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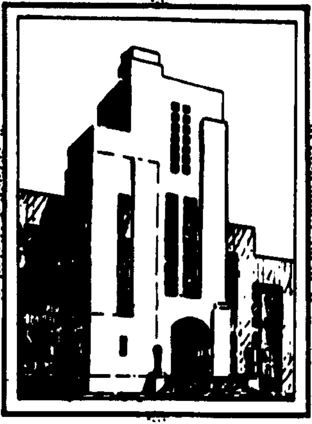

DEPARTMENT OF THE NAVY
DAVID TAYLOR MODEL BASIN

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MOTIONS OF A SPAR RAFT IN REGULAR WAVES

by

Robert Taussig

HYDROMECHANICS LABORATORY

RESEARCH AND DEVELOPMENT REPORT

May 1963

Report 1673

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NOTATION

A	Amplitude of incident surface wave
a_0	Radius of circle on which spar axes are located
$a_i(z_i)$	Radius of i^{th} spar as a function of z_i
g	Acceleration of gravity
I_i, J_i, K_i	Moment of inertia of i^{th} spar about x-, y-, z-axes, respectively
k	Wave number of incident waves $= \omega^2/g$
M	Mass of one spar (assumed equal for each spar)
$m(z_i')$	Mass per unit length of i^{th} spar
N	Number of spars
R	Space-fixed cylindrical polar coordinate
R'	Body-fixed cylindrical polar coordinate
S_n, T_n	Defined in Equations [13a], [13b]
$S_i(z_i)$	Cross-sectional area of i^{th} spar $= \pi a_i^2(z_i)$
X_i, Y_i, Z_i	Components of force on i^{th} spar
x, y, z	Space-fixed Cartesian coordinates
x', y', z'	Body-fixed Cartesian coordinates
x_0, y_0, z_0	Linear displacements of structure
A_i, B_i, Γ_i	Components of moment on i^{th} spar
α, β, γ	Angular displacements of structure about x-, y-, z-axes
θ	Space-fixed cylindrical angular coordinate

θ'	Body-fixed cylindrical angular coordinate
θ_i	Equilibrium anguiar position of ith spar axis
ν	Kinematic viscosity of water
ρ	Density of water
Φ	Complete velocity potential
Φ_0	Velocity potential of incident wave
Φ_1	$\Phi - \Phi_0$
ω	Circular frequency of incident waves

ABSTRACT

A theoretical analysis is constructed for the hydrodynamic forces acting on a system of interconnected vertical, slender, axisymmetric bodies which are floating in presence of incident waves. The theory is based on linearized water wave potential theory and the use of slender body techniques. The resulting expressions for the hydrodynamic forces are used to predict the motions of such a system. The effects of viscous damping are also estimated.

INTRODUCTION

A spar raft as defined here consists of several long thin bodies of revolution rigidly interconnected so that they will float vertically in the water and support a platform or submerged weight. When regular waves are incident on such a structure, it will generally oscillate in six degrees of freedom. The purpose of this report is to provide an approximative method for calculating such motions.

The assumptions are: (1) that the spars are identical, (2) that their interconnections are made in such a way that the mass and the hydrodynamic effects of the connecting members may be neglected, and (3) that the individual spars are far enough apart for their hydrodynamic interactions to be neglected. The motions of a single spar buoy have been treated by Newman;[1] here his method is extended to the case of N spars arranged in a circle. In addition to including the hydrodynamic and inertial forces on several spars, it is necessary to extend Newman's analysis to allow for all six degrees of freedom. (In his problem, only three degrees of freedom involved nontrivial results.)

The basic assumptions of Newman's analysis are used here. In particular, it is assumed that the wave amplitudes and body motions are small enough that linearized free surface theory may be applied and that

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[1] References are listed on page 30.

the spar radii are small enough compared to wavelength and spar separations that slender body theory may be used. Equations of motion are derived on these bases. These equations predict motions which are undamped; thus they are valid only for frequencies which are not near the resonance frequencies.

Near the resonance frequencies, it is necessary to consider the damping due to wave generation, and this report shows that forces of higher order in terms of spar radii must be included. The leading damping forces are found, thus providing equations of motion valid near resonance.

In addition, this report indicates that viscous forces depend linearly on the velocities for axial motions, and these forces are found explicitly.

GEOMETRY AND COORDINATES

It is convenient to define several coordinate systems. With the structure floating at rest, we place the origin of a space-fixed reference frame at the undisturbed free surface over the center of gravity of the structure. Let the Cartesian coordinates of a point in this system be (x, y, z), with the z-axis directed upwards. In this same system we define cylindrical coordinates (R, θ, z):

$$R^2 = x^2 + y^2 \quad ; \quad \theta = \tan^{-1} \frac{y}{x}$$

or

$$x = R \cos \theta \quad ; \quad y = R \sin \theta$$

z is here the same as the Cartesian coordinate z.

Let the undisturbed axis of the i^{th} spar be located at $R = a_0$, $\theta = \theta_i$. We define another set of space-fixed coordinates, (x_i, y_i, z_i), with origin at $R = a_0$, $\theta = \theta_i$, $z = 0$. Let the cylindrical coordinates of a point in this system be (R_i, λ_i, z_i), with the latter having the same orientation as the previous cylindrical system.

In the undisturbed condition, the surface of the i^{th} spar will be specified by the equation:

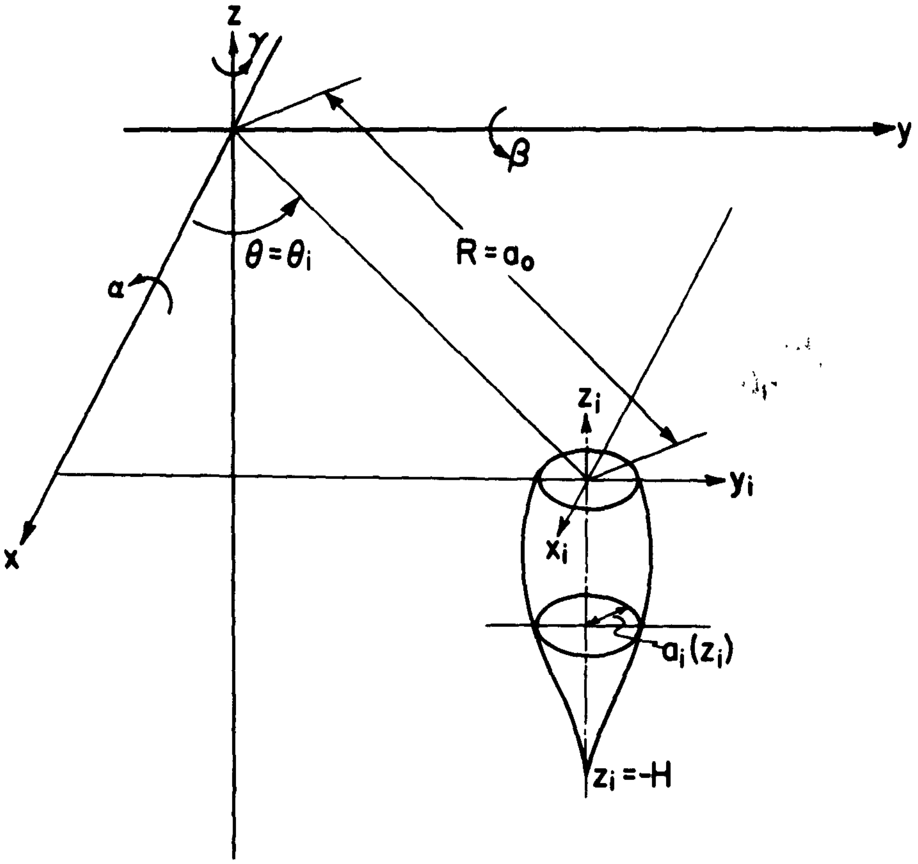

Figure 1 - The i^{th} Spar in Its Equilibrium Position

$$R_i = a_i(z_i) \quad ; \quad z_i \geq -H_i$$

These quantities are all shown in Figure 1.

Finally, we introduce primed coordinate systems which correspond to each of the systems just mentioned, but which are fixed in the body. When the body is in its equilibrium position, the primed and unprimed systems coincide.

The linear displacement of the raft will be described by three displacement variables, x_0, y_0, z_0; the angular displacement by α, β, γ, which are the positive rotations, respectively, about the x-, y-, z-axes.

Since the motions are assumed to be small enough that squares and products of these variables are negligible, the location of the raft is completely specified, and the three angular displacement variables can be treated as the components of a vector.

Let $\underline{r}'$ be the position vector of a point fixed in the raft, where $\underline{r}' = (x', y', z')$. In terms of the space-fixed coordinates, $\underline{r} = (x, y, z)$, we have, to first order in small quantities:

$$\underline{r} = \underline{r}' + \underline{r}_0 + \underline{\alpha} \times \underline{r}' \qquad [1]$$

where

$$\underline{r}_0 = (x_0, y_0, z_0)$$

$$\underline{\alpha} = (\alpha, \beta, \gamma)$$

In terms of components, this equation is equivalent to

$$\begin{aligned} x &= x' + x_0 + \beta z' - \gamma y' \\ y &= y' + y_0 + \gamma x' - \alpha z' \\ z &= z' + z_0 + \alpha y' - \beta x' \end{aligned} \qquad [1']$$

To first order also, it follows that

$$\begin{aligned} x' &= x - x_0 - \beta z + \gamma y \\ y' &= y - y_0 - \gamma x + \alpha z \\ z' &= z - z_0 - \alpha y + \beta x \end{aligned} \qquad [1'']$$

The unprimed coordinates are related by

$$\begin{aligned} x &= a_0 \cos \theta_i + x_i \\ y &= a_0 \sin \theta_i + y_i \\ z &= z_i \end{aligned} \qquad [2]$$

and the same equations hold if x, y, z and x_i, y_i, z_i are all primed.

It is assumed that there are incident waves which are described by the velocity potential

$$\Phi_0 (x, y, z, t) = \frac{\omega A}{k} e^{kz} \cos(kx - \omega t) \qquad [3]$$

where A is the amplitude of surface wave,

k is 2π/wavelength = wave number, and

ω is the circular frequency.

In the definition of the coordinates given above, the orientation of the x- and y-axes was not specified, except for the orientation of the plane which they defined. Now we specify that the x-axis points in the direction of propagation of the surface waves and the y-axis completes the right-hand system.

THE VELOCITY POTENTIAL

The surface of the i^{th} spar can be specified by the equation:

$$\begin{aligned} 0 &= F_i(x_i', y_i', z_i') \\ &= x_i'^2 + y_i'^2 - a_i^2(z_i') \\ &= [x_i - x_0 - \beta z_i + \gamma(a_0 \sin\theta_i + y_i)]^2 \\ &\quad + [y_i - y_0 - \gamma(a_0 \cos\theta_i + x_i) + \alpha z_i]^2 \\ &\quad - a_i^2 [z_i - z_0 - \alpha(a_0 \sin\theta_i + y_i) + \beta(a_0 \cos\theta_i + x_i)] \end{aligned}$$

The boundary condition on the i^{th} spar is then

$$\frac{\partial F_i}{\partial t} + (\nabla_i \Phi \cdot \nabla_i) F_i = 0 \qquad \text{on} \qquad F_i = 0$$

where ∇_i indicates the gradient in the (x_i, y_i, z_i) system, and $\Phi = \Phi(x_i, y_i, z_i, t)$ is the velocity potential (viz., Φ_0 plus a potential due to the presence and motion of the structure). After some simplification, we find that the boundary condition is

$$\begin{aligned} \frac{\partial\Phi}{\partial R_i} - a_i' \frac{\partial\Phi}{\partial z_i} = {} & [\dot{x}_0 + \dot{\beta}(z_i + a_i a_i') - \dot{\gamma} a_0 \sin\theta_i] \cos\lambda_i \\ & + [\dot{y}_0 - \dot{\alpha}(z_i + a_i a_i') + \dot{\gamma} a_0 \cos\theta_i] \sin\lambda_i \qquad [4] \\ & - [\dot{z}_0 + a_0(\dot{\alpha}\sin\theta_i - \dot{\beta}\cos\theta_i)] a_i' \quad \text{on} \quad R_i = a_i \end{aligned}$$

where $a_i' = \dfrac{da_i}{dz_i}$. Second and higher order terms in the motion variables have been consistently dropped.

Let $\Phi = \Phi_0 + \Phi_1$. We substitute this relation into the last equation to obtain a boundary condition on Φ_1. From Equations [2] and [3] we note that

$$\begin{aligned} \Phi_0 &= \frac{\omega A}{k} e^{kz} \cos(kx - \omega t) \\ &= \frac{\omega A}{k} e^{kz_i} \cos[k(a_0 \cos\theta_i + R_i \cos\lambda_i) - \omega t] \end{aligned} \qquad [5]$$

It follows that

$$\begin{aligned} \left(\frac{\partial\Phi_0}{\partial R_i} - a_i' \frac{\partial\Phi_0}{\partial z_i}\right)_{F_i=0} = \omega A e^{kz_i} \Big\{ & [(-1 + k a_i a_i' + \tfrac{3}{8} k^2 a_i^2) \cos\lambda_i \\ & + \tfrac{1}{8} k^2 a_i^2 \cos 3\lambda_i] \sin(k a_0 \cos\theta_i - \omega t) \\ & - [(a_i' + \tfrac{1}{2} k a_i) + \tfrac{1}{2} k a_i \cos 2\lambda_i] \cos(k a_0 \cos\theta_i - \omega t) \\ & + \dots \Big\} \end{aligned} \qquad [6]$$

where the omitted terms are of third and higher order in terms of the spar radius a_i and its derivative a_i', or of second and higher order in the motion variables. Neglecting now second-order terms in a_i and a_i', we find the condition on Φ_1:

$$\begin{aligned}\left(\frac{\partial\Phi_1}{\partial R_i} - a_i' \frac{\partial\Phi_1}{\partial z_i}\right)_{R_i=a_i} = &\left\{\dot{x}_0 + \dot{\beta} z_i - \dot{\gamma} a_0 \sin\theta_i \right.\\ &\left. + \omega A e^{kz_i} \sin(k a_0 \cos\theta_i - \omega t)\right\} \cos\lambda_i \\ &+ \left\{\dot{y}_0 - \dot{\alpha} z_i + \dot{\gamma} a_0 \cos\theta_i\right\} \sin\lambda_i \\ &+ \left\{\tfrac{1}{2} k a_i \omega A e^{kz_i} \cos(k a_0 \cos\theta_i - \omega t)\right\} \cos 2\lambda_i \\ &- \left\{\dot{z}_0 a_i' + a_0 a_i'(\dot{\alpha} \sin\theta_i - \dot{\beta} \cos\theta_i) \right. \qquad [7] \\ &\left. - (a_i' + \tfrac{1}{2} k a_i)\, \omega A e^{kz_i} \cos(k a_0 \cos\theta_i - \omega t)\right\}\end{aligned}$$

We note that the boundary condition is now applied at the surface of the undisturbed spar, and the right-hand side is evaluated in the space-fixed (x_i, y_i, z_i) coordinate system.

If this i^{th} spar were located alone in an infinite fluid with the above boundary condition valid for $-H < z_i < 0$, the solutions for Φ_1 by slender body theory would be

$$\begin{aligned}\Phi_1^* = \int_{-H}^{0} &\left\{\tfrac{1}{2} a_i \left[\dot{z}_0 a_i' + a_0 a_i'(\dot{\alpha} \sin\theta_i - \dot{\beta} \cos\theta_i)\right.\right. \\ &\left. - \omega A e^{k\zeta} (a_i' + \tfrac{1}{2} k a_i) \cos(k a_0 \cos\theta_i - \omega t)\right] \\ &+ \tfrac{1}{2} a_i^2 \left[\dot{x}_0 + \dot{\beta}\zeta - \dot{\gamma} a_0 \sin\theta_i + \omega A e^{k\zeta} \sin(k a_0 \cos\theta_i - \omega t)\right] \frac{\partial}{\partial x_i} \\ &+ \tfrac{1}{2} a_i^2 \left[\dot{y}_0 - \dot{\alpha}\zeta + \dot{\gamma} a_0 \cos\theta_i\right] \frac{\partial}{\partial y_i} \\ &\left. - \tfrac{1}{8} a_i^4 \left[\omega k A e^{k\zeta} \cos(k a_0 \cos\theta_i - \omega t)\right] \frac{\partial^2}{\partial x_i^2}\right\} \left[R_i^2 + (z_i - \zeta)^2\right]^{-\frac{1}{2}} d\zeta\end{aligned}$$

where $a_1 = a_i(\zeta)$ and $a_i' = \dfrac{da_i(\zeta)}{d\zeta}$ in the integrand. We adapt this type of solution to the present case as follows: (1) $[R_i^2 + (z_i - \zeta)^2]^{-\frac{1}{2}}$ is the potential for a source located at $(0, 0, \zeta)$ in an infinite fluid.

We replace this source potential by another source potential which, in addition, satisfies the free surface condition. (2) We now impose the condition that the radius a_i is much smaller than the distance between spars; i.e., $a_i/a_0 << 1$. Then the potential obtained by satisfying the boundary condition on the i^{th} spar will produce negligible fluid velocities at the other spars, and the total potential can be expressed as a sum of potentials, each satisfying the conditions on one spar. The resulting total potential is

$$\begin{aligned}
\Phi(x, y, z, t) = {} & \frac{\omega A}{k} e^{kz} \cos(kx - \omega t) \\
& + \sum_{n=1}^{N} \int_{-H}^{0} \Big\{ \tfrac{1}{2} a_i [\dot{z}_0 a_i' + a_0 a_i' (\dot{\alpha} \sin\theta_i - \dot{\beta} \cos\theta_i) \\
& - \omega A e^{k\zeta} (a_i' + \tfrac{1}{2} k a_i) \cos(k a_0 \cos\theta_i - \omega t)] \\
& + \tfrac{1}{2} a_i^2 [\dot{x}_0 + \dot{\beta}\zeta - \dot{\gamma} a_0 \sin\theta_i + \omega A e^{k\zeta} \sin(k a_0 \cos\theta_i - \omega t)] \frac{\partial}{\partial x_i} \\
& + \tfrac{1}{2} a_i^2 [\dot{y}_0 - \dot{\alpha}\zeta + \dot{\gamma} a_0 \cos\theta_i] \frac{\partial}{\partial y_i} \\
& - \tfrac{1}{8} a_i^4 [\omega k A e^{k\zeta} \cos(k a_0 \cos\theta_i - \omega t)] \frac{\partial^2}{\partial x_i^2} \Big\} \\
& \Big\{ [R_i^2 + (z_i - \zeta)^2]^{-\frac{1}{2}} + -\!\!\!\!\!\!\int_0^{\infty} \frac{\nu + k}{\nu - k} e^{\nu(z_i + \zeta)} J_0(\nu R_i) d\nu \Big\} d\zeta \\
& + \pi \omega k \sum_{i=1}^{N} \int_{-H}^{0} \Big\{ a_i [z_0 a_i' + a_0 a_i' (\alpha \sin\theta_i - \beta \cos\theta_i)
\end{aligned} \qquad [8]$$

[8]
continued

$$+ A e^{k\zeta} (a_i' + \tfrac{1}{2} k a_i) \sin (k a_0 \cos \theta_i - \omega t)]$$

$$+ a_i^2 [x_0 + \beta \zeta - \gamma a_0 \sin \theta_i + A e^{k\zeta} \cos (k a_0 \cos \theta_i - \omega t)] \frac{\partial}{\partial x_i}$$

$$+ a_i^2 [y_0 - \alpha \zeta + \gamma a_0 \cos \theta_i] \frac{\partial}{\partial y_i}$$

$$+ \tfrac{1}{4} a_i^4 [k A e^{k\zeta} \sin (k a_0 \cos \theta_i - \omega t)] \frac{\partial^2}{\partial x_i^2} \Bigg\} \Bigg\{ e^{k(z_i + \zeta)} J_0 (k R_i) \Bigg\} d\zeta$$

In accordance with the assumptions of slender body theory, we evaluate these terms as $R_i \rightarrow 0$ and identify the values so obtained with the potential on the body surface $R_i = a_i$. Because $a_i/a_0 << 1$, the value of Φ to lowest order on the i^{th} spar depends only on the first term above and one term in the first sum. By the same approximation procedure, we find that all terms in the second sum contribute amounts of higher order in terms of a_i. Later we shall reconsider the second sum when we calculate damping forces.

FIRST-ORDER FORCES AND MOMENTS

The pressure is obtained from Bernoulli's equation in linearized form

$$p = - \rho \frac{\partial \Phi}{\partial t} - \rho g z \qquad [9]$$

Thus it will be necessary to evaluate Φ_t on each spar and to integrate, in an appropriate way, the result over all spars to find the forces and moments.

Using slender body approximations again, we find that on the i^{th} spar

$$\begin{aligned}\Phi = {} & \frac{\omega A}{k} e^{kz_i} \cos (k a_0 \cos \theta_i - \omega t) - 2 \omega A e^{kz_i} a_i \cos \lambda_i \sin (k a_0 \cos\theta_i - \omega t) \\ & - [\dot{z}_0 a_i' + a_0 a_i' (\dot{\alpha} \sin \theta_i - \dot{\beta} \cos \theta_i) \\ & \qquad - \omega A e^{kz_i} (a_i' + \tfrac{1}{2} k a_i) \cos (k a_0 \cos \theta_i - \omega t)] a_i \log a_i \\ & - [\dot{x}_0 + \dot{\beta} z_i - \dot{\gamma} a_0 \sin \theta_i] a_i \cos \lambda_i - [\dot{y}_0 - \dot{\alpha} z_i + \dot{\gamma} a_0 \cos \theta_i] a_i \sin \lambda_i \\ & \qquad + 0(a_i^2) \end{aligned} \qquad [10]$$

Also, we note that on the surface of the i^{th} spar

$$z = z_i' + z_0 + \alpha(a_0 \sin \theta_i + a_i \sin \lambda_i') - \beta(a_0 \cos \theta_i + a_i \cos \lambda_i')$$

Thus the pressure on the i^{th} spar is

$$\begin{aligned} p = - \rho \Big\{ & g A e^{kz_i'} \sin (k a_0 \cos \theta_i - \omega t) \\ & \qquad + 2 g k A e^{kz_i'} a_i \cos \lambda_i' \cos (k a_0 \cos \theta_i - \omega t) \\ & - [\ddot{z}_0 a_i' + a_0 a_i' (\ddot{\alpha} \sin \theta_i - \ddot{\beta} \cos \theta_i) \\ & \qquad - g k A e^{kz_i'} (a_i' + \tfrac{1}{2} k a_i) \sin (k a_0 \cos \theta_i - \omega t)] a_i \log a_i \\ & - [\ddot{x}_0 + \ddot{\beta} z_i' - \ddot{\gamma} a_0 \sin \theta_i] a_i \cos \lambda_i' - [\ddot{y}_0 - \ddot{\alpha} z_i' + \ddot{\gamma} a_0 \cos \theta_i] a_i \sin \lambda_i' \\ & + g [z_i' + z_0 + \alpha(a_0 \sin \theta_i + a_i \sin \lambda_i') - \beta(a_0 \cos \theta_i + a_i \cos \lambda_i')] \Big\} \end{aligned} \qquad [11]$$

Let the force be resolved along the space-fixed axes which correspond to the coordinates (x, y, z). In particular, designate the components of hydrodynamic force on the i^{th} spar by X_i, Y_i, Z_i. Likewise, let the components of hydrodynamic moment on the i^{th} spar be denoted by A_i, B_i, Γ_i which correspond to the rotations α, β, γ. Note specifically that the moments are taken with respect to the space-fixed axes at the center of the whole structure. If we let $\underline{n}$ be a unit normal vector out of the fluid, then

$$X_i = \int_{S_i} p\cos(n,x)\,dS \tag{12a}$$

$$Y_i = \int_{S_i} p\cos(n,y)\,dS \tag{12b}$$

$$Z_i = \int_{S_i} p\cos(n,z)\,dS \tag{12c}$$

$$A_i = \int_{S_i} p\,[y\cos(n,z) - z\cos(n,y)]\,dS \tag{12d}$$

$$B_i = \int_{S_i} p\,[z\cos(n,x) - x\cos(n,z)]\,dS \tag{12e}$$

$$\Gamma_i = \int_{S_i} p\,[x\cos(n,y) - y\cos(n,x)]\,dS \tag{12f}$$

The integrals are taken over the instantaneous surface of the i^{th} spar. Here $\cos(n, x)$ is the cosine of the angle between $\underline{n}$ and the x-axis, etc.

We find readily that, to first order in small quantities,

$$\cos(n,x) = -\cos\lambda_i' + \gamma\sin\lambda_i' + \beta a_i'$$

$$\cos(n,y) = -\gamma\cos\lambda_i' - \sin\lambda_i' - \alpha a_i'$$

$$\cos(n,z) = \beta\cos\lambda_i' - \alpha\sin\lambda_i' + a_i'$$

For abbreviation, we also define two sets of integrals

$$S_n = \int_{-H}^{0} S(z_i')(z_i')^n\,dz_i' \tag{13a}$$

$$T_n = \int_{-H}^{0} e^{kz_i'}\,S(z_i')(z_i')^n\,dz_i' \tag{13b}$$

where $S(z_i') = \pi a_i^2(z_i')$ = cross-sectional area of i^{th} spar at z_i'.

The combination of these formulas and definitions with the previous pressure results yields for the force and moment components

$$X_i = 2\rho g k A \cos(k a_0 \cos\theta_i - \omega t) T_0 - \rho(\ddot{x}_0 - \ddot{\gamma} a_0 \sin\theta_i) S_0 - \rho \ddot{\beta} S_1 \qquad [14a]$$

$$Y_i = -\rho(\ddot{y}_0 + \ddot{\gamma} a_0 \cos\theta_i) S_0 + \rho \ddot{\alpha} S_1 \qquad [14b]$$

$$\begin{aligned} Z_i = {} & \rho g k A \sin(k a_0 \cos\theta_i - \omega t) T_0 + \rho g S_0 \\ & - \rho g [A \sin(k a_0 \cos\theta_i - \omega t) + z_0 + \alpha a_0 \sin\theta_i - \beta a_0 \cos\theta_i] S(0) \end{aligned} \qquad [14c]$$

$$\begin{aligned} A_i = {} & \rho g k a_0 \sin\theta_i A \sin(k a_0 \cos\theta_i - \omega t) T_0 \\ & + \rho g(y_0 + a_0 \sin\theta_i + \gamma a_0 \cos\theta_i) S_0 + \rho(\ddot{y}_0 + \ddot{\gamma} a_0 \cos\theta_i - g\alpha) S_1 - \rho \ddot{\alpha} S_2 \\ & - \rho g a_0 \sin\theta_i [z_0 + \alpha a_0 \sin\theta_i - \beta a_0 \cos\theta_i + A \sin(k a_0 \cos\theta_i - \omega t)] S(0) \end{aligned} \qquad [14d]$$

$$\begin{aligned} B_i = {} & -\rho g k a_0 \cos\theta_i A \sin(k a_0 \cos\theta_i - \omega t) T_0 + 2\rho g k A \cos(k a_0 \cos\theta_i - \omega t) T_1 \\ & - \rho g(x_0 + a_0 \cos\theta_i - \gamma a_0 \sin\theta_i) S_0 - \rho(\ddot{x}_0 - \ddot{\gamma} a_0 \sin\theta_i + g\beta) S_1 - \rho \ddot{\beta} S_2 \\ & + \rho g a_0 \cos\theta_i [z_0 + \alpha a_0 \sin\theta_i - \beta a_0 \cos\theta_i + A \sin(k a_0 \cos\theta_i - \omega t)] S(0) \end{aligned} \qquad [14e]$$

$$\begin{aligned} \Gamma_i = {} & -2\rho g k a_0 \sin\theta_i A \cos(k a_0 \cos\theta_i - \omega t) T_0 \\ & + \rho [a_0 \sin\theta_i \ddot{x}_0 - a_0 \cos\theta_i \ddot{y}_0 - a_0^2 \ddot{\gamma}] S_0 \\ & + \rho [a_0 \sin\theta_i \ddot{\beta} + a_0 \cos\theta_i \ddot{\alpha}] S_1 \end{aligned} \qquad [14f]$$

$S(0)$ is the cross-sectional area at $z_i = 0$ when the whole system is at rest and in equilibrium.

SIMPLIFIED INTERPRETATION OF FIRST-ORDER FORCES AND MOMENTS

These expressions for the forces and moments can be viewed from a simple point of view. Consider, for example, the x-component of force, Equation [14a], which when written out becomes

$$X_i = \rho \int_{-H}^{0} S(z_i') \left\{ 2\, g k A e^{kz_i'} \cos (k a_0 \cos \theta_i - \omega t) - \ddot{x}_0 + \ddot{\gamma} a_0 \sin \theta_i - z_i' \ddot{\beta} \right\} dz_i' \qquad [14a']$$

From Equation [5] we see that

$$\frac{\partial^2 \Phi_0}{\partial t\, \partial x} = g k A e^{kz_i'} \cos (k a_0 \cos \theta_i - \omega t)$$

on the equilibrium position of the i^{th} spar. Thus the first term in the bracket in Equation [14a′] is just twice the local acceleration that the water would have at the mean position of the spar axis if the spar were not present. The terms, $-\ddot{x}_0 + \ddot{\gamma} a_0 \sin \theta_i - z_i' \ddot{\beta}$, give the negative acceleration (in the x-direction) of the point on the spar axis. The quantity $\rho S(z_i')$ is the added mass per unit length of a cylinder accelerating normally to its axis. Thus the x-component of force is the integral over the mean spar length of

> (added mass per unit length) times (2 times local water acceleration on spar axis due to incident wave alone minus acceleration of point on axis of spar).

It may appear strange that the water particle acceleration is doubled in this formula. However, the cause is seen on examination of Equations [8] and [10]. In the latter equation, the terms containing the factor $(\cos \lambda_i)$ give rise to x-components of force. Here the term due to the incident waves (the second term) is already doubled. Half of this contribution comes directly from the first term of Equation [8] (i.e., directly from the incident wave potential) and half comes from the term

$$\int_{-H}^{0} \tfrac{1}{2} a_i^2 \omega A e^{k\zeta} \sin(k a_0 \cos\theta_i - \omega t) \frac{\partial}{\partial x_i} \left[R_i^2 + (z_i - \zeta)^2\right]^{-\frac{1}{2}} d\zeta$$

The latter is effectively a diffraction potential; it is part of the singularity potential which offsets the normal velocity component of the incident wave on the spar. These results can also be regarded as a special case of a general body, accelerating in a time-varying (but spatially constant) infinite field of fluid. It follows from consideration of the forces, both in the fixed and moving coordinate systems,[2] that the hydrodynamic force on the body is the added mass times the relative acceleration plus the displaced mass of fluid times the spatial acceleration of the (undisturbed) fluid. For a circular cylindrical section, the added mass and the displaced mass are equal, and the above relation for the x-component of the force on the spar follows immediately.

The z-component of force, Equation [14c], consists of three parts:

(a) $[\rho g S_0]$;

(b) $[-\rho g(z_0 + \alpha a_0 \sin\theta_i - \beta a_0 \cos\theta_i) S(0)]$;

(c) $\rho g A \sin(k a_0 \cos\theta_i - \omega t)[k T_0 - S(0)]$.

Part (a) is just the hydrostatic force. Part (b) is the decrease in buoyancy which occurs when the spar is raised an amount $(z_0 + \alpha a_0 \sin\theta_i - \beta a_0 \cos\theta_i)$. Part (c) is the integral over the undisturbed spar surface of the vertical pressure force due to the incident wave alone. This is easily seen by noting that

$$k T_0 - S(0) = -\int_{-H}^{0} e^{k z_i'} \frac{dS}{dz_i'} dz_i'$$

and, thus, that Part (c) is equal to

$$\int_{-H}^{0} \left[-\rho g A e^{k z_i'} \sin(k a_0 \cos\theta_i - \omega t)\right] \frac{dS}{dz_i'} dz_i' = \int_{S_i} p_0 \cos(n, z)\, dS$$

to first order, where p_0 is the first term of Equation [11].

The moments are obtained by calculating the force per unit length along each spar, multiplying by the appropriate lever arm, and integrating along the lengths of the spars. It should be noted again that the moments are calculated with respect to the space-fixed axes. Thus a point located at z_i' on the i^{th}-axis has space-fixed coordinates (see Equation [1′]):

$$x = a_0 \cos\theta_i + x_0 + \beta z_i' - \gamma a_0 \sin\theta_i ;$$

$$y = a_0 \sin\theta_i + y_0 + \gamma a_0 \cos\theta_i - \alpha z_i' ;$$

$$z = z_i' + z_0 + \alpha a_0 \sin\theta_i - \beta a_0 \cos\theta_i .$$

FIRST-ORDER EQUATIONS OF MOTION

Let M be the mass of a spar. (The N spars are assumed to be identical.) Let I_i, J_i, K_i be the moments of inertia of the i^{th} spar about the x-, y-, z-axes, respectively. Moreover, let M_0, I_0, J_0, and K_0 denote the mass and moments of inertia for any additional superstructure, and assume $(0, 0, z_0)$ to be the center of gravity thereof. Then the equations of motion are

$$(M_0 + NM)\ddot{x}_0 = \sum_{i=1}^{N} X_i \qquad [15a]$$

$$(M_0 + NM)\ddot{y}_0 = \sum_{i=1}^{N} Y_i \qquad [15b]$$

$$(M_0 + NM)\ddot{z}_0 = \sum_{i=1}^{N} Z_i - g(NM + M_0) \qquad [15c]$$

$$-M_0 g z_0 + \ddot{\alpha} \sum_{i=0}^{N} I_i = \sum_{i=1}^{N} A_i - g \sum_{i=1}^{N} \int_L m(z_i')y\, dz_i' \qquad [15d]$$

$$- M_0 g z_0 + \ddot{\beta} \sum_{i=0}^{N} J_i = \sum_{i=1}^{N} B_i + g \sum_{i=1}^{N} \int_L m(z_i')\,x\,dz_i' \qquad [15e]$$

$$\ddot{\gamma} \sum_{i=0}^{N} K_i = \sum_{i=1}^{N} \Gamma_i \qquad [15f]$$

$m(z_i')$ is the mass per unit length of the spar itself. The integral is taken over the length L of the spar. This length generally extends from $z_i' = -H$ to some value of z_i' greater than zero. x and y are the distances from the fixed reference frame to a point on the axis of the i^{th} spar.

The moment of inertia I_i about the x-axis is

$$I_i = \int_L m(z_i')(y^2 + z^2)\,dz_i'$$

where $y = a_0 \sin\theta_i + y_0 + \gamma a_0 \cos\theta_i - \alpha z_i' + \ldots$, and

$$z = z_i' + z_0 + \alpha a_0 \sin\theta_i - \beta a_0 \cos\theta_i + \ldots,$$

the omitted terms being of higher order in the small motion variables. Since I_i is multiplied by $\ddot{\alpha}$, we need keep only the zero-order terms in I_i. Clearly then,

$$I_i = \int_L m(z_i')\,[a_0^2 \sin^2\theta_i + z_i'^2]\,dz_i'$$

to the required order in small quantities; that is, I_i has the same value as in the equilibrium position. Similarly,

$$J_i = \int_L m(z_i')(x^2 + z^2)\,dz_i'$$

$$= \int_L m(z_i')\,[a_0^2 \cos^2\theta_i + z_i'^2]\,dz_i'$$

$$K_i = \int_L m(z_i')(x^2 + y^2)\,dz_i'$$

$$= \int_L m(z_i')\,a_0^2\,dz_i' = a_0^2\,M$$

The integral terms in Equations [15d] and [15e] can also be written explicitly by expanding x and y in the integrands. Thus

$$\int_L m(z_i')y\,dz_i' = \int_L m(z_i')[a_0 \sin\theta_i + y_0 + \gamma a_0 \cos\theta_i - \alpha z_i']\,dz_i'$$

$$= (a_0 \sin\theta_i + y_0 + \gamma a_0 \cos\theta_i)M - \alpha \int_L m(z_i')z_i'\,dz_i'$$

$$\int_L m(z_i')x\,dz_i' = \int_L m(z_i')[a_0 \cos\theta_i + x_0 + \beta z_i' - \gamma a_0 \sin\theta_i]\,dz_i'$$

$$= (a_0 \cos\theta_i + x_0 - \gamma a_0 \sin\theta_i)M + \beta \int_L m(z_i')z_i'\,dz_i'$$

We note that the final integral terms here would have vanished if z_i' had been measured from the spar center of gravity.

Now let us write the equations in full. Equation [15a] becomes

$$(M_0 + NM)\ddot{x}_0 = 2\rho g k A T_0 \sum_{i=1}^{N} \cos(k a_0 \cos\theta_i - \omega t) - \rho N \ddot{x}_0 S_0 - \rho N \ddot{\beta} S_1 + \rho a_0 \ddot{\gamma} S_0 \sum_{i=1}^{N} \sin\theta_i \qquad [15a']$$

Clearly, $M_0 + NM = \rho S_0 N$, since at rest the buoyancy of the total raft equals its weight. Also, we now impose the condition that the spars have a regular angular spacing. Thus

$$\theta_i = \theta_1 + \frac{2\pi}{N}(i-1) \quad ; \quad i = 1, 2, \ldots, N$$

If $N = 1$, then $a_0 = 0$, and the last term in Equation [15a′] vanishes. If $N > 1$,

$$\sum_{i=1}^{N} \sin \theta_i = \sum_{i=0}^{N-1} \sin \left(\theta_1 + \frac{2\pi i}{N}\right) = \frac{\sin\left(\theta_1 + \frac{N-1}{2} \cdot \frac{2\pi}{N}\right) \sin\left(\frac{N}{2} \cdot \frac{2\pi}{N}\right)}{\sin \frac{\pi}{N}} = 0$$

and again the last term in [15a′] vanishes.

Therefore, the equation of motion for x_0 is

$$2(M_0 + NM)\ddot{x}_0 + \rho N S_1 \ddot{\beta} = 2 \rho g k A T_0 \sum_{i=1}^{N} \cos (k a_0 \cos \theta_i - \omega t) \qquad [16a]$$

By similar arguments, the equation of motion for y_0, Equation [15b], becomes

$$2(M_0 + NM)\ddot{y}_0 - \rho N S_1 \ddot{\alpha} = 0 \qquad [16b]$$

The equation for z_0, written out, is

$$\begin{aligned}(M_0 + NM)(\ddot{z}_0 + g) = \rho g k A T_0 \sum_{i=1}^{N} \sin (k a_0 \cos \theta_i - \omega t) + N \rho g S_0 \\ - \rho g A S(0) \sum_{i=1}^{N} \sin (k a_0 \cos \theta_i - \omega t) - \rho g N z_0 S(0) \\ - \rho g \alpha a_0 S(0) \sum_{i=1}^{N} \sin \theta_i + \rho g \beta a_0 S(0) \sum_{i=1}^{N} \cos \theta_i\end{aligned}$$

Again we note that $M_0 + NM = \rho S_0 N$, which enables us to eliminate the gravity term on the left. Also,

$$\sum_{i=1}^{N} \cos \theta_i = 0 \quad ; \quad N > 1$$

and $a_0 = 0$ for $N = 1$. The equation becomes

$$(M_0+NM)\ddot{z}_0 = -N\rho g z_0 S(0) - \rho g A[-kT_0 + S(0)] \sum_{i=1}^{N} \sin(k a_0 \cos\theta_i - \omega t) \tag{16c}$$

The first term on the right-hand side is just the change in buoyancy which accompanies a vertical displacement of the raft. The remaining terms correspond to the vertical force obtained by integrating the dynamic pressure due to the incident wave over the surface of the spar. This is the Froude-Krylov hypothesis: To a first approximation, the presence of the body does not distort the incident wave or the pressure associated with it.

The α-equation is

$$\ddot{\alpha}\left\{M a_0^2 \sum_{i=1}^{N} \sin^2\theta_i + I_0 + \sum_{i=1}^{N} \int_L m(z_i')(z_i')^2\, dz_i'\right\}$$

$$= -\rho g a_0 A[S(0) - kT_0] \sum_{i=1}^{N} \sin\theta_i \sin(k a_0 \cos\theta_i - \omega t)$$

$$+ N\rho S_1 \ddot{y}_0 - N\rho g \alpha S_1 - N\rho\ddot{\alpha} S_2$$

$$- \rho g a_0^2 S(0) \sum_{i=1}^{N} [\alpha \sin^2\theta_i - \beta \sin\theta_i \cos\theta_i]$$

$$+ \alpha g\left[\sum_{i=1}^{N} \int_L m(z_i') z_i'\, dz_i' + M_0 z_0\right]$$

If $N > 2$,

$$\sum_{i=1}^{N} \sin^2\theta_i = \sum_{i=1}^{N} \cos^2\theta_i = \tfrac{1}{2} N$$

$$\sum_{i=1}^{N} \sin\theta_i \cos\theta_i = 0$$

Thus, for N > 2,

$$\ddot{\alpha}\left\{\tfrac{1}{2}NMa_0^2 + N\rho S_2 + I_0 + N\int_L m(z_i')(z_i')^2\,dz_i'\right\} + \alpha\left\{N\rho g S_1 + \tfrac{1}{2}N\rho g a_0^2 S(0) - M_0 g z_0 - Ng\int_L m(z_i')z_i'\,dz_i'\right\} - N\rho S_1\ddot{y}_0 = -\rho g a_0 A[S(0) - kT_0]\sum_{i=1}^{N}\sin\theta_i \sin(ka_0\cos\theta_i - \omega t) \qquad [16d]$$

Similarly, for N > 2, the β-equation becomes

$$\ddot{\beta}\left\{\tfrac{1}{2}NMa_0^2 + N\rho S_2 + J_0 + N\int_L m(z_i')(z_i')^2\,dz_i'\right\} + \beta\left\{N\rho g S_1 + \tfrac{1}{2}N\rho g a_0^2 S(0) - Ng\int_L m(z_i')z_i'\,dz_i'\right\} + N\rho S_1\ddot{x}_0 = \rho g a_0 A[S(0) - kT_0]\sum_{i=1}^{N}\cos\theta_i \sin(ka_0\cos\theta_i - \omega t) + 2\rho g k A T_1\sum_{i=1}^{N}\cos(ka_0\cos\theta_i - \omega t) \qquad [16e]$$

Under the same assumptions, we obtain for the last equation

$$\ddot{\gamma}\left\{K_0 + 2NMa_0^2\right\} = -2\rho g k a_0 A T_0\sum_{i=1}^{N}\sin\theta_i \cos(ka_0\cos\theta_i - \omega t) \qquad [16f]$$

In the case of N = 1, the above equations reduce to Newman's equation for a single spar. If N = 2, these equations do not hold. However, the special conditions that follow from $N=2$ can easily be applied here to obtain simple equations. The case is not considered sufficiently interesting to warrant writing out the equations here.

The heaving motion can be obtained immediately, if desired, since the z_0-equation contains no coupling terms. In addition, the equation for rotation about the z-axis is not coupled to the other equations. However, the y_0 and α motions are coupled; also the x_0 and β motions. Similarly, the couplings are simple enough for these equations to be solved directly.

We should note that there are two resonance frequencies. In heave, there is resonance when

$$\omega^2 = \frac{\rho g S(0)}{M + M_0/N} \qquad [17a]$$

In either of the coupled motions there is resonance when

$$\omega^2 = \frac{2Mg\left[\rho S_1 + \frac{1}{2}\rho a_0^2 S(0) - M_0 z_0/N \int_L m(z_i')z_i'\,dz_i'\right]}{2M\left[\frac{1}{2}Ma_0^2 + \rho S_2 + \begin{Bmatrix} I_0/N \\ K_0/N \end{Bmatrix} + \int_L m(z_i')(z_i')^2\,dz_i'\right] - \rho^2 S_1^2} \qquad [17b]$$

Since the equations contain no damping forces, infinite response amplitudes are predicted when resonance occurs. Of course, this is meaningless in the linearized model and so the above equations ([17a] and [17b]) can be valid only in frequency ranges, not including neighborhoods of the two exceptional frequencies. When such neighborhoods are excluded from consideration, the predictions should be fairly accurate if the small amplitude and slenderness restrictions are observed, since damping forces are of higher order than the forces considered. Near the resonance frequencies, however, the damping forces are important, even if small. This problem is considered in the next section.

EQUATIONS OF DAMPED MOTION

In the previous section, we considered only buoyancy and acceleration forces on the spars because, generally, these were the forces of lowest order in the small parameter $a_i(z)$. At resonance, these forces cancel each other, thus they are no longer the lowest order forces. We must re-examine the previous analysis and include terms of a higher order in $a_i(z)$ to obtain equations of motion which have meaning at and near resonance.

The boundary condition, Equation [7], was valid only to first order in $a_i(z)$. If we now include second-order quantities (in $a_i(z)$) in the boundary condition, Equation [7], and add the necessary corresponding terms in the potential function, Equation [8], we simply obtain more terms in the acceleration and buoyancy forces. Since these terms are much smaller than those already considered, they can alter the response only slightly, principally by changing the resonance frequencies somewhat. They still contribute no damping forces.

Nevertheless, the desired damping forces can be obtained from the second summation in the potential function, Equation [8]. These terms were discarded earlier because they contributed forces of higher order than those being considered. It is easily seen, however, that these terms do lead to damping forces, which will be the lowest order forces at resonance. We also see that the terms in this second sum which involve the incident wave amplitude A do not contribute damping effects. They simply affect the driving force, again by an amount of higher order.

So now we consider the potential

$$\begin{aligned}\Phi^* = \pi\omega k \sum_{i=1}^{N} \int_{-H}^{0} \Big\{ & a_i \left[z_0 a_i' + a_0 a_i' (\alpha \sin\theta_i - \beta \cos\theta_i) \right] \\ & + a_i^2 \left[x_0 + \beta\zeta - \gamma a_0 \sin\theta_i \right] \frac{\partial}{\partial x_i} \\ & + a_i^2 \left[y_0 - \alpha\zeta + \gamma a_0 \cos\theta_i \right] \frac{\partial}{\partial y_i} \Big\} \left[e^{k(z_i+\zeta)} J_0(kR_i) \right] d\zeta\end{aligned}$$

(See the second sum in Equation [8].) Here a_i and a_i' are functions of ζ. To calculate the associated force on the i^{th} spar, we must evaluate $\frac{\partial \Phi^*}{\partial t}$ for $R_i = a_i(z_i)$.

For small values of (kR_i), the Bessel function in Φ^* can be approximated by the beginning of its Taylor series, that is,

$$J_0(kR_i) = 1 - \tfrac{1}{4}(kR_i)^2 + \ldots$$

Similarly,

$$\frac{\partial}{\partial x_i} J_0(kR_i) = -\tfrac{1}{2}k^2 R_i \cos\lambda_i + \ldots$$

$$\frac{\partial}{\partial y_i} J_0(kR_i) = -\tfrac{1}{2}k^2 R_i \sin\lambda_i + \ldots$$

Keeping a one-term approximation in each case, we find

$$\begin{aligned}\Phi^*\Big|_{R_i=a_i} &= \tfrac{1}{2}\omega k[z_0 + a_0(\alpha\sin\theta_i - \beta\cos\theta_i)][S(0) - kT_0]e^{kz_i} \\ &\quad - \tfrac{1}{2}\omega k^3 a_i(z_i)e^{kz_i}\Big\{[(x_0 - \gamma a_0\sin\theta_i)T_0 + \beta T_1]\cos\lambda_i \\ &\qquad + [(y_0 + \gamma a_0\cos\theta_i)T_0 - \alpha T_1]\sin\lambda_i\Big\}\end{aligned}$$

The pressure due to this potential is, when evaluated on $R_i = a_i$,

$$\begin{aligned}p^*\Big|_{R_i=a_i} &= -\rho\,\Phi_t^*\Big|_{R_i=a_i} \\ &= -\tfrac{1}{2}\rho\omega k[\dot{z}_0 + a_0(\dot{\alpha}\sin\theta_i - \dot{\beta}\cos\theta_i)][S(0) - kT_0]e^{kz_i} \\ &\quad + \tfrac{1}{2}\rho\omega k^3 a_i(z_i)e^{kz_i}\Big\{[(\dot{x}_0 - \dot{\gamma}a_0\sin\theta_i)T_0 + \dot{\beta}T_1]\cos\lambda_i \\ &\qquad + [(\dot{y}_0 + \dot{\gamma}a_0\cos\theta_i)T_0 - \dot{\alpha}T_1]\sin\lambda_i\Big\}\end{aligned}$$

In the expressions for Φ^* and p^*, several terms which are of higher order in small variables have been omitted.

The forces and moments due to this pressure distribution are calculated from Equations [12a] through [12f], with asterisks inserted where appropriate. The results are as follows:

$$X_i^* = -\tfrac{1}{2}\rho\omega k^3 \left\{(\dot{x}_0 - \dot{\gamma} a_0 \sin\theta_i) T_0^2 + \dot{\beta} T_0 T_1\right\}$$

$$Y_i^* = -\tfrac{1}{2}\rho\omega k^3 \left\{(\dot{y}_0 + \dot{\gamma} a_0 \cos\theta_i) T_0^2 - \dot{\alpha} T_0 T_1\right\}$$

$$Z_i^* = -\tfrac{1}{2}\rho\omega k \left\{\dot{z}_0 + a_0(\dot{\alpha}\sin\theta_i - \dot{\beta}\cos\theta_i)\right\}[S(0) - kT_0]^2$$

$$\begin{aligned} A_i^* = {} & -\tfrac{1}{2}\rho\omega k a_0 \sin\theta_i \left\{\dot{z}_0 + a_0(\dot{\alpha}\sin\theta_i - \dot{\beta}\cos\theta_i)\right\}[S(0) - kT_0]^2 \\ & + \tfrac{1}{2}\rho\omega k^3 \left\{(\dot{y}_0 + \dot{\gamma} a_0 \cos\theta_i) T_0 T_1 - \dot{\alpha} T_1^2\right\} \end{aligned}$$

$$\begin{aligned} B_i^* = {} & \tfrac{1}{2}\rho\omega k a_0 \cos\theta_i \left\{\dot{z}_0 + a_0(\dot{\alpha}\sin\theta_i - \dot{\beta}\cos\theta_i)\right\}[S(0) - kT_0]^2 \\ & - \tfrac{1}{2}\rho\omega k^3 \left\{(\dot{x}_0 - \dot{\gamma} a_0 \sin\theta_i) T_0 T_1 + \dot{\beta} T_1^2\right\} \end{aligned}$$

There is no need for a damping moment Γ_i^*, since the γ-motion has no resonance in any case.

The modified equations of motion are obtained by adding $\sum_{i=1}^{N} X_i^*$, etc., to the right-hand sides of the previous equations, [15a] through [15f], or alternatively, to [16a] through [16f]. After simplification, the equations become, for $N > 2$,

$$\begin{aligned} & N\left\{\left[2(M + M_0/N)\ddot{x}_0 + \tfrac{1}{2}\rho\omega k^3 T_0^2 \dot{x}_0\right] + \left[\rho S_1 \ddot{\beta} + \tfrac{1}{2}\rho\omega k^3 T_0 T_1 \dot{\beta}\right]\right\} \\ & \qquad = 2\rho g k A T_0 \sum_{i=1}^{N} \cos(k a_0 \cos\theta_i - \omega t) \end{aligned} \qquad [18a]$$

$$N\left\{[2(M+M_0/N)\ddot{y}_0+\tfrac{1}{2}\rho\omega k^3T_0^2\,\dot{y}_0]-[\rho S_1\ddot{\alpha}+\tfrac{1}{2}\rho\omega k^3T_0T_1\,\dot{\alpha}]\right\}=0 \tag{18b}$$

$$\begin{aligned} N\left\{(M+M_0/N)\ddot{z}_0+\tfrac{1}{2}\rho\omega k[S(0)-kT_0]^2\,\dot{z}_0+\rho g S(0)z_0\right\} \\ = -\rho g A[S(0)-kT_0]\sum_{i=1}^{N}\sin(ka_0\cos\theta_i-\omega t) \end{aligned} \tag{18c}$$

$$\begin{aligned} N\Big\{[\tfrac{1}{2}Ma_0^2+\rho S_2+I_0/N+\int_L mz^2\,dz]\,\ddot{\alpha}+\Big(\tfrac{1}{4}\rho\omega k a_0^2[S(0)-kT_0]^2 \\ +\tfrac{1}{2}\rho\omega k^3T_1^2\Big)\dot{\alpha}+[\rho g S_1+\tfrac{1}{2}\rho g a_0^2S(0)-gM_0z_0/N-g\int_L mz\,dz]\,\alpha \\ -[\rho S_1]\,\ddot{y}_0-[\tfrac{1}{2}\rho\omega k^3T_0T_1]\,\dot{y}_0\Big\} \\ = -\rho g a_0A[S(0)-kT_0]\sum_{i=1}^{N}\sin\theta_i\,\sin(ka_0\cos\theta_i-\omega t) \end{aligned} \tag{18d}$$

$$\begin{aligned} N\Big\{[\tfrac{1}{2}Ma_0^2+\rho S_2+K_0/N+\int_L mz^2\,dz]\,\ddot{\beta}+\Big(\tfrac{1}{4}\rho\omega k a_0^2[S(0)-kT_0]^2 \\ +\tfrac{1}{2}\rho\omega k^3T_1^2\Big)\dot{\beta}+[\rho g S_1+\tfrac{1}{2}\rho g a_0^2S(0)-gM_0z_0/N-g\int_L mz\,dz]\,\beta \\ +[\rho S_1]\,\ddot{x}_0+[\tfrac{1}{2}\rho\omega k^3T_0T_1]\,\dot{x}_0\Big\} \\ = \rho g a_0A[S(0)-kT_0]\sum_{i=1}^{N}\cos\theta_i\,\sin(ka_0\cos\theta_i-\omega t) \\ +2\rho g k A T_1\sum_{i=1}^{N}\cos(ka_0\cos\theta_i-\omega t) \end{aligned} \tag{18e}$$

$$(K_0 + NMa_0^2)\ddot{y} = -\rho g k a_0 A T_0 \sum_{i=1}^{N} \sin\theta_i \cos(k a_0 \cos\theta_i - \omega t) \qquad [18f]$$

Of course these equations can be solved very easily by substitution of $x_0 = C_1 \sin(\omega t + \delta_1)$, $y_0 = C_2 \sin(\omega t + \delta_2)$, etc. The results will not be written because no additional perspicuity seems to follow.

VISCOUS DAMPING

All damping forces introduced so far correspond to the energy lost through radiation of surface waves. In addition, energy will be lost through the mechanism of viscosity. The viscous damping forces, in general, will be of second order in the motion variables. As an example, suppose that a right circular cylinder translates in a direction perpendicular to its axis. The viscous drag is proportional to the velocity squared, and so is negligible by the standards already assumed.

If the cylinder has an axial motion, however, the viscous force will be linear in the velocity. For example, if a right circular cylinder of radius a has an axial velocity $\operatorname{Re}\{We^{i\omega t}\} \equiv W_0 \cos(\omega t + \epsilon)$, then we can see easily from elementary fluid mechanics that the velocity anywhere in the fluid is

$$w(r,t) = \operatorname{Re}\left\{\frac{K_0\left(\sqrt{\frac{i\omega}{\nu}}\, r\right)}{K_0\left(\sqrt{\frac{i\omega}{\nu}}\, a\right)}\, We^{i\omega t}\right\}$$

$$= W_0 \operatorname{Re}\left\{\frac{\left[\ker\sqrt{\frac{\omega}{\nu}}\, r + i\,\mathrm{kei}\sqrt{\frac{\omega}{\nu}}\, r\right]}{\left[\ker\sqrt{\frac{\omega}{\nu}}\, a + i\,\mathrm{kei}\sqrt{\frac{\omega}{\nu}}\, a\right]}\, e^{i(\omega t+\epsilon)}\right\}$$

where $K_0(x)$ is the modified Bessel function of argument x and order zero, and the second expression gives $K_0(x)$ in the Kelvin representation of its real and imaginary parts. The only velocity component is that which is parallel to the cylinder axis, and this component depends in space only on the distance r from the axis.

A tangential (axial) stress on the cylinder surface is given by

$$\mu \left.\frac{\partial w}{\partial r}\right|_{r=a} = -\rho\sqrt{\omega\nu}\,\frac{W_0 b}{c}\cos\left(\omega t + \epsilon + \beta - \gamma + \frac{3\pi}{4}\right)$$

where b, c, β, γ are real numbers defined by

$$b\,e^{i\beta} = \mathrm{ker}_1 \sqrt{\frac{\omega}{\nu}}\,a + i\,\mathrm{kei}_1 \sqrt{\frac{\omega}{\nu}}\,a$$

$$c\,e^{i\gamma} = \mathrm{ker} \sqrt{\frac{\omega}{\nu}}\,a + i\,\mathrm{kei} \sqrt{\frac{\omega}{\nu}}\,a$$

Thus there is an axial force component per unit length on the cylinder:

$$F = -\frac{2\pi\rho a\sqrt{\omega\nu}\,W_0 b}{c}\cos\left(\omega t + \epsilon + \beta - \gamma + \frac{3\pi}{4}\right)$$

$$= -\frac{2\pi\rho a\sqrt{\omega\nu}\,W_0 b}{c}\left[\cos(\omega t + \epsilon)\cos\left(\beta - \gamma + \frac{3\pi}{4}\right) - \sin(\omega t + \epsilon)\sin\left(\beta - \gamma + \frac{3\pi}{4}\right)\right]$$

The first term in brackets in the last equation gives the drag force.

For the slender bodies considered in this report, it is consistent to calculate the viscous force in a stripwise manner. That is, at each cross section of the i^th^ spar, where the radius of the section is $a_i(z_i')$, we consider that particular section to be part of an infinitely long right circular cylinder translating axially, calculate the axial viscous force per unit length, and integrate such results over the length of the spar. To first order in the motion variables, the axial velocity of the ith spar is

$$\dot{z}_0 + a_0(\dot{\alpha}\sin\theta_i - \dot{\beta}\cos\theta_i)$$

Thus we add to Equations [14c], [14d], and [14e], respectively, the additional force and moments

$$Z_i^{**} = -2\pi\rho\sqrt{\omega\nu}\left\{\int_{-H}^{0}\frac{a_i(z_i)b_i(z_i)}{c_i(z_i)}\cos\left[\beta(z_i)-\gamma(z_i)+\frac{3\pi}{4}\right](\dot{z}_0 + a_0\dot{\alpha}\sin\theta_i - a_0\dot{\beta}\cos\theta_i)dz_i\right.$$

$$\left.+\int_{-H}^{0}\frac{a_i(z_i)b_i(z_i)}{\omega c_i(z_i)}\sin\left[\beta(z_i)-\gamma(z_i)+\frac{3\pi}{4}\right](\ddot{z}_0 + a_0\ddot{\alpha}\sin\theta_i - a_0\ddot{\beta}\cos\theta_i)dz_i\right\}$$

$$A_i^{**} = a_0 \sin\theta_i \, Z_i^{**}$$

$$B_i^{**} = -a_0 \cos\theta_i \, Z_i^{**}$$

If these additional quantities are inserted into the equations of motion, we can still obtain solutions by the same method used previously.

Although the viscous forces thus obtained are linear in the velocities, they do not fit properly into the perturbation scheme in terms of small radius. The modified Bessel functions encountered have singularities when the argument approaches zero. In fact,

$$\frac{b}{a} \sim \frac{1}{a\log a} \quad ; \quad \text{as} \quad a \to 0.$$

Thus, as the slenderness of the spars is accentuated, the viscous forces increase. This is in contrast to the potential flow forces, which become smaller and smaller. No general conclusion can be drawn concerning the relative importance of the viscous and nonviscous damping forces as far as dependence on radius is concerned. Calculations should be made in individual cases.

DISCUSSION

Equations of motion have now been obtained, based on the assumption that wave amplitude, body motions, and spar radii are small quantities. If damping is neglected, there are two resonance frequencies (as given by Equations [17a] and [17b]) at which infinite amplitudes of motion will occur. Of course, there does exist damping which prevents the responses from actually being infinite. That part of the damping due to generation of outgoing waves is small (of second order in terms of spar radii) and so is of importance only near resonance. Generally there is also a viscous damping which may be of importance throughout the interesting range of frequencies, or possibly only near resonance, or perhaps not at all. This damping is associated only with axial velocities of the spars, but its effects appear in each of the modes of motion for which there is a resonance. The relative importance of viscous damping can be determined in individual cases only by actually carrying out solutions of the equations of motion.

If it is desired to minimize the motions of the structure over a range of practical wave lengths, the most effective procedure is to attempt to remove the natural frequencies from the desired range. If this is not possible, the natural frequencies should be chosen as frequencies for which the incident wave amplitudes are smallest. In practice, this will generally be equivalent to reducing the natural frequencies as low as possible.

From Equation [17a] we see that the heave natural frequency is proportional to the radius of the spars at the undisturbed waterline. ($S(0)$ is the cross-sectional area at the equilibrium waterline.) Assuming that the total mass, M_0+NM, is approximately fixed, we have no other parameter to adjust; thus we would minimize $S(0)$ as far as possible.

In the case of the other resonance frequency, we see from Equation [17b] that, apparently, there are several parameters available for adjustment. We note that if the center of buoyancy and center of gravity coincide, then

$$\rho S_1 - M_0 z_0 - \int_L m(z) z \, dz = 0$$

By raising the center of gravity and/or lowering the center of buoyancy, we can make this difference negative and thereby possibly decrease this natural frequency. However, in general, the exact amount by which such adjustments will affect the natural frequency is not certain, since the denominator in Equation [17b] will also be affected. In any case, reductions in S(0) (for lowering the heave frequency) will also lower the pitch-surge frequency.

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