline{\overline{W}} \, \bar{\phi}_j \, dt = \overline{\overline{\delta_{ij}}} \qquad \text{(B.3)}^{***}$$

---

* The overbar indicates a matrix while no overbar indicates an element of a matrix.

** In general, w_{ij} is a function of i, j, and t. The t dependence is omitted in order to simplify the notation.

*** δ_{ij} is the $(n+1) \times (n+1)$ identidy matrix.

Here $\bar{\phi}_i$ and $\bar{\phi}_j$ are (n + 1) x (n + 1) diagonal matrices.

$$\bar{\phi}_k = \begin{bmatrix} \phi_0 & & & & \\ & \phi_1 & & \bigcirc & \\ & & \ddots & & \\ & & & \phi_k & \\ & \bigcirc & & & \ddots \\ & & & & \phi_n \end{bmatrix} \qquad \text{(B. 4)}$$

For clarity Eq. (B. 3) is written in expanded form, using Eqs. (B. 1) and (B. 4), in Eq. (B. 3a)below.

$$\int_0^\infty \begin{bmatrix} \phi_0^2 & w_{00} & \cdots & \phi_0\,\phi_j\,w_{0j} & \cdots & \phi_0\,\phi_n\,w_{0n} \\ \vdots & & & \vdots & & \vdots \\ \phi_i\,\phi_0\,w_{i0} & \cdots & & \phi_i\,\phi_j\,w_{ij} & \cdots & \phi_i\,\phi_n\,w_{in} \\ \vdots & & & \vdots & & \\ \phi_n\,\phi_0\,w_{n0} & \cdots & & \phi_n\,\phi_j\,w_{nj} & \cdots & \phi_n^2\,w_{nn} \end{bmatrix} dt = \overline{\delta_{ij}} \qquad \text{(B. 3a)}^*$$

In addition, if $\bar{W}$ is decomposed into the product of an (n + 1) x 1 column matrix, $\bar{W}_C$, and a 1 x (n + 1) row matrix, $\bar{W}_R$, in that order, where

$$\bar{W}_C = \begin{bmatrix} w_0 \\ w_1 \\ \vdots \\ w_i \\ \vdots \\ w_n \end{bmatrix} \quad \text{and} \quad \bar{W}_R = \left[w_0,\ w_1,\ \ldots\ w_j,\ \ldots,\ w_n \right] \qquad \text{(B. 5)}$$

---

* $\int_0^\infty [\quad] dt$ implies that <u>every</u> element in the matrix is integrated between the limits from 0 to ∞.

so that

$$\overline{\overline{W}} = \overline{\overline{W}}_C \; \overline{\overline{W}}_R \tag{B. 6}$$

then Eq. (B. 3) can be written as

$$\int_o^\infty (\phi_i w_i)(\phi_j w_j)\, dt = \delta_{ij} \tag{B. 7}$$

By introducing the matrix weighting function it has been possible to rewrite Eq. (1. 3) in a form where each element ϕ_k (t) is weighted by a different weighting function, w_k (t). The parentheses in Eq. (B. 7) imply that w_k(t) is to be associated with ϕ_k(t) and vice-versa. The significance of the matrix weighting function on the IE^2 criterion is discussed in the following section.

B.2 The IE^2 Redefined as a Matrix Error Criterion

In the light of the matrix weighting function Eq. (B.1) it can be shown that the integral-squared error criterion [Eq. (1.1)] becomes a matrix criterion. All of the results obtained by minimizing Eq. (1.1) hold in this case also provided one replaces $f_a(t)$ by $(f_a)^*_{ij}$ and C_k by $(C_k)_{ij}$ in Eqs. (1.1), (1.2) and (1.4). $\bar{f}_a$ and $\bar{C}_k$ are both $(n+1) \times (n+1)$ matrices since $\bar{W}$ is $(n+1) \times (n+1)$.

By postulating

$$(IE^2)_{ij} = \int_o^\infty \left[f - (f_a)_{ij} \right]^2 w_{ij} \, dt \tag{B.9}$$

where

$$(f_a)_{ij} = \sum_{k=0}^{n} (C_k)_{ij} \, \phi_k \tag{B.10}$$

**

Eqs. (1.2) and (1.4) become

$$(\min IE^2)_{ij} = \int_o^\infty f^2 \, w_{ij} \, dt - \sum_{k=0}^{n} \left[(C_k)_{ij} \right]^2 \tag{B.11}$$

$$(C_k)_{ij} = \int_o^\infty f \, w_{ij} \, \phi_k \, dt \tag{B.12}$$

An example will clarify the meaning of the subscripts in these equations.

Example.

Consider a two-term approximation - - n = 1.

$$\bar{f}_a = \sum_{k=0}^{1} \bar{C}_k \, \phi_k = \bar{C}_0 \, \phi_0 + \bar{C}_1 \, \phi_1 \tag{B.13}$$

$\bar{W}$, in this case, is a 2 x 2 matrix

$$\bar{W} = \begin{bmatrix} w_{00} & w_{01} \\ w_{10} & w_{11} \end{bmatrix} = \begin{bmatrix} w_0 \, w_0 & w_0 \, w_1 \\ w_1 \, w_0 & w_1 \, w_1 \end{bmatrix} \tag{B.14}$$

---

* To simplify the notation the t dependence is not indicated but is to be understood.

** The meaning of a matrix approximation will be clarified in the next section.

whereupon $\overline{C}_0$ and $\overline{C}_1$, in Eq. (B.12), are found to be

$$\overline{C}_0 = \int_o^\infty f\,\overline{W}\,\phi_0\,dt = \begin{bmatrix} \int_o^\infty f\,w_{00}\,\phi_0\,dt & \int_o^\infty f\,w_{01}\,\phi_0\,dt \\ \int_o^\infty f w_{10}\,\phi_0\,dt & \int_o^\infty f\,w_{11}\,\phi_0\,dt \end{bmatrix} \tag{B.15}$$

$$\overline{C}_1 = \int_o^\infty f\,\overline{W}\,\phi_1\,dt = \begin{bmatrix} \int_o^\infty f\,w_{00}\,\phi_1\,dt & \int_o^\infty f\,w_{01}\,\phi_1\,dt \\ \int_o^\infty f\,w_{10}\,\phi_1\,dt & \int_o^\infty f w_{11}\,\phi_1\,dt \end{bmatrix} \tag{B.16}$$

and $\overline{f}_a$, in Eq. (B.13), is found to be

$$\overline{f}_a = \begin{bmatrix} \phi_0 \int_o^\infty f\,\phi_0 w_{00}\,dt + \phi_1 \int_o^\infty f\,\phi_1\,w_{00}\,dt & \phi_0 \int_o^\infty f\,\phi_0\,w_{01}\,dt + \phi_1 \int_o^\infty f\,\phi_1\,w_{01}\,dt \\ \phi_0 \int_o^\infty f\,\phi_0\,w_{10}\,dt + \phi_1 \int_o^\infty f\,\phi_1\,w_{10}\,dt & \phi_0 \int_o^\infty f\,\phi_0\,w_{11}\,dt + \phi_1 \int_o^\infty f\,\phi_1\,w_{11}\,dt \end{bmatrix} \tag{B.17}$$

In order to find $\overline{(\min IE^2)}$ in Eq. (B.11) the products $\overline{C}_0\,\overline{C}_0$ and $\overline{C}_1\,\overline{C}_1$ must first calculated. Their sum will be an $(n+1) \times (n+1)$ matrix $\overline{c}$ where

$$\sum_{k=0}^{1} (\overline{C}_k)^2 = \overline{C}_0\,\overline{C}_0 + \overline{C}_1\,\overline{C}_1 = \begin{bmatrix} c_{00} & c_{01} \\ c_{10} & c_{11} \end{bmatrix} = \overline{c} \tag{B18}$$

From Eqs. (B.18) and (B.11)

$$\overline{(\min IE^2)} = \begin{bmatrix} \int_o^\infty f^2\,w_{00}\,dt - c_{00} & \int_o^\infty f^2\,w_{01}\,dt - c_{01} \\ \int_o^\infty f^2\,w_{10}\,dt - c_{10} & \int_o^\infty f^2\,w_{11}\,dt - c_{11} \end{bmatrix} \tag{B.19}$$

The addition of a third term to the approximation $\bar{f}_a$ not only adds a third term to Eq. (B.13), but, what is more important, also increases the dimensionality of all of the matrices in Eqs. (B.14) (B.15), (B.16), (B.17), (B.18) and (B.19) from two to three. This obviously occurs because each one of these is an $(n + 1) \times (n + 1)$ matrix, $(n + 1)$ being the number of terms in the approximation Eq. (B.10).

That $\bar{f}_a$, $\bar{C}_k$ and $\overline{(\min IE^2)}$ are symmetrical matrices is obvious from their dependency upon $\bar{\bar{W}}$ which is symmetrical [Eq. (B.2)].

B. 3 The Meaning of a Matrix Approximation

It has developed that, by weighting each one of the approximants ϕ_k (t) differently, $f_a(t)$ has become a matrix approximation function, $\overline{\overline{f}}_a(t)$. Basically, however, one seeks an approximation to f(t) which is not a matrix and, therefore, one must reinterpret $\overline{\overline{f}}_a(t)$ in the light of this requirement. To do this, refer to the two-term approximation worked out in detail in the preceding section. Eqs. (B.17) and (B.19) are the important results.

Quite obviously, associated with each element in $\overline{\overline{f}}_a$ is a corresponding element in (min IE^2), and vice-versa.

$$(f_a)_{ij} \quad \overset{ij}{\longleftrightarrow} \quad (\min IE^2)_{ij} \tag{B.20}$$

Due to the individuality of the weighting functions w_{ij} (t), $\overline{\overline{f}}_a$ in Eq. (B.17) will contain 3 completely different approximations, $(f_a)_{11}$, $(f_a)_{12}$, and $(f_a)_{22}$. Naturally there will be a different min IE^2 associated with each one of these approximations. If one agrees that the smallest numerical min IE^2 gives the best approximation, in the sense of a close fit between $f_a(t)$ and f(t), one must choose the one approximation whose min IE^2 is numerically smaller than all others from the elements of $\overline{\overline{f}}_a$. This means that one seeks the absolute minimum integral-squared error from a matrix of relative minimum integral-squared errors, that is,

$$f_a(t) \approx f(t) \longleftrightarrow \min (\min IE^2)_{ij} \tag{B.21}$$

Example $w_{ij} = w(t)$

By choosing all of the elements w_{ij} to be the same function the matrix approximation should degenerate to the non-matrix version discussed in the body of the dissertation. That this is indeed the case follows directly from a substitution of $w_{ij} = w(t)$ into Eqs. (B.12), (B.10), and (B.11).

$$(C_k)_{ij} = \int_0^\infty f \phi_k \, w \, dt \tag{B.22}$$

$$(f_a)_{ij} = \sum_{k=0}^{n} (C_k)_{ij} \phi_k = \sum_{k=0}^{n} C_k \phi_k \tag{B.23}$$

and

$$(\min IE^2)_{ij} = \int_0^\infty f^2 \, w \, dt - \sum_{k=0}^{n} C_k^2 \tag{B.24}$$

From Eqs. (B.23) and (B.24) it is obvious that all of the elements in $\overline{f_a}$ are identical and all of the elements in $(\min IE^2)$ are identical. One then argues that it doesn't matter which approximation, $(f_a)_{ij}$, is chosen to represent f(t) since they are all the same. Thus one sees that the matrix approximation reduces to the non-matrix approximation when Eqs. (B.12), (B.10) and (B.11) can be written as

$$\overline{C_k} = C_k \, \overline{U} \tag{B.25}$$

$$\overline{f_a} = f_a \, \overline{U} \tag{B.26}$$

and

$$\overline{(\min IE^2)} = (\min IE^2) \, \overline{U} \tag{B.27}$$

where $\overline{U}$ is the unit matrix.

B.4 $\phi_k(t)$: The Exponential Set.

Comparing Eqs. (3.2) and (B.7) one obtains the following recursive equation for the $\phi_k(t)$ in terms of the Jacobi Polynomials and a weighting function that is not only a function of time but is also a function of k:

$$\phi_k(t) = (-1)^k \sqrt{\frac{p}{K_k}} \frac{\epsilon^{-\frac{cp}{2}t}}{W_k(t)} \left(1 - \epsilon^{-pt}\right)^{\frac{a-c}{2}} \mathcal{F}_k \quad (a,\ c;\ \epsilon^{-pt}) \tag{B.28}$$

The properties of $\phi_k(t)$ when $w_k(t) = 1$ and $w_k(t) = \epsilon^{-\frac{\lambda_1 cp}{2}t} \left(1 - \epsilon^{-pt}\right)^{\frac{\lambda_2}{2}(a-c)}$ are discussed in chapters 1 and 3 respectively of this dissertation. To indicate what happens to the spacing of the poles of $F_a(s)$ when a weighting function is chosen that is a function of k, consider

$$w_k(t) = \epsilon^{-w(k)pt} \tag{B.29}$$

From Eqs. (B.29) and (B.28)

$$\phi_k(t) = (-1)^k \sqrt{\frac{p}{K_k}}\ \epsilon^{-\left[\frac{c}{2} - w(k)\right]pt} \left(1 - \epsilon^{-pt}\right)^{\frac{a-c}{2}} \mathcal{F}_k \quad (a, c;\ \epsilon^{-pt}) \tag{B.30}$$

This weighting function does not affect the results of Theorem 1 for A.O. (Φ_k).

B.5 The Spacing of the Poles of $\Phi_k(s)$ and $F_a(s)$.

Theorem 2(B) The poles of the $\Phi_k(s)$ are located along the negative - real axis in the s-domain at $-\left[q + \frac{c}{2} - w(k)\right]p$ where $0 \leq q \leq$ A. O. $(\Phi_k) + k - 1$ and is integral. These poles are spaced p units apart.

One sees that while the location of the poles of $\Phi_k(s)$ is now dependent upon k their spacing still is constant. The spacing of all the poles of $F_a(s)$, however, is not necessarily uniform. As a matter of fact $F_a(s)$, which is simply

$$F_a(s) = \sum_{k=0}^{n} C_k \Phi_k(s) \tag{B.31}$$

will contain (n + 1) unevenly-spaced clusters of equally-spaced poles. Fig. (B.5.1) illustrates this behavior for a three term approximation where A. O. $(\Phi_k) = 2$, $p = 1$, $c = 2$ and $w(k) = -(k+1)^2$. The negative sign in w(k) is necessary if all the poles of $F_a(s)$ are to be in the left-half of the s-plane.

Properly choosing w(k) allows one to achieve a variety of spacings for the pole clusters which appear in $F_a(s)$. One measure of this spacing is the distance from the center of one cluster to the center of an adjacent cluster. In this case, it can be shown that w(k) must satisfy the finite difference equation.

$$w(k+1) - w(k) = \frac{\Delta C}{p} + \frac{1}{2} \tag{B.32}$$

where ΔC is the centroidal-spacing of adjacent clusters. Table B.5.1 presents a number of w(k) which were obtained as the solutions of Eq. (B.32) for the ΔC listed in the first column of that table. With this type of spacing it is possible to lose the identity of the clusters since overlapping of the poles in adjacent clusters may occur; thus, the poles of $F_a(s)$ will appear to be unevenly-spaced.

On the other hand, if it is desirable to maintain the identidy of each one of the clusters then the results summarized in Table B.5.2 should be used for w(k) in the equation for the approximants $\phi_k(t)$, Eq. (B.30). Here w(k) is determined from the condition that the distance between the pole furthest to the left for one cluster and the pole furthest to the right for a second cluster, adjacent to the first on the left, be specified as ΔD. This condition can be expressed mathematically [from the results of Theorem 2 (B)] as

$$w(k+1) - w(k) = \frac{\Delta D}{p} - k + 1 - \text{A. O. } (\Phi_k) \tag{B.33}$$

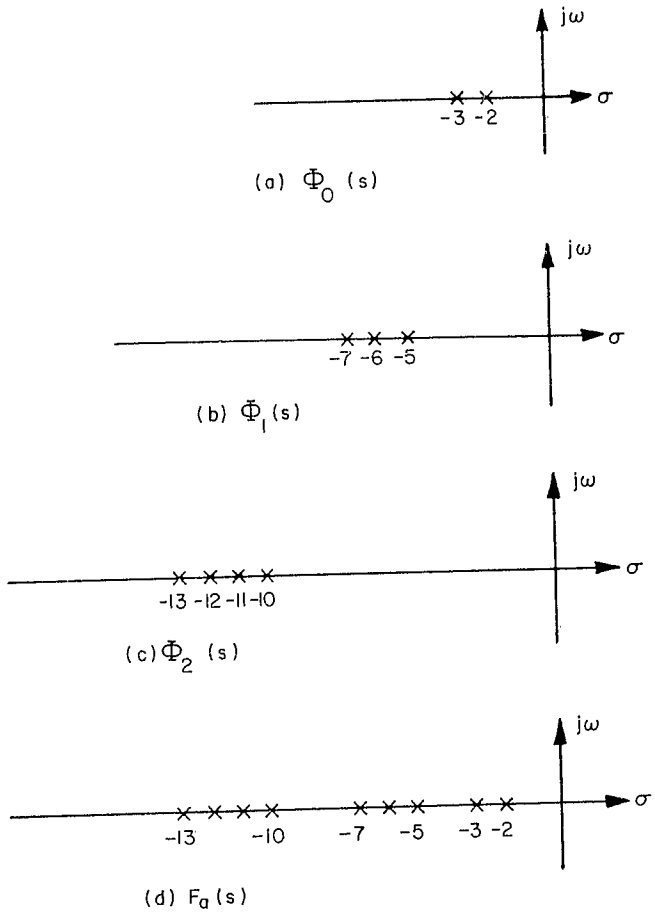

Figure B.5.1. Pole locations for a three term Expansion; (a), (b), and (c) Φ_k (s), (d), F_a(s).

The type of pole-spacing obtained through the use of the w(k) from Eq. (B. 33) is useful if one has a vague idea of the pole locations of F(s). By spacing the pole clusters of $F_a(s)$ at these locations it may be possible to improve the overall identification of f(t), as measured by the performance criteria listed on Fig. 4. 3. 6.

It is important to remember that the results of this section are based upon the choice of the weighting function $\epsilon^{-w(k)\,pt}$ in the preceding section, and that they can be rederived for other weighting functions as well. It is also important to remember that the two types of pole cluster-spacings discussed in this section - - the centroidal spacing and the non-overlapping spacing - - are not the only types of pole cluster-spacings that one might consider. Once a criterion for the pole cluster-spacing has been decided upon by the analyst, a finite difference equation, similar to Eqs. (B. 32) and (B. 33) will result, from which he can solve for w(k). In short then, the results of this section and the last, Section B. 4, have merely indicated the significance of a matrix weighting function on the pole-spacing of the poles of $F_a(s)$, for a very special weighting function and for special prechosen spacings of the pole clusters of $F_a(s)$. These results can be extended in many directions, as indicated at the beginning of this paragraph.

Table B. 5. 1 Summary of Results for the Pole Cluster-Centroidal - Spacing Design.

ΔC	$w(k) - w(o) \quad \left[w(o) < \frac{c}{2}\right]^*$
$-kp$	$k\left(\frac{1}{2} - K\right)$
$-kKp$	$\frac{k}{2}(1+K) - \frac{k^2}{2} K$
$-(k+\beta)^2 Kp$	$k\left[\frac{1}{2} - K(\beta^2 - \beta + \frac{1}{6})\right] - k^2 K(\beta - \frac{1}{2}) - k^3 \frac{K}{3}$
$-\epsilon^{\mu k} Kp$	$\frac{k}{2} - K\left(\frac{\epsilon^{\mu k} - 1}{\epsilon^{\mu} - 1}\right)$

* $w(o) < \frac{c}{2}$ ensures one that all of the poles of $F_a(s)$ are in the left-half of the s-plane, for all k.

Table B.5.2. Summary of Results for the Pole Cluster - Non-Overlapping Design.

ΔD	$w(k) - w(o) \quad \left[w(o) < \frac{c}{2}\right]^*$
$-Kp$	$k\left[\frac{3}{2} - K - A.O.\ (\Phi_k)\right] - \frac{1}{2}k^2$
$-kKp$	$\frac{k}{2}\left[2\ A.O.\ (\Phi_k) - 3 - K\right] - \frac{1}{2}(K+1)k^2$
$-(k+\beta)^2 Kp$	$k\left[\frac{3}{2} - A.O.\ (\Phi_k) - K(\beta^2 - \beta + \frac{1}{6})\right] - k^2\left[\frac{1}{2} + K(\beta - \frac{1}{2})\right] - \frac{K}{3}k^3$
$-\epsilon^{\mu k} Kp$	$k\left[\frac{3}{2} - A.O.\ (\Phi_k)\right] - \frac{1}{2}k^2 - K\left(\frac{\epsilon^{\mu k} - 1}{\epsilon^{\mu} - 1}\right)$

* $w(o) < \frac{c}{2}$ ensures one that all of the poles of $F_a(s)$ are in the left-half of the s-plane, for all k.

APPENDIX C. THE TWO-DIMENSIONAL APPROXIMATION PROBLEM AND THE ASSOCIATED TWO-DIMENSIONAL SCANNING ON A DIGITAL COMPUTER

C.1 Introduction

In this appendix two-term approximations of the signals $\sum_{k=0}^{N} A_k \epsilon^{-m_k t}$ (m_k real), determined from a minimization of any one of the error criteria in the AE - IT^n AE criteria family, are investigated; the approximations, $f_a(t)$, are of the form

$$f_a(t) = B \epsilon^{-a_o t} + C \epsilon^{-a_1 t} \tag{C.1}$$

For convenience Eqs. (4.3) and (4.4), which define the AE and the IT^n AE criteria respectively, are repeated below.

$$AE = \max_t \left| f - f_a \right| \tag{4.3}$$

$$IT^n\ AE = \int_o^\infty t^n \left| f - f_a \right| dt \tag{4.4}$$

From Eqs. (C.1), (4.3), and (4.4) one sees that the AE criterion is a function of three variables B, C, and t, while the IT^n AE criteria are functions of only B and C. To distinguish between the two cases, the following notation is introduced:

$$AE = \max_t E(B, C, t) = E\left(B, C, t^M_{B,C}\right) \tag{C.2}$$

and

$$IT^n\ AE = E'(B, C) \tag{C.3}$$

where $t^M_{B,C}$ is the time at which the absolute maximum of the AE occurs. The subscript B, C indicates that this time is also a function of the values B and C assume. Figure C.1.1. illustrates the relationship of $t^M_{B,C}$ to the AE for the arbitrarily chosen values of $B = B_1$ and $C = C_1$.

Since an extra dimension is involved in the AE criterion we shall first concentrate on its minimization. The reduction of the AE minimization techniques to techniques

useful in the minimization of a criterion from the IT^n AE criteria will then follow directly.

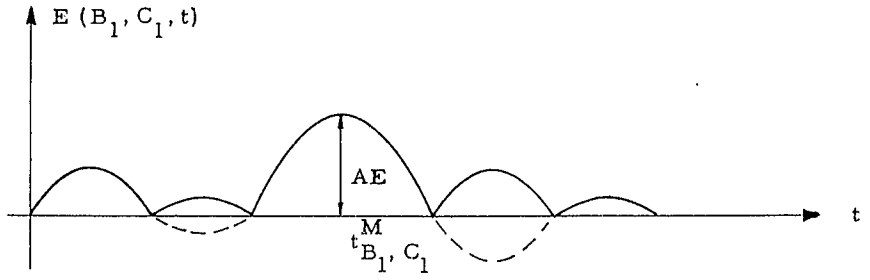

Figure C.1.1. Relationship of the AE to $t^M_{B,C}$ for $B = B_1$ and $C = C_1$.

C.2 Mathematical Formulation of the AE Minimization

Values of t, B, and C are sought such that $\max_t E(B, C, t)$ is a minimum with respect to B and C, that is, it is desired to find

$$\min_{B,C}\left[\max_t E(B, C, t)\right] = \min_{B,C} E\left(B, C, t^M_{B,C}\right) \tag{C.4}$$

The numerical value of Eq. (C.4) will be designated as $\min_{B,C} AE$.

Maximizing E(B, C, t) with respect to t is accomplished by differentiating it partially with respect to t, setting the result equal to zero, and solving the resulting equation for the time at which the absolute maximum of E(B, C, t) occurs. If this time is denoted $t^M_{B,C}$ then, it is quite obvious that, it will be a function of both B and C, that is,

$$t^M_{B,C} = f(B, C) \tag{C.5}$$

whereupon Eq. (C.2) becomes

$$AE = E\left[B,\ C,\ f(B, C)\right] \tag{C.6}$$

To minimize Eq. (C.6) with respect to B and C one may pursue the following course. By differentiating this equation partially with respect to B and setting the result equal to zero one should be able to solve for B in terms of C,

$$\frac{\partial E\left[B, C, f(B, C)\right]}{\partial B} = 0 \rightarrow B = g(C) \tag{C.7}$$

From Eqs. (C.7) and (C.6) one concludes that

$$\min_B AE = E\left\{g(C),\ C,\ f\left[g(C),\ C\right]\right\} \tag{C.8}$$

Finally, $\min_{B,C} AE$ is found by differentiating Eq. (C.8), setting the result equal to zero, and solving for C. Suppose $C = \eta$. In this case

$$\min_{B,C} AE = E\left\{g(\eta),\ \eta,\ f\left[g(\eta),\ \eta\right]\right\} \tag{C.9}$$

$$B = g(\eta) \tag{C.10}$$

and

$$t^{M}_{B,C} = f\left[g(\eta),\ \eta\right] \tag{C.11}$$

To perform the operations indicated in Eqs. (C.5)-(C.11) analytically is usually impossible in even the simplest of cases. This is due primarily to the facts that these equations are transcendental and E(B, C, t) is the absolute value of the error $f(t) - f_a(t)$. To solve this system of equations, we turn to a computer simulation of Eq. (C.4).

C. 3 Computer Formulation of the AE Minimization

The digital computer solution of Eq. (C. 4) for B, C and $t^{M}_{B,C}$ is a discrete solution in contrast to the continuous type of solution outlined in the preceding section. Before outlining the major computer operations needed in the solution, one must redefine the continuous variables B, C, and t as discrete variables. In the discussions that follow B, C, and t represent discrete sets of real numbers,

$$B = \left\{ b_\ell \right\} , \quad \ell = 0, 1, 2, \ldots \tag{C. 12}$$

$$C = \left\{ c_r \right\} , \quad r = 0, 1, 2, \ldots \tag{C. 13}$$

$$t = \left\{ t^q \right\} , \quad q = 0, 1, 2, \ldots \tag{C. 14}$$ *

where, in general, the elements b_ℓ and c_r may be positive, negative, or zero while the t^q must be positive or zero. Quite obviously the accuracy of the computer solutions will depend primarily upon the increments chosen for the b_ℓ, c_r, and t^q. For practical purposes the increments Δb and Δc are chosen the same, hence

$$\Delta b = b_\ell - b_{\ell-1} = \delta \tag{C. 15}$$

$$\Delta c = c_r - c_{r-1} = \delta \tag{C. 16}$$

$$\Delta t = t^q - t^{q-1} = T \tag{C. 17}$$

The major computer operations, in terms of this notation, are summarized below.

(1) The computer stores $\left\{ b_\ell \right\}$, $\left\{ c_r \right\}$, and $\left\{ t^q \right\}$.

(2) For an element of $\left\{ b_\ell \right\}$, say b_0, and an element of $\left\{ c_r \right\}$, say c_4, the computer evaluates $E\left[b_0 , c_4, \left\{ t^q \right\} \right]$ for all q, where $\Delta t = T$ as in Eq. (C. 17), and it stores these values.

(3) The data from step (2) is scanned by the computer in order to find $E(b_0, c_4, t^M)$ which corresponds to $\max_{\{t^q\}} E\left[b_0, c_4, \left\{ t^q \right\} \right]$. Since t^M will be a function of the b_ℓ and c_r it shall be designated as $t^M_{\ell,r}$. $t^M_{\ell,r}$ must be an element of $\left\{ t^q \right\}$.

* The use of the superscript notation will become clear in the development of this section.

(4) The computer repeats steps (2) and (3) for all combinations of b_ℓ and c_r. Naturally, the computational time in this step will be directly proportional to the increment δ chosen for the variables b_ℓ and c_r in Eqs. (C.15) and (C.16).

(5) The computer scans the complete set of stored numbers corresponding to $\max_{\{t^q\}} E\left[b_\ell, c_r, \{t^q\}\right]$ and chooses one of these numbers as the $\min_{B,C}$ AE. If this occurs for $\ell = \ell'$ and $r = r'$, for example, $f_a(t)$ will have been found to be

$$f_a(t) = b_{\ell'} \epsilon^{-a_o t} + c_{r'} \epsilon^{-a_1 t} \tag{C.18}$$

completing the problem.

In practice, one would like the computer to complete these five steps in a minimum amount of time. For a particular computer, this will depend upon the ranges chosen for B, C, and t, the increments in the b_ℓ, c_r, and t^q, and the development of a judicious implementation of step (4). In regards to this last phrase, one must admit that it may not be necessary to repeat steps (2) and (3) for <u>all</u> combinations of b_ℓ and c_r. The problem is, given a range for B and C how does one choose a minimum number of combinations of the b_ℓ and c_r such that the computer carries through the minimization procedure in a <u>minimum amount of time</u>? The following discussion presents one such method for accomplishing this.

C.4 A Practical Computer Implementation

By reducing the three-dimensional scanning procedure outlined in the last section to a manifold of two-dimensional scans, this scanning procedure can be systematized in such a manner that the computational time will be minimized. This can be accomplished by fixing either one of the constants b_ℓ or c_r at the beginning of a run. By varying the other constant over its range in step (4) of the scanning procedure, one will be led to a family of numbers from which the computer will pick the absolute minimum (the smallest number). As an example, suppose $b_\ell = b_4$ has been chosen ahead of time; it would then represent a portion of the solution of the following two-dimensional discrete version of Eq. (C.4):

$$\min_{\{c_r\}} \left[\max_{\{t^q\}} E\left(b_4,\ c_r,\ \{t^q\} \right) \right] \tag{C.19}$$

Associated with Eq. (C.19) is a value of c_r which, for example, might be c_7. This procedure is repeated for the elements of $\{b_\ell\}$ until the number in Eq. (C.19) is a minimum; thus, the numbers in Eq. (C.19) represent the locus of the relative minima of $\max_{\{t^q\}} E\left[b_\ell,\ c_r, \{t^q\} \right]$, b_ℓ fixed and c_r variable. Figure C.4.1 presents a typical locus, which, for clarity, is shown as a continuous curve. Actually, the locus of relative minima, plotted on the b_ℓ - c_r plane, is a set of discrete points whose b_ℓ coordinates are spaced uniformily - - δ units apart. Associated with each point on the locus is a number from Eq. (C.19). The point $(c_{20},\ b_9)$ is assumed to represent the

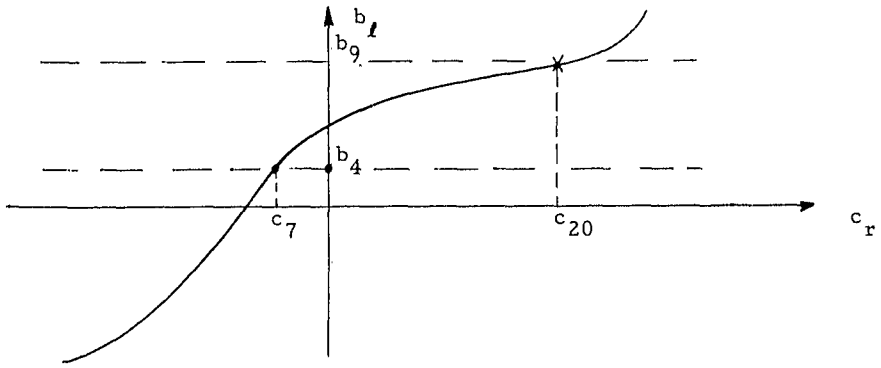

Figure C.4.1 A typical locus of the relative minima of $\max_{\{t^q\}} E\left[b_\ell,\ c_r, \{t^q\} \right]$, b_ℓ fixed and c_r variable.

solution of the three-dimensional discrete version of Eq. (C. 4),

$$\min_{\{b_\ell\}} \left\{ \min_{\{c_r\}} \left[\max_{\{t^q\}} E\left(b_\ell, c_r, \{t^q\}\right) \right] \right\} \tag{C. 20}$$

Of particular interest is the shape these loci may assume; for, if one can correctly predict the shape of a locus of relative minima, then the number of two-dimensional scans can be greatly reduced. The locus in Fig. C. 4. 2 is certainly one possible candidate for a locus of relative minima in the b_ℓ - c_r plane. For b_ℓ equal to

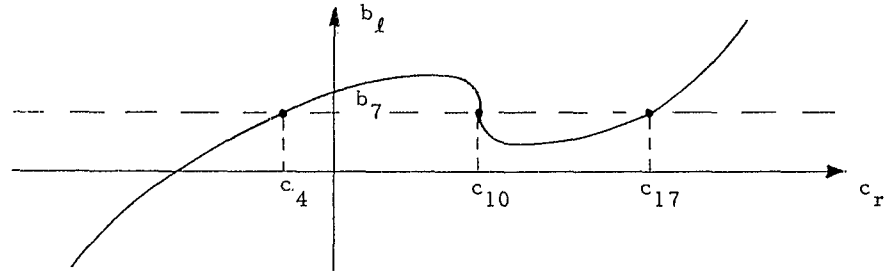

Figure C. 4. 2 A possible candidate for the locus of relative minima of $\max_{\{t^q\}} E\left[b_\ell, c_r, \{t^q\}\right]$, b_ℓ fixed and c_r variable.

b_7 there are three values of c_r each leading to exactly the same numerical relative minimum of the discrete version of Eq. (C. 4), with $b_\ell = b_7$. One sees, from Eq. (C. 19), that this means that

$$E\left(b_7, c_4, t^M_{7,4}\right) = E\left(b_7, c_{10}, t^M_{7,10}\right) = E\left(b_7, c_{17}, t^M_{7,17}\right) \tag{C. 21}$$

which is illustrated in Fig. C. 4. 3.

It seems highly improbable that this locus could be representative of physical reality, for the condition, Eq. (C. 21), is so stringent that one must seriously question its plausibility. A similar argument can be given for a parabolic locus. In this case Eq. (C. 21) reduces to an equality between two values of the error. Based upon these arguments we shall hypothesize the following: the locus of relative minima of Eq. C. 4) is either monotonically increasing or decreasing. While this has not been proven rigorously, it is interesting to note that in every case (32 in all) the computer results agreed with the hypothesis.

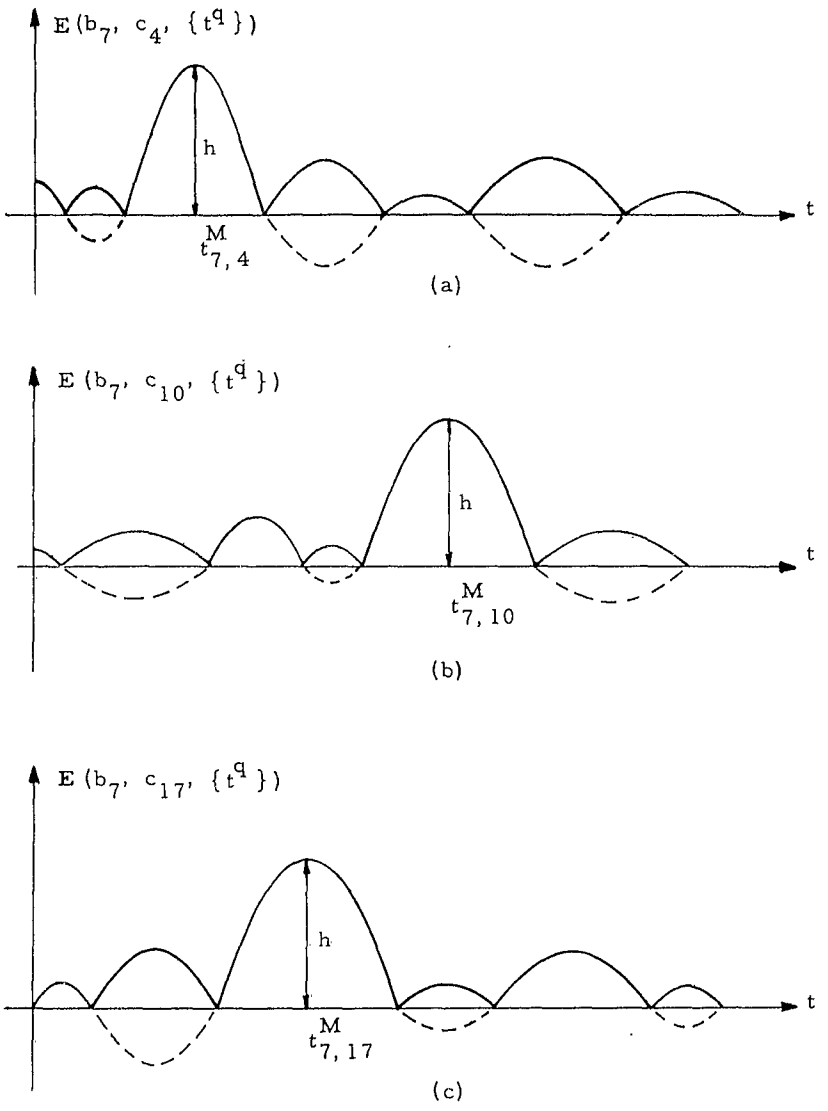

Figure C.4.3. Sketches illustrating Eq. (C.21).

By knowing the shape of the locus one is able to extrapolate it in either direction. Thus, computational time may be minimized if the procedure outlined below is used as the basis of the computer program.

(1) Obtain a rough idea of where the absolute minimum of Eq. (C. 4) is by following the procedure outlined in Section C. 2 for fairly large increments Δb and Δc. In this case, step (4) must be modified so that the computer will repeat steps (1) and (2) for a fixed value of b_ℓ (c_r) while scanning the c_r (b_ℓ) space. The result in step (5) will then correspond to a single point on the locus of relative minimia of Eq. (C. 4). This entire procedure must then be repeated for a number of the fixed b_ℓ . By properly spacing the b_ℓ it should be possible, after five or six runs, to get a rough idea of the shape of the locus. Associated with every point on this locus will be a numerical value for the error. By means of extrapolation this step can be repeated until the error gives the appearance of having passed through an absolute minimum.

(2) Fixing our attention on a region in the vicinity of the point of the apparant absolute minimum we can interpolate the results and then reprogram the computer so that it repeats the entire procedure outlined in (1) above for smaller increments in the b_ℓ and c_r . This procedure may have to be repeated several times, where in each case the Δb and Δc are made smaller. Of course, the number of times this is done is really dependent upon the desired accuracy.

A word of caution is necessary here to dispel any thoughts of skipping this step. It is possible that one may miss a very sharp absolute minimum by using the coarse scan in step (1) and that one would probably also miss it in this step if Δb and Δc were not chosen small enough. Examples as drastic as this are unlikely*; however, it is important to realize they can occur. In general, if one is not certain about the location of the absolute minimum, he should scan again.

Bellman[34] points out that the problem of distinguishing an absolute minimum from relative minima is one that plagues the optimization field. Computer results indicate that this problem usually arises in step (1) of the above procedure. If this does occur, step (2) must be repeated in the vicinity of each one of the relative minima. While this sounds like a great deal of work it certainly is far less than would be required had one scanned the entire two-dimensional b_ℓ - c_r domain in a haphazard way.

These results may be carried over en mass to the IT^nAE criteria; however, due to the fact that the IT^nAE criteria are only functions of Band C some of the computer operations in Section C. 2 must be modified accordingly.

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* See ref. 34 for a complete discussion.

APPENDIX D. TABULATION OF THE COEFFICIENTS C_k IN

$$f_a(t) = \sum_{k=0}^{m} C_k \phi_k(t).$$

Table D.1 $f(t) = e^{-2t} - e^{-4t}$, $f_a(t) = C_0 e^{-t}$, A. O. $(F_a) = 1$.

Error Criterion	Condition for determining C_0	C_0
IE	IE = 0	0.2500
ITE	ITE = 0	0.1875
IT^2E	$IT^2E = 0$	0.1092
$IT^{1/2}E$	$IT^{1/2}E = 0$	0.2070
IE^2	Orthonormalization	0.2660
ITE^2	Orthonormalization	0.2840
$IT^{1/2}E^2$	Orthonormalization	0.1829
AE	Digital computer solution of min AE C_0	0.15
IAE	Digital computer solution of min IAE C_0	0.28
ITAE	Digital computer solution of min ITAE C_0	0.19
$IT^{1/2}AE$	Digital computer solution of min $IT^{1/2}AE$ C_0	0.30

Table D.2 $f(t) = e^{-2t} - e^{-4t}$; $f_a(t) = C_0 e^{-3t}$, A. O. $(F_a) = 1$.

Error Criterion	Condition for determining C_0	C_0
IE	IE = 0	0.7500
ITE	ITE = 0	1.6850
IT^2E	$IT^2E = 0$	2.9500
$IT^{1/2}E$	$IT^{1/2}E = 0$	0.3580
IE^2	Orthonormalization	0.3432
ITE^2	Orthonormalization	0.7020
$IT^{1/2}E^2$	Orthonormalization	0.1683
AE	Digital computer solution of min AE C_0	0.19
IAE	Digital computer solution of min IAE C_0	0.60
ITAE	Digital computer solution of min ITAE C_0	1.20
$IT^{1/2}AE$	Digital computer solution of min $IT^{1/2}AE$ C_0	0.40

Table D.3 $f(t) = e^{-2t} - e^{-4t}$, $f_a(t) = C_0 e^{-5t}$, A. O. $(F_a) = 1$.

Error Criterion	Condition for determining C_0	C_0
IE	IE = 0	1.2500
ITE	ITE = 0	4.6800
IT^2E	$IT^2E = 0$	13.6800
$IT^{1/2}E$	$IT^{1/2}E = 0$	0.4650
IE^2	Orthonormalization	0.3200
ITE^2	Orthonormalization	0.8150
$IT^{1/2}E^2$	Orthonormalization	0.1413
AE	Digital computer solution of min AE C_0	0.21
IAE	Digital computer solution of min IAE C_0	0.60
ITAE	Digital computer solution of min ITAE C_0	1.20
$IT^{1/2}AE$	Digital computer solution of min $IT^{1/2}AE$ C_0	0.30

Table D.4 $f(t) = e^{-2t} - e^{-4t}$; $f_a(t) = C_0 e^{-t} + C_1 e^{-3t}$, A. O. $(F_a) = 1$.

Error Criterion	Conditions for determining C_0 and C_1	C_0	C_1
IE	IE = 0, $f_a(\frac{1}{2}) = f(\frac{1}{2})$	-1.0700	3.9700
ITE	ITE = 0, $f_a(\frac{1}{2}) = f(\frac{1}{2})$	0.1019	0.7710
IT^2E	$IT^2E = 0$, $f_a(\frac{1}{2}) = f(\frac{1}{2})$	0.1110	0.7460
$IT^{1/2}E$	$IT^{1/2}E = 0$, $f_a(\frac{1}{2}) = f(\frac{1}{2})$	0.6950	-0.8480
IE^2	Orthonormalization	0.3740	-0.2160
ITE^2	Orthonormalization	0.2454	0.1518
$IT^{1/2}E^2$	Orthonormalization	0.3350	-0.215
AE	Digital computer solution of min AE C_0, C_1	0.45	-0.10
IAE	Digital computer solution of min IAE C_0, C_1	0.25	0.10
ITAE	Digital computer solution of min ITAE C_0, C_1	0.10	0.30
$IT^{1/2}AE$	Digital computer solution of min $IT^{1/2}AE$ C_0, C_1	0.40	-0.20

Table D. 5 $f(t) = \epsilon^{-2t} - \epsilon^{-4t}$, $f_a(t) = C_0\epsilon^{-t} + C_1\epsilon^{-5t}$, A. O. $(F_a) = 1$.

Error Criterion	Conditions for determining C_0 and C_1	C_0	C_1
IE	$IE = 0$, $f_a(\frac{1}{2}) = f(\frac{1}{2})$	0.6666	-2.0800
ITE	$ITE = 0$, $f_a(\frac{1}{2}) = f(\frac{1}{2})$	0.1040	2.0800
IT^2E	$IT^2E = 0$, $f_a(\frac{1}{2}) = f(\frac{1}{2})$	0.0320	2.1700
$IT^{-1/2}E$	$IT^{-1/2}E = 0$, $f_a(\frac{1}{2}) = f(\frac{1}{2})$	0.4625	-0.5730
IE^2	Orthonormalization	2.7660	-7.5000
ITE^2	Orthonormalization	0.2810	-0.0256
$IT^{-1/2}E^2$	Orthonormalization	0.3958	-0.3680
AE	Digital computer solution of $\min_{C_0,C_1}$ AE	0.40	-0.40
IAE	Digital computer solution of $\min_{C_0,C_1}$ IAE	0.30	-0.10
ITAE	Digital computer solution of $\min_{C_0,C_1}$ ITAE	0.15	0.70
$IT^{-1/2}AE$	Digital computer solution of $\min_{C_0,C_1} IT^{-1/2}AE$	0.40	-0.30

Table D. 6 $f(t) = \epsilon^{-2t} - \epsilon^{-4t}$; $f_a(t) = C_0\epsilon^{-3t} + C_1\epsilon^{-5t}$, A. O. $(F_a) = 1$.

Error Criterion	Conditions for determining C_0 and C_1	C_0	C_1
IE	$IE = 0$, $f_a(\frac{1}{2}) = f(\frac{1}{2})$	1.5080	-1.2650
ITE	$ITE = 0$, $f_a(\frac{1}{2}) = f(\frac{1}{2})$	33.0900	-87.3000
IT^2E	$IT^2E = 0$, $f_a(\frac{1}{2}) = f(\frac{1}{2})$	4.2800	-8.8100
$IT^{-1/2}E$	$IT^{-1/2}E = 0$, $f_a(\frac{1}{2}) = f(\frac{1}{2})$	1.6670	-1.6950
IE^2	Orthonormalization	1.6930	-1.8000
ITE^2	Orthonormalization	1.9800	2.2700
$IT^{-1/2}E^2$	Orthonormalization	1.2470	-1.2440
AE	Digital computer solution of $\min_{C_0,C_1}$ AE	1.50	-1.50
IAE	Digital computer solution of $\min_{C_0,C_1}$ IAE	1.90	-2.20
ITAE	Digital computer solution of $\min_{C_0,C_1}$ ITAE	2.10	-2.50
$IT^{-1/2}AE$	Digital computer solution of $\min_{C_0,C_1} IT^{-1/2}AE$	1.80	-2.0

Table D. 7 $f(t) = \epsilon^{-2t} - \epsilon^{-4t}$; $f_a(t) = C_0\epsilon^{-t} + C_1\epsilon^{-3t} + C_2\epsilon^{-5t}$, A. O. $(F_a) = 1$.

Error Criterion	Conditions for determining C_0, C_1, and C_2	C_0	C_1	C_2
IE	$IE = 0$, $f_a(\frac{1}{2}) = f(\frac{1}{2})$, $f_a(t_m) = f_m(t_m)$*	0.1140	0.8940	-0.8070
ITE	$ITE = 0$, $f_a(\frac{1}{2}) = f(\frac{1}{2})$, $f_a(t_m) = f_m(t_m)$	0.1230	0.8510	-0.7580
IT^2E	$IT^2E = 0$, $f_a(\frac{1}{2}) = f(\frac{1}{2})$, $f_a(t_m) = f_m(t_m)$	0.0715	1.6300	-2.1100
$IT^{-1/2}E$	$IT^{-1/2}E = 0$, $f_a(\frac{1}{2}) = f(\frac{1}{2})$, $f_a(t_m) = f_m(t_m)$	0.2740	0.6860	-1.0460
IE^2	Orthonormalization	0.1940	0.8640	-1.0800
ITE^2	Orthonormalization	0.3245	-0.6842	1.1750
$IT^{-1/2}E^2$	Orthonormalization	0.1980	0.4022	-0.5203

* t_m is the time at which the maximum of $f(t)$ occurs. The value of $f(t)$ at t_m is designated $f_m(t_m)$.

Table D. 8 $f(t) = \epsilon^{-2t} - \epsilon^{-4t}$; $f_a(t) = C_0(\epsilon^{-t} - \epsilon^{-3t})$, A. O. $(F_a) = 2$.

Error Criterion	Condition for determining C_0	C_0
IE	$IE = 0$	0.3750
ITE	$ITE = 0$	0.2110
IT^2E	$IT^2E = 0$	0.1136
$IT^{-1/2}E$	$IT^{-1/2}E = 0$	0.4880
IE^2	Orthonormalization	0.4570
ITE^2	Orthonormalization	0.3290
$IT^{-1/2}E^2$	Orthonormalization	0.5220
AE	Digital computer solution of $\min_{C_0}$ AE	0.60
IAE	Digital computer solution of $\min_{C_0}$ IAE	0.40
ITAE	Digital computer solution of $\min_{C_0}$ ITAE	0.17
$IT^{-1/2}AE$	Digital computer solution of $\min_{C_0} IT^{-1/2}AE$	0.50

Table D. 9 $f(t) = \epsilon^{-2t} - \epsilon^{-4t}$; $f_a(t) = C_0 (\epsilon^{-t} - \epsilon^{-5t})$, A. O. $(F_a) = 2$.

Error Criterion	Condition for determining C_0	C_0
IE	IE = 0	0.3130
ITE	ITE = 0	0.1950
IT^2E	$IT^2E = 0$	0.1100
$IT^{-1/2}E$	$IT^{-1/2}E = 0$	0.3730
IE^2	Orthonormalization	0.3790
ITE^2	Orthonormalization	0.3040
$IT^{-1/2}E^2$	Orthonormalization	0.3976
AE	Digital computer solution of min AE C_0	0.40
IAE	Digital computer solution of min IAE C_0	0.35
ITAE	Digital computer solution of min ITAE C_0	0.18
$IT^{-1/2}AE$	Digital computer solution of min $IT^{-1/2}AE$ C_0	0.40

Table D. 10 $f(t) = \epsilon^{-2t} - \epsilon^{-4t}$; $f_a(t) = C_0(\epsilon^{-3t} - \epsilon^{-5t})$, A. O. $(F_a) = 2$.

Error Criterion	Condition for determining C_0	C_0
IE	IE = 0	1.8750
ITE	ITE = 0	2.6400
IT^2E	$IT^2E = 0$	3.7600
$IT^{-1/2}E$	$IT^{-1/2}E = 0$	1.5700
IE^2	Orthonormalization	1.5000
ITE^2	Orthonormalization	1.7390
$IT^{-1/2}E^2$	Orthonormalization	1.4380
AE	Digital computer solution of min AE C_0	1.50
IAE	Digital computer solution of min IAE C_0	1.50
ITAE	Digital computer solution of min ITAE C_0	2.00
$IT^{-1/2}AE$	Digital computer solution of min $IT^{-1/2}AE$ C_0	1.40

Table D. 11 $f(t) = \epsilon^{-2t} - \epsilon^{-4t}$; $f_a(t) = C_0(\epsilon^{-t} - \epsilon^{-3t}) + C_1(\epsilon^{-3t} - \epsilon^{-5t})$, A. O. $(F_a) = 2$.

Error Criterion	Conditions for determining C_0 and C_1	C_0	C_1
IE	IE = 0, $f_a(t_m) = f_m(t_m)$*	0.1500	1.1300
ITE	ITE = 0, $f_a(t_m) = f_m(t_m)$	0.1150	1.2000
IT^2E	$IT^2E = 0$, $f_a(t_m) = f_m(t_m)$	0.0745	1.2810
$IT^{-1/2}E$	$IT^{-1/2}E = 0$, $f_a(t_m) = f_m(t_m)$	0.1080	1.2150
IE^2	Orthonormalization	0.7170	-1.0400
ITE^2	Orthonormalization	0.3879	-0.3990
$IT^{-1/2}E^2$	Orthonormalization	0.1730	1.0970
AE	Digital computer solution of min AE C_0, C_1	0.25	0.85
IAE	Digital computer solution of min IAE C_0, C_1	0.20	1.05
ITAE	Digital computer solution of min ITAE C_0, C_1	0.20	1.10
$IT^{-1/2}AE$	Digital computer solution of min $IT^{-1/2}AE$ C_0, C_1	0.25	0.90

*t_m is the time at which the maximum of f(t) occurs. The value of f(t) at t_m is designated $f_m(t_m)$.

Table D. 12 $f(t) = \epsilon^{-2t} - \epsilon^{-4t}$; $f_a(t) = C_0(\epsilon^{-t} - \epsilon^{-3t}) + C_1(\epsilon^{-3t} - \epsilon^{-5t}) + C_2(\epsilon^{-5t} - \epsilon^{-7t})$; A. O. $(F_a) = 2$.

Error Criterion	Conditions for determining C_0, C_1, and C_2	C_0	C_1	C_2
IE	IE = 0, $f_a(t_m) = f_m(t_m)$ * , $f_a(1) = f(1)$	-3.2200	86.9700	-161.0000
ITE	ITE = 0, $f_a(t_m) = f_m(t_m)$, $f_a(1) = f(1)$	0.1025	2.6000	-4.5700
IT^2E	$IT^2E = 0$, $f_a(t_m) = f_m(t_m)$, $f_a(1) = f(1)$	0.2590	0.1735	-2.8600
$IT^{-1/2}E$	$IT^{-1/2}E = 0$, $f_a(t_m) = f_m(t_m)$, $f_a(1) = f(1)$	-0.0100	5.4000	-9.8000
IE^2	Orthonormalization	0.6890	-0.7600	0.4200
ITE^2	Orthonormalization	0.2867	1.2317	3.1900
$IT^{-1/2}E^2$	Orthonormalization	-0.1920	3.1980	2.0720

* t_m is the time at which the maximum of f(t) occurs. The value of f(t) at t_m is designated $f_m(t_m)$.

Table D. 13 $f(t) = \epsilon^{-2t} - \epsilon^{-4t}$; $f_a(t) = C_0(\epsilon^{-t} - 2\epsilon^{-3t} + \epsilon^{-5t})$, A. O. $(F_a) = 3$.

Error Criterion	Condition for determining C_0	C_0
IE	$IE = 0$	0.4710
ITE	$ITE = 0$	0.2290
IT^2E	$IT^2E = 0$	0.1162
$IT^{-1/2}E$	$IT^{-1/2}E = 0$	0.7010
IE^2	Orthonormalization	0.5000
ITE^2	Orthonormalization	0.3790
$IT^{-1/2}E^2$	Orthonormalization	0.5890
AE	Digital computer solution of min AE C_0	0.90
IAE	Digital computer solution of min IAE C_0	0.30
ITAE	Digital computer solution of min ITAE C_0	0.20
$IT^{-1/2}AE$	Digital computer solution of min $IT^{-1/2}AE$ C_0	0.45

Table D. 14 $f(t) = \epsilon^{-2t} - \epsilon^{-4t}$; $f_a(t) = C_0(\epsilon^{-t} - 2\epsilon^{-3t} + \epsilon^{-5t}) + C_1(\epsilon^{-3t} - 2\epsilon^{-5t} + \epsilon^{-7t})$, A. O. $(F_a) = 3$.

Error Criterion	Conditions for determining C_0 and C_1	C_0	C_1
IE	$IE = 0$, $f_a(t_m) = f_m(t_m)$*	0.9350	1.0000
ITE	$ITE = 0$, $f_a(t_m) = f_m(t_m)$	0.0250	2.8100
IT^2E	$IT^2E = 0$, $f_a(t_m) = f_m(t_m)$	0.0500	2.7500
$IT^{-1/2}E$	$IT^{-1/2}E = 0$, $f_a(t_m) = f_m(t_m)$	0.3830	2.0900
IE^2	Orthonormalization	0.9480	-2.6800
ITE^2	Orthonormalization	0.8230	-0.1025
$IT^{-1/2}E^2$	Orthonormalization	0.0660	2.3170
AE	Digital computer solution of min AE C_0, C_1	-0.20	3.90
IAE	Digital computer solution of min IAE C_0, C_1	0.10	2.40
ITAE	Digital computer solution of min ITAE C_0, C_1	0.10	2.40
$IT^{-1/2}AE$	Digital computer solution of min $IT^{-1/2}AE$ C_0, C_1	1.00	2.50

* t_m is the time at which the maximum of $f(t)$ occurs. The value of $f(t)$ at t_m is designated $f_m(t_m)$.

Table D. 15 $f(t) = \epsilon^{-2t} - \epsilon^{-4t}$; $f_a(t) = C_0(\epsilon^{-t} - 2\epsilon^{-3t} + \epsilon^{-5t}) + C_1(\epsilon^{-3t} - 2\epsilon^{-5t} + \epsilon^{-7t}) + C_2(\epsilon^{-5t} - 2\epsilon^{-7t} + \epsilon^{-9t})$, A. O. $(F_a) = 3$.

Error Criterion	Conditions for determining C_0, C_1, and C_2	C_0	C_1	C_2
IE	$IE = 0$, $f_a(t_m) = f_m(t_m)$*, $f_a(1) = f(1)$	0.1180	2.0880	1.0500
ITE	$ITE = 0$, $f_a(t_m) = f_m(t_m)$, $f_a(1) = f(1)$	0.0580	2.8160	-0.5640
IT^2E	$IT^2E = 0$, $f_a(t_m) = f_m(t_m)$, $f_a(1) = f(1)$	0.2448	-0.5620	0.0417
$IT^{-1/2}E$	$IT^{-1/2}E = 0$, $f_a(t_m) = f_m(t_m)$, $f_a(1) = f(1)$	0.2650	0.7500	3.1900
IE^2	Orthonormalization	1.1800	-5.9700	6.5500
ITE^2	Orthonormalization	0.9288	-1.5444	4.5300
$IT^{-1/2}E^2$	Orthonormalization	-0.5910	4.3120	2.8980

* t_m is the time at which the maximum of $f(t)$ occurs. The value of $f(t)$ at t_m is designated $f_m(t_m)$.

Table D. 16 $f(t) = \epsilon^{-2t} + \epsilon^{-4t}$; $f_a(t) = C_0\epsilon^{-t}$, A. O. $(F_a) = 1$.

Error Criterion	Condition for determining C_0	C_0
IE	$IE = 0$	0.7500
ITE	$ITE = 0$	0.3130
IT^2E	$IT^2E = 0$	0.1405
$IT^{-1/2}E$	$IT^{-1/2}E = 0$	1.2070
IE^2	Orthonormalization	1.0660
ITE^2	Orthonormalization	0.6040
$IT^{-1/2}E^2$	Orthonormalization	1.4450
AE	Digital computer solution of min AE C_0	1.60
IAE	Digital computer solution of min IAE C_0	0.60
ITAE	Digital computer solution of min ITAE C_0	0.21
$IT^{-1/2}AE$	Digital computer solution of min $IT^{-1/2}AE$ C_0	1.00

Table D.17 $f(t) = e^{-2t} + e^{-4t}$; $f_a(t) = C_0 e^{-3t}$, A. O. $(F_a) = 1$.

Error Criterion	Condition for determining C_0	C_0
IE	$IE = 0$	2.2500
ITE	$ITE = 0$	2.8100
IT^2E	$IT^2E = 0$	3.8000
$IT^{-1/2}E$	$IT^{-1/2}E = 0$	2.0900
IE^2	Orthonormalization	2.0500
ITE^2	Orthonormalization	2.1780
$IT^{-1/2}E^2$	Orthonormalization	2.0100
AE	Digital computer solution of $\min_{C_0} AE$	2.00
IAE	Digital computer solution of $\min_{C_0} IAE$	2.10
ITAE	Digital computer solution of $\min_{C_0} ITAE$	2.30
$IT^{-1/2}AE$	Digital computer solution of $\min_{C_0} IT^{-1/2}AE$	2.05

Table D.1b $f(t) = e^{-2t} + e^{-4t}$, $f_a(t) = C_0 e^{-5t}$, A. O. $(F_a) = 1$.

Error Criterion	Condition for determining C_0	C_0
IE	$IE = 0$	3.7500
ITE	$ITE = 0$	7.8100
IT^2E	$IT^2E = 0$	17.5600
$IT^{-1/2}E$	$IT^{-1/2}E = 0$	2.7200
IE^2	Orthonormalization	2.5400
ITE^2	Orthonormalization	3.2850
$IT^{-1/2}E^2$	Orthonormalization	2.2510
AE	Digital computer solution of $\min_{C_0} AE$	2.20
IAE	Digital computer solution of $\min_{C_0} IAE$	3.00
ITAE	Digital computer solution of $\min_{C_0} ITAE$	3.90
$IT^{-1/2}AE$	Digital computer solution of $\min_{C_0} IT^{-1/2}AE$	2.55

Table D.19 $f(t) = e^{-2t} + e^{-4t}$; $f_a(t) = C_0 e^{-t} + C_1 e^{-3t}$, A. O. $(F_a) = 1$.

Error Criterion	Conditions for determining C_0 and C_1	C_0	C_1
IE	$IE = 0$, $f_a(0) = f(0)$	0.1250	1.8750
ITE	$ITE = 0$, $f_a(0) = f(0)$	0.1000	1.9000
IT^2E	$IT^2E = 0$, $f_a(0) = f(0)$	0.0700	1.9300
$IT^{-1/2}E$	$IT^{-1/2}E = 0$, $f_a(0) = f(0)$	0.1250	1.8750
IE^2	Orthonormalization	0.1520	1.8300
ITE^2	Orthonormalization	0.1890	1.8600
$IT^{-1/2}E^2$	Orthonormalization	0.2360	1.7120
AE	Digital computer solution of $\min_{C_0, C_1} AE$	0.10	1.90
IAE	Digital computer solution of $\min_{C_0, C_1} IAE$	0.15	1.80
ITAE	Digital computer solution of $\min_{C_0, C_1} ITAE$	0.10	2.00
$IT^{-1/2}AE$	Digital computer solution of $\min_{C_0, C_1} IT^{-1/2}AE$	0.15	1.80

Table D.20 $f(t) = e^{-2t} + e^{-4t}$; $f_a(t) = C_0 e^{-t} + C_1 e^{-5t}$, A. O. $(F_a) = 1$.

Error Criterion	Conditions for determining C_0 and C_1	C_0	C_1
IE	$IE = 0$, $f_a(0) = f(0)$	0.4370	1.5650
ITE	$ITE = 0$, $f_a(0) = f(0)$	0.2400	1.7600
IT^2E	$IT^2E = 0$, $f_a(0) = f(0)$	0.1250	1.8750
$IT^{-1/2}E$	$IT^{-1/2}E = 0$, $f_a(0) = f(0)$	0.5700	1.4300
IE^2	Orthonormalization	0.4940	1.7150
ITE^2	Orthonormalization	0.3480	2.3000
$IT^{-1/2}E^2$	Orthonormalization	0.6144	1.4370
AE	Digital computer solution of $\min_{C_0, C_1} AE$	0.50	1.50
IAE	Digital computer solution of $\min_{C_0, C_1} IAE$	0.35	2.30
ITAE	Digital computer solution of $\min_{C_0, C_1} ITAE$	0.15	3.40
$IT^{-1/2}AE$	Digital computer solution of $\min_{C_0, C_1} IT^{-1/2}AE$	0.45	1.80

Table D. 21 $f(t) = e^{-2t} - e^{-4t}$; $f_a(t) = C_0 e^{-3t} + C_1 e^{-5t}$, A. O. $(F_a) = 1$.

Error Criterion	Conditions for determining C_0 and C_1	C_0	C_1
IE	$IE = 0$, $f_a(0) = f(0)$	2.6250	-0.6250
ITE	$ITE = 0$, $f_a(0) = f(0)$	3.2700	-1.2700
IT^2E	$IT^2E = 0$, $f_a(0) = f(0)$	4.2800	-2.2800
$IT^{-1/2}E$	$IT^{-1/2}E = 0$, $f_a(0) = f(0)$	2.3900	-0.3950
IE^2	Orthonormalization	2.4310	-0.5080
ITE^2	Orthonormalization	2.7540	-1.0200
$IT^{-1/2}E^2$	Orthonormalization	2.2910	-0.3240
AE	Digital computer solution of $\min_{C_0, C_1} AE$	2.30	0.30
IAE	Digital computer solution of $\min_{C_0, C_1} IAE$	2.60	-0.80
ITAE	Digital computer solution of $\min_{C_0, C_1} ITAE$	2.75	-1.00
$IT^{-1/2}AE$	Digital computer solution of $\min_{C_0, C_1} IT^{-1/2}AE$	2.50	-0.60

Table D. 22 $f(t) = e^{-2t} + e^{-4t}$; $f_a(t) = C_0 e^{-t} + C_1 e^{-3t} + C_2 e^{-5t}$, A. O. $(F_a) = 1$.

Error Criterion	Conditions for determining C_0, C_1, and C_2	C_0	C_1	C_2
IE	$IE = 0$, $f_a(0) = f(0)$, $f_a(1) = f(1)$	-20.0310	21.4000	0.6310
ITE	$ITE = 0$, $f_a(0) = f(0)$ $f_a(1) = f(1)$	-0.0932	3.1200	-1.0270
IT^2E	$IT^2E = 0$, $f_a(0) = f(0)$, $f_a(1) = f(1)$	4.8750	-38.1750	35.3000
$IT^{-1/2}E$	$IT^{-1/2}E = 0$, $f_a(0) = f(0)$, $f_a(1) = f(1)$	0.1960	1.5750	0.2290
IE^2	Orthonormalization	0.1520	1.8300	0
ITE^2	Orthonormalization	0.1802	1.7668	0.1308
$IT^{-1/2}E^2$	Orthonormalization	0.2910	1.4650	0.2080

APPENDIX E. DATA EVALUATION RESULTS

The results tabulated in this appendix supplement the discussion of "data evaluation" presented in Section 4. 3. Tables E. 1-E. 9 are the basis for the summary figures, Figures4. 3. 7 - 4. 3. 18, from which the results in Section 4. 4 are obtained.

Tables E. 1 - E. 8 are based upon only the one-and two-term approximations tabulated in Appendix D, while Table E. 9 is based upon the three-term approximations listed in Appendix D which were calculated for only seven of the eleven error criteria. More specifically, Tables E. 1 and E. 5 are based upon the six one-and two-term approximations listed in Tables D. 1 - D. 6; Tables E. 2 and E. 6 are based upon the four one-and two-term approximations listed in Tables D. 8 - D. 11; Tables E. 3 and E. 7 are based upon the two one-and two-term approximations listed in Tables D. 13 and D. 14; Tables E. 4 and E. 8 are based upon the six one-and two-term approximations listed in Tables D. 16 - D. 21, and; Table E. 9 is based upon the three-term approximations listed in Tables D. 7, D. 12, D. 15, and D. 22.

Strictly speaking, the sum of the numbers, for each family of error criteria, in each row of Tables E. 1, E. 2, E. 3, and E. 4 should be 6, 4, 2, and 6 respectively; the sum of the numbers in each row of Tables E. 5, E. 6, E. 7, and E. 8 should be 6, 4, 2, and 6 respectively, and; the sum of the numbers in each row of Table E. 9 should be 4. These numbers correspond to the number of approximations used in the formation of Tables E. 1 - E. 9, as discussed in the preceding paragraph. There are many places in these tables, however, where the row sums either exceed or are less than the expected total (which is listed in a footnote at the bottom of each table). The excess occurred when more than one error criterion led to an optimum approximation, in the judgement of the author. In these cases, the error criterion corresponding to each one of the optimum approximations each received a full point; thus, it was possible, in this way, for the actual row sums to exceed the expected total. The defecit occurred when, in the opinion of the author, no error criterion led to an optimum approximation useful from an engineering point of view. This does not mean that an optimum approximation, based upon a comparison of a number of approximations with the actual function, did not exist. It means that, in the opinion of the author, the optimum approximation was too poor to be considered "optimum" from an engineering point of view. Thus, it was possible, in this way, for the actual row sums to be less than the expected total.

As can be seen from the preceding discussion, many human factors such as judgement, opinion, and even bias and prejudice enter into the evaluation of data where "optimum" results are sought. The results presented in Section 4. 4 naturally include

the various human factors that entered into the author' s evaluation of the data. These results should be accepted bearing the preceding discussion in mind. To be more conclusive, this entire problem should be repeated by a group of randomly chosen investigators, in order to average out the effects of judgement, opinion, bias, and prejudice.

Table E.1 Frequency with which an error criterion leads to approximations of best performance between $f_a(t)$ and $f(t)$. Sample-space restricted to families of error criteria (IT^nE, IT^nE^2, and AE-IT^nAE families). $f(t) = \epsilon^{-2t} - \epsilon^{-4t}$; A.O. ($F_a$) = 1; $f_a(t)$ restricted to one-and two-term approximations (Tables D.1 - D.6).

	IT^nE family				IT^nE^2 family			AE - IT^nAE family			
Frequency with which an error criterion leads to approximations of:	IE	ITE	IT^2E	$IT^{-1/2}E$	IE^2	ITE^2	$IT^{-1/2}E^2$	AE	IAE	ITAE	$IT^{-1/2}AE$
Minimum $E_M(j\omega)$ (1)	0*	0	0	4	0	0	4	1	0	0	2
Minimum $E\phi(j\omega)$ (2)	1	1	1	1	2	0	3	3	0	0	1
Best absolute-stability behavior, $K \geq 0$ (3)	2	2	2	1	1	1	3	3	0	0	1
Best absolute-stability behavior, $K \leq 0$ (4)	0	0	0	4	1	1	3	3	0	0	2
Best relative-stability behavior, $K \geq 0$ (5)	1	0	0	3	3	0	1	2	0	0	1
Best fit between $f_a(t)$ and $f(t)$ (6)	0	0	1	4	1	2	2	0	0	1	2
Best combination of (1) and (2)	0	0	0	1	1	0	4	1	0	0	2
Best combination of (3) and (4)	1	2	2	1	0	1	3	3	0	0	1
Best combination of (6) and (1)	0	0	0	4	0	0	2	0	0	0	2
Best combination of (6) and (2)	0	0	1	1	1	0	2	0	0	0	0
Best combination of (6) and (5)	0	0	0	3	1	0	1	0	0	0	1
Best combination of (6), (1) and (2)	0	0	0	1	1	0	2	0	0	0	2
Best combination of (6), (5), (1), and (2)	0	0	0	1	1	0	1	0	0	0	1

* Each one of the numbers listed in this table is a number "out of a maximum total of six".

Table E.2 Frequency with which an error criterion leads to approximations of best performance between $f_a(t)$ and $f(t)$. Sample-space restricted to families of error criteria (IT^nE, IT^nE^2, and AE - IT^nAE families). $f(t) = \epsilon^{-2t} - \epsilon^{-4t}$; A.O. ($F_a$) = 2; $f_a(t)$ restricted to one-and two-term approximations (Tables D.8 - D.11).

	IT^nE family				IT^nE^2 family			AE - IT^nAE family			
Frequency with which an error criterion leads to approximations of:	IE	ITE	IT^2E	$IT^{-1/2}E$	IE^2	ITE^2	$IT^{-1/2}E^2$	AE	IAE	$ITAE$	$IT^{-1/2}AE$
Minimum $E_M(j\omega)$ (1)	1*	1	0	3	1	1	3	1	0	0	3
Minimum $E\phi(j\omega)$ (2)	2	0	0	0	0	0	2	0	0	1	0
Best absolute-stability behavior, $K \geq 0$ (3)	1	1	1	2	0	0	2	1	1	1	1
Best absolute-stability behavior, $K \leq 0$ (4)	1	1	2	2	0	0	2	1	1	1	1
Best relative-stability behavior, $K \geq 0$ (5)	0	2	0	4	2	1	2	4	1	0	0
Best fit between $f_a(t)$ and $f(t)$ (6)	2	0	0	3	0	1	4	2	0	1	2
Best combination of (1) and (2)	2	0	0	3	1	0	3	1	0	0	3
Best combination of (3) and (4)	1	1	1	2	0	0	2	1	1	1	1
Best combination of (6) and (1)	1	0	0	3	0	0	2	1	0	0	1
Best combination of (6) and (2)	2	0	0	0	0	0	2	0	0	0	0
Best combination of (6) and (5)	0	0	0	3	0	0	2	2	0	0	0
Best combination of (6), (1) and (2)	2	0	0	3	0	0	2	1	0	0	3
Best combination of (6), (5), (1), and (2)	0	0	0	3	0	0	1	0	0	0	0

* Each one of the numbers listed in this table is a number "out of a maximum total of four".

Table E. 3 Frequency with which an error criterion leads to approximations of best performance between $f_a(t)$ and $f(t)$. Sample-space restricted to families of error criteria (IT^nE, IT^nE^2, and AE - IT^nAE families). $f(t) = \epsilon^{-2t} - \epsilon^{-4t}$; A.O. $(F_a) = 3$; $f_a(t)$ restricted to one-and two-term approximations (Tables D.13 and D.14).

	$IT^n E$ family				IT^nE^2 family			AE - IT^nAE family			
Frequency with which an error criterion leads to approximations of:	IE	ITE	IT^2E	$IT^{-1/2}E$	IE^2	ITE^2	$IT^{-1/2}E^2$	AE	IAE	ITAE	$IT^{-1/2}AE$
Minimum $E_M(j\omega)$ (1)	0*	0	1	2	1	1	1	0	1	1	1
Minimum $E\phi(j\omega)$ (2)	0	1	0	1	0	0	2	0	1	1	0
Best absolute-stability behavior, $K \geq 0$ (3)	0	2	1	0	0	1	1	1	0	0	0
Best absolute-stability behavior, $K \leq 0$ (4)	0	0	1	1	0	1	1	0	1	1	1
Best relative-stability behavior, $K \geq 0$ (5)	2	0	0	1	0	0	2	1	0	0	2
Best fit between $f_a(t)$ and $f(t)$ (6)	0	0	1	2	0	1	2	1	1	1	0
Best combination of (1) and (2)	0	1	0	2	1	1	1	0	1	1	1
Best combination of (3) and (4)	0	1	0	1	0	1	1	0	1	1	1
Best combination of (6) and (1)	0	0	1	2	0	1	1	0	1	1	0
Best combination of (6) and (2)	0	0	0	1	0	0	1	0	1	1	0
Best combination of (6) and (5)	0	0	0	1	0	0	2	1	0	0	0
Best combination of (6), (1) and (2)	0	0	0	2	0	1	1	0	1	1	0
Best combination of (6), (5), (1), and (2)	0	0	0	1	0	0	1	0	0	0	0

* Each one of the numbers listed in this table is a number "out of a maximum total of two".

Table E.4 Frequency with which an error criterion leads to approximations of best performance between $f_a(t)$ and $f(t)$. Sample-space restricted to families of error criteria (IT^nE, IT^nE^2, and AE - IT^nAE families). $f(t) = \epsilon^{-2t} + \epsilon^{-4t}$; A.O. $(F_a) = 1$; $f_a(t)$ restricted to one-and two-term approximations (Tables D.16 - D.21).

	IT^nE family				IT^nE^2 family			AE - IT^nAE family			
Frequency with which an error criterion leads to approximations of:	IE	ITE	IT^2E	$IT^{-1/2}E$	IE^2	ITE^2	$IT^{-1/2}E^2$	AE	IAE	ITAE	$IT^{-1/2}AE$
Minimum $E_M(j\omega)$ (1)	1*	0	0	3	2	0	2	0	1	0	3
Minimum $E\phi(j\omega)$ (2)	2	0	0	2	3	0	1	2	1	0	1
Best absolute-stability behavior, $K \geq 0$ (3)	1	0	0	4	0	0	4	3	1	1	2
Best absolute-stability behavior, $K \leq 0$ (4)	1	1	0	2	0	2	3	1	2	1	2
Best relative-stability behavior, $K \geq 0$ (5)	1	0	1	3	0	1	3	2	0	1	0
Best fit between $f_a(t)$ and $f(t)$ (6)	1	0	0	4	3	1	0	1	1	0	2
Best combination of (1) and (2)	2	0	0	2	3	0	1	1	1	0	3
Best combination of (3) and (4)	1	1	0	3	0	1	4	2	2	1	3
Best combination of (6) and (1)	1	0	0	3	2	0	1	0	1	0	2
Best combination of (6) and (2)	1	0	0	2	2	0	0	1	1	0	1
Best combination of (6) and (5)	0	0	0	3	0	0	1	1	0	0	0
Best combination of (6), (1) and (2)	1	0	0	2	2	0	0	1	1	0	2
Best combination of (6), (5), (1), and (2)	0	0	0	2	0	0	0	1	0	0	0

* Each one of the numbers listed in this table is a number "out of a maximum total of six".

Table E. 5 Frequency with which an error criterion leads to approximations of best performance between $f_a(t)$ and $f(t)$. Sample-space includes all eleven error criteria. $f(t) = \epsilon^{-2t} - \epsilon^{-4t}$; A. O. $(F_a) = 1$; $f_a(t)$ restricted to one-and two-term approximations (Tables D.1 - D.6).

Frequency with which an error criterion leads to approximations of:	IE	ITE	IT^2E	$IT^{-1/2}E$	IE^2	ITE^2	$IT^{-1/2}E^2$	AE	IAE	ITAE	$IT^{-1/2}AE$
Minimum $E_M(j\omega)$ (1)	0*	0	0	2	0	0	1	0	0	0	0
Minimum $E_\phi(j\omega)$ (2)	0	0	0	0	0	0	1	3	0	0	0
Best absolute-stability behavior, $K \geq 0$ (3)	0	0	0	0	1	0	3	3	0	0	2
Best absolute-stability behavior, $K \leq 0$ (4)	0	0	0	3	0	0	3	3	0	0	1
Best relative-stability behavior, $K \geq 0$ (5)	1	0	0	1	0	0	0	1	0	0	0
Best fit between $f_a(t)$ and $f(t)$ (6)	0	0	0	1	0	0	0	0	0	0	2
Best combination of (1) and (2)	0	0	0	0	0	0	2	0	0	0	1
Best combination of (3) and (4)	0	0	0	0	0	0	3	3	0	0	1
Best combination of (6) and (1)	0	0	0	1	0	0	0	0	0	0	2
Best combination of (6) and (2)	0	0	1	0	0	0	2	0	0	0	0
Best combination of (6) and (5)	0	0	0	1	0	0	0	0	0	0	0
Best combination of (6), (1) and (2)	0	0	0	0	0	0	2	0	0	0	1
Best combination of (6), (5), (1), and (2)	0	0	0	0	0	0	0	0	0	0	0

* Each one of the numbers listed in this table is a number "out of a maximum total of six".

Table E.6 Frequency with which an error criterion leads to approximations of best performance between $f_a(t)$ and $f(t)$. Sample-space includes all eleven error criteria. $f(t) = \epsilon^{-2t} - \epsilon^{-4t}$; A.O. $(F_a) = 2$; $f_a(t)$ restricted to one-and two-term approximations (Tables D.8 - D.11).

Frequency with which an error criterion leads to approximations of:	IE	ITE	IT^2E	$IT^{-1/2}E$	IE^2	ITE^2	$IT^{-1/2}E^2$	AE	IAE	ITAE	$IT^{-1/2}AE$
Minimum $E_M(j\omega)$ (1)	0*	0	0	2	0	0	0	0	0	0	2
Minimum $E\phi(j\omega)$ (2)	1	0	0	0	0	0	1	0	0	1	0
Best absolute-stability behavior, $K \geq 0$ (3)	1	1	1	1	0	0	1	1	1	1	1
Best absolute-stability behavior, $K \leq 0$ (4)	1	1	1	1	0	0	1	1	1	1	1
Best relative-stability behavior, $K \geq 0$ (5)	0	1	0	2	0	0	1	2	0	0	0
Best fit between $f_a(t)$ and $f(t)$ (6)	0	0	0	1	0	1	1	2	0	0	2
Best combination of (1) and (2)	1	0	0	2	0	0	0	0	0	0	1
Best combination of (3) and (4)	1	1	1	1	0	0	1	1	1	1	1
Best combination of (6) and (1)	0	0	0	1	0	0	1	0	0	0	1
Best combination of (6) and (2)	0	0	0	0	0	0	0	0	0	0	0
Best combination of (6) and (5)	0	0	0	1	0	0	0	2	0	0	0
Best combination of (6), (1) and (2)	0	0	0	1	0	0	1	0	0	0	1
Best combination of (6), (5), (1), and (2)	0	0	0	0	0	0	0	0	0	0	0

* Each one of the numbers listed in this table is a number "out of a maximum total of four".

Table E.7 Frequency with which an error criterion leads to approximations of best performance between $f_a(t)$ and $f(t)$. Sample-space includes all eleven error criteria. $f(t) = \epsilon^{-2t} - \epsilon^{-4t}$; A.O. $(F_a) = 3$; $f_a(t)$ restricted to one-and two-term approximations (Tables D.13 and D.14).

Frequency with which an error criterion leads to approximations of:	IE	ITE	IT^2E	$IT^{-1/2}E$	IE^2	ITE^2	$IT^{-1/2}E^2$	AE	IAE	ITAE	$IT^{-1/2}AE$
Minimum $E_M(j\omega)$ (1)	0*	0	1	0	1	0	0	0	0	0	1
Minimum $E\phi(j\omega)$ (2)	0	1	0	0	0	0	1	0	1	1	0
Best absolute-stability behavior, $K \geq 0$ (3)	0	1	1	0	0	0	1	1	0	0	0
Best absolute-stability behavior, $K \leq 0$ (4)	0	0	0	1	0	1	1	0	1	1	1
Best relative-stability behavior, $K \geq 0$ (5)	0	0	0	1	0	0	0	0	0	0	1
Best fit between $f_a(t)$ and $f(t)$ (6)	0	0	0	0	0	0	0	1	1	1	0
Best combination of (1) and (2)	0	1	0	0	1	0	0	0	0	0	1
Best combination of (3) and (4)	0	0	0	0	0	0	1	0	1	1	1
Best combination of (6) and (1)	0	0	0	0	0	0	0	0	1	1	0
Best combination of (6) and (2)	0	0	0	0	0	0	0	0	1	1	0
Best combination of (6) and (5)	0	0	0	0	0	0	0	1	0	0	0
Best combination of (6), (1) and (2)	0	0	0	0	0	0	0	0	1	1	0
Best combination of (6), (5), (1), and (2)	0	0	0	0	0	0	0	0	0	0	0

* Each one of the numbers listed in this table is a number "out of a maximum total of two".

Table E. 8 Frequency with which an error criterion leads to approximations of best performance between $f_a(t)$ and $f(t)$. Sample-space includes all eleven error criteria. $f(t) = \epsilon^{-2t} + \epsilon^{-4t}$; A. O. $(F_a) = 1$; $f_a(t)$ restricted to one-and two-term approximations (Tables D.16 - D.21).

Frequency with which an error criterion leads to approximations of:	IE	ITE	IT^2E	$IT^{-1/2}E$	IE^2	ITE^2	$IT^{-1/2}E^2$	AE	IAE	ITAE	$IT^{-1/2}AE$
Minimum $E_M(j\omega)$ (1)	0*	0	0	2	1	0	1	0	0	0	1
Minimum $E\phi(j\omega)$ (2)	1	0	0	1	2	0	1	0	1	0	1
Best absolute-stability behavior, $K \geq 0$ (3)	1	1	1	2	1	1	2	3	1	1	2
Best absolute- stability behavior, $K \leq 0$ (4)	0	1	0	2	0	1	2	1	2	1	2
Best relative-stability behavior, $K \geq 0$ (5)	0	0	0	1	0	0	1	1	0	1	0
Best fit between $f_a(t)$ and $f(t)$ (6)	1	0	0	2	1	1	0	0	1	0	1
Best combination of (1) and (2)	0	0	0	1	2	0	1	0	0	0	0
Best combination of (3) and (4)	0	1	0	3	0	1	3	1	1	0	2
Best combination of (6) and (1)	0	0	0	2	1	0	0	0	0	0	1
Best combination of (6) and (2)	0	0	0	1	1	0	0	0	1	0	1
Best combination of (6) and (5)	0	0	0	1	0	0	0	0	0	0	0
Best combination of (6), (1) and (2)	0	0	0	1	1	0	0	0	0	0	0
Best combination of (6), (5), (1), and (2)	0	0	0	0	0	0	0	0	0	0	0

* Each one of the numbers listed in this table is a number "out of a maximum total of six".

Table E. 9 Frequency with which an error criterion leads to three-term approximations of best performance between $f_a(t)$ and $f(t)$. Sample-space includes seven error criteria. $f(t) = \epsilon^{-2t} - \epsilon^{-4t}$, A. O. (F_a) = 1, 2, and 3 (Tables D. 7, D. 12, and D. 15); $f(t) = \epsilon^{-2t} + \epsilon^{-4t}$, A. O. $(F_a)=1$ (Table D. 22).

Frequency with which an error criterion leads to approximations of:	IE	ITE	IT^2E	$IT^{-1/2}E$	IE^2	ITE^2	$IT^{-1/2}E^2$
Minimum $E_M(j\omega)$ (1)	0*	1	0	3	0	0	0
Minimum $E\phi(j\omega)$ (2)	1	0	0	1	2	0	0
Best absolute-stability behavior, $K \geq 0$ (3)	1	0	0	2	0	2	1
Best absolute-stability behavior, $K \leq 0$ (4)	1	0	2	1	1	1	2
Best relative-stability behavior, $K \geq 0$ (5)	1	0	0	1	0	1	1
Best fit between $f_a(t)$ and $f(t)$ (6)	1	0	0	3	1	0	0
Best combination of (1) and (2)	1	0	0	3	1	0	0
Best combination of (3) and (4)	1	0	0	2	0	1	1
Best combination of (6) and (1)	0	0	0	3	0	0	0
Best combination of (6) and (2)	1	0	0	2	1	0	0
Best combination of (6) and (5)	0	0	0	1	0	0	0
Best combination of (6), (1) and (2)	1	0	0	3	1	0	0
Best combination of (6), (5), (1), and (2)	0	0	0	1	0	0	0

* Each one of the numbers listed in this table is a number "out of the maximum total of four".

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