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DECOMPOSITIONS OF FINITE AUTOMATA

by

MICHAEL YOELI

Technion, Israel Institute of Technology, Haifa

Technical Report No. 10

This report was prepared for the
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under Contract No. N62558-3510

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INTRODUCTION

One purpose of this report is to serve as a short, unified introduction into the algebraic decomposition theory of finite automata (sequential machines). This theory has been established by J. Hartmanis in a series of papers [JH 1-5], extending available results from the decomposition theory of (abstract) algebras (cf. [GB], Foreword on Algebra, and Ch. VI). Furthermore, a variety of aspects of sequential machine decompositions are discussed in [SG2], [AG], [SH], [KC], and [MY 1, 2].

A further purpose of this report is the study of overlapping partitions of automata state sets in connection with a generalized approach to the decomposition problem. Previously only output-consistent overlapping partitions of incompletely specified sequential machines have been considered in connection with the problem of state reduction (cf. [SG1] and [PU]).

Throughout this report the emphasis is on the algebraic rather than engineering aspects of the problem. The applicability of the results to the actual synthesis of sequential machines will be discussed in a subsequent report.

I. PRELIMINARIES

In this section we summarize the basic concepts, notations, and theorems concerning binary relations and partitions, to be used in the

sequel. For detailed expositions of most of this material the reader is referred to [GB], [PD], or [HH].

I.1 BINARY RELATIONS

Given two binary relations R_1 and R_2 over the set S , we denote inclusion, intersection, union, and relative product by $R_1 \subseteq R_2$, $R_1 \cap R_2$, $R_1 \cup R_2$ and $R_1 R_2$, respectively. For any positive integer i the i -th power R^i of the binary relation R over S is defined recursively by $R^1 = R$, $R^i = R^{i-1} R$, $R^0 = I$. The transitive closure \bar{R} of R is defined by $\bar{R} = \bigcup_{i=1}^{\infty} R^i$. The identity relation over S will be denoted as I ($sIt \Leftrightarrow s=t$), the universal relation as U (sUt for every s and t in S).

Given a finite set S , we denote the number of its elements as $\#S$.

We now state the following well-known, easily proved

Lemma 1.1 Let S be a finite set, $\#S = n$, and R is a binary relation over S , such that $R \supseteq I$. Then

$$\bar{R} = R^m \quad \text{for every } m \geq n-1.$$

A convenient method of representing binary relations over finite sets is by means of Boolean matrices. However, in this report we shall not make use of such a representation. Let now $\Sigma(S)$ be the set of all equivalence relations over a given set S . We shall need the following

Theorem 1.1 $\Sigma(S)$ forms a lattice with respect to inclusion. The meet (lattice product) and join (lattice sum) of R_1 and R_2 in $\Sigma(S)$ are given

by $R_1 \wedge R_2$ and $R_1 \sqcup R_2 = \overline{R_1 \cup R_2}$, respectively.

For a proof see, e. g. [HH], § 17. If S is finite and $\#S=n$, we have, by lemma 1.1, for any R_1 and R_2 in $\mathcal{L}(S)$:

$$R_1 \sqcup R_2 = (R_1 \cup R_2)^{n-1}.$$

1.2 DECOMPOSITIONS AND PARTITIONS

A decomposition π of a given set S is a family of non-empty subsets of S (the π -blocks) whose set union is S . If the π -blocks are non-overlapping, π is a partition of S . If $s \in S$ and π is a partition of S , $\pi(s)$ denotes the π -block containing s . If S is finite, and π any decomposition of S , we denote by $|\pi|$ the number of elements of the largest π -block.

With a given decomposition π of the set S we associate a binary relation R_π over S such that $sR_\pi t$ whenever there exists a π -block containing both s and t . R_π is an equivalence relation, if and only if π is a partition. In this case π is the quotient set of S relative to R_π (notation: $\pi = S/R_\pi$). If π is a partition of S , the notation $s \equiv t(\pi)$ instead of $sR_\pi t$ will also be used.

The set of all partitions of a given set S will be denoted as $\Pi(S)$. For π and ρ in $\Pi(S)$ let $\pi \leq \rho$ mean that π is a subpartition of ρ , i. e. $R_\pi \subseteq R_\rho$. Clearly, by Thm. 1.1, the system $\langle \Pi(S), \leq \rangle$ forms a lattice isomorphic to $\langle \mathcal{L}(S), \leq \rangle$. Denoting the lattice product and sum

of the partitions π and ρ by $\pi \cdot \rho$ and $\pi + \rho$, respectively, we thus have:

$$\pi \cdot \rho = S / R_{\pi} \cap R_{\rho}$$

$$\pi + \rho = S / R_{\pi} \cup R_{\rho}$$

The lattice $\Pi(S)$ includes the zero element $0 = S/I$ and the 1-element $1 = S/U$.

Given two partitions π and ρ of S , the quotient decomposition ρ/π of π is defined as follows:

$$\rho/\pi = \{ \pi(s) \mid s \in H \mid H \in \rho \}$$

Clearly if $\rho \geq \pi$, then ρ/π is a partition of π defined as follows:

$$\tau(s) \equiv \pi(t) \quad (\rho/\pi) \Leftrightarrow s \equiv t \quad (\rho). \quad (1)$$

II. X-AUTOMATA

In this study we are only concerned with the transition (next-state) mappings of finite automata (sequential machines). We, therefore, introduce the following

Definition. Let X be a finite alphabet. An X -automaton is a system $A = \langle S, \Delta \rangle$ where S is a finite non-empty set (the states of A) and Δ a single-valued mapping (next state or transition map) of a subset of the Cartesian product $S \times X$ into S . The symbols of X are the inputs of A . If Δ is defined for each $\langle s, x \rangle$ -pair, $s \in S$ and $x \in X$, A is complete, otherwise partial.

We shall use the (operational) notation $s \Delta x = t$, rather than the (functional) notation $\Delta(s, x) = t$. If $s \Delta x$ is defined, s will be said

to admit x .

Definition. The X -automata $A = \langle S, \Delta \rangle$ and $A' = \langle S', \Delta' \rangle$ are isomorphic (notation $A \cong A'$) if there exists a one-one mapping η of S onto S' , such that, for each $s \in S$, s and $s\eta$ admit the same inputs, and furthermore $(s \Delta x)\eta = s\eta \Delta' x$ for each s in S and each x admitted by s .

We now extend the concept of admissible partition (cf. [MY 1]) or partition with substitution property [JH 1-5] to decompositions.

Definition. Let $A = \langle S, \Delta \rangle$ be an X -automaton. The decomposition π of S is admissible by A , if for each π -block H and each input x there exists a π -block K such that $s \Delta x \in K$ for each s in H admitting x .

Next, we introduce the concept of " π -factor", closely connected with the concept of quotient-algebra (or factor-algebra) of an abstract algebra (cf. [GB], p. IX).

Definition. Let $A = \langle S, \Delta \rangle$ be an X -automaton and π a decomposition of S . The X -automaton $\bar{A} = \langle \bar{S}, \bar{\Delta} \rangle$ is a π -factor of A , if i) $\bar{S} = \pi$, ii) any π -block H admits any input x if and only if there exists an S in H admitting x , and iii) $s \Delta x \in H \bar{\Delta} x$ for each $H \in \bar{S}$, each $s \in H$, and each input x admitted by s .

The following two lemmata (2.1 and 2.2) are immediate consequences of the above definitions.

Lemma 2.1 Let $\bar{A} = \langle \pi, \bar{\Delta} \rangle$ be a π -factor of the X-automaton $A = \langle S, \Delta \rangle$. Then the decomposition π is admissible by A. Conversely, if the decomposition π is admissible by the X-automaton $A = \langle S, \Delta \rangle$, there exists a π -factor $\bar{A} = \langle \pi, \bar{\Delta} \rangle$ of A.

Lemma 2.2 If π is an admissible partition of the X-automaton A, there exists exactly one π -factor \bar{A} of A (notation: $\bar{A} = A/\pi$).

If A is a complete X-automaton, and π an admissible partition of A, R_π is a congruence relation of A. In this case the theory of congruence relations of abstract algebras (cf. [GB] and [HH]) directly applies.

In this connection we have the following

Theorem 2.1 If π and ρ are partitions admissible by the complete X-automaton $A = \langle S, \Delta \rangle$, then $\pi \cdot \rho$ and $\pi + \rho$ are also admissible by A. Thus the set $\Pi(A)$ of all partitions admissible by A forms a sublattice of $\Pi(S)$.

For a proof of the corresponding theorem on abstract algebras see e.g. [GB], p. 23, or [HH], p. 95. The application of this result from the theory of algebras to finite complete automata is due to J. Hartmanis (cf. [JH 1]).

Next, we apply a well-known isomorphism theorem on abstract algebras (cf. [GB], p. IX, Ex. 2) to both complete (cf. [KC]) and partial automata.

Theorem 2.2 If π_1 and π_2 are admissible partitions of the X-automaton A and $\pi_1 \geq \pi_2$, then the partition π_1/π_2 is admissible by A/π_2 and $A/\pi_1 \cong A/\pi_2/\pi_1/\pi_2$.

Conversely, let π_2 be an admissible partition of the X-automaton A. Then any admissible partition π of A/π_2 uniquely determines an admissible partition $\pi_1 (\geq \pi_2)$ of A, such that $\pi = \pi_1/\pi_2$.

Proof. Let H_2 be a state of $A/\pi_2 = \langle \pi_2, \Delta^2 \rangle$ admitting the input x , and let $H_{12} = (\pi_1/\pi_2)(H_2)$, $K_{12} = (\pi_1/\pi_2)(H_2 \Delta^2 x)$. Given now another π_2 -block H'_2 which belongs to H_{12} and also admits the input x , it follows from (1), that $(\pi_1/\pi_2)(H'_2 \Delta^2 x) = K_{12}$, i.e. π_1/π_2 is admissible by A/π_2 .

Let now s be an arbitrary state of A, and let $H_1 = \pi_1(s)$, $H_2 = \pi_2(s)$. We define the mapping ν of π_1 onto π_1/π_2 as follows:

$$H_1 \nu = \pi_1/\pi_2 (H_2) \quad \text{for each } s.$$

Due to $\pi_1 \geq \pi_2$, ν is one-one. Furthermore ν is an isomorphism of A/π_1 onto $A/\pi_2/\pi_1/\pi_2$.

Conversely, let π be an admissible partition of A/π_2 . We define the partition π_1 of A by $\pi_1/\pi_2 = \pi$, i.e.

$$s \equiv t (\pi_1) \iff \pi_2(s) \equiv \pi_2(t) \quad (\pi).$$

The partition $\pi_1 (\geq \pi_2)$ is thus uniquely determined and, furthermore, π_1 is admissible by A.

So far, the theory of parallel (direct) or cascade decompositions of sequential machines (cf. [JH 1-5], [MY 1, 2]) was based on the study of admissible partitions. In the sequel it will be shown how this theory may be extended by also considering admissible decompositions of X-automata. In this connection we shall need the following two theorems.

Theorem 2.3 Let $A = \langle S, \Delta \rangle$ be an X-automaton, π an admissible decomposition of A, and $\bar{A} = \langle \pi, \bar{\Delta} \rangle$ a π -factor of A. Then there exists an X-automaton $A' = \langle S', \Delta' \rangle$ admitting partitions ϱ and π' , such that $|\pi'| = |\pi|$ and

$$A'/\varrho \cong A \quad (2)$$

$$A'/\pi' \cong \bar{A} \quad (3)$$

Proof. Let $S' = \{ \langle s, H \rangle \mid s \in S, H \in \pi \}$

We define $\langle s, H \rangle \Delta' x$ for each s in S , $s \in H \in \pi$, and each input x admitted by s by

$$\langle s, H \rangle \Delta' x = \langle s \Delta x, H \bar{\Delta} x \rangle$$

Now, let

$$\langle s, H \rangle \equiv \langle t, K \rangle \quad (\varrho) \iff s=t$$

$$\langle s, H \rangle \equiv \langle t, K \rangle \quad (\pi') \iff H=K$$

Evidently, ϱ and π' are admissible partitions of the X-automaton $A' = \langle S', \Delta' \rangle$ satisfying (2) and (3), and $|\pi'| = |\pi|$.

Theorem 2.4 Let $A = \langle S, \Delta \rangle$ be a complete X-automaton, π and ϱ admissible partitions of A, such that $\#(\varrho/\pi) = \#\varrho$. Then there exists a ϱ/π -factor of A/π isomorphic to A/ϱ .

Proof. Let η be the mapping of \mathcal{S} onto \mathcal{S}/π defined as follows:

$$H\eta = \{ \pi(s) \mid s \in H \} \quad \text{for each } H \in \mathcal{S}$$

$\#(\mathcal{S}/\pi) = \#\mathcal{S}$ implies that η is 1-1.

Let $A/\mathcal{S} = \langle \mathcal{S}, \Delta^{\mathcal{S}} \rangle$ and $A/\pi = \langle \pi, \Delta^{\pi} \rangle$.

We now define an X-automaton $\bar{A} = \langle \mathcal{S}/\pi, \bar{\Delta} \rangle$ such that $\bar{A} \cong A/\mathcal{S}$, i.e.

$$H\eta \bar{\Delta} x = (H\Delta^{\mathcal{S}} x)\eta \quad \text{for each } H \in \mathcal{S} \text{ and each } x \in X.$$

Next, we have to show that \bar{A} is a \mathcal{S}/π -factor of A/π . Indeed, let K be an arbitrary \mathcal{S}/π -block, and $x \in X$. Let $H = K\eta^{-1}$ and $L = K\bar{\Delta} x = H\eta \bar{\Delta} x = (H\Delta^{\mathcal{S}} x)\eta$. If $M \in K$, there exists, due to $H\eta = K$, an element $s \in H$, such that $M = \pi(s)$. Then

$$s \in H \Rightarrow s\Delta x \in H\Delta^{\mathcal{S}} x \Rightarrow \pi(s\Delta x) \in (H\Delta^{\mathcal{S}} x)\eta = L.$$

Furthermore

$$M = \pi(s) \Rightarrow M\Delta^{\pi} x = \pi(s\Delta x).$$

Thus, $M\Delta^{\pi} x \in L = K\bar{\Delta} x$ for each π -block $M \in K$, i.e. \bar{A} is a \mathcal{S}/π -factor of A/π .

Theorem 2.4 is thus proved.

The condition $\#(\mathcal{S}/\pi) = \#\mathcal{S}$ of Thm. 2.4 is satisfied, if $\mathcal{S} \gg \pi$.

Thus Thm 2.4 generalizes the Isomorphism Theorem 2.2.

In the sequel we shall also need the following two definitions.

Definition. The X-automaton $A = \langle S, \Delta \rangle$ is a subsystem of the X-automaton $A' = \langle S', \Delta' \rangle$ (notation: $A \subseteq A'$) if i) $S \subseteq S'$, ii) every input x admitted by any state s in A is also admitted by s in A' , and iii) $s\Delta x = s\Delta' x$ for each s in S and each input x admitted by s .

Definition. Let $A = \langle S, \Delta \rangle$ and $A' = \langle S', \Delta' \rangle$ be X -automata. A' covers A (notation: $A' \gg A$) if there exists a single-valued mapping η of a subset S' onto S , such that each s' in S' admits any input x admitted by $s'\eta$ and, furthermore, $s'\eta \Delta x = (s' \Delta' x) \eta$.

The relation \gg is clearly reflexive and transitive, i. e. a weak ordering of the system of all X -automata. Furthermore, if $A \gg A'$ and $A' \gg A$, then $A \cong A'$. Obviously, $A' \supset A \Rightarrow A' \gg A$.

The concept of covering introduced above is especially important from an engineering point of view. Given the specifications of a Moore or Mealy type sequential machine M , engineers will frequently construct a larger machine M' , which will cover M , i. e. will perform at least as much as the specified machine M (cf. [PU] and [SG 1]). If the X -automaton corresponds to M , and $A' \gg A$, it is easily seen that A' can be converted into an M' -machine covering M by a suitable specification of its output function.

III. DIRECT PRODUCTS OF X-AUTOMATA

The concept of direct product of universal algebras is directly applicable to X -automata:

Definition. The complete X -automaton $A = \langle S, \Delta \rangle$ is the direct product of the complete X -automata $A_i = \langle S_i, \Delta^i \rangle$, $i=1, \dots, r$, (notation: $A = A_1 \times \dots \times A_r$) if $S = S_1 \times \dots \times S_r$ and if, for each $s = \langle s_1, \dots, s_r \rangle$

$(s \in S, s_i \in S_i)$ and each $x \in X$, $s \Delta x = \langle s_1 \Delta^1 x, \dots, s_r \Delta^r x \rangle$.

Evidently $(A_1 \times A_2) \times A_3 \cong A_1 \times (A_2 \times A_3) \cong A_1 \times A_2 \times A_3$ and $A_1 \times A_2 \cong A_2 \times A_1$.

Let now $A = A_1 \times \dots \times A_r$, where $A = \langle S, \Delta \rangle$ and $A_i = \langle S_i, \Delta^i \rangle$,
 $i = 1, \dots, r$.

Let us define the partition π_i of S by

$$\langle s_1, \dots, s_n \rangle \equiv \langle t_1, \dots, t_r \rangle \quad (\pi_i) \Leftrightarrow s_i = t_i.$$

Obviously, π_i is admissible by A and $A / \pi_i \cong A_i$.

Furthermore, $\pi_1 \cdot \pi_2 \cdot \dots \cdot \pi_r = 0$ and $(\# \pi_1) \cdot \dots \cdot (\# \pi_r) = \# S$.

Conversely, we have the following

Theorem 3.1 Let π_1, \dots, π_r be admissible partitions of the complete X -automaton $A = \langle S, \Delta \rangle$ such that

$$\pi_1 \cdot \pi_2 \cdot \dots \cdot \pi_r = 0 \quad (4)$$

$$\text{and } (\# \pi_1) \cdot \dots \cdot (\# \pi_r) = \# S \quad (5)$$

then $A \cong A / \pi_1 \times \dots \times A / \pi_r$.

Proof. Considering any state s of A , let $s_i = \pi_i(s)$, $i=1, \dots, r$.

We define the mapping η of S into $\bar{S} = \pi_1 \times \dots \times \pi_r$, by

$$s \eta = \langle s_1, \dots, s_r \rangle.$$

Due to (4), η is one-one, and due to (5) η is onto \bar{S} . One now easily verifies that η is an isomorphism of A onto $A / \pi_1 \times \dots \times A / \pi_r$.

As mentioned at the end of section II, we are frequently interested in finding an X -automaton A' covering a specified X -automaton A , in as far as A' is preferable to A from some engineering point of view.

In this connection the following theorems are rather useful.

Theorem 3.2 Let $A = \langle S, \Delta \rangle$ be a (complete or partial) X -automaton, and π_1, \dots, π_r admissible partitions of A , satisfying

$$\pi_1 \cdot \dots \cdot \pi_r = 0.$$

Let $\bar{A}_i = \langle \bar{S}_i, \bar{\Delta}^i \rangle$, $i = 1, \dots, r$ be complete X -automata such that

$$\bar{A}_i \geq A_i = A / \pi_i.$$

Then $A \leq \bar{A}_1 \times \dots \times \bar{A}_r$.

Proof. Let η_i be the mapping corresponding to $\bar{A}_i \geq A / \pi_i$.

Let $s \in S$ and $s_i = \pi_i(s)$, $i = 1, \dots, r$. We now define a single-valued mapping η of a subset of $\bar{S}_1 \times \dots \times \bar{S}_r$ onto S as follows:

$$s \eta^{-1} = s_1 \eta_1^{-1} \times \dots \times s_r \eta_r^{-1}.$$

It is easily seen that $\bar{A}_1 \times \dots \times \bar{A}_r$ covers A with respect to η .

Theorem 3.3 Let $A = \langle S, \Delta \rangle$ be a complete X -automaton, π an admissible decomposition, and ψ an admissible partition of A , such that $R_\pi \cap R_\psi = I$. Then $A \leq \bar{A} \times A / \psi$, where $\bar{A} = \langle \pi, \bar{\Delta} \rangle$ is a π -factor of A .

Proof. By Thm. 2.3 there exists an X -automaton $A' = \langle S', \Delta' \rangle$, admitting partitions ϱ and π' , such that $A' / \varrho \cong A$, i.e. $A' \geq A$, and $A' / \pi' \cong \bar{A}$. S' , ϱ , and π' were defined as follows (see proof of Thm 2.3):

$$S' = \{ \langle s, H \rangle \mid s \in H, H \in \pi \}$$

$$\langle s, H \rangle \bar{\Delta} \langle t, K \rangle (\varrho) \iff s = t$$

$$\langle s, H \rangle \equiv \langle t, K \rangle (\pi') \iff H=K$$

We now define the partition $\psi' \triangleright \mathcal{Q}$ of A' by

$$\langle s, H \rangle \equiv \langle t, K \rangle (\psi') \iff s \equiv t (\psi).$$

ψ is admissible by A , hence ψ' is admissible by A' . Furthermore (cf. Thm 2.2), $A/\psi \cong A'/\mathcal{Q} / \psi'/\mathcal{Q} \cong A'/\psi'$.

Let now $\langle s, H \rangle$ and $\langle t, K \rangle$ be elements of S' , i.e. $s \in H \in \pi$, $t \in K \in \pi$. If $\langle s, H \rangle \equiv \langle t, K \rangle (\pi' \circ \psi')$, then $H=K$, by definition of π' , and $s \equiv t (\psi)$, by the definition of ψ' . $H=K$ implies $sR_{\pi}t$, whence, due to $s \equiv t (\psi)$ and $R_{\pi} \cap R_{\psi} = I$, $s=t$. Thus, $\langle s, H \rangle = \langle t, K \rangle$, i.e. $\pi' \circ \psi' = 0$. Applying Thm 3.2 we obtain:

$$A' \leq A'/\pi' \times A'/\psi'.$$

Now $A'/\pi' \cong \bar{A}$ and $A'/\psi' \cong A/\psi$. Hence

$$A \leq A' \leq \bar{A} \times A/\psi.$$

IV. CASCADE PRODUCTS OF X-AUTOMATA.

Cascade compositions of sequential machines are discussed in [SG 2], [AG], [MY 1] and [JH 2-5]. Extending these considerations to X-automata, we introduce the following

Definition. Let $A_i = \langle S_i, \Delta^i \rangle$ be complete X_i -automata, $i = 1, \dots, r$, such that

$$X_{i+1} = S_i \times X_i, \quad i = 1, \dots, r-1.$$

The cascade product A of the A_i (notation: $A = A_1 \circ \dots \circ A_r$) is the complete X-automaton $A = \langle S, \Delta \rangle$ defined as follows:

i) $X = X_1$.

ii) $S = S_1 \times \dots \times S_r$.

iii) for each $s = \langle s_1, \dots, s_r \rangle$ in S and each x in X

we have $s \Delta x = \langle s_1 \Delta^1 x_1, \dots, s_r \Delta^r x_r \rangle$

where $x_1 = x$ and $x_{i+1} = \langle s_i, x_i \rangle$, $i=1, \dots, r-1$.

Let us now assume that $A = \langle S, \Delta \rangle = A_1 \circ \dots \circ A_r$, where $A_i = \langle S_i, \Delta^i \rangle$.

We define the partitions π_i of S ($i = 1, \dots, r-1$) by:

$$\langle s_1, \dots, s_i \rangle \equiv \langle t_1, \dots, t_i \rangle \ (\pi_i) \iff s_1 = t_1, \dots, s_i = t_i$$

Clearly $\pi_1 \supseteq \pi_2 \supseteq \dots \supseteq \pi_r = 0$. Furthermore, π_i is admissible by A , and

$$A / \pi_i \cong A_1 \circ \dots \circ A_i.$$

Conversely, we first consider the case $r = 2$. For this case we have the following

Theorem 4.1. Let $A = \langle S, \Delta \rangle$ be a complete X -automaton and π an admissible partition of A . Then there exists a complete $\pi \times X$ -automaton

$$A_2 = \langle S_2, \Delta^2 \rangle \text{ such that } A_1 \circ A_2 \supseteq A,$$

where $A_1 = A / \pi = \langle \pi, \Delta^1 \rangle$ and $\#S_2 = |\pi|$.

Proof. Let τ be a partition of S , such that $\pi \cdot \tau = 0$, and $\#\tau = |\pi|$.

Such a partition τ obviously exists. We now construct a suitable

$\pi \times X$ -automaton $A_2 = \langle \tau, \Delta^2 \rangle$ as follows:

Let $H \in \pi$, $K \in \tau$, such that $H \cap K \neq \emptyset$. Due to $\pi \cdot \tau = 0$, the

intersection $H \cap K$ includes a single element $s \in S$. Let x be any

input of A . Then

$$K \Delta^2 \langle H, x \rangle = \tau (s \Delta x).$$

If $H \cap K = \emptyset$, $K \Delta^2 \langle H, x \rangle$ may be arbitrarily determined.

Let now $\bar{S} = \{ \langle H, K \rangle \mid H \in \pi, K \in \tau, H \cap K \neq \emptyset \}$ and η the mapping of \bar{S} onto S determined by

$$\langle H, K \rangle \eta = s \iff s \in H \cap K .$$

We shall now show that $A_1 \circ A_2 = A' = \langle S', \Delta' \rangle$ is a cover of A with respect to η . Indeed, let $x \in X$, $s \in S$, and let $H = \pi(s)$, $K = \tau(s)$.

Then $H \cap K = \{s\}$. Hence $\langle H, K \rangle \eta = s$. Now, $\langle H, K \rangle \Delta' x = \langle H \Delta^1 x, K \Delta^2 \langle H, x \rangle \rangle = \langle H \Delta^1 x, \tau(s \Delta x) \rangle$.

$s \in H$ implies $s \Delta x \in H \Delta^1 x$. Thus $s \Delta x$ is common to $H \Delta^1 x$ and $\tau(s \Delta x)$. Therefore, $\langle H, K \rangle \Delta' x = \langle H \Delta^1 x, \tau(s \Delta x) \rangle$ belongs to \bar{S} and its η -image is $s \Delta x$. This completes the proof that $A' \geq A$.

Let now π_1 and π_2 be admissible partitions of the complete X -automaton $A = \langle S, \Delta \rangle$, where $\pi_1 \succ \pi_2 \succ 0$. Applying Thm 4.1 to π_2 , we deduce the existence of a $\pi_2 \times X$ -automaton $A' = \langle S', \Delta' \rangle$ such that $\#S' = |\pi_2|$ and

$$A / \pi_2 \circ A' \geq A. \quad (6)$$

By Thm. 2.2, π_1 / π_2 is an admissible partition of A / π_2 , and $A / \pi_1 \cong A_1 = A / \pi_2 / \pi_1 / \pi_2$. Applying now Thm 4.1 to the partition

π_1 / π_2 of A / π_2 , we derive the existence of an X_2 -automaton

$A_2 = \langle S_2, \Delta^2 \rangle$, where $\#S_2 = |\pi_1 / \pi_2|$ such that $A_1 \circ A_2$ is defined and

$$A_1 \circ A_2 \geq A / \pi_2. \quad (7)$$

Combining (6) and (7) one easily derives the existence of an X_3 -automaton

$A_3 = \langle S', \Delta^3 \rangle$, such that $A_1 \circ A_2 \circ A_3$ is defined and

$$A_1 \circ A_2 \circ A_3 \geq A.$$

By induction on r we immediately obtain the following Cascade Decompo-

sition Theorem:

Theorem 4.2. Let π_1, \dots, π_{r-1} be admissible partitions of the complete X-automaton $A = \langle S, \Delta \rangle$, where $\pi_1 > \pi_2 \dots \pi_{r-1} > \pi_r = 0$. Then there exist complete X_i-automata $A_i = \langle S_i, \Delta^i \rangle$, $i = 1, \dots, r$, such that $A_1 \circ \dots \circ A_r$ is defined and

$$A_1 \circ \dots \circ A_r \geq A,$$

where $A_1 = A / \pi_1$, $A_1 \circ \dots \circ A_i \geq A / \pi_i$ ($i = 2, \dots, r-1$) and $\#S_i = |\pi_{i-1} / \pi_i|$, $i = 2, \dots, r$.

Finally we wish to show that cascade decompositions of X-automata may be derived from their admissible decompositions in accordance with the following

Theorem 4.3 Let $A = \langle S, \Delta \rangle$ be a complete X-automaton, π an admissible decomposition of A , and \bar{A} a π -factor of A . Then there exist automata A_1 and A_2 , such that $A_1 \cong \bar{A}$, $\#A_2 = |\pi|$ and $A_1 \circ A_2 \geq A$.

Proof. By Thm 2.3 there exists an X-automaton $A' = \langle S', \Delta' \rangle$ admitting partitions ϱ and π' such that $A' / \varrho \cong A$, $A' / \pi' \cong \bar{A}$ and $|\pi'| = |\pi|$. Applying Thm 4.1 to the partition π' of A' we derive the existence of automata A_1 and A_2 such that $A_1 = A' / \pi'$, $\#A_2 = |\pi'|$ and $A_1 \circ A_2 \geq A'$. Clearly $A' / \varrho \cong A$ implies $A' \geq A$. Thus $A_1 \circ A_2 \geq A$, where $A_1 = A' / \pi' \cong \bar{A}$ and $\#A_2 = |\pi'| = |\pi|$. Thm. 4.3 is thus proved.

An important step toward the efficient realization of a sequential machine is usually considered to be state reduction, by which the corresponding X-automaton A is replaced by A / π , where π is an admissible (and output consistent) partition of A . Recently J. Hartmanis has pointed

out certain negative effects of state reduction [JH 4]. Namely (cf. Thm 2.2), only those admissible partitions of A which include π are preserved in A/π . Thus, state reduction may destroy possibilities of machine decompositions, especially if only admissible partitions of the reduced machine are considered.

However, by also taking into consideration admissible decompositions of the reduced machine this danger of state reduction is considerably diminished. To illustrate this point, let us assume that the X -automaton $B = A/\pi$ is obtained from the complete X -automaton A by state reduction, and that ρ is an admissible partition of A , which does not include π . If, however, $\# \rho \neq \#(\rho/\pi)$, Thm 2.4 applies, i.e. there exists a ρ/π -factor B of $B = A/\pi$. Thus, Thm. 4.3 leads to a cascade-decomposition of the reduced automaton B , although the admissible partition ρ of A has been destroyed by state reduction.

CONCLUSION

The basic ideas of an algebraic decomposition theory of finite automata, essentially due to J. Hartmanis, have been presented. Furthermore, these ideas have been generalized by also considering admissible decompositions (overlapping partitions) of finite automata.

Further research is required in order to derive from the basic theory, presented in this report, efficient techniques for the synthesis of sequential machine networks.

On the other hand, the extension of some of the results obtained in this report to abstract algebras in general might be of some interest.

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