63-3-5

405839

RADC-TDR-63-10, Suppl 1

Supplement to Third Quarterly Report

OPTIMUM APERTURE STUDY

Technical Documentary Report No. RADC-TDR-63-10, Suppl 1
May 1963

ROME AIR DEVELOPMENT CENTER
Research and Technology Division
Air Force Systems Command
United States Air Force
Griffiss Air Force Base
New York

Project No. 4506, Task No. 450604

(Prepared under Contract No. AF30(602)-2676
by D. Lee, Electronic Systems and Products
Division, Martin Company, Baltimore 3, Md.)

RADC-TDR-63-10, Suppl 1

Supplement to Third Quarterly Report

OPTIMUM APERTURE STUDY

Technical Documentary Report No. RADC-TDR-63-10, Suppl 1
May 1963

ROME AIR DEVELOPMENT CENTER
Research and Technology Division
Air Force Systems Command
United States Air Force
Griffiss Air Force Base
New York

Project No. 4506, Task No. 450604

(Prepared under Contract No. AF30(602)-2676
by D. Lee, Electronic Systems and Products
Division, Martin Company, Baltimore 3, Md.)

CONTENTS

ABSTRACT

The object of this contract is to study the applicability of the Wiener-Spencer Theorem to antennas. This theorem states that minimum standard deviation of the far-field pattern occurs when the illumination function corresponds to the lowest mode of vibration of a membrane stretched across the aperture opening.

This report presents the investigation of four selected nonoptimum illuminations for the elliptical apertures. Approximations are used to obtain expressions for far-field power patterns, and second moments are tabulated. In addition, illuminations and far-field power patterns are plotted.

Title of Report RADC-TDR-63-10, Suppl 1

PUBLICATION REVIEW

This report has been reviewed and is approved.

Approved: ARTHUR J. FROHLICH
Chief, Techniques Laboratory
Directorate of Aerospace Surveillance & Control

Approved: WILLIAM T. POPE
Acting Director
Director of Aerospace
Surveillance & Control

I. INTRODUCTION

The Third Quarterly Report states that a second group of nonoptimum illuminations for elliptical apertures of the form

$$F(\xi, \eta) = \left[(1 + a \sin^2 \eta) \cos \frac{\pi\xi}{2\xi_0}\right]^N \quad N = 1, 2, 3, 4$$

will be investigated. It further states that a comparison will be made between the optimum and nonoptimum illuminations. The following work has been accomplished:

(1) The far-field power patterns of elliptical apertures with four selected nonoptimum illuminations were derived through approximation.

(2) An IBM 1620 program was written to tabulate the moments.

(3) Investigation was made between optimum and nonoptimum illuminations to the degree of improvement in terms of the second moments, the side lobes and the beamwidth.

(4) Far-field power patterns of four selected nonoptimum illuminations were plotted along major axes.

II. ELLIPTICAL APERTURE WITH NONOPTIMUM ILLUMINATION

The far-field voltage power pattern of elliptical aperture is given by

$$G(u, v) = \int\int e^{i(ux + vy)} F \, dxdy.$$

For the optimum case, F is a product of two Mathieu Functions

$$[Ce(q, \xi)] \; [ce(q, \eta)]$$

In elliptical coordinates,

$$G(u, v) = \frac{h^2}{2} \int_0^{2\pi} \int_0^{\xi_0} e^{ih[u \cosh \xi \cos \eta + v \sinh \xi \sin \eta]} (\cosh 2\xi - \cos 2\eta) F(\xi, \eta) \, d\xi d\eta$$

where $F(\xi, \eta)$ is the illumination distribution.

The zeroth moment is given by

$$\mu_0 = \frac{h^2}{2} \int_0^{2\pi} \int_0^{\xi_0} F^2(\xi, \eta) (\cosh 2\xi - \cos 2\eta) \, d\xi d\eta,$$

and the second moment is given by

$$\mu_2 = \int_0^{2\pi} \int_0^{\xi_0} \left[\left(\frac{\partial F}{\partial \xi}\right)^2 + \left(\frac{\partial F}{\partial \eta}\right)^2\right] d\xi d\eta.$$

For a nonoptimum illumination, let

$$F(\xi, \eta) = \left[(1 + a \sin^2 \eta) \cos \frac{\pi \xi}{2\xi_0}\right]^N$$

The illumination satisfies the conditions

$$F(\xi, \eta) = F(\xi, \eta + 2\pi)$$
$$F(\xi_0, \eta) = 0.$$

Thus,

$$G(u, v) = \frac{h^2}{2} \int_0^{2\pi} \int_0^{\xi_0} e^{ih\left[u \cosh \xi \cos \eta + v \sinh \xi \sin \eta\right]} (\cosh 2\xi - \cos 2\eta) \left[(1 + a \sin^2 \eta) \cos \frac{\pi\xi}{2\xi_0}\right]^N d\xi d\eta$$

$$= \frac{h^2}{2} \int_0^{2\pi} \int_0^{\xi_0} e^{ih\left[u \cosh \xi \cos \eta + v \sinh \xi \sin \eta\right]} \cosh 2\xi \left[(1 + a \sin^2 \eta) \cos \frac{\pi\xi}{2\xi_0}\right]^N d\xi d\eta$$

$$- \frac{h^2}{2} \int_0^{2\pi} \int_0^{\xi_0} e^{ih\left[u \cosh \xi \cos \eta + v \sinh \xi \sin \eta\right]} \cos 2\eta \left[(1 + a \sin^2 \eta) \cos \frac{\pi\xi}{2\xi_0}\right]^N d\xi d\eta.$$

The preceding integrals do not appear to be solvable in closed form. Instead, we examine G (u, 0).

$$G(u, 0) = \frac{h^2}{2} \int_0^{2\pi} \int_0^{\xi_0} e^{ih\, u \cosh \xi \cos \eta} \cosh 2\xi \left[(1 + a \sin^2 \eta) \cos \frac{\pi\xi}{2\xi_0}\right]^N d\xi d\eta$$

$$- \frac{h^2}{2} \int_0^{2\pi} \int_0^{\xi_0} e^{ih\, u \cosh \xi \cos \eta} \cos 2\eta \left[(1 + a \sin^2 \eta) \cos \frac{\pi\xi}{2\xi_0}\right]^N d\xi d\eta.$$

For the given aperture $\xi_0 = 0.277$, $0 \le \xi \le \xi_0$

$\cos \xi \sim 1$

$\sinh \xi \sim \xi$.

Thus,

$$G(u, 0) = \frac{h^2}{2} \int_0^{2\pi} \int_0^{\xi_0} e^{ih\,u \cos \eta} \cosh 2\xi \left[(1 + a \sin^2 \eta) \cos \frac{\pi \xi}{2\xi_0} \right]^N d\xi\, d\eta$$

$$- \frac{h^2}{2} \int_0^{2\pi} \int_0^{\xi_0} e^{ih\,u \cos \eta} \cos 2\eta \left[(1 + a \sin^2 \eta) \cos \frac{\pi \xi}{2\xi_0} \right]^N d\xi\, d\eta.$$

Recall that

$$e^{ix \cos \theta} = J_0(x) + 2 \sum_{K=1}^{\infty} (-)^K J_{2K}(x) \cos 2K\theta + 2i \sum_{K=1}^{\infty} (-)^{K-1} J_{2K-1}(x) \cos (2K-1)\theta.$$

Thus,

$$G(u, 0) = \frac{h^2}{2} \int_0^{2\pi} \int_0^{\xi_0} \left[J_0(hu) + 2 \sum_{K=1}^{\infty} (-)^K J_{2K}(hu) \cos 2K\eta + 2i \sum_{K=1}^{\infty} (-)^{K-1} J_{2K-1}(hu) \cos (2K-1)\eta \right] \cosh 2\xi \left[(1 + a \sin^2 \eta) \cos \frac{\pi \xi}{2\xi_0} \right]^N d\xi\, d\eta$$

$$- \frac{h^2}{2} \int_0^{2\pi} \int_0^{\xi_0} \Big[J_0 (hu) + 2 \sum_{K=1}^{\infty} (-)^K J_{2K} (hu) \cos 2K \eta$$

$$+ 2i \sum_{K=1}^{\infty} (-)^{K-1} J_{2K-1} (hu) \cos (2K-1) \eta \Big] \cos 2\eta \, \Big[(1$$

$$+ a \sin^2 \eta) \cos \frac{\pi\xi}{2\xi_0} \Big]^N \, d\xi \, d\eta.$$

For $N = 1$,

$$G_1 (u, 0) = \frac{h^2}{2} \int_0^{2\pi} \int_0^{\xi_0} \Big[J_0 (hu) + 2 \sum_{K=1}^{\infty} (-)^K J_{2K} (hu) \cos 2K \eta$$

$$+ 2i \sum_{K=1}^{\infty} (-)^{K-1} J_{2K-1} (hu) \cos (2K-1) \eta \Big] \cosh 2\xi_0 \, \Big[(1$$

$$+ a \sin^2 \eta) \cos \frac{\pi\xi}{2\xi_0} \Big] \, d\xi \, d\eta$$

$$- \frac{h^2}{2} \int_0^{2\pi} \int_0^{\xi_0} \Big[J_0 (hu) + 2 \sum_{K=1}^{\infty} (-)^K J_{2K} (hu) \cos 2K \eta$$

$$+ 2i \sum_{K=1}^{\infty} (-)^{K-1} J_{2K-1} (hu) \cos (2K-1) \eta \Big] \cos 2\eta \, \Big[(1$$

$$+ a \sin^2 \eta) \cos \frac{\pi\xi}{2\xi_0} \Big] \, d\xi \, d\eta$$

$$= \frac{h^2}{2} \left(\int_0^{2\pi} J_0 (hu) (1 + a \sin^2 \eta) \, d\eta \right) \left(\int_0^{\xi_0} \cosh 2\xi \cos \frac{\pi \xi}{2\xi_0} \, d\xi \right)$$

$$+ \frac{h^2}{2} \left(\int_0^{2\pi} 2 \sum_{K=1}^{\infty} (-)^K J_{2K} (hu) \cos 2K \eta (1$$

$$+ a \sin^2 \eta) \, d\eta \right) \left(\int_0^{\xi_0} \cosh 2\xi \cos \frac{\pi \xi}{2\xi_0} \, d\xi \right)$$

$$+ \frac{h^2}{2} \left(\int_0^{2\pi} 2i \sum_{K=1}^{\infty} (-)^{K-1} J_{2K-1} (hu) \cos (2K-1) \eta (1$$

$$+ a \sin^2 \eta) \, d\eta \right) \left(\int_0^{\xi_0} \cosh 2\xi \cos \frac{\pi \xi}{2\xi_0} \, d\xi \right)$$

$$- \frac{h^2}{2} \left(\int_0^{2\pi} J_0 (hu) \cos 2 \eta \left[(1 + a \sin^2 \eta) \right] d\eta \right) \left(\int_0^{\xi_0} \cos \frac{\pi \xi}{2\xi_0} \, d\xi \right)$$

$$- \frac{h^2}{2} \left(\int_0^{2\pi} 2 \sum_{K=1}^{\infty} (-)^K J_{2K} (hu) \cos 2K \eta \cos 2\eta (1$$

$$+ a \sin^2 \eta d\eta \right) \left(\int_0^{\xi_0} \cos \frac{\pi \xi}{2\xi_0} \, d\xi \right)$$

$$- \frac{h^2}{2} \left(\int_0^{2\pi} 2i \sum_{K=1}^{\infty} (-)^{K-1} J_{2K-1} (hu) \cos (2K-1) \eta \cos 2 \eta (1$$

$$+ a \sin^2 \eta) \, d\eta\Big) \left(\int_0^{\xi_0} \cos \frac{\pi\xi}{2\xi_0} \, d\xi\right).$$

Here,

$$\int_0^{\xi_0} \cosh 2\xi \cos \frac{\pi\xi}{2\xi_0} \, d\xi = \frac{2\xi_0 \, \pi \cosh 2\xi_0}{16 \, \xi_0{}^2 + \pi^2}$$

$$\int_0^{2\pi} J_0 \, (hu) \, (1 + a \sin^2 \eta) \, d\eta = 2\pi \left(1 + \frac{a}{2}\right) J_0 \, (hu)$$

$$\int_0^{2\pi} 2 \sum_{K=1}^{\infty} (-)^K J_{2K} \, (hu) \cos 2K \, \eta \, (1 + a \sin^2 \eta) \, d\eta = a \, \pi \, J_2 \, (hu)$$

$$\int_0^{2\pi} 2i \sum_{K=1}^{\infty} (-)^{K-1} J_{2K-1} \, (hu) \cos (2K-1) \, \eta \, (1 + a \sin^2 \eta) \, d\eta = 0$$

$$\int_0^{\xi_0} \cos \frac{\pi\xi}{2\xi_0} \, d\xi = \frac{2\xi_0}{\pi}$$

$$\int_0^{2\pi} J_0 \, (hu) \cos 2 \, \eta \, \left[(1 + a \sin^2 \eta)\right] \, d\eta = - \frac{a}{2} \, \pi \, J_0 \, (hu)$$

$$\int_0^{2\pi} 2 \sum_{K=1}^{\infty} (-)^K J_{2K} \, (hu) \cos 2K \, \eta \cos 2 \, \eta \, \left[(1 + a \sin^2 \eta)\right] \, d\eta$$

$$= - \, 2\pi \left(1 + \frac{a}{2}\right) J_2 \, (hu) - \frac{a}{2} \, J_4 \, (hu)$$

$$\int_0^{2\pi} 2i \sum_{K=1}^{\infty} (-)^{K-1} J_{2K-1}(hu) \cos(2K-1)\eta \cos 2\eta \left[(1 + a \sin^2 \eta)\right] d\eta = 0.$$

Therefore,

$$G_1(u, 0) = h^2 \xi_0 \left[\left\{\frac{a}{2} + \left(1 + \frac{a}{2}\right) \frac{2\pi^2 \cosh 2\xi_0}{16\xi_0^2 + \pi^2}\right\} J_0(hu) + \left\{2\left(1 + \frac{a}{2}\right) + \frac{a\pi^2 \cosh 2\xi_0}{16\xi_0^2 + \pi^2}\right\} J_2(hu) + \frac{a}{2} J_4(hu)\right]$$

For N = 2,

$$G_2(u, 0) = \frac{h^2}{2} \int_0^{2\pi} \int_0^{\xi_0} \left[J_0(hu) + 2 \sum_{K=1}^{\infty} (-)^K J_{2K}(hu) \cos 2K\eta + 2i \sum_{K=1}^{\infty} (-)^{K-1} J_{2K-1}(hu) \cos(2K-1)\eta\right] \cdot \cosh 2\xi \left[(1 + a \sin^2 \eta) \cos \frac{\pi\xi}{2\xi_0}\right]^2 d\xi d\eta$$

$$- \frac{h^2}{2} \int_0^{2\pi} \int_0^{\xi_0} \left[J_0(hu) + 2 \sum_{K=1}^{\infty} (-)^K J_{2K}(hu) \cos 2K\eta + 2i \sum_{K=1}^{\infty} (-)^{K-1} J_{2K-1}(hu) \cos(2K-1)\eta\right] \cdot \cos 2\eta \left[(1\right.$$

$$+ a \sin^2 \eta) \cos \frac{\pi \xi}{2\xi_0}\Big]^2 \quad d\xi d\eta$$

$$= \frac{h^2}{2} \left(\int_0^{2\pi} J_0 (hu) (1 + a \sin^2 \eta)^2 \; d\eta\right) \left(\int_0^{\xi_0} \cosh 2\xi \cos^2 \frac{\pi \xi}{2\xi_0} \; d\xi\right)$$

$$+ \frac{h^2}{2} \left(\int_0^{2\pi} 2 \sum_{K=1}^{\infty} (-)^K J_{2K} (hu) \cos 2K \eta (1\right.$$

$$\left. + a \sin^2 \eta)^2 \; d\eta\right) \left(\int_0^{\xi_0} \cosh 2\xi \cos^2 \frac{\pi \xi}{2\xi_0} \; d\xi\right)$$

$$+ \frac{h^2}{2} \left(\int_0^{2\pi} 2i \sum_{K=1}^{\infty} (-)^{K-1} J_{2K-1} (hu) \cos (2K-1) \eta (1\right.$$

$$\left. + a \sin^2 \eta)^2 \; d\eta\right) \left(\int_0^{\xi_0} \cosh 2\xi \cos^2 \frac{\pi \xi}{2\xi_0} \; d\xi\right)$$

$$- \frac{h^2}{2} \left(\int_0^{2\pi} J_0 (hu) \cos 2\eta \; \left[(1 + a \sin^2 \eta)\right]^2 \; d\eta\right) \left(\int_0^{\xi_0} \cos^2 \frac{\pi \xi}{2\xi_0} \; d\xi\right)$$

$$- \frac{h^2}{2} \left(\int_0^{2\pi} 2 \sum_{K=1}^{\infty} (-)^K J_{2K} (hu) \cos 2K \eta \cos 2\eta \; \left[(1\right.\right.$$

$$\left.\left. + a \sin^2 \eta)\right]^2 \; d\eta\right) \left(\int_0^{\xi_0} \cos^2 \frac{\pi \xi}{2\xi_0} \; d\xi\right)$$

$$- \frac{h^2}{2} \left(\int_0^{2\pi} 2i \sum_{K=1}^{\infty} (-)^{K-1} J_{2K-1}(hu) \cos(2K-1)\eta \cos 2\eta \left[(1 + a \sin^2 \eta) \right]^2 d\eta \right) \left(\int_0^{\xi_0} \cos^2 \frac{\pi \xi}{2\xi_0} \, d\xi \right).$$

Here,

$$\int_0^{\xi_0} \cosh 2\xi \cos^2 \frac{\pi \xi}{2\xi_0} \, d\xi = \frac{\pi^2 \sinh 2\xi_0}{4\left(4\xi_0^2 + \pi^2\right)}$$

$$\int_0^{2\pi} J_0(hu)(1 + a \sin^2 \eta)^2 \, d\eta = 2\pi \left(1 + a + \frac{3}{8} a^2\right) J_0(hu)$$

$$\int_0^{2\pi} 2 \sum_{K=1}^{\infty} (-)^K J_{2K}(hu) \cos 2K\eta (1 + a \sin^2 \eta)^2 \, d\eta = 2\pi a \left(1 + \frac{a}{2}\right) J_2(hu) + \frac{a^2}{4} \pi J_4(hu)$$

$$\int_0^{2\pi} 2i \sum_{K=1}^{\infty} (-)^{K-1} J_{2K-1}(hu) \cos(2K-1)\eta (1 + a \sin^2 \eta)^2 \, d\eta = 0$$

$$\int_0^{\xi_0} \cos^2 \frac{\pi \xi}{2\xi_0} \, d\xi = \frac{\xi_0}{2}$$

$$\int_0^{2\pi} J_0(hu) \cos 2\eta (1 + a \sin^2 \eta)^2 \, d\eta = -a \left(1 + \frac{a}{2}\right) \pi J_0(hu)$$

$$\int_0^{2\pi} 2 \sum_{K=1}^{\infty} (-)^K J_{2K} (hu) \cos 2K\eta \cos 2\eta (1 + a \sin^2 \eta)^2 \, d\eta$$

$$= - \left(2 + 2a + \frac{7}{8} a^2\right) \pi J_2 (hu) - a \left(1 + \frac{a}{2}\right) \pi J_4 (hu) - \frac{a^2}{8} \pi J_6 (hu)$$

$$\int_0^{2\pi} 2i \sum_{K=1}^{\infty} (-)^{K-1} J_{2K-1} (hu) \cos (2K-1) \eta \cos 2\eta (1$$

$$+ a \sin^2 \eta)^2 \, d\eta = 0.$$

Therefore,

$$G_2 (u, 0) = h^2 \pi \left[\left\{\left(1 + a + \frac{3}{8} a^2\right) \frac{\pi^2 \sinh 2\xi_0}{4 (4\xi_0^2 + \pi^2)} + \frac{\xi_0 a}{4} \left(1\right.\right.\right.$$

$$\left.\left. + \frac{a}{2}\right)\right\} J_0 (hu) + \left\{a \left(1 + \frac{a}{2}\right) \frac{\pi^2 \sinh 2\xi_0}{4 (4\xi_0^2 + \pi^2)} + \frac{\xi_0}{4} \left(2\right.\right.$$

$$\left.\left. + 2a + \frac{7}{8} a^2\right)\right\} J_2 (hu) + \left\{\frac{a^2}{8} \cdot \frac{\pi^2 \sinh 2\xi_0}{4 (4\xi_0^2 + \pi^2)}\right.$$

$$\left.\left. + \frac{\xi_0 a}{4} \left(1 + \frac{a}{2}\right)\right\} J_4 (hu) + \frac{\xi_0 a^2}{32} J_6 (hu)\right] .$$

For N = 3,

$$G_3 (u, 0) = \frac{h^2}{2} \int_0^{2\pi} \int_0^{\xi_0} \left[J_0 (hu) + 2 \sum_{K=1}^{\infty} (-)^K J_{2K} (hu) \cos 2K\eta\right.$$

$$\left. + 2i \sum_{K=1}^{\infty} (-)^{K-1} J_{2K-1} (hu) \cos (2K-1) \eta\right] . \cosh 2\xi \left[(1\right.$$

$$\left. + a \sin^2 \eta) \cos \frac{\pi\xi}{2\xi_0}\right]^3 d\xi \, d\eta$$

$$- \frac{h^2}{2} \int_0^{2\pi} \int_0^{\xi_0} \left[J_0 (hu) + 2 \sum_{K=1}^{\infty} (-)^K J_{2K} (hu) \cos 2K \eta \right.$$

$$\left. + 2i \sum_{K=1}^{\infty} (-)^{K-1} J_{2K-1} (hu) \cos (2K-1) \eta \right] \cdot \cos 2\eta \left[(1 \right.$$

$$\left. + a \sin^2 \eta) \cos \frac{\pi \xi}{2\xi_0} \right]^3 d\xi \, d\eta$$

$$= \frac{h^2}{2} \left(\int_0^{2\pi} J_0 (hu) (1 + a \sin^2 \eta)^3 \, d\eta \right) \left(\int_0^{\xi_0} \cosh 2\xi \cos^3 \left(\frac{\pi \xi}{2\xi_0} \right) d\xi \right)$$

$$+ \frac{h^2}{2} \left(\int_0^{2\pi} 2 \sum_{K=1}^{\infty} (-)^K J_{2K} (hu) \cos 2K \eta \, (1 \right.$$

$$\left. + a \sin^2 \eta)^3 \, d\eta \right) \left(\int_0^{\xi_0} \cosh 2\xi \cos^3 \frac{\pi \xi}{2\xi_0} \, d\xi \right)$$

$$+ \frac{h^2}{2} \left(\int_0^{2\pi} 2i \sum_{K=1}^{\infty} (-)^{K-1} J_{2K-1} (hu) \cos (2K-1) \eta \, (1 \right.$$

$$\left. + a \sin^2 \eta)^3 \, d\eta \right) \left(\int_0^{\xi_0} \cosh 2\xi \cos^3 \frac{\pi \xi}{2\xi_0} \, d\xi \right)$$

$$- \frac{h^2}{2} \left(\int_0^{2\pi} J_0 (hu) \cos 2\eta \, (1 + a \sin^2 \eta)^3 \, d\eta \right) \left(\int_0^{\xi_0} \cos^3 \frac{\pi \xi}{2\xi_0} \, d\xi \right)$$

$$- \frac{h^2}{2} \left(\int_0^{2\pi} 2 \sum_{K=1}^{\infty} (-)^K J_{2K} (hu) \cos 2K \eta \cos 2\eta \, (1 \right.$$

$$\left. + a \sin^2 \eta)^3 \, d\eta \right) \left(\int_0^{\xi_0} \cos^3 \frac{\pi \xi}{2\xi_0} \, d\xi \right)$$

$$- \frac{h^2}{2} \left(\int_0^{2\pi} 2i \sum_{K=1}^{\infty} (-)^{K-1} J_{2K-1}(hu) \cos(2K-1)\eta \cos 2\eta \, (1 + a \sin^2 \eta)^3 \, d\eta \right) \left(\int_0^{\xi_0} \cos^3 \frac{\pi \xi}{2\xi_0} \, d\xi \right) .$$

Here,

$$\int_0^{\xi_0} \cosh 2\xi \cos^3 \frac{\pi\xi}{2\xi_0} \, d\xi = \frac{3}{2} \xi_0 \, \pi \cosh 2\xi_0 \left[\frac{1}{16\xi_0^2 + \pi^2} - \frac{1}{16\xi_0^2 + 9\pi^2} \right]$$

$$\int_0^{2\pi} J_0(hu) (1 + a \sin^2 \eta)^3 \, d\eta = \left(1 + \frac{3}{2} a + \frac{9}{8} a^2 + \frac{5}{16} a^3 \right) 2\pi J_0(hu)$$

$$\int_0^{2\pi} 2 \sum_{K=1}^{\infty} (-)^K J_{2K}(hu) \cos 2K\eta \, (1 + a \sin^2 \eta)^3 \, d\eta =$$

$$\left(\frac{3}{2} a + \frac{3}{2} a^2 + \frac{15}{32} a^3 \right) 2\pi J_2(hu) + \frac{3}{4} a^2 \left(1 + \frac{a}{2} \right) \pi J_4(hu) + \frac{1}{16} a^3 \pi J_6(hu)$$

$$\int_0^{2\pi} 2i \sum_{K=1}^{\infty} (-)^{K-1} J_{2K-1}(hu) \cos(2K-1)\eta \, (1 + a \sin^2 \eta)^3 \, d\eta = 0$$

$$\int_0^{\xi_0} \cos^3 \frac{\pi \xi}{2\xi_0} \, d\xi = \frac{4\xi_0}{3\pi}$$

$$\int_0^{2\pi} J_0 (hu) \cos 2\eta \, (1 + a \sin^2 \eta)^3 \, d\eta = - \left(\frac{3}{2} a + \frac{3}{2} a^2 + \frac{15}{32} a^3\right) \pi J_0 (hu)$$

$$\int_0^{2\pi} 2 \sum_{K=1}^{\infty} (-)^K J_{2K} (hu) \cos 2K\eta \cos 2\eta \, (1 + a \sin^2 \eta)^3 \, d\eta =$$

$$- \left(2 + 3a + \frac{21}{8} a^2 + \frac{13}{16} a^3\right) \pi J_2 (hu) - a\left(\frac{3}{2} + \frac{3}{2} a + \frac{1}{2} a^2\right) \pi J_4 (hu)$$

$$- \frac{3}{8} a^2 \left(1 + \frac{a}{2}\right) \pi J_6 (hu) - \frac{1}{32} a^3 \pi J_8 (hu)$$

$$\int_0^{2\pi} 2i \sum_{K=1}^{\infty} (-)^{K-1} J_{2K-1} (hu) \cos (2K-1) \eta \cos 2\eta \, (1 + a \sin^2 \eta)^3 \, d\eta = 0.$$

Therefore,

$$G_3 (u, 0) = h^2 \xi_0 \sum_{r=0}^{4} \alpha_{2r} J_{2r} (hu)$$

where

$$\alpha_0 = \frac{3}{2} \pi^2 \cosh 2\xi_0 \left[\frac{1}{16 \xi_0^2 + \pi^2} - \frac{1}{16 \xi_0^2 + 9\pi^2}\right] \Bigg(1$$

$$+ \frac{3}{2} a + \frac{9}{8} a^2 + \frac{5}{16} a^3\Bigg) + \frac{2}{3} a \left(\frac{3}{2} + \frac{3}{2} a + \frac{15}{32} a^2\right)$$

$$\alpha_2 = \frac{3}{2}\,\pi^2 \cosh 2\xi_0 \left[\frac{1}{16\,\xi_0^{\,2} + \pi^2} - \frac{1}{16\,\xi_0^{\,2} + 9\,\pi^2}\right]\left(\frac{3}{2}\,a + \frac{3}{2}\,a^2\right.$$

$$\left. + \frac{15}{32}\,a^3\right) + \frac{2}{3}\left(2 + 3a + \frac{21}{8}\,a^2 + \frac{13}{16}\,a^3\right)$$

$$\alpha_4 = \frac{9}{16}\,\pi^2 \cosh 2\xi_0 \left[\frac{1}{16\,\xi_0^{\,2} + \pi^2} - \frac{1}{16\,\xi_0^{\,2} + 9\,\pi^2}\right] a^2\left(1 + \frac{a}{2}\right)$$

$$+ \frac{2}{3}\,a\left(\frac{3}{2} + \frac{3}{2}\,a + \frac{1}{2}\,a^2\right)$$

$$\alpha_6 = \frac{3}{64}\,\pi^2 \cosh 2\xi_0 \left[\frac{1}{16\,\xi_0^{\,2} + \pi^2} - \frac{1}{16\,\xi_0^{\,2} + 9\,\pi^2}\right] a^3$$

$$+ \frac{1}{4}\,a^2\left(1 + \frac{a}{2}\right)$$

$$\alpha_8 = \frac{1}{48}\,a^3.$$

For N = 4,

$$G_4(u, 0) = \frac{h^2}{2}\int_0^{2\pi}\int_0^{\xi_0}\left[J_0(hu) + 2\sum_{K=1}^{\infty}(-)^K J_{2K}(hu)\cos 2K\eta\right.$$

$$\left. + 2i\sum_{K=1}^{\infty}(-)^{K-1} J_{2K-1}(hu)\cos(2K-1)\eta\right]\cdot \cosh 2\xi \left[(1\right.$$

$$\left. + a\sin^2\eta)\cos\frac{\pi\xi}{2\xi_0}\right]^4 d\xi\, d\eta$$

$$- \frac{h^2}{2} \int_0^{2\pi} \int_0^{\xi_0} \left[J_0 (hu) + 2 \sum_{K=1}^{\infty} (-)^K J_{2K} (hu) \cos 2 K\eta \right.$$

$$\left. + 2i \sum_{K=1}^{\infty} (-)^{K-1} J_{2K-1} (hu) \cos (2K - 1) \eta \right] \cos 2\eta \left[(1 \right.$$

$$\left. + a \sin^2 \eta) \cos \frac{\pi\xi}{2\xi_0} \right]^4 d\xi \, d\eta$$

$$= \frac{h^2}{2} \left(\int_0^{2\pi} J_0 (hu) (1 + a \sin^2 \eta)^4 d\eta \right) \left(\int_0 \cosh 2\xi \cos^4 \frac{\pi\xi}{2\xi_0} d\xi \right)$$

$$+ \frac{h^2}{2} \left(\int_0^{2\pi} 2 \sum_{K=1}^{\infty} (-)^K J_{2K} (hu) \cos 2K \eta (1 \right.$$

$$\left. + a \sin^2 \eta)^4 d\eta \right) \left(\int_0^{\xi_0} \cosh 2\xi_0 \cos^4 \frac{\pi \xi}{2\xi_0} d\xi \right)$$

$$+ \frac{h^2}{2} \left(\int_0^{2\pi} 2i \sum_{K=1}^{\infty} (-)^{K-1} J_{2K-1} (hu) \cos (2K - 1) \eta (1 \right.$$

$$\left. + a \sin^2 \eta)^4 d\eta \right) \left(\int_0^{\xi_0} \cosh 2\xi \cos^4 \frac{\pi\xi}{2\xi_0} d\xi \right)$$

$$- \frac{h^2}{2} \left(\int_0^{2\pi} J_0 (hu) \cos 2\eta (1 + a \sin^2 \eta)^4 d\eta \right) \left(\int_0^{\xi_0} \cos^4 \frac{\pi\xi}{2\xi_0} d\xi \right)$$

$$- \frac{h^2}{2} \left(\int_0^{2\pi} 2 \sum_{K=1}^{\infty} (-)^K J_{2K} (hu) \cos 2K\eta \cos 2\eta (1 \right.$$

$$\left. + a \sin^2 \eta)^4 d\eta \right) \left(\int_0^{\xi_0} \cos^4 \frac{\pi \xi}{2\xi_0} d\xi \right)$$

$$- \frac{h^2}{2} \left(\int_0^{2\pi} 2i \sum_{K=1}^{\infty} (-)^{K-1} J_{2K-1}(hu) \cos(2K-1)\eta \cos 2\eta (1 + a \sin^2 \eta)^4 \, d\eta \right) \left(\int_0^{\xi_0} \cos^4 \frac{\pi\xi}{2\xi_0} \, d\xi \right) .$$

Here,

$$\int_0^{\xi_0} \cosh 2\xi \cos^4 \frac{\pi\xi}{2\xi_0} \, d\xi = \left(\frac{3}{16} - \frac{\xi_0^{\ 2}}{4\xi_0^{\ 2} + \pi^2} + \frac{\xi_0^{\ 2}}{16(\xi_0^{\ 2} + \pi^2)} \right) \sinh 2\xi_0$$

$$\int_0^{2\pi} J_0(hu)(1 + a \sin^2 \eta)^4 \, d\eta = 2\pi \left(1 + 2a + \frac{9}{4} a^2 + \frac{5}{4} a^3 + \frac{35}{128} a^4 \right) J_0(hu)$$

$$\int_0^{2\pi} 2 \sum_{K=1}^{\infty} (-)^K J_{2K}(hu) \cos 2K\eta (1 + a \sin^2 \eta)^4 \, d\eta = 2\pi \left(2a + 3a^2 + \frac{15}{8} a^3 + \frac{7}{16} a^4 \right) J_2(hu) + 2\pi \left(\frac{3}{4} a^2 + \frac{3}{4} a^3 + \frac{7}{32} a^4 \right) J_4(hu) + 2\pi \left(\frac{1}{8} a^3 + \frac{1}{16} a^4 \right) J_6(hu) + 2\pi \frac{1}{128} a^4 J_8(hu)$$

$$\int_0^{2\pi} 2i \sum_{K=1}^{\infty} (-)^{K-1} J_{2K-1}(hu) \cos(2K-1)\eta (1 + a \sin^2 \eta)^4 \, d\eta = 0$$

$$\int_0^{\xi_0} \cos^4 \frac{\pi\xi}{2\xi_0} \, d\xi = \frac{3}{8} \xi_0$$

$$\int_0^{2\pi} J_0(hu) \cos 2\eta (1 + a \sin^2 \eta)^4 \, d\eta = -\left(2a + 3a^2 + \frac{15}{8} a^3 + \frac{7}{16} a^4\right) \pi J_0(hu)$$

$$\int_0^{2\pi} 2 \sum_{K=1}^{\infty} (-)^K J_{2K}(hu) \cos 2K\eta \cos 2\eta (1 + a \sin^2 \eta)^4 \, d\eta = -\left(2 + 4a + \frac{21}{4} a^2 + \frac{13}{4} a^3 + \frac{49}{64} a^4\right) \pi J_2(hu) - \left(2a + 3a^2 + 2a^3 + \frac{1}{2} a^4\right) \pi J_4(hu) - \left(\frac{3}{4} a^2 + \frac{3}{4} a^3 + \frac{29}{128} a^4\right) \pi J_6(hu) - \left(\frac{1}{8} a^3 + \frac{1}{16} a^4\right) \pi J_8(hu) - \frac{1}{128} a^4 \pi J_{10}(hu)$$

$$\int_0^{2\pi} 2i \sum_{K=1}^{\infty} (-)^{K-1} J_{2K-1}(hu) \cos (2K-1)\eta \cos 2\eta (1 + a \sin^2 \eta)^4 \, d\eta = 0.$$

Therefore,

$$G_4(u, 0) = h^2 \pi \sum_{r=0}^{5} \beta_{2r} J_{2r}(hu)$$

where

$$\beta_0 = \left(1 + 2a + \frac{9}{4} a^2 + \frac{5}{4} a^3 + \frac{35}{128} a^4\right)\left(\frac{3}{16} - \frac{\xi_0^2}{4\xi_0^2 + \pi^2} + \frac{\xi_0^2}{16(\xi_0^2 + \pi^2)}\right) \sinh 2\xi_0 + \frac{3}{16} \xi_0 \left(2a + 3a^2 + \frac{15}{8} a^3 + \frac{7}{16} a^4\right)$$

$$\beta_2 = \left(2a + 3a^2 + \frac{15}{8}\,a^3 + \frac{7}{16}\,a^4\right)\left(\frac{3}{16} - \frac{\xi_0^{\,2}}{4\,\xi_0^{\,2} + \pi^2}\right.$$

$$\left. + \frac{\xi_0^{\,2}}{16\,(\xi_0^{\,2} + \pi^2)}\right) \sinh 2\,\xi_0 + \frac{3}{16}\,\xi_0\,\left(2 + 4a + \frac{21}{4}\,a^2\right.$$

$$\left. + \frac{13}{4}\,a^3 + \frac{49}{64}\,a^4\right)$$

$$\beta_4 = \left(\frac{3}{4}\,a^2 + \frac{3}{4}\,a^3 + \frac{7}{32}\,a^4\right)\left(\frac{3}{16} - \frac{\xi_0^{\,2}}{4\,\xi_0^{\,2} + \pi^2} + \frac{\xi_0^{\,2}\quad\eta}{16\,(\xi_0^{\,2} + \pi^2)}\right) \sinh 2\,\xi_0$$

$$+ \frac{3}{16}\,\xi_0\,\left(2a + 3a^2 + 2a^3 + \frac{1}{2}\,a^4\right)$$

$$\beta_6 = \frac{1}{8}\,a^3\left(1 + \frac{1}{2}\,a\right)\left(\frac{3}{16} - \frac{\xi_0^{\,2}}{4\,\xi_0^{\,2} + \pi^2} + \frac{\xi_0^{\,2}}{16\,(\xi_0^{\,2} + \pi^2)}\right) \sinh 2\,\xi_0$$

$$+ \frac{3}{16}\,\xi_0\,\left(\frac{3}{4}\,a^2 + \frac{3}{4}\,a^3 + \frac{29}{128}\,a^4\right)$$

$$\beta_8 = \frac{1}{128}\,a^4\,\left(\frac{3}{16} - \frac{\xi_0}{4\,\xi_0^{\,2} + \pi^2} + \frac{\xi_0^{\,2}}{16\,(\xi_0^{\,2} + \pi^2)}\right) \sinh 2\,\xi_0$$

$$+ \frac{3}{128}\,\xi_0\,a^3\left(1 + \frac{1}{2}\,a\right)$$

$$\beta_{10} = \frac{3}{2048}\,\xi_0\,a^4.$$

For the zeroth moment in elliptical coordinates,

$$\mu_0 = \frac{h^2}{2} \int_0^{2\pi} \int_0^{\xi_0} (\cosh 2\xi - \cos 2\eta)\, F^2(\xi, \eta) \quad d\xi\, d\eta$$

$$= \frac{h^2}{2} \int_0^{2\pi} \int_0^{\xi_0} (\cosh 2\xi - \cos 2\eta) \left[(1 + a \sin^2 \eta) \cos \frac{\pi \xi}{2\xi_0}\right]^{2N} \quad d\xi\, d\eta$$

$$= \frac{h^2}{2} \int_0^{2\pi} \int_0^{\xi_0} \cosh 2\xi \left[(1 + a \sin^2 \eta) \cos \frac{\pi \xi}{2\xi_0}\right]^{2N} \quad d\xi\, d\eta$$

$$- \frac{h^2}{2} \int_0^{2\pi} \int_0^{\xi_0} \cos 2\eta \left[(1 + a \sin^2 \eta) \cos \frac{\pi \xi}{2\xi_0}\right]^{2N} \quad d\xi\, d\eta.$$

For $N = 1$,

$$\mu_{0,1} = \frac{h^2}{2} \left(\int_0^{2\pi} (1 + a \sin^2 \eta)^2 \quad d\eta\right) \left(\int_0^{\xi_0} \cosh 2\xi \cos^2 \frac{\pi\xi}{2\xi_0} \quad d\xi\right)$$

$$- \frac{h^2}{2} \left(\int_0^{2\pi} \cos 2\eta\, (1 + a \sin^2 \eta)^2 \quad d\eta\right) \left(\int_0^{\xi_0} \cos^2 \frac{\pi\xi}{2\xi_0} \quad d\xi\right).$$

Here,

$$\int_0^{2\pi} (1 + a \sin^2 \eta)^2 \quad d\eta = 2\pi \left(1 + a + \frac{3}{8}\, a^2\right)$$

$$\int_0^{\xi_0} \cosh 2\xi \cos^2 \frac{\pi\xi}{2\xi_0} \quad d\xi = \frac{\pi^2 \sinh 2\xi_0}{4\,(4\xi_0^{\,2} + \pi^2)}$$

$$\int_0^{2\pi} \cos 2\eta \,(1 + a \sin^2 \eta)^2 \, d\eta = - a\left(1 + \frac{a}{2}\right)\pi$$

$$\int_0^{\xi_0} \cos^2 \frac{\pi\xi}{2\xi_0} \, d\xi = \frac{1}{2} \xi_0 \, .$$

Therefore,

$$\mu_{0,1} = h^2 \pi \left[\left(1 + a + \frac{3}{8} a^2\right) \frac{\pi^2 \sinh 2\xi_0}{4\,(4\xi_0^{\;2} + \pi^2)} + \frac{1}{4} \xi_0 \, a \left(1 + \frac{1}{2} a\right)\right] . \tag{1}$$

For N = 2,

$$\mu_{0,2} = \frac{h^2}{2} \left(\int_0^{2\pi} (1 + a \sin^2 \eta)^4 \, d\eta\right) \left(\int_0^{\xi_0} \cosh 2\xi \cos^4 \frac{\pi\xi}{2\xi_0} \, d\xi\right)$$

$$- \frac{h^2}{2} \left(\int_0^{2\pi} \cos 2\eta \,(1 + a \sin^2 \eta)^4 \, d\eta\right) \left(\int_0^{\xi_0} \cos^4 \left(\frac{\pi\xi}{2\xi_0}\right) d\xi\right)$$

Here,

$$\int_0^{2\pi} (1 + a \sin^2 \eta)^4 \, d\eta = \left(1 + 2a + \frac{9}{4} a^2 + \frac{5}{4} a^3 + \frac{35}{128} a^4\right) 2\pi$$

$$\int_0^{\xi_0} \cosh 2\xi \cos^4 \frac{\pi\xi}{2\xi_0} \, d\xi = \left(\frac{3}{16} - \frac{\xi_0^{\;2}}{4\xi_0^{\;2} + \pi^2} + \frac{\xi_0^{\;2}}{16\,(\xi_0^{\;2} + \pi^2)}\right) \sinh 2\xi_0$$

$$\int_0^{2\pi} \cos 2\eta \,(1 + a \sin^2 \eta)^4 \, d\eta = -\left(2a + 3a^2 + \frac{15}{8} a^3 + \frac{7}{16} a^4\right) \pi$$

$$\int_0^{\xi_0} \cos^4 \frac{\pi\xi}{2\xi_0} \, d\xi = \frac{3}{8} \xi_0 .$$

Therefore,

$$\mu_{0,2} = h^2 \pi \left[\left(1 + 2a + \frac{9}{4} a^2 + \frac{5}{4} a^3 + \frac{35}{128} a^4\right)\left(\frac{3}{16} - \frac{\xi_0^{\,2}}{4\xi_0^{\,2} + \pi^2} + \frac{\xi_0^{\,2}}{16(\xi_0^{\,2} + \pi^2)}\right) \sinh 2\xi_0 + \frac{3}{16} \xi_0 \left(2a + 3a^2 + \frac{15}{8} a^3 + \frac{7}{16} a^4\right)\right]. \qquad (2)$$

For N = 3,

$$\mu_{0,3} = \frac{h^2}{2} \left(\int_0^{2\pi} (1 + a \sin^2 \eta)^6 \, d\eta\right)\left(\int_0^{\xi_0} \cosh 2\xi \cos^6 \frac{\pi\xi}{2\xi_0} \, d\xi\right) - \frac{h^2}{2}\left(\int_0^{2\pi} \cos 2\eta (1 + a \sin^2 \eta)^6 \, d\eta\right)\left(\int_0^{\xi_0} \cos^6 \frac{\pi\xi}{2\xi_0} \, d\xi\right).$$

Here,

$$\int_0^{2\pi} (1 + a \sin^2 \eta)^6 \, d\eta = 2\pi \left(1 + 3a + \frac{45}{8} a^2 + \frac{25}{4} a^3 + \frac{525}{128} a^4 + \frac{189}{128} a^5 + \frac{231}{1024} a^6\right)$$

$$\int_0^{\xi_0} \cosh 2\xi \cos^6 \frac{\pi\xi}{2\xi_0} \, d\xi = \left(\frac{5}{32} - \frac{15\xi_0^{\,2}}{16(4\xi_0^{\,2} + \pi^2)} + \frac{3\xi_0^{\,2}}{32(\xi_0^{\,2} + \pi^2)} - \frac{\xi_0^{\,2}}{16(\xi_0^{\,2} + 9\pi^2)}\right) \sinh 2\xi_0$$

$$\int_0^{2\pi} \cos 2\eta \, (1 + a \sin^2 \eta)^6 \, d\eta = -\pi a \left(3 + \frac{15}{2} a + \frac{75}{8} a^2 + \frac{105}{16} a^3 + \frac{315}{128} a^4 + \frac{99}{256} a^5\right)$$

$$\int_0^{\xi_0} \cos^6 \frac{\pi \xi}{2\xi_0} \, d\xi = \frac{5}{16} \xi_0 .$$

Therefore,

$$\mu_{0,3} = \frac{1}{8} h^2 \pi \left[\left(1 + 3a + \frac{45}{8} a^2 + \frac{25}{4} a^3 + \frac{525}{128} a^4 + \frac{189}{128} a^5 + \frac{231}{1024} a^6\right)\left(\frac{5}{4} - \frac{15 \xi_0^2}{2(4\xi_0^2 + \pi^2)} + \frac{3\xi_0^2}{4(\xi_0^2 + \pi^2)} - \frac{\xi_0^2}{2(4\xi_0^2 + 9\pi^2)}\right) \sinh 2\xi_0 + \frac{5}{4} \xi_0 a \left(3 + \frac{15}{2} a + \frac{75}{8} a^2 + \frac{105}{16} a^3 + \frac{315}{128} a^4 + \frac{99}{256} a^5\right)\right]. \tag{3}$$

For N = 4,

$$\mu_{0,4} = \frac{h^2}{2} \left(\int_0^{2\pi} (1 + a \sin^2 \eta)^8 \, d\eta\right)\left(\int_0^{\xi_0} \cosh 2\xi \cos^8 \frac{\pi\xi}{2\xi_0} \, d\xi\right) - \frac{h^2}{2} \left(\int_0^{2\pi} \cos 2\eta \, (1 + a \sin^2 \eta)^8 \, d\eta\right)\left(\int_0^{\xi_0} \cos^8 \frac{\pi\xi}{2\xi_0} \, d\xi\right).$$

Here,

$$\int_0^{2\pi} (1 + a \sin^2 \eta)^8 \, d\eta = 2\pi \left(1 + 4a + \frac{21}{2} a^2 + \frac{35}{2} a^3 + \frac{1225}{64} a^4 + \frac{441}{32} a^5 + \frac{1617}{128} a^6 + \frac{429}{128} a^7 + \frac{6335}{32768} a^8\right)$$

$$\int_0^{\xi_0} \cosh 2\xi \cos^8 \frac{\pi\xi}{2\xi_0} \, d\xi = \left(\frac{35}{256} - \frac{7\xi_0^2}{32(4\xi_0^2 + \pi^2)} + \frac{7\xi_0^2}{64(\xi_0^2 + \pi^2)} - \frac{\xi_0^2}{8(4\xi_0^2 + 9\pi^2)} + \frac{\xi_0^2}{256(\xi_0^2 + 4\pi^2)}\right) \sinh 2\xi_0$$

$$\int_0^{2\pi} \cos 2\eta (1 + a \sin^2 \eta)^8 \, d\eta = -\pi a \left(1 + \frac{a}{2}\right)\left(4 + 12a + \frac{81}{4} a^2 + \frac{41}{2} a^3 + \frac{407}{32} a^4 + \frac{143}{32} a^5 + \frac{715}{1024} a^6\right)$$

$$\int_0^{\xi_0} \cos^8 \frac{\pi\xi}{2\xi_0} \, d\xi = \frac{35}{128} \xi_0 .$$

Therefore,

$$\mu_{0,4} = \frac{1}{4} h^2 \pi \left[\left(1 + 4a + \frac{21}{2} a^2 + \frac{35}{2} a^3 + \frac{1225}{64} a^4 + \frac{441}{32} a^5 + \frac{1617}{128} a^6 + \frac{429}{128} a^7 + \frac{6335}{32768} a^8\right)\left(\frac{35}{64} - \frac{7\xi_0^2}{8(4\xi_0^2 + \pi^2)} + \frac{7\xi_0^2}{16(\xi_0^2 + \pi^2)} - \frac{\xi_0^2}{2(4\xi_0^2 + \pi^2)} + \frac{\xi_0^2}{64(\xi_0^2 + 4\pi^2)}\right) \sinh 2\xi_0 + \frac{35}{64} \xi_0 a \left(1 + \frac{a}{2}\right)\right.$$

$$\cdot\left(4 + 12a + \frac{81}{4} a^2 + \frac{41}{2} a^3 + \frac{407}{32} a^4 + \frac{143}{32} a^5 + \frac{715}{1024} a^6\right)\Bigg]. \qquad (4)$$

For the second moment in elliptical coordinates,

$$\mu_2 = \int_0^{2\pi} \int_0^{\xi_0} \left[\left(\frac{\partial F}{\partial \xi}\right)^2 + \left(\frac{\partial F}{\partial \eta}\right)^2\right] d\xi d\eta$$

$$= \int_0^{2\pi} \int_0^{\xi_0} \left[\frac{N^2 \pi^2}{4 \xi_0^2} (1 + a \sin^2 \eta)^{2N} \cos^{2(N-1)} \frac{\pi\xi}{2\xi_0} \sin^2 \frac{\pi\xi}{2\xi_0}\right] d\xi d\eta$$

$$+ \int_0^{2\pi} \int_0^{\xi_0} \left[a^2 N^2 (1 + a \sin^2 \eta)^{2(N-1)} \sin^2 2\eta \cos^{2N} \frac{\pi\xi}{2\xi_0}\right] d\xi d\eta .$$

For $N = 1$,

$$\mu_{2,1} = \frac{\pi^2}{4 \xi_0^2} \left(\int_0^{2\pi} (1 + a \sin^2 \eta) \, d\eta\right) \left(\int_0^{\xi_0} \sin^2 \frac{\pi\xi}{2\xi_0} \, d\xi\right)$$

$$+ a^2 \left(\int_0^{2\pi} \sin^2 2\eta \, d\eta\right) \left(\int_0^{\xi_0} \cos^2 \frac{\pi\xi}{2\xi_0} \, d\xi\right) .$$

Here,

$$\int_0^{2\pi} (1 + a \sin^2 \eta)^2 \, d\eta = 2\pi \left(1 + a + \frac{3}{8} a^2\right)$$

$$\int_0^{\xi_0} \sin^2 \frac{\pi\xi}{2\xi_0} \, d\xi = \frac{1}{2} \xi_0 \quad .$$

$$\int_0^{2\pi} \sin^2 2\eta \quad d\eta = \pi$$

$$\int_0^{\xi_0} \cos^2 \frac{\pi\xi}{2\xi_0} \quad d\xi = \frac{1}{2}\,\xi_0 \,.$$

Therefore,

$$\mu_{2,1} = \frac{\pi^3}{4\,\xi_0{}^2}\left(1 + a + \frac{3}{8}\,a^2\right) + \frac{\pi}{2}\,a^2\,\xi_0 \,. \qquad (5)$$

For N = 2,

$$\mu_{2,2} = \frac{\pi^2}{\xi_0{}^2}\left(\int_0^{2\pi} (1 + a \sin^2 \eta)^4 \quad d\eta\right)\left(\int_0^{\xi_0} \frac{1}{4}\,\sin^2 \frac{\pi\xi}{\xi_0} \quad d\xi\right)$$

$$+ 4a^2\left(\int_0^{2\pi} (1 + a \sin^2 \eta)^2 \sin^2 2\eta \quad d\eta\right)\left(\int_0^{\xi_0} \cos^4 \frac{\pi\xi}{2\xi_0} \quad d\xi\right).$$

Here,

$$\int_0^{2\pi} (1 + a \sin^2 \eta)^4 \quad d\eta = 2\pi\left(1 + 2a + \frac{9}{4}\,a^2 + \frac{5}{4}\,a^3 + \frac{35}{128}\,a^4\right)$$

$$\int_0^{\xi_0} \frac{1}{4}\,\sin^2 \frac{\pi\xi}{\xi_0} \quad d\xi = \frac{1}{8}\,\xi_0$$

$$\int_0^{2\pi} (1 + a \sin^2 \eta)^2 \sin^2 2\eta \quad d\eta = \pi\left(1 + a + \frac{5}{16}\,a^2\right)$$

$$\int_0^{\xi_0} \cos^4 \frac{\pi\xi}{2\xi_0} \quad d\xi = \frac{3}{8}\ \xi_0 \,.$$

Therefore,

$$\mu_{2,2} = \frac{\pi^3}{4\,\xi_0^{\,2}} \left(1 + 2a + \frac{9}{4}\ a^2 + \frac{5}{4}\ a^3 + \frac{35}{128}\ a^4\right) + \frac{3}{2}\ \pi\,\xi_0 \left(1 + a + \frac{5}{16}\ a^2\right) a^2 \,. \qquad (6)$$

For N = 3,

$$\mu_{2,3} = \frac{9\,\pi^2}{4\,\xi_0^{\,2}} \left(\int_0^{2\pi} (1 + a \sin^2 \eta)^6 \ d\eta\right) \left(\int_0^{\xi_0} \cos^4 \frac{\pi\xi}{2\xi_0} \ \sin^2 \frac{\pi\xi}{2\xi_0} \ d\xi\right) + 9\,a^2 \left(\int_0^{2\pi} (1 + a \sin^2 \eta)^4 \sin^2 2\eta \ d\eta\right) \left(\int_0^{\xi_0} \cos^6 \frac{\pi\xi}{2\xi_0} \ d\xi\right).$$

Here,

$$\int_0^{2\pi} (1 + a \sin^2 \eta)^6 \ d\eta = 2\pi \left(1 + 3a + \frac{45}{8}\ a^2 + \frac{25}{4}\ a^3 + \frac{525}{128}\ a^4 + \frac{189}{128}\ a^5 + \frac{231}{1024}\ a^6\right)$$

$$\int_0^{\xi_0} \cos^4 \frac{\pi\xi}{2\xi_0} \ \sin^2 \frac{\pi\xi}{2\xi_0} \quad d\xi = \frac{1}{16}\ \xi_0$$

$$\int_0^{2\pi} (1 + a \sin^2 \eta)^4 \sin^2 2\eta \quad d\eta = \pi \left(1 + 2a + \frac{15}{8}\ a^2 + \frac{7}{8}\ a^3 + \frac{21}{32}\ a^4\right)$$

$$\int_0^{\xi_0} \cos^6 \frac{\pi\xi}{2\xi_0} \, d\xi = \frac{5}{16} \xi_0 .$$

Therefore,

$$\mu_{2,3} = \frac{9\pi^3}{32\xi_0^2} \left(1 + 3a + \frac{45}{8} a^2 + \frac{25}{4} a^3 + \frac{525}{128} a^4 + \frac{189}{128} a^5 + \frac{231}{1024} a^6\right) + \frac{45}{16} \xi_0 a^2 \pi \left(1 + 2a + \frac{15}{8} a^2 + \frac{7}{8} a^3 + \frac{21}{32} a^4\right). \tag{7}$$

For N = 4,

$$\mu_{2,4} = \frac{4\pi^2}{\xi_0^2} \left(\int_0^{2\pi} (1 + a \sin^2 \eta)^8 \, d\eta\right)\left(\int_0^{\xi_0} \cos^6 \frac{\pi\xi}{2\xi_0} \sin^2 \frac{\pi\xi}{2\xi_0} \, d\xi\right) + 16a^2 \left(\int_0^{2\pi} (1 + a \sin^2 \eta)^6 \sin^2 2\eta \, d\eta\right) \left(\int_0^{\xi_0} \cos^8 \frac{\pi\xi}{2\xi_0} \, d\xi\right) .$$

Here,

$$\int_0^{2\pi} (1 + a \sin^2 \eta)^8 \, d\eta = 2\pi \left(1 + 4a + \frac{21}{2} a^2 + \frac{35}{2} a^3 + \frac{1225}{64} a^4 + \frac{441}{32} a^5 + \frac{1617}{128} a^6 + \frac{429}{128} a^7 + \frac{6335}{32,768} a^8\right)$$

$$\int_0^{\xi_0} \cos^6 \frac{\pi\xi}{2\xi_0} \sin^2 \frac{\pi\xi}{2\xi_0} \, d\xi = \frac{5}{128} \xi_0$$

$$\int_0^{2\pi} (1 + a \sin^2 \eta)^6 \sin^2 2\eta \, d\eta = \left(1 + 3a + \frac{75}{16} a^2 + \frac{35}{8} a^3 + \frac{75}{128} a^4 + \frac{99}{128} a^5 + \frac{429}{4096} a^6\right) \pi$$

$$\int_0^{\xi_0} \cos^8 \frac{\pi \xi}{2\xi_0} \, d\xi = \frac{55}{16} \xi_0.$$

Therefore,

$$\mu_{2,4} = \frac{5}{16} \frac{\pi^3}{\xi_0} \left(1 + 4a + \frac{21}{2} a^2 + \frac{35}{2} a^3 + \frac{1225}{64} a^4 + \frac{441}{32} a^5 + \frac{1617}{128} a^6 + \frac{429}{128} a^7 + \frac{6335}{32,768} a^8\right) + 55 \xi_0 \pi a^2 \left(1 + 3a + \frac{75}{16} a^2 + \frac{35}{8} a^3 + \frac{75}{128} a^4 + \frac{99}{128} a^5 + \frac{429}{4096} a^6\right). \tag{8}$$

III. GRAPHS

Figures 1 through 5 are plots of illuminations and far-field power patterns. All illuminations are plotted with peak amplitude equal to unity. For all far-field powers, the logarithm of the power is plotted with the center of the main lobe normalized to zero decibels.

FIGURES:

Fig. 1 = $\left[(1 + a \sin^2 \eta) \cos \frac{\pi \xi}{2 \xi_0}\right]^N$ Illumination of Elliptical Aperture Along Major Axis. N = 1, 2, 3, 4

Fig. 2 = Far-Field Power for $\left[(1 + a \sin^2 \eta) \cos \frac{\pi \xi}{2 \xi_0}\right]$ Elliptical Illumination Major Axis

Fig. 3 = Far-Field Power for $\left[(1 + a \sin^2 \eta) \cos \frac{\pi \xi}{2 \xi_0}\right]^2$ Elliptical Illumination Along Major Axis.

Fig. 4 = Far-Field Power for $\left[(1 + a \sin^2 \eta) \cos \frac{\pi \xi}{2 \xi_0}\right]^3$ Elliptical Illumination Along Major Axis.

Fig. 5 = Far-Field Power for $\left[(1 + a \sin^2 \eta) \cos \frac{\pi \xi}{2 \xi_0}\right]^4$ Elliptical Illumination Along Major Axis.

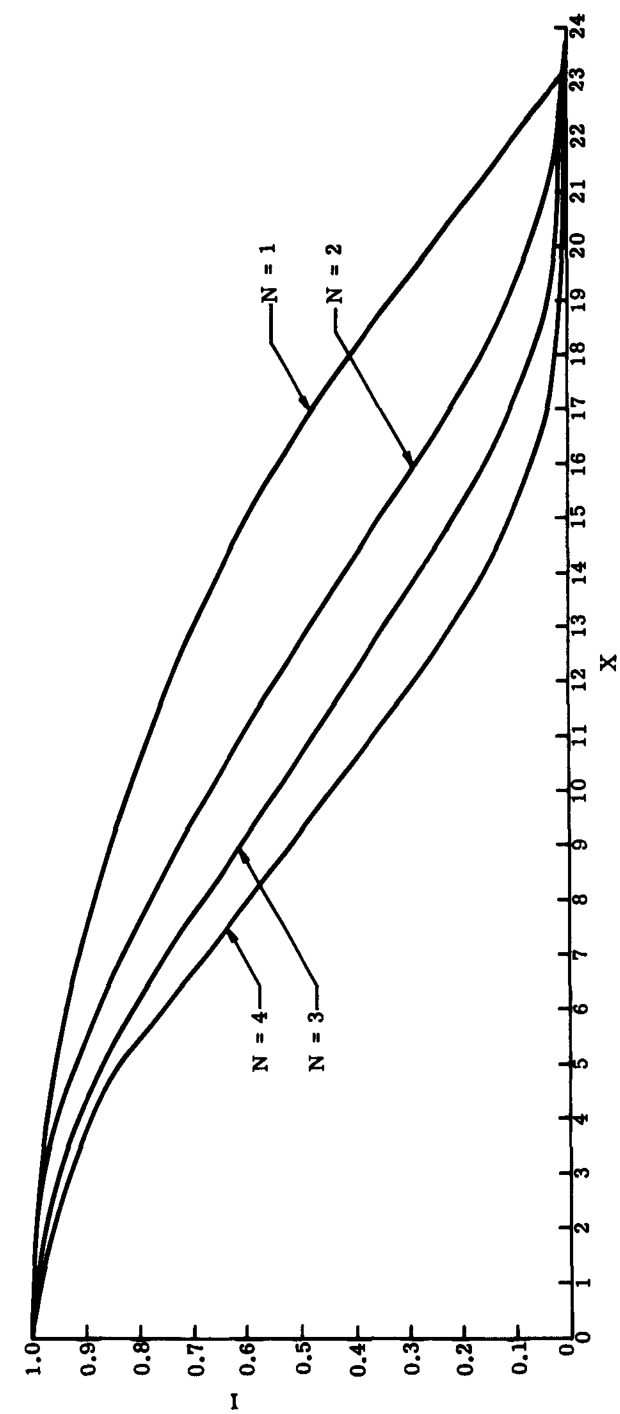

Fig. 1. $\left[(1 + a \sin^2 \eta) \cos \frac{\pi\xi}{2\xi_0}\right]^N$ Illumination of Elliptical Aperture Along Major Axis (N = 1, 2, 3, 4)

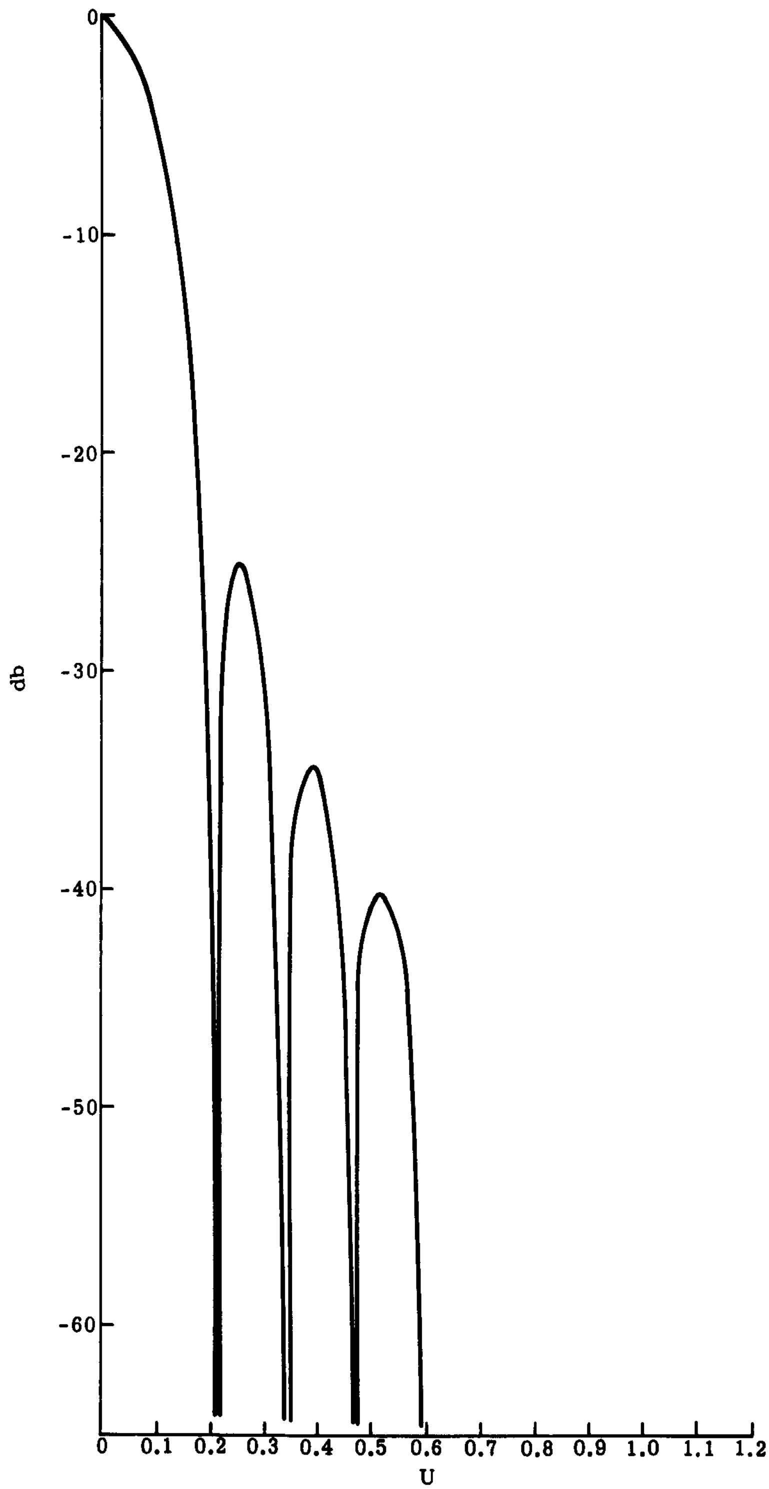

Fig. 2. Far-Field Power for $\left[(1 + a \sin^2 \eta) \cos \frac{\pi\xi}{2\xi_0}\right]$ Elliptical Illumination Along Major Axis

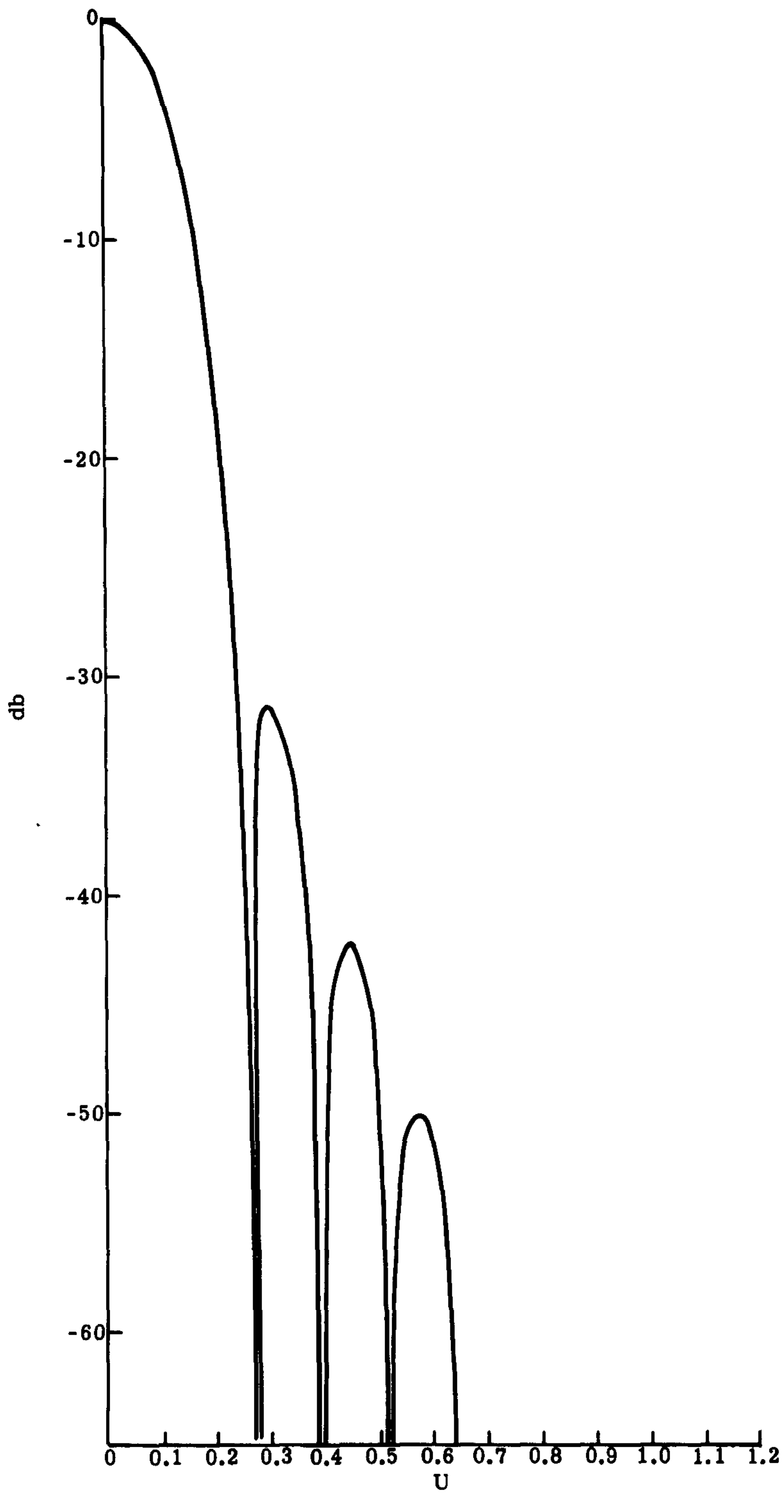

Fig. 3. Far-Field Power for $\left[(1 + a \sin^2 \eta) \cos \frac{\pi\xi}{2\xi_0}\right]^2$ Elliptical Illumination Along Major Axis

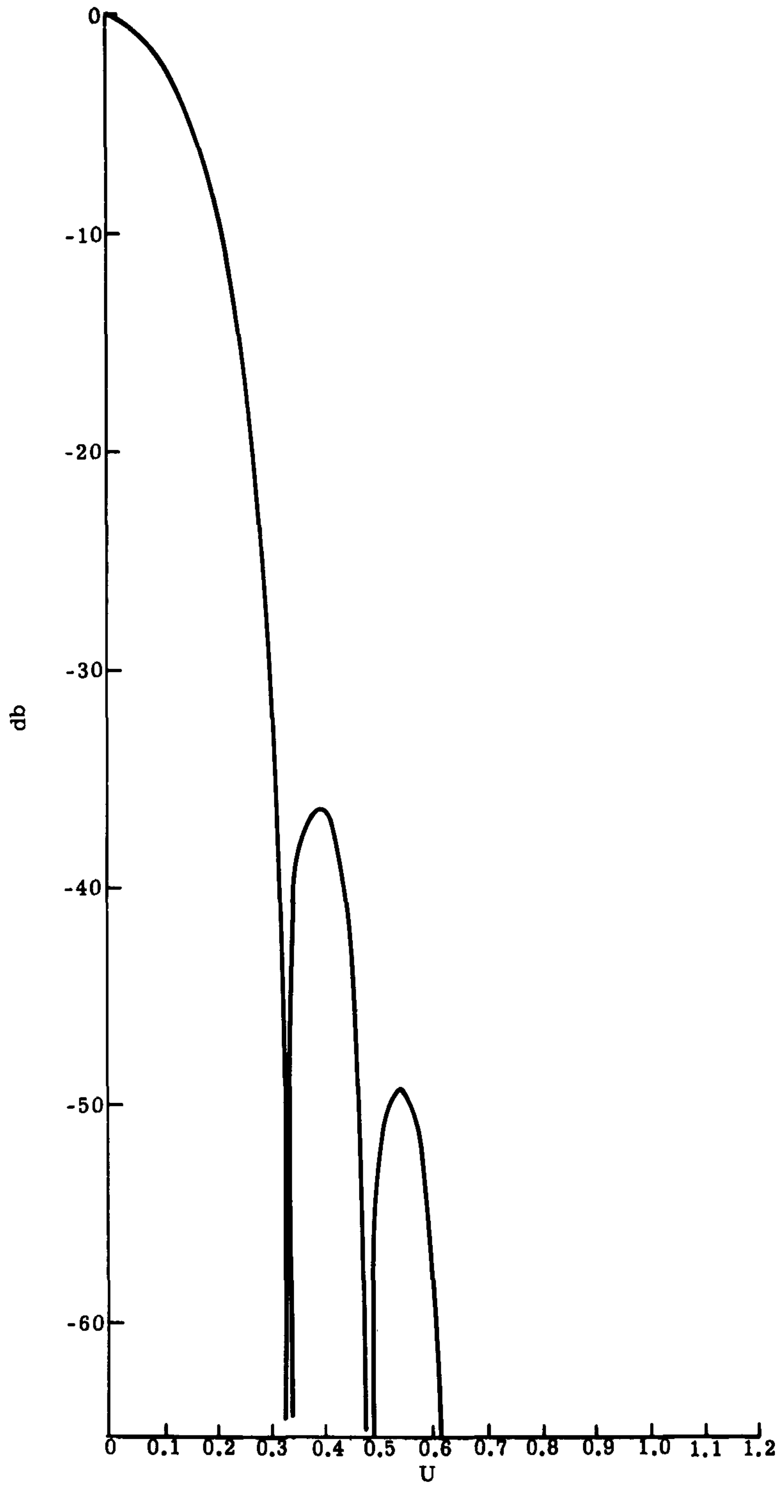

Fig. 4. Far-Field Power for $\left[(1 + a \sin^2 \eta) \cos \frac{\pi\xi}{2\xi_0}\right]^3$ Elliptical Illumination Along Major Axis

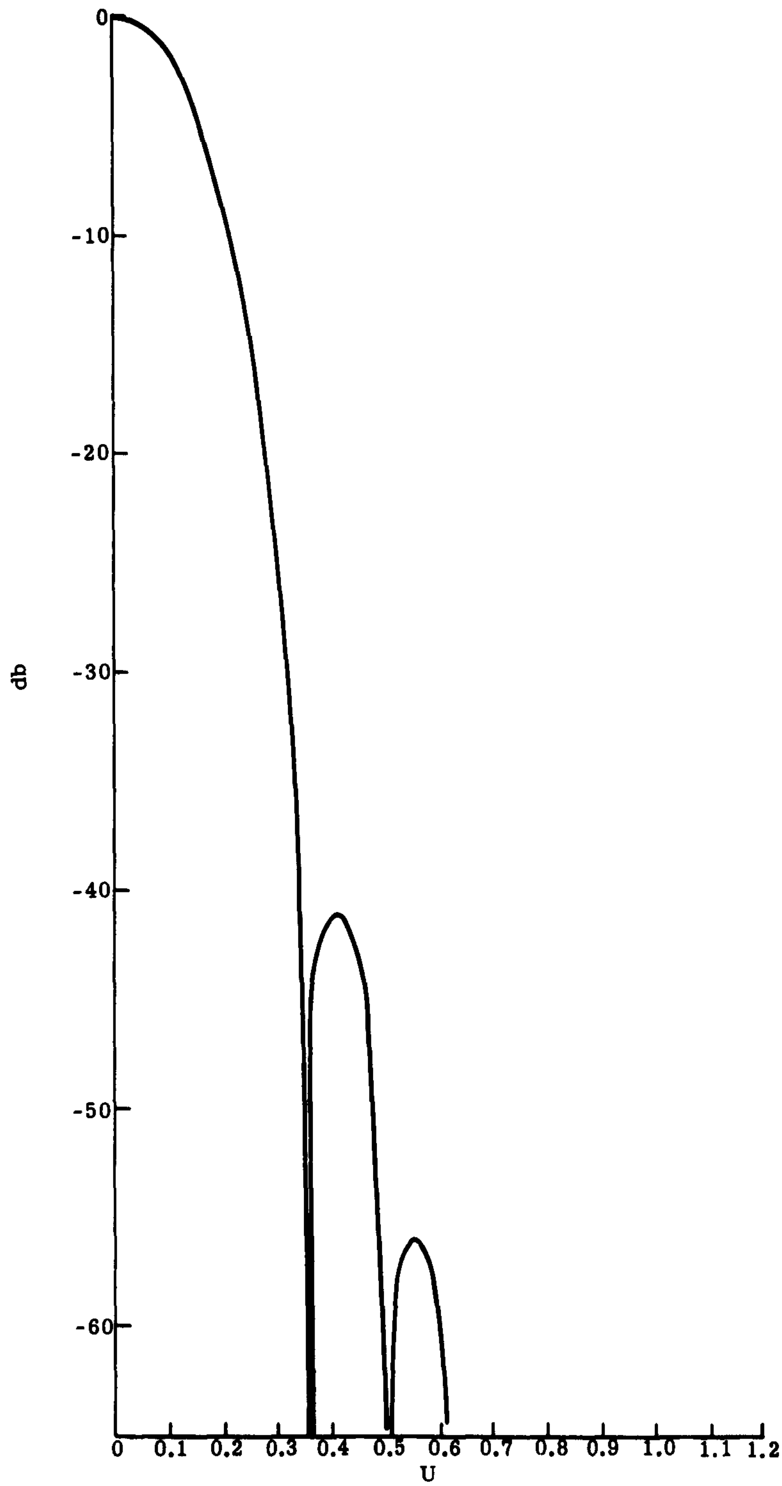

Fig. 5. Far-Field Power for $\left[(1 + a \sin^2 \eta) \cos \frac{\pi\xi}{2\xi_0}\right]^4$ Elliptical Illumination Along Major Axis

IV. TABLE OF COMPARISON

Type	Illumination	Function	Beamwidth	Sidelobe	Moment
Circular	Optimum	$J_0 (K_{00} r)$	2.2	-28.4	5.794
Circular	Uniform	A constant	1.6	-16.8	∞
Circular	Nonoptimum	$\cos \left(\frac{\pi r}{2a}\right)$	2.1	-25.6	5.83
Circular	Nonoptimum	$\cos^2 \left(\frac{\pi r}{2a}\right)$	2.3	-34	7.17
Circular	Nonoptimum	$\cos^3 \left(\frac{\pi r}{2a}\right)$	2.6	-41	9.41
Elliptical	Optimum	$Ce_0 (\xi, q)\, ce_0 (\eta, q)$	0.11	-36	0.0718
Elliptical	Uniform	A constant	0.075	-17.5	∞
Elliptical	Nonoptimum	$(1 + a \sin^2 \eta) \cos \frac{\pi \xi}{2\xi_0}$	0.08	-24	0.0745
Elliptical	Nonoptimum	$\left[(1 + a \sin^2 \eta) \cos \frac{\pi \xi}{2\xi_0}\right]^2$	0.095	-30	0.0962
Elliptical	Nonoptimum	$\left[(1 + a \sin^2 \eta) \cos \frac{\pi \xi}{2\xi_0}\right]^3$	0.115	-36.25	0.1455
Elliptical	Nonoptimum	$\left[(1 + a \sin^2 \eta) \cos \frac{\pi \xi}{2\xi_0}\right]^4$	0.135	-41	0.2583
Square	Optimum	$\cos \frac{\pi x}{2a} \cos \frac{\pi y}{2a}$	2.0	-22.8	4.9868

APPENDIX A

COMPUTER PROGRAMS

Program 1.

This program is designed to compute moments of the four selected nonoptimum illuminations for elliptical antennas.

$$\left(\left[(1 + a \sin^2 \eta) \cos \frac{\pi \xi}{2\xi_0}\right]^N ; N = 1, 2, 3, 4\right)$$

The value of a is chosen to be 99.

The program calculates Eqs (1) through (8) and the moments are obtained by

$$\frac{(5)}{(1)}, \quad \frac{(6)}{(2)}, \quad \frac{(7)}{(3)}, \quad \frac{(8)}{(4)}, \text{ respectively.}$$

```
07300        PRINT2
07324      2 FORMAT(19H   TABLE OF MOMENTS/)
07392        AH=23.1
07416        PI=3.14159
07440        SI=0.277
07464        A=99.0
07488        R=(AH**2)*PI*A*SI*(1.+.5*A)/4.
07632        S=(.1457*PI**2)/(4.*SI**2+PI**2)
07788        T=(AH**2)*PI*(1.+A*(1.+0.375*A))
07920        ZU01=T*S+R
07968        PRINT 22,ZU01
07992     22 FORMAT(11H   MU(0,1)=E15.8)
08044        ZU21=(PI**3/(4.*SI))*(1.+A*(1.+ 0.375*A)) +1.5708*(A**2)*SI
08284        PRINT 21,ZU21
08308     21 FORMAT(11H   MU(2,1)=E15.8)
08360        V=ZU21/ZU01
08396        PRINT 1,V
08420      1 FORMAT(9H  MOMENT=E15.8///)
08482        P=PI**2
08518        Q=SI**2
08554        PA=(1.+A*(2.+A*(2.25+A*(1.25+(35.*A)/128.))))
08686        PSI=SI*(.375-SI*.5828*(1./(4.*Q+P)-.0625/(P+Q)))
08878        P1A=A*(2.+A*(3.+A*(1.875+A*0.4375)))
08986        PA1=SI*0.1875*P1A
09034        YU02=AH**2*(PI)*(PA*PSI+PA1)
09142        PRINT 31,YU02
09166     31 FORMAT(11H   MU(0,2)=E15.8)
09218        RA=1.5*PI*SI*(1.+A*(1.+0.3125*A))*(A**2)
09398        R1A=(0.25*(PI**3)/SI)*PA+RA
09482        PRINT 32,R1A
09506     32 FORMAT(11H   MU(2,2)=E15.8)
09558        W=R1A/YU02
09594        PRINT 1,W
09618        H=(2.5-.5828*SI*(7.5/(4.*Q+P)-.75/(P+Q)+.5/(4.*Q+9.*P)))
09918        Q=(3.+A*(7.5+A*(9.375+A*(6.5626+A*(2.38125+0.38671*A)))))
10062        E=(1.+A*(3.+A*(5.625+A*(6.25+A*(4.101+A*(1.476+.2255*A))))))
10230        U03=(AH**2)*PI*SI*(0.125*E*H+.15625*A*Q)
10410        PRINT 42,U03
10434     42 FORMAT(11H   MU(0,3)=E15.8)
10486        E1=(0.28125*(PI**3)/SI)*E
10558        F=(1.+A*(2.+A*(1.875+A*(.875+0.65625*A))))
10678        F1=2.8125*PI*SI*(A**2)*F
10786        U23=E1+F1
10822        PRINT 41,U23
10846     41 FORMAT(11H   MU(2,3)=E15.8)
10898        O=U23/U03
10934        PRINT 1,O
10958        C=10.5
10982        D=17.5
11006        E=19.140625
11030        F=13.78125
11054        G=12.6328125
11078        H=3.3515625
11102        P=6335./32768.
11138        SA=(1.+A*(4.+A*(C+A*(D+A*(E+A*(F+A*(G+A*(H+A*P))))))))
11354        S1=7.0/(4.*SI**2+PI**2)/2.
11462        S2=0.875/(SI**2+PI**2)
11558        S3=1.00/(4.*SI**2+9.*PI**2)
11678        S4=1./(32.*(SI**2+4.*PI**2))
11798        C=4.
11822        D=12.
11846        E=20.25
11870        F=20.5
11894        G=12.71875
11918        H=4.46875
11942        P=0.6982422
11966        TA=.5465*A*50.5*(C+A*(D+A*(E+A*(F+A*(G+A*(H+P*A))))))
12194        EXTRA=1.09375-.2914*SI*(S1-S2+S3-S4)
12326        U04=.25*(AH**2)*PI*SI*(SA*EXTRA+TA)
12458        PRINT 51,U04
12482     51 FORMAT(11H   MU(0,4)=E15.8)
12534        C=1.
12558        D=3.
12582        E=75./16.
12618        F=35./8.
12654        G=75./128.
12690        H=99./128.
12726        P=429./4096.
12762        RA=55.*SI*PI*(A**2)*(C+A*(D+A*(E+A*(F+A*(G+A*(H+P*A))))))
13038        TT=0.3125*(PI**3/SI)*SA+RA
13122        PRINT 52,TT
13146     52 FORMAT(11H   MU(2,4)=E15.8)
13198        T=TT/U04
13234        PRINT 1,T
13258        END
```

<u>Program 2.</u>

This program is divided into two parts. Part I is a program to compute far-field powers of

G_1 (u, o) [page 9] and G_2 (u, o) [page 12]

The increment of u is approximately 0. 01. Part II, a similar program, computes

G_3 (u, o) [page 15] and G_4 (u, o) [page 1[illegible]]

The value of a is chosen to be 99.

Part I

```
07300     3 FORMAT(18H   VOLTAGE-POWER 1/)
07366     4 FORMAT(18H   VOLTAGE-POWER 2/)
07432     5 FORMAT(F10.4)
07454     7 FORMAT(12H    G1(0,0)=E15.8)
07508     8 FORMAT(16H    LOG G1(0,0)=E15.8)
07570    10 FORMAT(4H    F10.5)
07608    11 FORMAT(3H   ///)
07654    12 FORMAT(10H   G(U,0)=E15.8)
07704    17 FORMAT(12H    G2(0,0)=E15.8)
07758    18 FORMAT(16H    LOG G2(0,0)=E15.8)
07820       DIMENSION Y0(60),Y1(60),Y2(60),Y3(60),Y4(60),Y5(60),Y6(60)
07820       DO 6 I=1,60
07832     6 READ 5, Y0(I)
07916       DO 70 J=1,60
07928    70 READ 5, Y1(J)
08012       PRINT 3
08036       A=99.
08072       PI=22./7.
08120       SI=0.277
08156       B=PI*(1.+SI**2)/(16.*SI**2+PI**2)
08336       A0=49.5+101.0*PI*B
08408       A2=101.0+A*PI*B
08480       A4=49.5
08516       PRINT 7,A0
08540       IF(A0) 37,38,38
08596    37 A0=-A0
08644    38 C=LOGF(A0)
08680       PRINT 8,C
08704       I=0
08740       AX=0.0
08776   100 I=I+1
08824       AX=AX+.25
08872       Y2(I)=(2./AX)*Y1(I)-Y0(I)
09016       Y3(I)=(4./AX)*Y2(I)-Y1(I)
0916C       Y4(I)=(6./AX)*Y3(I)-Y2(I)
09304       POWER=A0*Y0(I)+A2*Y2(I)+A4*Y4(I)
09520       IF(POWER) 57,58,58
09576    57 POWER=-POWER
09624    58 P=LOGF(POWER)
09660       POWER=8.6858*(P-C)
09720       PRINT 10,POWER
09744       IF(I-60)100,108,108
09812   108 PRINT 11
09836       PRINT 4
09860       P=PI**2
09908       Q=SI**2
09956       B=0.5*P/(4.*Q+P)
10064       A0=(1.+A*(1.+.375*A))*B+.25*A*(1.+.5*A)
10268       PRINT 17,A0
10292       IF(A0) 97,98,98
10348    97 A0=-A0
10396    98 T=LOGF(A0)
10432       PRINT 18,T
10456       A2=(1.+.5*A)*A*B+.25*(2.+A*(2.+7.*A/8.))
10648       A4=.125*A*(A*B+2.*(1.+.5*A))
10816       A6=(A**2)/32.
10876       I=0
10912       AX=0.0
10948   200 I=I+1
10996       AX=AX+.25
11044       Y2(I)=(2./AX)*Y1(I)-Y0(I)
11188       Y3(I)=(4./AX)*Y2(I)-Y1(I)
11332       Y4(I)=(6./AX)*Y3(I)-Y2(I)
11476       Y5(I)=(8./AX)*Y4(I)-Y3(I)
11620       Y6(I)=(10./AX)*Y5(I)-Y4(I)
11764       POWER=A0*Y0(I)+A2*Y2(I)+A4*Y4(I)+A6*Y6(I)
12052       IF(POWER) 87,88,88
12108    87 POWER=-POWER
12156    88 P=LOGF(POWER)
12192       POWER=8.8658*(P-T)
12252       PRINT 10,POWER
12276       IF(I-60)200,208,208
12344   208 PAUSE
12356       END
```

Part II

```
07300     5 FORMAT(18H    VOLTAGE-POWER 3/)
07366     6 FORMAT(18H    VOLTAGE-POWER 4/)
07432    13 FORMAT(13H    LN G(0,0)=E15.8)
07488    15 FORMAT(19H                    F10.5/)
07560    16 FORMAT(11H    G4(0,0)=E15.8)
07612    17 FORMAT(F10.4)
07634       DIMENSION Y0(60),Y1(60),Y2(60),Y3(60),Y4(60),Y5(60),Y6(60)
07634       DIMENSION Y7(60),Y8(60),Y9(60),Y10(60)
07634       A=99.
07670       PI=22./7.
07718       SI=.277
07754       P=PI**2
07802       Q=SI**2
07850       DO 1 I=1,60
07862     1 READ 17, Y0(I)
07946       DO 2 J=1,60
07958     2 READ 17, Y1(J)
08042       PRINT 5
08066       B1=1./(16.*Q+P)
08138       B2=1./(16.*Q+9.*P)
08246       B=B1-B2
08294       PONE=1.+A*(1.5+A*(1.125+A*0.3125))
08402       PTWO=A*(1.5+A*(1.5+15.*A/32.))
08510       SQ=1.+(.277*.277)
08570       A0=1.5*P*SQ*B*PONE+2.*PTWO/3.
08714       A2=1.5*P*SQ*B*PTWO+2.*(2.+A*(3.+A*(21./8.+13.*A/16.)))/3.
08978       A4=.5625*P*SQ*50.5*(A**2)*B+2.*A*(1.5+A*(1.5+.5*A))/3.
09254       A6=(3./64.)*P*SQ*(A**3)*B+.25*50.5*A*A
09458       A8=A**3/48.
09518       A10=0.0
09554       IF(A0) 37,38,38
09610    37 A0=-A0
09658    38 POWER=LOGF(A0)
09694       PRINT 13,POWER
09718       J=0
09754   200 I=0
09790       AX=0.0
09826       J=J+1
09874   100 I=I+1
09922       AX=AX+.25
09970       Y2(I)=(2./AX)*Y1(I)-Y0(I)
10114       Y3(I)=(4./AX)*Y2(I)-Y1(I)
10258       Y4(I)=(6./AX)*Y3(I)-Y2(I)
10402       Y5(I)=(8./AX)*Y4(I)-Y3(I)
10546       Y6(I)=(10./AX)*Y5(I)-Y4(I)
10690       Y7(I)=(12./AX)*Y6(I)-Y5(I)
10834       Y8(I)=(14./AX)*Y7(I)-Y6(I)
10978       Y9(I)=(16./AX)*Y8(I)-Y7(I)
11122       Y10(I)=(18./AX)*Y9(I)-Y8(I)
11266       G=A0*Y0(I)+A2*Y2(I)+A4*Y4(I)+A6*Y6(I)+A8*Y8(I)+A10*Y10(I)
11698       IF(G) 57,58,58
11754    57 G=-G
11802    58 GG=LOGF(G)
11838       R=8.6858*(GG-POWER)
11898       PRINT 15,R
11922       IF(I-60) 100,206,206
11990   206 IF(J-2) 207,208,208
12058   208 PAUSE
12070   207 PRINT 6
12094       SS=(.277*.277)*(2./(4.*Q+P)-.125/(P+Q))
12286       C=35./128.
12334       TTT=(1.+A*(2.+A*(2.25+A*(1.25+C*A))))*(.375-SS)
12514       A0=TTT+.1875*A*(2.+A*(3.+A*(1.875+7.*A/16.)))
12694       PRINT 16,A0
12718       IF(A0) 67,68,68
12774    67 A0=-A0
12822    68 POWER=LOGF(A0)
12858       PRINT 13,POWER
12882       C=5.25
12918       D=3.25
12954       E=49./64.
13002       T=0.1375*(2.+A*(4.+A*(C+A*(D+A*E))))
13146       A2= A*(2.+A*(3.+A*(1.875+0.4375*A)))*(0.375-SS)+T
13326       C=7./32.
13374       R=(A**2)*(0.75+A*(0.75+A*C))
13506       T=0.1875*A*(2.+A*(3.+A*(2.+0.5*A)))
13662       A4=R*(0.375-SS)+T
13734       R= 0.125*(A**3)*(1.+0.5*A)
13854       T=(A*A)*0.1875*(0.75+A*(0.75+29.*A/128.))
14010       A6=R*(0.375-SS)+T
14082       A8=(A**4/128.)*(0.375-SS)+(3./128.)*A**3*(1.+0.5*A)
14346       A10=3.*A**4/2048.
14418       GO TO 200
14426       END
```