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A NUMERICAL INVERSION OF THE LAPLACE TRANSFORM

Richard Bellman, Robert Kalaba and Bernard Shiffman

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The RAND Corporation
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PREFACE

Part of the RAND research program consists of basic support studies in mathematics. In the present Memorandum, the authors investigate computer methods of solving systems of differential equations through the so-called Laplace-transformation technique.

Applications of this technique are important in connection with the use of invariant imbedding in the study of time-dependent transport processes.

SUMMARY

The usual analytic methods of inverting the Laplace transformation—such as the Bromwich-Wagner integral in the complex plane—are mostly impractical for numerical work. This Memorandum discusses a method applicable to the numerical analysis of the inverse Laplace transform and includes numerical examples illustrating this method.

CONTENTS

A NUMERICAL INVERSION OF THE LAPLACE TRANSFORM

1. INTRODUCTION

When one is interested in obtaining numerical solutions of systems of differential equations, one often finds numerical integration techniques to be too time-consuming, even when employing high-speed electronic computers. It would then be desirable to be able to use other techniques for solving such a system of differential equations. The Laplace transformation immediately suggests itself, and its satisfactory use in numerical work depends on the numerical analyst's ability to recover the function $f(t)$, given its Laplace transformation $F(s)$.

The usual analytic methods of inverting the Laplace transformation—such as the Bromwich-Wagner integral of $F(s)$ in the complex plane—are mostly impractical for numerical work. This paper discusses a method applicable to the numerical analysis of the inverse Laplace transform and includes numerical examples illustrating this method. We shall represent $f(t)$ by a series of orthogonal functions and then find the coefficients of this expansion from the values of $F(s)$ on a set of equidistant points on the positive real line. For examples of this technique applied to other sets of orthogonal functions, see Papoulis's original article [1]. (Further examples can be found in [3].)

2. PRELIMINARY DISCUSSION

Throughout this paper we shall assume that $f(t)$ is continuous on $[0,\infty)$ and approaches a finite limit as $t \to \infty$. Letting

$$(2.1) \qquad F(s) = \mathrm{Lap}(f) = \int_0^\infty e^{-st} f(t)\,dt,$$

we can make the change of variables $r = e^{-t}$; then (2.1) can be written as

$$(2.2) \qquad F(s) = \int_0^1 r^{s-1} f(\ell n\ 1/r)\,dr.$$

The function $g(r)$, defined by

$$g(r) = f(\ell n\ 1/r),$$

is continuous on $[0,1]$, and we can expand $g(r)$ in a series of polynomials that are complete and orthogonal on $[0,1]$. Let*

$$(2.3) \qquad g(r) \sim \sum_{k=0}^{\infty} c_k \Phi_k(r),$$

where $\{\Phi_k\}$ is the set of these complete orthogonal polynomials. If we write

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*The notation of equation (2.3) means that $\sum_{k=0}^{n} c_k \Phi_k \to g$ in $\mathcal{L}^2[0,1]$; that is,

$$\lim_{n\to\infty} \int_0^1 \left[\sum_{k=0}^{n} c_k \Phi_k(r) - g(r)\right]^2 dr = 0.$$

(2.4) $$\Phi_n(x) = \sum_{j=0}^{n} a_{jn} x^j,$$

and let

(2.5) $$A_n = \int_0^1 [\Phi_n(x)]^2 dx,$$

(Of course, one could pick an orthonormal set $\{\Phi_n\}$, but it is more convenient to use an unnormalized form, as we shall see later.), then we have

(2.6) $$\begin{aligned} c_k A_k &= \int_0^1 \Phi_k(r) g(r) dr \\ &= \sum_{j=0}^{k} a_{jk} \int_0^1 r^j f(\ln 1/r) dr \\ &= \sum_{j=0}^{k} a_{jk} F(j+1). \end{aligned}$$

Therefore, we can determine the coefficients c_k—and thus the function $f(t) = g(e^{-t})$—by knowing $F(s)$ only at the integral points $1, 2, \ldots$.

In general, by making the change of variable $r = e^{-\sigma t}$, where σ is any positive constant, we obtain

(2.7) $$F(k\sigma) = \frac{1}{\sigma} \int_0^1 r^{k-1} f(1/\sigma \ln 1/r) dr,$$

and by repeating the above procedure we can recover the function $f(t)$ by knowing its Laplace transform only at the points $\sigma, 2\sigma, \ldots$. While equations (2.6) and (2.7) motivate the use of orthogonal polynomials and demonstrate that $f(t)$ can be recovered by knowing its Laplace transform at only a sequence of equally spaced points on the real line, we shall not use equation (2.6) but instead shall make use of a more practical technique based on the orthogonal set $\{\Phi_n\}$. Equation (2.6) is clumsy for numerical work since the coefficients a_{jk} get very large in magnitude and oscillate in sign.

3. THE ORTHOGONAL POLYNOMIALS

The Legendre polynomials $P_n(x)$ are complete and orthogonal on $[-1, 1]$; that is, they satisfy

$$\int_{-1}^{1} P_j(x)P_k(x)dx = \frac{2}{2k+1}\,\delta_{jk}.$$

They are easily computed by the recursive relation

$$(3.1)\qquad (n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x),$$

where $P_0(x) = 1$, $P_1(x) = x$. The Legendre polynomials are described and tabulated in [4].

We then define the "shifted Legendre polynomials" $\Phi_n(x)$ by

$$(3.2)\qquad \Phi_n(x) = P_n(2x-1),$$

so that $\{\Phi_n\}$ is a complete orthogonal set on $[0,1]$, and

$$(3.3)\qquad A_k = \int_{0}^{1} \Phi_k(x)\Phi_k(x)dx = \frac{1}{2k+1}\,.$$

Furthermore, we have the recursive relation

$$(3.4) \qquad (n+1)\Phi_{n+1}(x) = (2n+1)(2x-1)\Phi_n(x) - n\Phi_{n-1}(x),$$

where $\Phi_0(x) = 1$, $\Phi_1(x) = 2x - 1$.

We now derive a useful property of the shifted Legendre polynomials, Φ_n. Define

$$(3.5) \qquad G_n(\sigma,s) = \int_0^\infty \Phi_n(e^{-\sigma t})e^{-st}dt = \mathrm{Lap}[\Phi_n(e^{-\sigma t})],$$

for $\sigma > 0$ and $n = 0,1,2,\ldots$. To derive an expression for G_n in closed form, we proceed as follows. From (2.4), we see that

$$(3.6) \qquad \Phi_n(e^{-\sigma t}) = \sum_{m=0}^{n} a_{mn} e^{-m\sigma t},$$

and hence

$$(3.7) \qquad G_n(\sigma,s) = \sum_{m=0}^{n} a_{mn}\, \mathrm{Lap}(e^{-m\sigma t}) = \sum_{m=0}^{n} \frac{a_{mn}}{s+m\sigma}$$

$$= \frac{Q_n}{s(s+\sigma)\cdots(s+n\sigma)},$$

where Q_n is a polynomial of degree $\leq n$.

Consider the case $n \geq 1$. (For $n = 0$, we calculate immediately $G_0(\sigma,s) = \mathrm{Lap}(1) = 1/s$.) By the change of variables $x = e^{-\sigma t}$, we get

$$(3.8) \qquad G_n(\sigma,k\sigma) = \frac{1}{\sigma}\int_0^1 x^{k-1}\Phi_n(x)dx.$$

For $1 \leq k \leq n$, x^{k-1} is a linear combination of $\Phi_0,\ldots,\Phi_{n-1}$; further, because of the orthogonality of $\{\Phi_n\}$ on

$[0,1]$, the integral in (3.8) vanishes. Thus we obtain

$$G_n(\sigma,k\sigma) = 0 \quad \text{for} \quad 1 \le k \le n,$$

and Q_n has zeros at $s = \sigma, 2\sigma, \ldots, n\sigma$. Hence, we have

$$Q_n = B_n(s-\sigma)(s-2\sigma)\cdots(s-n\sigma),$$

where B_n is some constant, and

$$(3.9) \qquad G_n(\sigma,s) = \frac{B_n}{s}\prod_{m=1}^{n}\frac{s-m\sigma}{s+m\sigma}.$$

From (3.9), it follows that

$$\lim_{s\to 0} sG_n(\sigma,s) = (-1)^n B_n,$$

and from (3.7) we have

$$\lim_{s\to 0} sG_n(\sigma,s) = a_{0n} = \Phi_n(0).$$

The recursive formula (3.4) implies that $\Phi_n(0) = (-1)^n$, and therefore $B_n = 1$, and we have our desired result:

$$(3.10) \qquad G_n(\sigma,s) = \frac{1}{s}\prod_{m=1}^{n}\frac{s-m\sigma}{s+m\sigma} \quad \text{for} \quad n = 1,2,\ldots,$$

$$G_0(\sigma,s) = \frac{1}{s}.$$

4. NUMERICAL INVERSION FORMULA

If

$$(4.1) \qquad f\left(-\frac{1}{\sigma}\ln r\right) \sim \sum_{n=0}^{\infty} c_n\Phi_n(r),$$

then by letting $r = e^{-\sigma t}$ in (4.1) and taking the Laplace transformation of the result, we would expect that

$$(4.2) \qquad F(k\sigma) = \sum_{n=0}^{k-1} c_n G_n(\sigma, k\sigma),$$

since $G_n(\sigma, k\sigma)$ vanishes for $k \leq n$. A rigorous justification of equation (4.2) is given in Sec. 7 of this Memorandum. From (3.10) we have

$$(4.3) \qquad G_n(\sigma, k\sigma) = \frac{1}{k\sigma} \cdot \frac{k-1}{k+1} \cdot \frac{k-2}{k+2} \cdot \; \cdots \; \frac{k-n}{k+n} \text{ for } n = 1, \ldots, k-1,$$

$$G_0(\sigma, k\sigma) = \frac{1}{k\sigma}.$$

Therefore, from (4.2), we obtain

$$(4.4) \qquad F(\sigma) = \frac{c_0}{\sigma},$$

$$F(k\sigma) = \frac{c_0}{k\sigma} + \sum_{i=1}^{k-1} \frac{c_i}{k\sigma} \frac{k-1}{k+1} \cdots \frac{k-i}{k+i} \text{ for } k = 2, 3, \ldots .$$

Equation (4.4) gives us the following formula for sequentially finding the coefficients c_n:

$$(4.5) \qquad c_0 = \sigma F(\sigma),$$

$$c_{n-1} = n\sigma F(n\sigma) \frac{n+1}{n-1} \cdot \frac{n+2}{n-2} \cdots \frac{2n-1}{1}$$

$$- \sum_{i=0}^{n-2} c_i \frac{n+(i+1)}{n-(i+1)} \cdot \frac{n+(i+2)}{n-(i+2)} \cdots \frac{2n-1}{1}$$

$$= n\sigma F(n\sigma) \binom{2n-1}{n} - \sum_{i=0}^{n-2} \binom{2n-1}{n+i} c_i .$$

Therefore, to determine the first n coefficients— $c_0, c_1, \ldots, c_{n-1}$—it is necessary to know the values of the Laplace transform only at the points

$$\sigma, 2\sigma, \ldots, n\sigma.$$

5. METHOD OF NUMERICAL INVERSION

When inverting the Laplace transform numerically, we wish to represent $f(t)$ by the sum $\Sigma\, c_n \Phi_n(e^{-\sigma t})$, and then determine the coefficients c_n using formula (4.5). One should first determine the optimum σ to use. Although in some practical problems it is often convenient to select σ before computing the Laplace transform at $\sigma, 2\sigma, \ldots,$ we shall assume that the Laplace transform we are using is easy to calculate at all points on the positive real line and that we have complete freedom in selecting σ. We shall consider the case in which we know that $f(t)$, in addition to being continuous and finite on $[0,\infty]$, has the following properties:

$$f(0) = 0,$$

$f(t)$ is monotonic on the positive real line.

In other words, $f(t)$ behaves very much like

$$\alpha(1 - e^{-\sigma t}),$$

where α and σ are constants, $\sigma > 0$.

Since we wish to approximate $f(t)$ with as few terms as possible in the $\Phi_n(e^{-\sigma t})$ series, $f(t)$ should be well represented as a linear combination of a few terms of the form $e^{-n\sigma t}$. Let us then write

(5.1) $f(t) = \alpha(1 - e^{-\sigma t}) + h(t)$,

where $h(t) \sim \alpha_2 e^{-2\sigma t} + \alpha_3 e^{-3\sigma t} + \cdots$.

We therefore wish to minimize the contribution of $h(t)$. Writing

(5.2) $F(s) = \frac{\alpha}{s} - \frac{\alpha}{s+\sigma} + H(s)$,

we shall minimize $H(s)$ for positive s. We therefore desire that

$$F(s) \approx \frac{\alpha}{s} - \frac{\alpha}{s+\sigma},$$

or

(5.3) $sF(s)(s+\sigma) \approx \alpha\sigma$.

Picking σ (and α) such that (5.3) is a best least-squares approximation for some set $s_1, s_2, \ldots, s_N$ of positive real points, we obtain

(5.4) $\sigma = \frac{AB - ND}{NC - A^2}$,

where

$$A = \sum_{i=1}^{N} s_i F(s_i), \quad B = \sum_{i=1}^{N} s_i^2 F(s_i),$$

$$C = \sum_{i=1}^{N} s_i^2 F^2(s_i), \quad D = \sum_{i=1}^{N} s_i^3 F^2(s_i).$$

(If we did not have the condition $f(0) = 0$, we would instead determine σ, α, α' such that

$$F(s) \approx \frac{\alpha}{s} + \frac{\alpha'}{s+\sigma} .)$$

After using (5.4) to obtain σ, we then can use (4.5) to determine the coefficients c_n in the representation

$$f(t) \sim \Sigma\, c_n \Phi_n(r), \quad \text{where} \quad t = -\frac{1}{\sigma} \ln r.$$

We use the recursive relation (3.4) to evaluate the Φ_n numerically.

Since $f(-\frac{1}{\sigma} \ln r)$ is continuous on the closed set $[0,1]$, it is known (see [5]) that, although $\Sigma\, c_n \Phi_n(r)$ may not converge for all r, the second mean of the partial sums of the series will always converge pointwise to the function; that is, letting

$$(5.5) \qquad s_n(t) = \sum_{k=0}^{n} c_k \Phi_k(e^{-\sigma t}),$$

$$\sigma_n^{(1)}(t) = \frac{1}{n+1} \sum_{k=0}^{n} s_n(t),$$

$$\sigma_n^{(2)}(t) = \frac{1}{n+1} \sum_{k=0}^{n} \sigma_n^{(1)}(t),$$

we have

$$(5.6) \qquad \lim_{n\to\infty} \sigma_n^{(2)}(t) = f(t)$$

for all t.

For numerical work, $\sigma_n^{(2)}$ converges too slowly for accurate determination of f. Rapid convergence is necessary, since it is difficult to determine the c_n accurately for large n. A good procedure is to consider the weighted mean,

$$(5.7) \qquad \hat{\sigma}_n(t) = \frac{\sum_{k=1}^{n} k s_k(t)}{\sum_{k=1}^{n} k} = \frac{2}{n(n+1)} \sum_{k=1}^{n} k s_k(t),$$

which will converge reasonably rapidly for most practical problems in which $f(t)$ is well behaved. The next section of this Memorandum includes examples of numerical computations of the inverse of the Laplace transformation using the method described here.

6. NUMERICAL EXAMPLES

In practical applications, the accuracy of the numerically obtained inverse of $F(s)$ is limited by the accuracy with which $F(s)$ is known. As can be seen from equation (4.5), the number of coefficients c_n that can be determined with reasonable accuracy depends on the accuracy with which the quantities $F(n\sigma)$ are known. The accuracy of the numerically obtained value of $f(t)$ will then depend on the rapidity of convergence of the sequence $\hat{\sigma}_n(t)$.

Our numerical examples will then be taken from results obtained from computation of the inverse of the

Laplace transform of known functions using the IBM-7090 computer. We consider the following functions to illustrate the results of our method:

$$(6.1)\quad F(s) = \frac{s + e^{-\pi/2\ s}}{s(s^2+1)}, \text{ for which } f(t) = \begin{cases} \sin t & \text{for } t \le \pi/2, \\ 1 & \text{for } t > \pi/2; \end{cases}$$

$$(6.2)\quad F(s) = \frac{1 - e^{-s}}{s^2}, \text{ for which } f(t) = \begin{cases} t & \text{for } t \le 1, \\ 1 & \text{for } t > 1. \end{cases}$$

We use the least-squares formula (5.4) at five points—$s_i = i$, $i = 1,\ldots,5$—to determine the σ used in equations (4.5). The $F(s)$ we used were computed on the IBM-7090 in two ways: (a) $F(s)$ was calculated in "single precision" arithmetic, which allows eight significant decimal digits accuracy; (b) $F(s)$ was calculated as above, and a random fractional error in the interval $[-10^{-5}, +10^{-5}]$ was introduced into the result to provide a final $F(s)$ accurate to only five significant decimal digits. In both of these cases, we then computed the sequence $\hat{\sigma}_n(t)$ using the full "single-precision" accuracy of the IBM-7090. By computing the inverses of several Laplace transforms, we empirically discovered that $\hat{\sigma}_8$ usually gives the best results when $F(s)$ is known to within eight significant decimal digits, and $\hat{\sigma}_5$ usually gives the best results for $F(s)$ known to five significant digits.

The functions (6.1) and (6.2) above test our inversion method when $f(t)$ has the desired properties of monotonicity and boundedness but does not resemble an exponential. Function (6.2) has the added difficulty of having a "sharp corner." For function (6.1) we numerically obtain $\sigma = 0.87826$, and for function (6.2) we get $\sigma = 0.84565$. Results of our numerical inversion of these functions are shown in Tables 1 and 2, and the second is exhibited graphically in Fig. 1.

7. A MATHEMATICAL DISCUSSION

The purpose of this section is to justify equation (4.2) on a rigorous basis. Again making the change of variables $r = e^{-\sigma t}$, let

$$g(r) = f(-\frac{1}{\sigma} \ln r),$$

so that (4.1) becomes

$$g(r) \sim \sum_{n=0}^{\infty} c_n \Phi_n(r).$$

Let $<\ ,\ >$ be the standard inner product in $\mathcal{L}^2[0,1]$; that is,

$$\text{(7.1)} \qquad <f_1,f_2> = \int_0^1 f_1(x) f_2(x) dx.$$

Hence, from the orthogonality of $\{\Phi_n\}$, we get

$$\text{(7.2)} \qquad c_n A_n = <\Phi_n, g>,$$

Table 1

INVERSION OF THE FUNCTION (6.1)

t	f(t)	$\hat{\sigma}_5^{(a)}$	$\hat{\sigma}_8^{(a)}$	$\hat{\sigma}_5^{(b)}$
0.1	0.09983	0.10018	0.09950	0.09982
0.2	0.19867	0.20070	0.20007	0.20044
0.3	0.29552	0.29985	0.29787	0.29995
0.4	0.38942	0.39668	0.39198	0.39701
0.5	0.47943	0.48887	0.48219	0.48925
0.6	0.56464	0.57423	0.56779	0.57449
0.7	0.64422	0.65122	0.64746	0.65129
0.8	0.71736	0.71908	0.71966	0.71896
0.9	0.78333	0.77769	0.78318	0.77740
1.0	0.84147	0.82743	0.83736	0.82703
1.1	0.89121	0.86896	0.88218	0.86853
1.2	0.93204	0.90316	0.91815	0.90274
1.3	0.96356	0.93093	0.94616	0.93057
1.4	0.98545	0.95319	0.96732	0.95293
1.5	0.99749	0.97081	0.98277	0.97067
1.6	1.00000	0.98456	0.99364	0.98455
1.7	1.00000	0.99514	1.0010	0.99526
1.8	1.00000	1.0031	1.0056	1.0034
1.9	1.00000	1.0091	1.0082	1.0094
2.0	1.00000	1.0133	1.0095	1.0138
3.0	1.00000	1.0175	1.0041	1.0178
4.0	1.00000	1.0102	1.0037	1.0097
5.0	1.00000	1.0063	1.0052	1.0053
6.0	1.00000	1.0045	1.0062	1.0033
7.0	1.00000	1.0038	1.0067	1.0025
8.0	1.00000	1.0035	1.0069	1.0021
9.0	1.00000	1.0034	1.0070	1.0020
10.0	1.00000	1.0034	1.0070	1.0019
∞	1.00000	1.0033	1.0070	1.0019

(a) Results obtained with F(s) given to 8 significant digits.

(b) Results obtained with F(s) given to only 5 significant digits.

Table 2

INVERSION OF THE FUNCTION $F(s) = \frac{1 - e^{-s}}{s^2}$

t	f(t)	$\hat{\sigma}_5^{(a)}$	$\hat{\sigma}_8^{(a)}$	$\hat{\sigma}_5^{(b)}$
0.1	0.1	0.09503	0.09971	0.09527
0.2	0.2	0.19919	0.20221	0.19932
0.3	0.3	0.31035	0.30070	0.31023
0.4	0.4	0.42236	0.40346	0.42210
0.5	0.5	0.52943	0.51122	0.52918
0.6	0.6	0.62734	0.61827	0.62720
0.7	0.7	0.71354	0.71736	0.71357
0.8	0.8	0.78702	0.80294	0.78721
0.9	0.9	0.84784	0.87224	0.84815
1.0	1.0	0.89684	0.92498	0.89722
1.1	1.0	0.93526	0.96268	0.93565
1.2	1.0	0.96456	0.98779	0.96492
1.3	1.0	0.98624	1.0031	0.98652
1.4	1.0	1.0017	1.0112	1.0019
1.5	1.0	1.0122	1.0144	1.0123
1.6	1.0	1.0189	1.0146	1.0188
1.7	1.0	1.0227	1.0130	1.0224
1.8	1.0	1.0243	1.0107	1.0240
1.9	1.0	1.0244	1.0082	1.0240
2.0	1.0	1.0236	1.0060	1.0231
3.0	1.0	1.0074	1.0041	1.0070
4.0	1.0	1.0058	1.0060	1.0062
5.0	1.0	1.0087	1.0034	1.0098
6.0	1.0	1.0109	1.0009	1.0123
7.0	1.0	1.0120	0.99959	1.0136
8.0	1.0	1.0126	0.99895	1.0141
9.0	1.0	1.0128	0.99866	1.0144
10.0	1.0	1.0129	0.99854	1.0145
∞	1.0	1.0130	0.99844	1.0146

(a) Results obtained with F(s) given to 8 significant digits.

(b) Results obtained with F(s) given to only 5 significant digits.

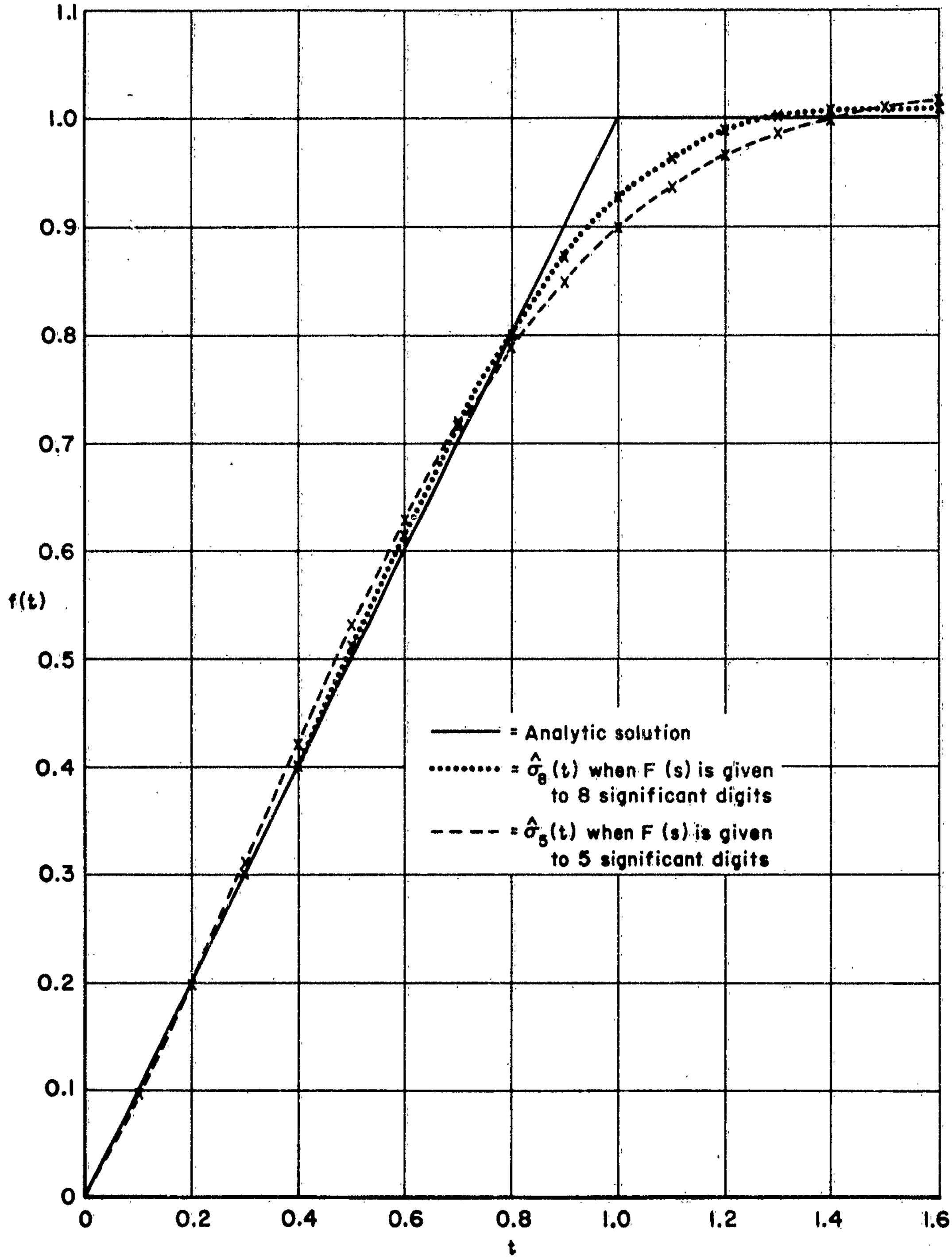

Fig. 1 — Inversion of the function $F(s) = \frac{1-e^{-s}}{s^2}$

recalling from equation (3.3) that

$$A_n = \langle \Phi_n, \Phi_n \rangle = \frac{1}{2n+1} .$$

Equation (3.8) takes the algebraic form

$$(7.3) \qquad \sigma G_n(\sigma, k\sigma) = \langle x^{k-1}, \Phi_n \rangle,$$

and (2.7) takes the form

$$(7.4) \qquad \sigma F(k\sigma) = \langle x^{k-1}, g \rangle.$$

Since $\{\Phi_n\}$ is complete and orthogonal in $\mathcal{L}^2[0,1]$, and since

$$G_n(\sigma, k\sigma) = 0 \quad \text{for} \quad n \geq k,$$

equation (7.3) implies that

$$(7.5) \qquad x^{k-1} = \sum_{n=0}^{k-1} \frac{\sigma G_n(\sigma, k\sigma)}{A_n} \Phi_n(x)$$

for $k = 1, 2, \ldots$.

Therefore, we have

$$F(k\sigma) = \frac{1}{\sigma} \langle \sum_{n=0}^{k-1} \frac{\sigma G_n(\sigma, k\sigma)}{A_n} \Phi_n, g \rangle$$

$$= \sum_{n=0}^{k-1} \frac{G_n(\sigma, k\sigma)}{A_n} \langle \Phi_n, g \rangle = \sum_{n=0}^{k-1} c_n G_n(\sigma, k\sigma),$$

which is our desired result.

Equations (7.3) and (7.5) yield an interesting result. We let

$$G_n(k) = G_n(1,k) = \sigma G_n(\sigma, k\sigma),$$

and (7.5) becomes

$$x^k = \sum_{n=0}^{k} \frac{G_n(k+1)}{A_n} \Phi_n(x).$$

We therefore have obtained the formula for the expansion of x^k in terms of the shifted Legendre polynomials:

(7.6) $$x^{k-1} = \frac{1}{k}\Phi_0(x) + \frac{3}{k}\cdot\frac{k-1}{k+1}\Phi_1(x) + \frac{5}{k}\cdot\frac{k-1}{k+1}\cdot\frac{k-2}{k+2}\Phi_2(x) + \cdots + \frac{2n+1}{k}\cdot\frac{k-1}{k+1}\cdot\frac{k-2}{k+2}\cdots\frac{k-n}{k+n}\Phi_n(x) + \cdots + \frac{2k-1}{k}\cdot\frac{k-1}{k+1}\cdot\frac{k-2}{k+2}\cdots\frac{1}{2k-1}\Phi_{k-1}(x).$$

The functions G_n are related in the following way to the coefficients a_{nm} introduced in (2.4). Letting

$$b_{nm} = \frac{G_n(m+1)}{A_n},$$

so that the quantities b_{nm} are the coefficients in (7.6), we have the following two equations of the "basis change" between $\{x^n\}$ and $\{\Phi_n\}$:

(7.7) $$x^m = \sum_{n=0}^{m} b_{nm}\Phi_n, \quad \Phi_m = \sum_{n=0}^{m} a_{nm}x^n.$$

We therefore can use the functions $G_n(m)$ to determine the coefficients a_{nm}, and conversely.

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