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TRANSLATION

SOME METHODS FOR INCREASING THE RATE OF CALCULATION OF ELEMENTARY FUNCTIONS ON DIGITAL ELECTRONIC COMPUTERS

By

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FOREIGN TECHNOLOGY DIVISION

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NEKOTORYYE PUTI POVYSHENIYA SKOROSTI VYCHISLENIYA ELEMENTARNYKH FUNKTSIY NA TSIFROVYKH ELEKTRONNYKH VYCHISLITEL'NYKH MASHINAKH

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CONTENTS

SOME METHODS FOR INCREASING THE RATE OF CALCULATION OF ELEMENTARY FUNCTIONS ON DIGITAL ELECTRONIC COMPUTERS

Algorithms for table-polynomial approximation of elementary functions on digital computers are discussed. At the expense of a certain increase in the number of computer memory cells occupied with sub-routines for calculating elementary functions, these methods permit a significant increase in the rate of calculation of these functions.

INTRODUCTION

The use of digital electronic computers as devices to control complex technical instruments leads to the creation of computers with a high operating speed. In the literature such computers are called computers which operate in the real time scale.

Besides the increase in the rate of performing individual operations in computers which operate in the real time scale, the problem of creating mathematical methods which permit the rapid performance of the required calculations becomes quite real. Here the expediency of increasing as much as possible the rate for such basic elementary functions as sinx, cosx, tgx, arcsinx, arctgx, lnx, e^x and $\sqrt{x}$ is immediately apparent. For many devices, where the digital computer is the controlling complex, the calculation of elementary functions can occupy up to 40 or even 60 percent of the time required for the solution of a control problem. Therefore, with a reduction in the time for computing elementary functions, the total solution time may be reduced significantly, and the freed time can be used for correcting the control of the actual instrument.

Algorithms with which elementary functions are calculated on digital computers intended for the solution of engineering problems are poorly suited for computers which operate in the real time scale. This is explained by the fact that, for usual digital

computers, the choice of algorithms is based primarily on considerations of the space occupied by their programming in the memory unit.

Algorithms which are presently in use are constructed in such a way that they give the values of elementary functions on the entire interval of variation of the argument with a given degree of accuracy.

Algorithms used for the calculation of elementary functions can be divided into four groups:

1) Algorithms which permit the calculation of elementary functions with polynomials which approximate the given function (polynomial approximation);

2) Algorithms which permit the calculation of elementary functions with rational functions which approximate a given function (rational approximation);

3) Algorithms which permit the calculation of elementary functions with tables which contain reference values of the function on a given interval (table approximation);

4) Algorithms which permit the calculation of elementary functions with the method of calculation of values of the function "digit by digit."

The enumerated algorithms for calculating elementary functions have one part in common: they require a reduction, using appropriate formulas, of the entire interval of variation of the argument of a given elementary function to some minimum interval, where the approximation is accomplished by a given algorithm. We

shall consider this common part in the enumerated algorithms to be identical, and we shall see the distinction between algorithms in the methods through which the calculation of the function is accomplished in the reduced interval.

In widest use today are algorithms which permit a polynomial approximation of elementary functions. Therefore, these algorithms will be of greatest interest for us.

Algorithms which use rational approximation are not yet in wide use, since in the majority of existing computers the operation of division exceeds the operation of multiplication in the time required for its performance by two or three times. They are used only in computers in which the operation of division is to a certain extent equivalent to the operation of multiplication in the time required for its performance.

Algorithms which use tables of reference values for calculating elementary functions are even less widely used. This is explained by the fact that the tables are constructed to use the most simple interpolation formulas which contain a minimum number of multiplications, and this leads to extremely large tables of reference values when the condition of high accuracy of calculations is imposed. A large volume of tables increases the time required to locate the necessary values and requires a large memory unit. Tables of reference values of functions can be considered fully adequate for lower accuracy of calculation not exceeding $0.5 \cdot 10^{-6}$, [illegible] when a sufficiently large memory unit is available.

Algorithms which permit calculation of elementary functions "digit by digit" are suitable for use in a computer. Here certain elementary functions, for example lnx and e^x, can be calculated in the time of one multiplication. The inadequacy of these algorithms is the fact that not all elementary functions can be calculated with this method. In addition each elementary function requires expensive additional equipment for its approximation by this method.

Experience in the operation of digital electronic computers in use today has shown that algorithms which permit polynomial or rational approximation of elementary functions are most suitable in practice.

(Here we shall examine rational approximation in the reduced form, i.e. as the ratio of two polynomials.)

Thus we may assume that in computers which operate in the real time scale these algorithms may be employed with certain variations in the method of application. With an increase in sub-routines, they may be redesigned in such a way that the approximation of any elementary function in the reduced interval is conducted not with one polynomial which insures a given accuracy on the entire interval, but by a certain set of polynomials applicable on sub-intervals of this reduced interval. Here the degrees of the approximating polynomials increase from one sub-interval to the next, and only in the last sub-interval of the reduced intervalcan the polynomial have the form which is now used over the entire length of the inverval of reduction.

Thus a combined table-polynomial algorithm, as it were, is being used to approximate the elementary functions. The same considerations apply to rational approximation represented as the ratio of two polynomials.

I. THE TABLE-POLYNOMIAL APPROXIMATION METHOD

The possibility and expediency of employing the method of table-polynomial approximation of elementary functions may be based on the following considerations.

As is generally known, a computer operates with digital information with a certain degree of accuracy.

This accuracy is determined not only by the error in approximating a given function, but also by the fact that the computer performs the necessary operations on numbers which lie in a fully defined range and are discretely distributed within this range. The accuracy of calculation of elementary functions according to any algorithm is determined also by this discreteness. For values of functions near zero in modulus, polynomials which give a good approximation near the values of these functions which are maximum in modulus have a large number of terms of higher orders in the sub-intervals near the zero values. However, the terms of higher orders in sub-intervals near the zero values of the function take on values which are not caught by the computer, as a result of its limited discrete perception of these numbers, but require a large number of multiplications. Herein lies the basis for hope that the entire interval of reduc-

tion can judiciously be divided into smaller sub-intervals in which an approximation of functions can be accomplished with polynomials of lower orders.

Let us assume that we have succeeded in dividing the interval of reduction (a,b) into the segments $\Delta_1, \Delta_2, \ldots, \Delta_k$, such that $\Delta_i \cap \Delta_j = 0$, $\bigcup_{i=1}^{k} = (a, b)$.

Relative to the time required for their accomplishment all arithmetic operations performed by the computer can be examined from the point of view of the time required for the performance of the operations of addition and multiplication, so that the operation of subtraction is in this sense equivalent to the operation of addition, and the operation of division can be thought of as several multiplications, etc.

Let the polynomial which gives an approximation of a particular elementary function on the entire interval (a, b) require m additions and n multiplications.

The time expended by the computer to accomplish one or another algorithm is determined as the number of arithmetic operations of addition and multiplication, multiplied by the corresponding time required for the performance of these operations on the computer.

Further, let n_i and m_i be the number of multiplications and additions on the segment Δ_i. Here we assume that $n_i \leq n_{i+1}$, $m_i \leq m_{i+1}$. Then in the case of application of one polynomial the number of multiplications on (a,b) is

$$M_1(n) = n \ , \qquad (1)$$

while in the case of application of several polynomials the number of multiplications is

$$M_2(n) = \sum_{i=1}^{k} P(\Delta_i) n_i \ . \qquad (2)$$

where $P(\Delta_i)$ are the probabilities that the argument for whose values the elementary function is being calculated will fall in Δ_i.

We may obtain exactly the same expressions for the mathematical expectation of the number of additions when calculating an elementary function:

$$M_1(m) = m \qquad (3)$$

$$M_2(m) = \sum_{i=1}^{k} P(\Delta_i) \cdot m_i \ . \qquad (4)$$

The time required for the accomplishment of the algorithm on the computer in each of these cases will be:

$$T_1 = \tau_1 M_1(n) + \tau_2 M_1(m) = (\tau_1 n + \tau_2 m) \qquad (5)$$

$$T_2 = \tau_1 M_2(n) + \tau_2 M_2(m) \ , \qquad (6)$$

where τ_1 is the time of one multiplication on the computer,

τ_2 is the time of one addition on the computer.

Let us assume that a polynomial of appropriate degree which approximates the given elementary function on Δ_i is given

by its coefficients. Then the selection of the polynomial reduces to the selection of the appropriate coefficients from the table. Let the time required to locate these coefficients on the table equal τ_0. Then the ratio

$$A = \frac{\tau_1 N_3(n) + \tau_2 N_3(m) + \tau_0}{(\tau_1 n + \tau_2 m)} \qquad (7)$$

permits an evaluation of the advantage of the table-polynomial method of calculation of elementary functions.

An evaluation of the character of the distribution of probabilities P(x) that the values of the argument x will lie in the corresponding segment Δ_i of the reduced interval is difficult even in the case of control units of relatively slight complexity. We shall assume that P(x) is determined by uniform distribution and that the occurence of x in one or another sub-interval of the reduced interval is an independent event.

Then from (7) we easily find that

$$A = \frac{\tau_1 \sum_i \Delta_i n_i + \tau_2 \sum_i \Delta_i m_i}{(b-a)(n\tau_1 + m\tau_2)} + \frac{\tau_0}{n\tau_1 + m\tau_2}. \qquad (8)$$

Thus we see that A consists of the two parts

$$A_1 = \frac{\tau_1 \sum_i \Delta_i n_i + \tau_2 \sum_i \Delta_i m_i}{(b-a)(n\tau_1 + m\tau_2)} \qquad (8')$$

$$A_2 = \frac{\tau_0}{n\tau_1 + m\tau_2}. \qquad (8'')$$

The ratio A_1, as we shall show subsequently for specific elementary functions, can be made less than 1 through an appropriate choice of approximating polynomials. It will be near 0.4 - 0.7.

We shall consider briefly the question of how to organize the search in the table in such a way that we may determine the corresponding Δ_i according to the argument x reduced to the basic interval (more precisely, those coefficients a_i of the polynomial which approximates the given function on the sub-interval Δ_i) with the condition that the ratio $A_2 = \frac{\tau_0}{n\tau_1 + m\tau_2}$ be as small as possible.

We may consider the usual program methods used during the choice of one of k possible branches of the calculation process as one method for solving this problem.

The division of the interval of reduction into the system of sub-intervals Δ_i can be organized in such a way that the sub-interval Δ_i is determined by several significant digits of the binary representation of the argument x, or that the argument x is associated with an appropriately selected constant prior to the selection of the sub-interval, or that a somewhat more complex function of x has been formed which contains certain characteristic indications for each sub-interval Δ_i. Then the choice of the sub-interval Δ_i and, consequently, the choice of an appropriate approximating polynomial $P_n(x)$ can be expediently accomplished by a very simple deciphering unit which will control the calculation of the polynomial with the required number of

coefficients. In this case the time τ_0 required for the search of the table can be made very small in comparison to the time $\tau_1+\tau_2$. It is true that in this case certain sub-routines will require individual elementary modifications, which do not complicate the design of the computer, in order for a very significant advantage in rate of operation to be obtained.

In essence, we may provide in the computer a special command which would determine the appropriate address of the sub-routine for determining $P_n(x)$ according to a given x. Then this command could select several digits from x and add to them a certain constant, and the quantity thus derived could be the address of the sub-routine. We could provide other, more complex commands, specially adapted to this problem, which would permit a maximum reduction of the time required to search the interval to which the given x belongs. This would permit a branching to a special sub-routine for calculation of the appropriate approximation of the elementary function.

II. THE TABLE-POLYNOMIAL APPROXIMATION OF ELEMENTARY FUNCTIONS

If on the interval [a,b] we are given an arbitrary differentiable function f(x), then the best piecewise approximation of this function on the given interval can be performed with polynomials or rational fraction functions of any degree, with the assumption that the interval [a,b] is divided into a

series of smaller intervals.

We shall perform the division of the interval $[a,b]$ in one of the following two ways.

If the function $f(x)$ has the property of evenness or oddness, then in order not to lose these properties of the function we shall divide the interval $[a,b]$ into a series of nested intervals with the common point a. In this case the length of the nested intervals into which the interval $[a,b]$ is divided are determined in such a way that the degree of the polynomials which approximate the function $f(x)$ with a given accuracy increases with an increase in the length of the segment.

If the function $f(x)$ does not have the property of evenness or oddness, the interval $[a,b]$ is divided into a series of small non-overlapping intervals. In this case we may design in advance a system of easily calculated best polynomials of low degree[1] which approximate an arbitrary differentiable function on the interval $[x_i, x_{i+1}] \in [a,b]$. These polynomials will be required later during the construction of formulas for approximating the functions $\sqrt{x}$ and e^x.

We shall expand the given function $f(x)$ into a Taylor series at the point a_i, which is the mid-point of the interval $[x_i, x_{i+1}]$:

$$f(x)=f(a_i)+\frac{x-a_i}{1!}f'(a_i)+\frac{(x-a_i)^2}{2!}f''(a_i)+$$

$$+\frac{(x-a_i)^3}{3!}f'''(a_i)+\cdots . \qquad (9)$$

Using the substitution $x-a_i = \frac{x_{i+1}-x_i}{2}u$ and Chebyshev polynomials of appropriate degrees, and returning again to the old variable, we obtain the best polynomial of the necessary degree which approximates the function f(x) on the interval $[x_i, x_{i+1}]$.

Thus the best polynomial which approximates the function f(x) on the interval $[x_i, x_{i+1}]$ will be:

of the second degree

$$f(x) \approx A_2 + B_2(x-a_i) + C_2(x-a_i)^2, \qquad (10)$$

where the coefficients of the polynomial are defined by the following expressions:

$$A_2 = [f(a_i) + \frac{1}{2^5}\lambda_2^4 R_2 - \frac{1}{2^3}\lambda_2^2 M_2],$$

$$E_2 = [f'(a_i) - \frac{1}{2^4}\lambda_2^2 S_2 + \frac{3}{2^2} N_2],$$

$$C_2 = [\frac{1}{2!} f''(a_i) - \frac{9}{2^4}\lambda_2^2 R_2 + M_2],$$

$$M_2 = [\frac{1}{4!} f^{IV}(a_i) + \frac{3}{2} R_2]\lambda_2^2,$$

$$N_2 = [\frac{1}{3!} f'''(a_i) + \frac{1}{2^2} S_2]\lambda_2^2,$$

$$R_2 = \frac{1}{6!}\lambda_2^2 f^{VI}(a_i),$$

$$S_2 = \frac{1}{4!}\lambda_2^2 f^{V}(a_i),$$

$$\lambda_2 = \frac{x_{i+1}-x_i}{2}.$$

of the third degree

$$f(x) \approx A_3 + B_3(x-a_i) + C_3(x-a_i)^2 + D_3(x-a_i)^3 , \quad (11)$$

where the coefficients of the polynomial are:

$$A_3 = [f(a_i) + \frac{1}{2^5}\lambda_3^4 S_3 - \frac{1}{2^3}\lambda_3^2 N_3],$$

$$B_3 = [f'(a_i) + \frac{1}{2^6}\lambda_3^4 R_3 - \frac{5}{24}\lambda_3^2 N_3],$$

$$C_3 = [\frac{1}{2!}f''(a_i) - \frac{9}{24}\lambda_3^2 S_3 + N_3],$$

$$D_3 = [\frac{1}{3!}f'''(a_i) - \frac{1}{2^3}\lambda_3^2 R_3 + \frac{5}{2^2}N_3],$$

$$N_3 = [\frac{1}{4!}f^{IV}(a_i) + \frac{3}{2}S_3]\lambda_3^2 ,$$

$$N_3 = [\frac{1}{5!}f^{V}(a_i) + \frac{1}{2^2}R_3]\lambda_3^2 ,$$

$$R_3 = \frac{1}{6!}\lambda_3^2 f^{VII}(a_i),$$

$$S_3 = \frac{1}{6!}\lambda_3^2 f^{VI}(a_i),$$

$$\lambda_3 = \frac{x_{i+1} - x_i}{2}.$$

of the fourth degree

$$f(x) \approx A_4 + B_4(x-a_i) + C_4(x-a_i)^2 + D_4(x-a_i)^3 + E_4(x-a_i)^4 \quad (12)$$

where the coefficients of the polynomial are:

$$A_4 = [f(a_i) - \frac{1}{2^7}\lambda_4^6 R_4 + \frac{1}{2^5}\lambda_4^4 N_4],$$

$$B_4 = [f'(a_i) + \frac{1}{2^6}\lambda_4^4 S_4 - \frac{5}{2^4}\lambda_4^2 N_4],$$

$$C_4 = [\frac{1}{2!}f''(a_i) - \frac{1}{2^2}\lambda_4^4 R_4 - \frac{9}{2^4}\lambda_4^2 N_4],$$

$$D_4 = [\frac{1}{3!}f'''(a_i) - \frac{1}{2^3}\lambda_4^2 S_4 + \frac{5}{2^2}N_4],$$

$$E_4 = [\frac{1}{4!}f^{IV}(a_i) - \frac{5}{2^3}\lambda_4^2 R_4 + \frac{3}{2} M_4] ,$$

$$M_4 = [\frac{1}{6!}f^{VI}(a_i) + 2R_4]\lambda_4^2 ,$$

$$N_4 = [\frac{1}{5!}f^{V}(a_i) + \frac{1}{2^3} S_4]\lambda_4^2 ,$$

$$R_4 = \frac{1}{8!}\lambda_4^2 f^{VIII}(a_i) ,$$

$$S_4 = \frac{1}{6!}\lambda_4^2 f^{VII}(a_i) ,$$

$$\lambda_4 = \frac{x_{i+1} - x_i}{2} .$$

of the fifth degree

$$f(x) \approx A_5 + B_5(x-a_i) + C_5(x-a_i)^2 + D_5(x-a_i)^3 + E_5(x-a_i)^4 + F_5(x-a_i)^5 , \qquad (13)$$

where the coefficients of the polynomial are:

$$A_5 = [f(a_i) - \frac{1}{2^7}\lambda_5^6 S_5 + \frac{1}{2^5}\lambda_5^4 N_5],$$

$$B_5 = [f'(a_i) - \frac{1}{2^8}\lambda_5^6 R_5 + \frac{7}{2^6}\lambda_5^4 N_5],$$

$$C_5 = [\frac{1}{2!}f''(a_i) + \frac{1}{2^3}\lambda_5^4 S_5 - \frac{9}{2^4}\lambda_5^2 N_5] ,$$

$$D_5 = [\frac{1}{3!}f'''(a_i) + \frac{15}{2^5}\frac{1}{9}\lambda_5^4 R_5 - \frac{7}{2^3}\lambda_5^2 N_5] ,$$

$$E_5 = [\frac{1}{4!} f^{IV}(a_i) - \frac{5}{2^3}\lambda_5^2 S_5 + \frac{3}{2} N_5],$$

$$F_5 = [\frac{1}{5!} f^{V}(a_i) - \frac{3}{2^4}\lambda_5^2 R_5 + \frac{7}{2^3} N_5],$$

$$N_5 = [\frac{1}{7!} f^{VII}(a_i) + \frac{1}{2^3} R_5]\lambda_5^2,$$

$$M_5 = [\frac{1}{6!} f^{VI}(a_i) + 2S_5]\lambda_5^2,$$

$$R_5 = \frac{1}{8!}\lambda_5^2 f^{IX}(a_i),$$

$$S_5 = \frac{1}{8!}\lambda_5^2 f^{VIII}(a_i),$$

$$\lambda_5 = \frac{x_{i+1} - x_i}{2}.$$

We now proceed with the direct derivation of formulas for specific elementary functions.

Hereafter it is assumed that all formulas constructed for the approximation of elementary functions insure the derivation of 10 significant decimal places; we therefore will not evaluate them in each separate case.

Here we shall pause to analyze the formulas given in [1]. These formulas are used for the calculation of elementary functions in the computers BESM-1 and BESM-II. Some of these use rational approximation; others use polynomial approximation.

It has been shown in [1] that, in comparison with pure polynomial approximation, the use of these formulas permits a significant reduction in the time required for calculation of elementary functions. Thus a further refinement of these formulas is desirable for the purpose of obtaining a further economy in calculating time.

The following considerations, which concern the calculation of elementary functions in the smaller sub-intervals of the reduced interval with polynomials of lower degrees, relate equally to any other formulas which use polynomial or rational approximation.

1. Calculation of the functions sinx, cosx and tgx

As in [1], the expansion of the function ctgz into a Laurent series is used for calculation of the values of the functions sinx, cosx and tgx:

$$zctgz = 1 - \sum_{\kappa=1}^{\infty} \frac{2^{2\kappa}|B_{2\kappa}|}{(2\kappa)!} z^{2\kappa-1} , \qquad (14)$$

which converges for $|z| < \pi$.

Setting $z = \frac{x}{2}$, we obtain the expression

$$2tg\frac{x}{2} = \frac{x}{1 - \sum_{\kappa=1}^{\infty} \frac{|B_{2\kappa}| . x^{2\kappa}}{(2\kappa)!}} . \qquad (15)$$

It is assumed that, before calculation of the tangent of half of the argument according to formula (15), the original

argument was reduced to a quantity less than $\frac{\pi}{4}$, and from this quantity a whole and a fractional part were extracted. The argument x in this formula is the fractional part of a quantity less than $\frac{\pi}{4}$, and thus varies in the interval $0 \leq x \leq 1$.

We shall divide the interval of variation of the independent variable x into four partial sub-intervals with the points $\frac{1}{8}$, $\frac{2}{5}$ and $\frac{3}{5}$.

For each of the sub-intervals thus obtained we construct its best polynomial which approximates in it the denominator of the function defined by relation(15). Then we shall have the following series of approximating expressions.

In the interval $[0; \frac{1}{8}]$ the function $2\operatorname{tg}\frac{x}{2}$ is approximated by the expression:

$$2\operatorname{tg}\frac{x}{2} = \frac{x}{A_1 - B_1 x^2 - C_1 x^4} \quad , \qquad (16)$$

where the coefficients A_1, B_1, and C_1 are defined as follows:

$$A_1 = [1 + \frac{1}{2^7}\lambda_1^6 R_1 - \frac{1}{2^5}\lambda_1^4 N_1] \, ,$$

$$B_1 = [\frac{1}{2!}\frac{1}{6} + \frac{1}{2^3}\lambda_1^4 R_1 + \frac{9}{2^4}\lambda_1^2 N_1] \, ,$$

$$C_1 = [\frac{1}{4!}\frac{1}{30} - \frac{5}{2^3}\lambda_1^2 R_1 - \frac{3}{2} N_1] \, ,$$

$$N_1 = (\frac{1}{6!}\frac{1}{42} + 2R_1) \lambda_1^2 \, .$$

$$R_1 = \frac{1}{8!}\frac{1}{30}\lambda_1^8 ,$$

$$\lambda_1 = \frac{1}{2^3} .$$

Formula (16) is used to calculate the values of $2\mathrm{tg}\frac{x}{2}$ for any x on the interval $[0;\frac{1}{8}]$.

For the interval $[0;\frac{2}{5}]$ we find the expression which approximates the function $2\mathrm{tg}\frac{x}{2}$;

$$2\mathrm{tg}\frac{x}{2} = \frac{x}{A_2 - B_2x^2 - C_2x^4 - D_2x^6} , \qquad (17)$$

where the coefficients A_2, B_2, C_2 and D_2 are defined as follows:

$$A_2 = [1 - \frac{1}{2^9}\lambda_2^8 R_2 + \frac{1}{2^7}\lambda_2^6 M_2] ,$$

$$B_2 = [\frac{1}{2!}\frac{1}{6} - \frac{25}{2^8}\lambda_2^6 R_2 + \frac{1}{2^3}\lambda_2^4 M_2],$$

$$C_2 = [\frac{1}{4!}\frac{1}{30} + \frac{25}{2^5}\lambda_2^4 R_2 - \frac{5}{2^3}\lambda_2^2 M_2] ,$$

$$D_2 = [\frac{1}{6!}\frac{1}{42} - \frac{35}{24}\lambda_2^2 R_2 + 2M_2],$$

$$M_2 = (\frac{1}{8!}\frac{1}{30} + \frac{5}{2}R_2)\lambda_2^2,$$

$$R_2 = \frac{1}{10!}\frac{5}{66}\lambda_2^2 ,$$

$$\lambda_2 = \frac{2}{5}.$$

Formula (17) is used to calculate the values of $2\mathrm{tg}\frac{x}{2}$ only for the argument in the interval $\frac{1}{8}\leqslant x\leqslant\frac{2}{5}$, since calculation of the values of the function for the argument x on the interval $0\leqslant x\leqslant\frac{1}{8}$ is accomplished more economically according to formula (16).

In the interval $[0,\frac{3}{5}]$ the following expression, which approximates the function $2\mathrm{tg}\frac{x}{2}$ in the indicated interval, holds:

$$2\mathrm{tg}\frac{x}{2}=\frac{x}{A_3-B_3x^2-C_3x^4-D_3x^6-E_3x^8}, \qquad (18)$$

where the coefficients A_3, B_3, C_3, D_3 and E_3 are defined as follows:

$$A_3=\left[1+\frac{1}{2^{11}}\lambda_3^{10}R_3-\frac{1}{2^9}\lambda_3^8M_3\right],$$

$$B_3=\left[\frac{1}{21}\frac{1}{6}+\frac{9}{2^8}\lambda_3^8R_3-\frac{25}{2^8}\lambda_3^6M_3\right],$$

$$C_3=\left[\frac{1}{41}\frac{1}{30}-\frac{105}{2^8}\lambda_3^6R_3+\frac{25}{2^5}\lambda_3^4M_3\right],$$

$$D_3=\left[\frac{1}{61}\frac{1}{42}+\frac{7}{2^2}\lambda_3^4R_3-\frac{35}{2^4}\lambda_3^2M_3\right],$$

$$E_3=\left[\frac{1}{81}\frac{1}{30}-\frac{27}{2^3}\lambda_3^2R_3+\frac{5}{2}M_3\right],$$

$$M_3=\left(\frac{1}{101}\frac{5}{66}-3R_3\right)\lambda_3^2,$$

$$R_3=\frac{1}{121}\frac{691}{2730}\lambda_3^2,$$

$$\lambda_3=\frac{3}{5}.$$

Formula (18) is used to calculate the values of $2\mathrm{tg}\frac{x}{2}$ only for those x which lie in the interval $\frac{2}{5}\leqslant x\leqslant\frac{3}{5}$.

For x from the interval $0 \leq x \leq \frac{2}{5}$ the values of the function are calculated according to formulas (16) and (17).

In the interval $[0; 1]$ the next expression, which approximates the function $2\operatorname{tg}\frac{x}{2}$, holds:

$$2\operatorname{tg}\frac{x}{2} \approx \frac{x}{A_4 - B_4x^2 - C_4x^4 - D_4x^6 - E_4x^8 - F_4x^{10}}, \quad (19)$$

where the coefficients of the denominator are defined as follows:

$$A_4 = \left[1 - \frac{1}{2!3}R_4 + \frac{1}{2^{11}}N_4\right],$$

$$B_4 = \left[\frac{1}{2!}\frac{1}{6} - \frac{49}{2^{12}}R_4 + \frac{9}{2^8}N_4\right],$$

$$C_4 = \left[\frac{1}{4!}\frac{1}{30} + \frac{49}{2^8}R_4 - \frac{105}{2^8}N_4\right],$$

$$D_4 = \left[\frac{1}{6!}\frac{1}{42} - \frac{147}{2^7}R_4 + \frac{7}{2^3}N_4\right],$$

$$E_4 = \left[\frac{1}{8!}\frac{1}{30} + \frac{105}{2^5}R_4 - \frac{27}{2^3}N_4\right]$$

$$F_4 = \left[\frac{1}{10!}\frac{5}{66} - \frac{77}{2^4}R_4 + 3N_4\right],$$

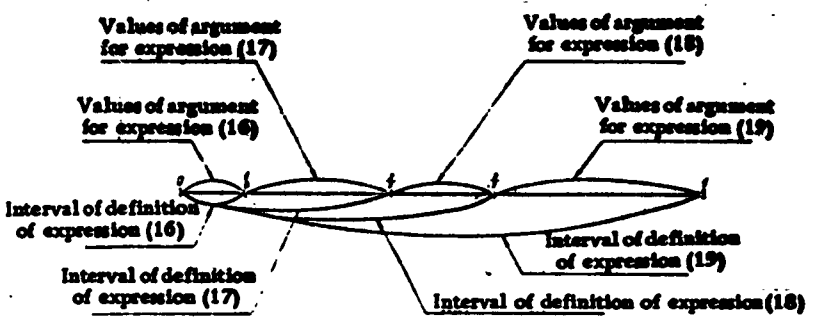
$$N_4 = \frac{1}{12!}\frac{691}{2730} + \frac{7}{2}R_4,$$

$$R_4 = \frac{1}{14!}\frac{7}{6}.$$

Formula (19) is used, not for the entire interval for which it was derived, but only for values of the argument from the interval $\frac{3}{5} \leq x \leq 1$.

Graphically the region of existence of each of expressions

(16), (17), (18) and (19) and the intervals of variation of the independent variable for each of these expressions can be depicted as follows:



After the value of $2\mathrm{tg}\frac{x}{2}$ has been found, the absolute value of the functions sinx, cosx and tgx are easily calculated according to the formulas:

$$\left.\begin{aligned} \sin x &= \frac{2\mathrm{tg}\frac{x}{2}}{1+\mathrm{tg}^2\frac{x}{2}}, \quad \cos x = \frac{1-\mathrm{tg}^2\frac{x}{2}}{1+\mathrm{tg}^2\frac{x}{2}} \\ \mathrm{tg}x &= \frac{\sin x}{\cos x} \end{aligned}\right\}. \quad (20)$$

For calculation of the functions sinx, cosx and tgx, it would have been possible to construct still other approximating formulas based on their expansion into a Taylor series and Chebyshev polynomials of appropriate degree.

The expansion of ctgz into a Laurent series was used here as the most economical for the simultaneous determination of all three functions. Hence we have demonstrated a method for constructing approximating functions which accelerate the calcu-

lation process based on this expansion.

2. Calculation of the function lnx

During calculation of the natural logarithm, the argument x, is assumed to be given in normalized form, i.e. in the form

$$x = 2^p \cdot x_1 , \qquad (21)$$

where p is the order of the number x, x_1 is the decimal part of the number x which is contained in the interval $\frac{1}{2} \leqslant x_1 < 1$.

Taking the logarithm of relation (21), we obtain

$$\ln x = p \ln 2 + \ln x_1 , \qquad (22)$$

where $\ln x_1$ in formula (22) can be expressed by means of the series

$$\ln x_1 = \sum_{n=0}^{\infty} \left(\frac{x_1 - 1}{x_1 + 1} \right)^{2n+1} \cdot \frac{2}{2n+1} . \qquad (23)$$

Using relation (23), the identity

$$\ln x_1 = \ln \mu_i x_1 - \ln \mu_i$$

with certain assumptions relative to the particular values of μ_i, and Chebyshev polynomials of appropriate degree, we easily obtain the best polynomials which approximate the function lnx on the given interval with a given degree of accuracy.

We shall divide the interval $\left[\frac{1}{2}, 1\right]$ of variation of the

independent variable x_1 by the points $x_1=\frac{17}{32}$, $x=\frac{5}{8}$ and $x_1=\frac{13}{16}$ into four partial nested sub-intervals with the common point $x_1=\frac{1}{2}$, and in each of the intervals thus obtained, including the entire interval itself, we shall construct its best polynomial which approximates the function lnx.

Then for the interval $\left[\frac{1}{2};\ \frac{17}{32}\right]$ we will have the following relation containing the best polynomial of the third degree which approximates function lnx:

$$\ln x = (p-3)\ln 2 + \frac{1}{2}\ln 17 + A_1 u + B_1 u^3 , \qquad (24)$$

where the coefficients are defined as follows:

$$A_1 = \left[2 + \frac{1}{2^5}\lambda_1^6 - \frac{5}{2^4}\lambda_1^2 \ N_1\ \right],$$

$$B_1 = \left[\frac{2}{3} - \frac{1}{2^2}\lambda_1^4 + \frac{5}{2^3}\ N_1\right] ,$$

$$N_1 = \left(\frac{2}{5} + \frac{1}{2}\lambda_1^2\right)\lambda_1^2 ,$$

$$\lambda_1 = \frac{\sqrt{17}}{256 + 33\sqrt{17}} ,$$

$$\mu_1 = \frac{8}{\sqrt{17}} ,$$

$$u = \frac{x_1 - \frac{\sqrt{17}}{8}}{x_1 + \frac{\sqrt{17}}{8}} .$$

Formula (24) is used for calculation of values of the function lnx for all x_1 lying in the interval $\frac{1}{2} \leq x_1 \leq \frac{17}{32}$.

The following relation, which contains the best polynomial of the fifth degree, is obtained for the interval $\left[\frac{1}{2};\ \frac{5}{8}\right]$:

$$\ln x = (p-2)\ln 2 + \frac{1}{2}\ln 5 + A_2 u + B_2 u^3 + C_2 u^5 , \quad (25)$$

where the coefficients are defined as follows:

$$A_2 = \left[2 - \frac{1}{2^7}\lambda_2^8 + \frac{7}{2^6}\lambda_2^4 N_2 \right] ,$$

$$B_2 = \left[\frac{2}{3} + \frac{5}{3}\frac{1}{2^4}\lambda_2^8 - \frac{7}{2^3}\lambda_2^2 N_2 \right] ,$$

$$C_2 = \left[\frac{2}{5} - \frac{3}{2^3}\lambda_2^4 + \frac{7}{2^3}N_2 \right] ,$$

$$N_2 = \left(\frac{2}{7} + \frac{1}{2}\lambda_2^2 \right)\lambda_2^2 ,$$

$$\lambda_2 = \frac{\sqrt{5}}{20 + 9\sqrt{5}} ,$$

$$\mu_2 = \frac{4}{\sqrt{5}} ,$$

$$u = \frac{x_1 - \frac{\sqrt{5}}{4}}{x_1 + \frac{\sqrt{5}}{4}} .$$

Formula (25) is used for calculation of the values of the function $\ln x$ only for those values of x_1 lying in the interval $\frac{17}{32} \leq x_1 \leq \frac{5}{8}$.

For $x_1 \leq \frac{17}{32}$ it is more expedient to use formula (24).

The following relation with the best polynomial of the seventh degree is constructed for the interval $\left[\frac{1}{2}; \frac{13}{16}\right]$:

$$\ln x = (p-3)\ln 2 + \frac{1}{2}\ln 26 + A_3 u + B_3 u^3 + C_3 u^5 + D_3 u^7 , \quad (26)$$

where the coefficients of the polynomial are defined as follows:

$$A_3 = \left[2 + \frac{1}{2^9}\lambda_3^{10} - \frac{9}{2^8}\lambda_3^6 N_3\right],$$

$$B_3 = \left[\frac{2}{3} - \frac{5}{2^7}\lambda_3^8 + \frac{15}{2^5}\lambda_3^4 N_3\right].$$

$$C_3 = \left[\frac{2}{5} + \frac{7}{2^5}\lambda_3^6 - \frac{27}{2^4}\lambda_3^2 N_3\right].$$

$$D_3 = \left[\frac{2}{7} - \frac{1}{2}\lambda_3^4 + \frac{9}{2^2}N_3\right],$$

$$N_3 = \left(\frac{2}{9} + \frac{1}{2}\lambda_3^2\right)\lambda_3^2,$$

$$\lambda_3 = \frac{5\sqrt{26}}{104 + 21\sqrt{26}},$$

$$\mu_3 = \frac{8}{\sqrt{26}},$$

$$u = \frac{x_1 - \frac{\sqrt{26}}{8}}{x_1 + \frac{\sqrt{26}}{8}}.$$

Formula (26) is used to calculate the values of the function $\ln x$ for values of x_1 lying in the interval $\frac{5}{8} \leq x_1 \leq \frac{13}{16}$.

The following expression containing the best polynomial of the ninth degree is constructed on the interval $\left[\frac{1}{2};\ 1\right]$:

$$\ln x = \left(p - \frac{1}{2}\right)\ln 2 + A_4 u + B_4 u^3 + C_4 u^5 + D_4 u^7 + E_4 u^9, \quad (27$$

where the coefficients of the polynomial are defined as follows:

$$A_4 = \left[2 - \frac{1}{2^{11}}\lambda_4^{12} + \frac{11}{2^{10}}\lambda_4^8 N_4\right],$$

$$B_4 = \left[\frac{2}{3} + \frac{7}{2^9}\lambda_4^{10} - \frac{55}{2^8}\lambda_4^6 N_4\right],$$

$$C_4 = \left[\frac{2}{5} - \frac{7}{2^6}\lambda_4^6 + \frac{77}{2^6}\lambda_4^4 M_4\right],$$

$$D_4 = \left[\frac{2}{7} + \frac{3}{2^3}\lambda_4^6 - \frac{11}{2^3}\lambda_4^2 M_4\right],$$

$$E_4 = \left[\frac{2}{9} - \frac{5}{2^3}\lambda_4^4 + \frac{11}{2^3}M_4\right],$$

$$M_4 = \left(\frac{2}{11} + \frac{1}{2}\lambda_4^2\right)\lambda_4^2,$$

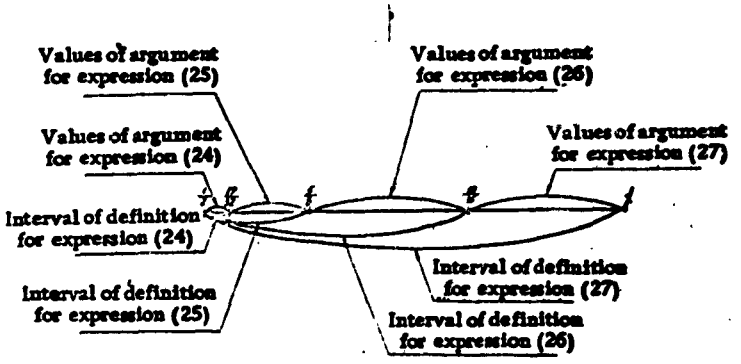
$$\lambda_4 = \frac{\sqrt{2}-1}{\sqrt{2}+1},$$

$$\mu_4 = \sqrt{2},$$

$$u = \frac{x_1 - \frac{\sqrt{2}}{2}}{x_1 + \frac{\sqrt{2}}{2}}.$$

Formula (27) is used for calculation of the values of the function $\ln x$ for x_1 lying in the interval $\frac{13}{16} \leq x_1 < 1$.

The regions where polynomials (24) - (27) approximate $\ln x$ with the given accuracy may be presented graphically as follows:



From the very construction of the approximating best

polynomials introduced here, it follows that each successive polynomial overlaps the preceeding ones and permits the derivation of the value lnx with the given accuracy in those intervals more to the left than the region of variation of the argument for which this polynomial was constructed. Although the utilization of all of these polynomials will require additional loading of the computer's memory unit, it will nonetheless lead to a significant economy of machine time during repeated calculation of the values of lnx.

3. Calculation of the function e^x

The calculation of the function e^x begins with the representation of this function in the form:

$$e^x = 2^{\frac{x}{\ln 2}} = 2^{[\frac{x}{\ln 2}]} \cdot 2^{\{\frac{x}{\ln 2}\}}, \qquad (28)$$

where $[\frac{x}{\ln 2}]$ is the whole part of the index $\frac{x}{\ln 2}$, and $\{\frac{x}{\ln 2}\}$ is the fractional part of the index $\frac{x}{\ln 2}$.

The part of the quantity e^x which is expressed by the factor $2^{[\frac{x}{\ln 2}]}$ we assume to be calculated and ready to add to the other part of e^x, i.e. to $2^{\{\frac{x}{\ln 2}\}}$, which remains to be calculated.

Here we note the applicability of the equality

$$2^{\{\frac{x}{\ln 2}\}} = e^{\{\frac{x}{\ln 2}\}\ln 2} = e^v, \qquad (29)$$

Where

$$v = \{\frac{x}{\ln 2}\}\ln 2.$$

Clearly the variable v varies within the limits $|v| \leq \ln 2$, so that the quantity $\{\frac{x}{\ln 2}\}$ varies between 0 and 1.

The limits of variation of the index function can be reduced further if we multiply and divide it by $\sqrt{2}$. Then we obtain:

$$\sqrt{2}\cdot e^{v-\frac{\ln 2}{2}} = e^{u}\cdot\sqrt{2}\,, \qquad (30)$$

where $u = \left\{\frac{x}{\ln 2}\right\}\cdot\ln 2 - \frac{\ln 2}{2}$ varies within the limits $|u| \leq \frac{\ln 2}{2}$.

Calculating e^{u}, we at the same time find the value of the quantity $2^{\left\{\frac{x}{\ln 2}\right\}}$ according to (29).

The values of the function e^{u} can be calculated in two ways.

First method.

As the initial series for calculation of the function e^{u} we take its expansion into a Maclaurin series:

$$e^{u} = 1 + \frac{u}{1!} + \frac{u^2}{2!} + \frac{u^3}{3!} + \frac{u^4}{4!} + \frac{u^5}{5!} + \ldots \qquad (31)$$

We shall divide the interval $\left[0;\ \frac{\ln 2}{2}\right]$ of variation of the modulus of the independent variable u by the three points $\frac{31\ln 2}{4096}$, $\frac{391\ln 2}{1024}$ and $\frac{51\ln 2}{24}$ into four nested intervals with the common point 0, and we shall construct for each of these intervals the best polynomial which approximates the function e^{u}.

Using series (31) and Chebyshev polynomials, for the interval $\left[0;\ \frac{31\ln 2}{4096}\right]$ we obtain the following best polynomial:

$$e^{u} = A_1 + B_1 u + C_1 u^2\,, \qquad (32)$$

where the coefficients are defined as follows:

$$A_1 = \left[1 + \frac{1}{2^5}\lambda_1^4 S_1 - \frac{1}{2^3}\lambda_1^2 N_1\right],$$

$$B_1 = \left[1 - \frac{1}{24}\lambda_1^2 R_1 + \frac{3}{2^3} N_1\right],$$

$$C_1 = [\frac{1}{2!} - \frac{9}{2^4}\lambda_1^2 S_1 + H_1] ,$$

$$H_1 = (\frac{1}{3!} + \frac{1}{2^3} R_1)\lambda_1^2 ,$$

$$H_1 = (\frac{1}{4!} + \frac{3}{2} S_1)\lambda_1^2 ,$$

$$R_1 = \frac{1}{4!}\lambda_1^2 ,$$

$$S_1 = \frac{1}{6!}\lambda_1^2 ,$$

$$\lambda_1 = \frac{3\ln 2}{4096} .$$

The best polynomial (32) is used for calculation of the values of the function e^u for all values of u in the interval $[0; \frac{3\ln 2}{4096}]$.

The following best polynomial is constructed for approximation of the function e^u on the interval $[0; \frac{39\ln 2}{1024}]$:

$$e^u = A_2 + B_2 u + C_2 u^2 + D_2 u^3 + E_2 u^4 , \qquad (33)$$

where the coefficients are defined as follows:

$$A_2 = [1 - \frac{1}{2^7}\lambda_2^6 S_2 + \frac{1}{2^5}\lambda_2^4 H_2] ,$$

$$B_2 = [1 + \frac{1}{2^6}\lambda_2^4 R_2 - \frac{5}{2^4}\lambda_2^2 H_2] ,$$

$$C_2 = [\frac{1}{2!} + \frac{1}{2^3}\lambda_2^4 S_2 - \frac{9}{2^4}\lambda_2^2 H_2] ,$$

$$D_2 = [\frac{1}{3!} - \frac{1}{2^3}\lambda_2^2 R_2 + \frac{5}{2^3} H_2] ,$$

$$E_2 = [\frac{1}{4!} - \frac{5}{2^3}\lambda_2^2 S_2 + \frac{3}{2^3} H_2] ,$$

$$H_2 = \left(\frac{1}{5!} + \frac{1}{2^2}R_2\right)\lambda_2^2 ,$$

$$N_2 = \left(\frac{1}{6!} + 2S_2\right)\lambda_2^2 ,$$

$$R_2 = \frac{1}{6!}\lambda_2^2 ,$$

$$S_2 = \frac{1}{8!}\lambda_2^2 ,$$

$$\lambda_2 = \frac{39\ln 2}{1024} .$$

The polynomial (33) is used for calculation of the values of e^u only for those values of the argument which lie in the interval

$$\frac{3\ln 2}{4096} \leqslant u \leqslant \frac{39\ln 2}{1024} .$$

For the interval $\left[0; \frac{5\ln 2}{24}\right]$ we obtain the following best polynomial to approximate the function e^u:

$$e^u = A_3 + B_3 u + C_3 u^2 + D_3 u^3 + E_3 u^4 + F_3 u^5 + G_3 u^6 ; \quad (34)$$

where the coefficients are found from the relations:

$$A_3 = \left[1 + \frac{1}{2^9}\lambda_3^8 S_3 - \frac{1}{2^7}\lambda_3^6 N_3\right],$$

$$B_3 = \left[1 - \frac{1}{2^8}\lambda_3^6 R_3 + \frac{7}{2^6}\lambda_3^4 N_3\right],$$

$$C_3 = \left[\frac{1}{2!} - \frac{25}{2^8}\lambda_3^6 S_3 + \frac{1}{2^3}\lambda_3^4 N_3\right] ,$$

$$D_3 = \left[\frac{1}{3!} + \frac{5}{2^5}\cdot\frac{1}{3}\lambda_3^4 R_3 - \frac{7}{2^3}\lambda_3^2 N_3\right] ,$$

$$E_3 = \left[\frac{1}{4!} + \frac{25}{2^5}\lambda_3^4 S_3 - \frac{5}{2^3}\lambda_3^2 N_3\right],$$

$$F_3 = \left[\frac{1}{5!} - \frac{3}{2^4}\lambda_3^2 R_3 + \frac{7}{2^3} M_3\right],$$

$$G_3 = \left[\frac{1}{6!} - \frac{35}{2^4}\lambda_3^2 S_3 + 2N_3\right],$$

$$K_3 = \left(\frac{1}{7!} + \frac{1}{2^3} R_3\right)\lambda_3^2,$$

$$M_3 = \left(\frac{1}{8!} + \frac{5}{2} S_3\right)\lambda_3^2,$$

$$R_3 = \frac{1}{8!}\lambda_3^2,$$

$$S_3 = \frac{1}{10!}\lambda_3^2,$$

$$\lambda_3 = \frac{51n2}{24}.$$

The polynomial (34) is used for calculation of the values of e^u for values of the argument contained in the interval $\left[\frac{39\ln 2}{1024}\right] \leqslant u \leqslant \left[\frac{5\ln 2}{24}\right]$.

For the interval $\left[0; \frac{\ln 2}{2}\right]$ we obtain the following best polynomial to give the values of the function e^u:

$$e^u = A_4 + B_4 u + C_4 u^2 + D_4 u^3 + E_4 u^4 + F_4 u^5 + G_4 u^6 + H_4 u^7 + J_4 u^8, \qquad (35)$$

where the coefficients of the polynomial are defined by:

$$A_4 = [1 - \frac{1}{2^{11}}\lambda_4^{10}S_4 + \frac{1}{2^9}\lambda_4^8 H_4] ,$$

$$B_4 = [1 + \frac{1}{2^{10}}\frac{1}{11}\lambda_4^8 R_4 - \frac{9}{2^8}\lambda_4^6 H_4] ,$$

$$C_4 = [\frac{1}{2!} + \frac{9}{2^8}\lambda_4^8 S_4 - \frac{25}{2^8}\lambda_4^6 H_4] ,$$

$$D_4 = [\frac{1}{3!} - \frac{5}{2^8}\lambda_4^6 R_4 + \frac{15}{2^5}\lambda_4^4 H_4] ,$$

$$E_4 = [\frac{1}{4!} - \frac{105}{28}\lambda_4^6 S_4 + \frac{25}{25}\lambda_4^4 H_4],$$

$$F_4 = [\frac{1}{5!} + \frac{7}{2^6}\lambda_4^4 S_4 - \frac{27}{2^4}\lambda_4^2 H_4],$$

$$G_4 = [\frac{1}{6!} + \frac{7}{2^3}\lambda_4^4 S_4 - \frac{35}{2^4}\lambda_4^2 H_4] ,$$

$$H_4 = [\frac{1}{7!} - \frac{1}{2^2}\lambda_4^2 R_4 + \frac{9}{2^2} H_4] ,$$

$$I_4 = [\frac{1}{8!} - \frac{27}{2^3}\lambda_4^2 S_4 + \frac{5}{2} H_4],$$

$$M_4 = (\frac{1}{9!} + \frac{1}{2^2}R_4)\lambda_4^2 ,$$

$$N_4 = (\frac{1}{10!} + 3S_4)\lambda_4^2 ,$$

$$\bar{R}_4 = \frac{1}{10!}\lambda_4^2 ,$$

$$S_4 = \frac{1}{12!}\lambda_4^2 ,$$

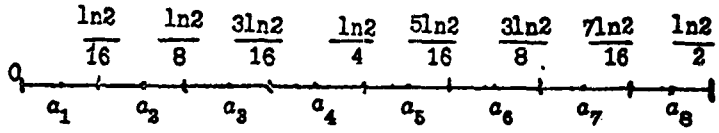
$$\lambda_4 = \frac{\ln 2}{2} .$$

Polynomial (35) is used for calculation of the values of e^u only for those values of the argument lying in the interval

$$\frac{5\ln 2}{24} \leqslant u \leqslant \frac{\ln 2}{2}.$$

Second method.

We shall divide the positive part of the interval of variation of the independent variable u into 8 equal parts and determine the midpoint of each of these segments.



Having established that

$$\left.\begin{aligned} a_1 &= \frac{\ln 2}{32}, & a_5 &= \frac{9\ln 2}{32}, \\ a_2 &= \frac{3\ln 2}{32}, & a_6 &= \frac{11\ln 2}{32}, \\ a_3 &= \frac{5\ln 2}{32}, & a_7 &= \frac{13\ln 2}{32}, \\ a_4 &= \frac{7\ln 2}{32}; & a_8 &= \frac{15\ln 2}{32}. \end{aligned}\right\} \qquad (36)$$

and that $\lambda = \frac{\ln 2}{32}$ identically for all of the segments, we shall find the values of the function e^u and its successive derivatives at the points a_1 (36).

Through estimations we find that, with an accuracy not less than $0.3 \cdot 10^{-11}$, a polynomial of the fourth degree approximates the function e^u on each of the segments.

Thus to find the values of the function e^u on each of the eight segments we can use a best polynomial of the form (12), find-

ing the coefficients of these polynomials according to the same formulas as for (12). Here we need to find only the values of the function and its derivatives to the eighth order, inclusively, at each of the points a_i ($i = 1, 2, \ldots, 8$).

With this method of calculating e^u 5 additions and 4 multiplications must be expended in the derivation of the values of the function. The necessary interval, depending on the value of u, is found by means of 3 comparisons.

The foregoing discussion relative to $u > 0$ relates equally to $u < 0$, which also require approximating polynomials of the 4th degree. Thus one more conditional step is added for recognition of the sign of u.

Thus 96 constants of 16 approximating polynomials must be stored in the computer's memory unit to determine the values of e^u for any u.

After the quantity e^u has been determined, the value of the function e^x is found by means of relation (26).

4. Calculation of the square root $\sqrt{x}$

In the calculation of the square root it is assumed that the argument is always given in normalized form, i.e. that all values of x_1 lie in the interval $\frac{1}{2} \leqslant x_1 < 1$. Then calculation of the square root of the number x reduces to the extraction of the square root of the two factors:

$$x = \sqrt{2^p} \cdot \sqrt{x_1} .$$

Extraction of the square root of the first factor poses no particular problems since

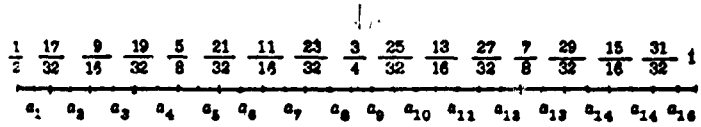
$$\sqrt{2^p} = \begin{cases} 2^\kappa, & \text{if } p = 2\kappa \\ 2^\kappa\sqrt{2}, & \text{if } p = 2\kappa+1. \end{cases}$$

To extract the square root of the second factor we divide the region of definition of the argument into 16 equal segments, and construct for each of them the best polynomial to approximate the function $f(x) = \sqrt{x}$ in that interval.

We shall denote by a_i the midpoints of the segments. Then, with the above division of the total interval, we shall have the following values for the points a_i:

$$\left.\begin{array}{llll} a_1 = \frac{33}{64}, & a_5 = \frac{41}{64}, & a_9 = \frac{49}{64}, & a_{13} = \frac{57}{64} \\ a_2 = \frac{35}{64}, & a_6 = \frac{43}{64}, & a_{10} = \frac{51}{64}, & a_{14} = \frac{59}{64} \\ a_3 = \frac{37}{64}, & a_7 = \frac{45}{64}, & a_{11} = \frac{53}{64}, & a_{15} = \frac{61}{64} \\ a_4 = \frac{39}{64}, & a_8 = \frac{47}{64}, & a_{12} = \frac{55}{64}, & a_{16} = \frac{63}{64} \end{array}\right\} \quad (37)$$

The distribution of the intervals in which polynomials to approximate the function will be constructed can be depicted graphically as follows:



In accomplishing this division we at the same time have determined the quantity $\lambda = \frac{x_{i+1} - x_i}{2}$. In the present case λ is the same for all segments and equals $\frac{1}{26}$.

Calculating the values of the derivatives for the points a_i ($i = 1, 2, \ldots, 16$) from (37) according to the formulas:

$$
\left.
\begin{aligned}
&f(a_i) = \sqrt{a_i}\,, && f^{III}(a_i) = -\frac{3!!}{2^3}\frac{1}{(\sqrt{a_i})^5}\,,\\
&f^{I}(a_i) = \frac{1}{2}\frac{1}{\sqrt{a_i}}\,, && f^{IV}(a_i) = -\frac{5!!}{2^4}\frac{1}{(\sqrt{a_i})^7}\,,\\
&f^{II}(a_i) = -\frac{1}{2^2}\frac{1}{(\sqrt{a_i})^3}, && f^{V}(a_i) = \frac{7!!}{2^5}\frac{1}{(\sqrt{a_i})^9}\,,\\
&f^{VI}(a_i) = -\frac{9!!}{2^6}\frac{1}{(\sqrt{a_i})^{11}}\,; && f^{VII}(a_i) = -\frac{11!!}{2^7}\frac{1}{(\sqrt{a_i})^{13}}\,;\\
&f^{VIII}(a_i) = -\frac{13!!}{2^8}\frac{1}{(\sqrt{a_i})^{15}}\,; && f^{IX}(a_i) = -\frac{15!!}{2^9}\frac{1}{(\sqrt{a_i})^{17}}\,.
\end{aligned}
\right\} \quad (36)
$$

and estimating the size of the remainder terms, we find that, with an accuracy not less than $0.35 \cdot 10^{-12}$, the function $\sqrt{x}$ is approximated in the first five segments from the left with best polynomials of the fifth degree of the form (13). The coefficients are determined in the same way as for the polynomial (13). Here the values of the function and its derivatives are determined according to formula (38) for each point a_i ($i = 1,2,3,4,5$) from (37).

With an accuracy not less than $0.8 \cdot 10^{-13}$, the function $f(x) = \sqrt{x}$ is approximated on all of the remaining segments by 4-th degree polynomials of the form (12), where the coefficients are determined in the same way as for polynomial (12). The values of the derivatives are found according to formula (38) for each point a_i ($i = 6, 7, \ldots, 16$) from (37).

Thus, in dividing the interval $[\frac{1}{2}, 1]$ into 16 partial segments, we are able to calculate the root of x_1 with one of 16 polynomials, requiring in the worst case 6 additions and 5 multiplications, and in the best case 5 additions and 4 multiplications. Finding the necessary interval requires 4 comparisons.

In the computer's memory unit 101 coefficients must be stored.

It is possible that the following method of calculating the values of the square root with an accuracy to ten decimal places would be more economical.

As before, we divide the interval of definition of the independent variable into 16 segments.

We shall use second-degree polynomials of the form (10). Estimates indicate that the accuracy of the square root calculation with these polynomials will not be lower than $0.3 \cdot 10^{-6}$ on any of the 16 segments. Adding one iteration, which doubles the accuracy, we obtain the value of the square root in 4 additions, 3 multiplications and 1 division.

If calculation of the value of the square root is performed with second-degree polynomials with one iteration for the 8 left segments, and with fourth-degree polynomials for the 8 right segments, the number of coefficients to be stored reduces to 60.

If the value of the square root in all 16 intervals is calculated with second-degree polynomials with one subsequent iteration, the number of coefficients to be stored will be 64.

5. Calculation of the function arctgx

The function arctgx can be calculated for all numbers x which can be represented in the given computer.

However, the formulas with which arctgx is calculated hold only for $|x| \leq 1$. Therefor, if $|x| > 1$, calculation of the function arctgx is accomplished with the formula

$$\operatorname{arctg} x = \frac{\pi}{2} - \operatorname{arctg}\frac{1}{x}, \qquad (39)$$

i.e. it is reduced to the calculation of the arctangent of an argument less than unity.

The point λ isolates from the interval $-1 \leq x \leq 1$ the interval $|x| \leq \lambda$ such that the polynomial obtained by expanding the function arctgx into a Maclaurin series and rotated by means of Chebyshev polynomials gives the values of the arctangent on the interval $|x| \leq \lambda$ with the required accuracy when the polynomial of a given degree is used. Thus we are able to calculate the values of the arctangent on the interval $|x| \leq \lambda$. To calculate arctgx on the interval $[\lambda; 1]$, we use the familiar formula for the sum of two arctangents:

$$\operatorname{arctg} u + \operatorname{arctg} v_t = \operatorname{arctg}\frac{u+v_t}{1-uv_t}. \qquad (40)$$

If we use the substitution

$$u = \frac{|x| - v_t}{1+|x|v_t}, \qquad (41)$$

from relation (40) we easily obtain:

$$\operatorname{arctg} u + \operatorname{arctg} v_t = \operatorname{arctg} x. \qquad (42)$$

Thus, to calculate the values of arctgx at any point of the interval $\lambda \leq |x| \leq 1$, for a given x in this interval we must calculate, according to formula (41), the u which corresponds to it, taking the value of v_1 necessary for this, and substitute this value of u in place of x in the derived best polynomial. Then we add the value of arctgu thus obtained with the value of arctgv_1 calculated earlier, which agrees with relation (42).

In order to be able to use the best polynomial which holds on the interval $|x| \leq \lambda$ to calculate the values of the function on the interval $\lambda \leq |x| \leq 1$, we must insure that the u which corresponds to a given x is not greater than λ. Hence the interval $[\lambda; 1]$ is divided into the k intervals $[x_1; x_2]$, $[x_2, x_3]$, ..., $[x_k; 1]$, in each of which $|u| \leq \lambda$. The dividing points are determined from the formula

$$x_{i+1} = \frac{x_i + v_1}{1 - v_1 x_i}, \quad (i = 1, 2, \ldots, k). \qquad (43)$$

where $x_1 = \lambda$, $x_{k+1} \leq 1$. In each of the intervals $[x_i; x_{i+1}]$ the constants v_i are found from the formula

$$v_i = \frac{x_1 + x_i}{1 - x_i x_1}. \qquad (44)$$

The values of arctgv_i are calculated in advance.

Now we shall consider the construction of approximating polynomials on the interval $|x| \leq \lambda$.

If we set $\lambda_1 = \frac{\sqrt{3}-1}{3+1}$, the best polynomial which provides the indicated accuracy on the interval $|x| \leq \lambda_1$ will be

$$\text{arctgx} = A_1 x - B_1 x^3 + C_1 x^5 - D_1 x^7 + E_1 x^9 - F_1 x^{11}, \quad (45)$$

where

$$A_1 = [1 - \frac{1}{2^{14}}\lambda_1^{14} + \frac{13}{2^{12}}\lambda_1^{12} N_1],$$

$$B_1 = [\frac{1}{3} - \frac{1}{2^{10}}\frac{7}{3}\lambda_1^{12} + \frac{91}{2^{10}}\lambda_1^{10} N_1],$$

$$C_1 = [\frac{1}{5} - \frac{1}{2^9}\frac{63}{5}\lambda_1^{10} - \frac{91}{2^7}\lambda_1^8 N_1],$$

$$D_1 = [\frac{1}{7} - \frac{15}{2^7}\lambda_1^8 + \frac{39}{2^4}\lambda_1^6 N_1],$$

$$E_1 = [\frac{1}{9} - \frac{1}{2^6}\frac{55}{3}\lambda_1^6 - \frac{65}{2^4}\lambda_1^4 N_1],$$

$$F_1 = [\frac{1}{11} - \frac{3}{2^3}\lambda_1^4 + \frac{13}{2^3}\lambda_1^2 N_1],$$

$$N_1 = \frac{1}{13} - \frac{1}{2^3}\lambda_1^2 .$$

The interval $\lambda \leq |x| \leq 1$ is not divided into smaller intervals. For it $v_1 = \frac{\sqrt{3}}{3}$ and $\operatorname{arctg} v_1 = \frac{\pi}{6}$.

The degree of the polynomial which approximates the function arctgx on the interval $|x| \leq \lambda$, can be made lower than the degree of polynomial (45) if we set $\lambda_2 = \frac{17}{128}$. Then the best polynomial which approximates arctgx on the interval $|x| \leq \lambda_2$ will be

$$\operatorname{arctg} x = A_2 x - B_2 x^3 + C_2 x^5 - D_2 x^7 , \qquad (46)$$

where

$$A_2 = [1 - \frac{1}{2^{10}}\lambda_2^{10} - \frac{9}{2^8}\lambda_2^8 N_2],$$

$$B_2 = [\frac{1}{3} - \frac{5}{2^8}\lambda_2^8 - \frac{15}{2^5}\lambda_2^6 N_2],$$

$$C_2 = [\frac{1}{5} - \frac{7}{2^5}\lambda_2^6 - \frac{27}{2^4}\lambda_2^4 N_2],$$

$$D_2 = [\frac{1}{7} - \frac{1}{2^2}\lambda_2^4 - \frac{9}{2^2}\lambda_2^2 N_2],$$

$$N_2 = \frac{1}{9} - \frac{1}{2^2}\lambda_2^2.$$

For calculation of arctgx on the remaining interval, the interval $[\lambda_2;1]$ is divided into the three intervals $[\lambda_2;x_2]$, $[x_2;x_3]$, $[x_3;1]$. The values of v_1, v_2 and v_3 are found from formula (44). Then the values of $\text{arctg}v_1$, $\text{arctg}v_2$ and $\text{arctg}v_3$ are calculated. The values of arctgx on the interval $[\lambda_2;1]$ are calculated with formulas (46), (40) and (41) using one of the three pairs of final constants.

Without sacrificing accuracy, we may lower the degree of the approximating polynomial still further if we take $\lambda_3 = \frac{1}{16}$. In this case the best polynomial which approximates the function arctgx on the interval $|x| \leq \lambda_3$ with the same accuracy will have the form

$$\text{arctg}x = A_3 x - B_3 x^3 + C_3 x^5, \qquad (47)$$

where

$$A_3 = [1 - \frac{1}{2^8}\lambda_3^8 - \frac{7}{2^6}\lambda_3^6 N_3],$$

$$B_3 = [\frac{1}{3} - \frac{1}{2^5}\frac{5}{3}\lambda_3^6 - \frac{7}{2^3}\lambda_3^4 N_3],$$

$$C_3 = \left[\frac{1}{5} - \frac{3}{2^4}\lambda_3^4 - \frac{7}{2^3}\lambda_3^2 N_3\right].$$

$$N_3 = \frac{1}{7} - \frac{1}{2^3}\lambda_3^2 .$$

To calculate the function on the remaining interval, we divide the segment $[\lambda_3;1]$ into 6 segments: $[\lambda_3;x_2]$, $[x_2;x_3]$, ..., $[x_6;1]$, for which the values v_1, v_2, ..., v_6 and $\operatorname{arctg} v_1$, $\operatorname{arctg} v_2$, ..., $\operatorname{arctg} v_6$ are found. When these values are used, the value of u does not exceed λ_3 on each of the segments, and formula (47), together with formula (40), will give the values of arctgx on the entire interval $[0;1]$.

A polynomial of lower degree will approximate the function arctgx on the interval $|x| \leq \lambda$ with the same accuracy if we set $\lambda_4 = \frac{7}{512}$. In this case the best polynomial which gives the values of the function arctgx on the interval $|x| \leq \lambda_4$ will be:

$$\operatorname{arctg} x = A_4 x - B_4 x^3 , \qquad (48)$$

where

$$A_4 = \left[1 - \frac{1}{2^6}\lambda_4^6 - \frac{5}{2^4}\lambda_4^4 N_4\right],$$

$$B_4 = \left[\frac{1}{3} - \frac{1}{2^3}\lambda_4^4 - \frac{5}{2^3}\lambda_4^2 N_4\right],$$

$$N_4 = \frac{1}{5} - \frac{1}{2^3}\lambda_4^2 .$$

The interval $[\lambda_4;1]$ is divided by the points x_1, determined by formula (43), into the 29 segments $[\lambda_4;x_2]$, $[x_2;x_3]$, ..., $[x_{29};1]$, for which the values of v_1, v_2, ..., v_{29} and $\operatorname{arctg} v_1$,

$\operatorname{arctg} v_2$, ..., $\operatorname{arctg} v_{29}$ are calculated. With these constants we may make the absolute value of u less than or equal to λ_4 on any of the segments, and compute the values of arctgx on the entire interval [0;1] using formulas (49) and (40).

Finally, the degree of the approximating polynomial can be reduced to unity if we reduce further the interval on which this polynomial is defined.

Setting $\lambda_5 = \frac{1}{2048}$, for any x on the interval $|x| \leq \lambda_5$ we obtain the values of the function arctgx, accurate to ten decimal places, using the following best polynomial of the first degree:

$$\operatorname{arctg} x \approx A_5 x \,, \qquad (49)$$

where

$$A_5 = [(1 - \frac{1}{2^6}\lambda_5^6 - \frac{5}{2^4}\lambda_5^2 N_5) + \frac{3}{2^2}\lambda_5^2(\frac{1}{3} - \frac{1}{2^3}\lambda_5^4 - \frac{5}{2^2} N_5)] \,,$$

$$N_5 = (\frac{1}{5} - \frac{1}{2^2}\lambda_5^2)\,\lambda_5^2 \,.$$

Clearly, it is not possible to obtain a more simple polynomial than polynomial (49) for calculation of arctgx at any point of the interval $|x| \leq \lambda$.

The interval $[\lambda_5; 1]$ is divided by the points x_i into more than 100 segments.

If for these segments we determine the constants v_i and $\operatorname{arctg} v_i$ from formula (44), then we may use formulas (49) and (44) to calculate the values of arctgx on the entire interval [0;1].

6. Calculation of the function arcsinx

During calculation of the function arcsinx, it is assumed that the argument is contained in the interval $-1 \leq x \leq 1$, i.e. that only the principal values of the function are calculated.

As the initial series we take the series

$$\arcsin x = x + \sum_{n=1}^{\infty} \frac{(2n-1)!!}{(2n)!!} \frac{x^{2n+1}}{2n+1} , \qquad (50)$$

defined on the interval $|x| \leq 1$.

We shall divide the interval of variation of the independent variable into four partial sub-intervals by the points 1/10, 2/5, and 3/4, and construct in each of these sub-intervals $[0;1/10]$, $[0;2/5]$, $[0;3/4]$ and $[0;1]$ the best polynomial which approximates the function arcsinx in it.

Using series (50) and Chebyshev polynomials of appropriate degree, for the interval $[0;1/10]$ we obtain the following best polynomial:

$$\arcsin x = A_1 x + B_1 x^3 + C_1 x^5 + D_1 x^7 , \qquad (51)$$

where the coefficients are defined as follows:

$$A_1 = [1 + \frac{1}{2^{10}} \lambda_1^6 M_1 - \frac{9}{2^8} \lambda_1^6 M_1] ,$$

$$B_1 = [\frac{1}{6} - \frac{5}{2^8} \lambda_1^4 M_1 + \frac{15}{2^5} \lambda_1^4 M_1] ,$$

$$C_1 = [\frac{3!!}{4!!} \frac{1}{5} + \frac{7}{2^6} \lambda_1^2 M_1 - \frac{27}{2^4} \lambda_1^2 M_1] ,$$

$$D_1 = [\frac{5!!}{6!!} \frac{1}{7} - \frac{1}{2^3} M_1 + \frac{9}{2^3} M_1] ,$$

$$N_1 = \left(\frac{7!!}{8!!}\frac{1}{9} + \frac{1}{2^3}\frac{9!!}{10!!}\lambda_1^2\right)\lambda_1^2 .$$

$$N_1 = \frac{9!!}{10!!}\lambda_1^4 .$$

$$\lambda_1 = \frac{1}{10} .$$

The best polynomial (51) is used to calculate the values of the function arcsinx on the interval [0;1/10]. For the interval [0;2/5] , a best polynomial of higher degree is constructed:

$$\arcsin x = A_2x + B_2x^3 + C_2x^5 + D_2x^7 + E_2x^9 + F_2x^{11} + G_2x^{13}, \qquad (52)$$

where the coefficients are defined as follows:

$$A_2 = \left[1 - \frac{1}{2^{16}}\lambda_2^{12}N_2 + \frac{15}{2^{14}}\lambda_2^{12}N_2\right] ,$$

$$B_2 = \left[\frac{1}{6} + \frac{3}{2^{12}}\lambda_2^{10}N_2 - \frac{35}{2^{10}}\lambda_2^{10}N_2\right] ,$$

$$C_2 = \left[\frac{3!!}{4!!}\frac{1}{5} - \frac{21}{2^{11}}\lambda_2^8N_2 + \frac{189}{2^9}\lambda_2^8N_2\right] ,$$

$$D_2 = \left[\frac{5!!}{6!!}\frac{1}{7} + \frac{33}{2^9}\lambda_2^6N_2 - \frac{225}{2^7}\lambda_2^6N_2\right] ,$$

$$E_2 = \left[\frac{7!!}{8!!}\frac{1}{9} - \frac{55}{2^8}\lambda_2^4N_2 + \frac{275}{2^6}\lambda_2^4N_2\right] ,$$

$$F_2 = \left[\frac{9!!}{10!!}\frac{1}{11} + \frac{13}{2^5}\lambda_2^2N_2 - \frac{45}{2^3}\lambda_2^2N_2\right] ,$$

$$G_2 = \left[\frac{11!!}{12!!}\frac{1}{13} - \frac{7}{2^4}N_2 + \frac{15}{2^3}N_2\right] ,$$

$$N_2 = \left(\frac{13!!}{14!!}\,\frac{1}{15} + \frac{1}{2^3}\,\frac{15!!}{16!!}\,\lambda_2^2\right)\lambda_2^2 ,$$

$$N_3 = \frac{15!!}{16!!}\,\lambda_2^4 ,$$

$$\lambda_2 = \frac{2}{5} .$$

Polynomial (52) is used to calculate values of the function arcsinx not on the entire interval of definition, but only for those values of the argument x which lie in the interval $1/10 \leq |x| \leq 2/5$. For the values $|x| < 1/10$, calculation of the function arcsinx with the same accuracy is much more easily accomplished with polynomial (51).

A best polynomial of 21st degree is constructed to approximate the function arcsinx on the interval $[0;3/4]$:

$$\left.\begin{aligned} \arcsin x = {} & K_1 x + K_2 x^3 + K_3 x^5 + K_4 x^7 + \\ & + K_5 x^9 + K_6 x^{11} + K_7 x^{13} + K_8 x^{15} + K_9 x^{17} + \\ & + K_{10} x^{19} + K_{11} x^{21} \end{aligned}\right\}, \quad (53)$$

where the coefficients are defined by the expressions:

$$K_1 = \left[1 - \frac{1}{2^{24}}\,\lambda_3^{20} N_3 + \frac{23}{2^{22}}\,\lambda_3^{20} N_3\right] ,$$

$$K_2 = \left[\frac{1}{6} + \frac{13}{2^{21}}\,\lambda_3^{18} N_3 - \frac{253}{2^{19}}\,\lambda_3^{18} N_3\right] .$$

$$K_3 = \left[\frac{3!!}{4!!}\,\frac{1}{5} - \frac{1001}{2^{20}}\,\frac{1}{5}\,\lambda_3^{16} N_3 + \frac{3289}{2^{18}}\,\lambda_3^{16} N_3\right] ,$$

$$K_4 = \left[\frac{5!!}{6!!}\,\frac{1}{7} + \frac{715}{2^{18}}\,\lambda_3^{14} N_3 + \frac{9867}{2^{16}}\,\lambda_3^{14} N_3\right] .$$

$$K_5 = \left[\frac{7!!}{8!!}\,\frac{1}{9} - \frac{715}{2^{15}}\,\lambda_3^{12} N_3 + \frac{16445}{2^{14}}\,\lambda_3^{12} N_3\right] ,$$

$$K_6 = \left[\frac{9!!}{10!!}\frac{1}{11} + \frac{221}{2^{11}}\lambda_3^{10}H_3 - \frac{2093}{2^9}\lambda_3^{10}H_3\right],$$

$$K_7 = \left[\frac{11!!}{12!!}\frac{1}{13} - \frac{357}{2^{10}}\lambda_3^8 H_3 + \frac{2737}{2^8}\lambda_3^8 H_3\right],$$

$$K_8 = \left[\frac{13!!}{14!!}\frac{1}{15} + \frac{969}{2^8}\frac{1}{5}\lambda_3^6 H_3 - \frac{1173}{2^6}\lambda_3^6 H_3\right],$$

$$K_9 = \left[\frac{15!!}{16!!}\frac{1}{17} - \frac{285}{2^8}\lambda_3^4 H_3 + \frac{1311}{2^6}\lambda_3^4 H_3\right],$$

$$K_{10} = \left[\frac{17!!}{18!!}\frac{1}{19} + \frac{35}{2^5}\lambda_3^2 H_3 - \frac{115}{2^3}\lambda_3^2 H_3\right],$$

$$K_{11} = \left[\frac{19!!}{20!!}\frac{1}{21} - \frac{11}{2^4}H_3 + \frac{23}{2^2}H_3\right],$$

$$H_3 = \left(\frac{21!!}{22!!} + \frac{1}{2^2}\frac{23!!}{24!!}\lambda_3^2\right)\lambda_3^3,$$

$$H_3 = \frac{23!!}{24!!}\lambda_3^4,$$

$$\lambda_3 = \frac{3}{4}.$$

The best polynomial (53) is used to calculate values of the function arcsinx for values of x lying in the interval $2/5 \leq |x| \leq 3/4$.

For $|x| \leq 2/5$, calculations are accomplished with formulas (51) and (52). To calculate the values of the function arcsinx with the required accuracy for x on the interval

$\frac{3}{4} \leq |x| < 1$, we could construct the best polynomial for the interval [0;1]. However, this polynomial will contain terms of the 29th degree and thus 14 operations of addition and 16 operations of multiplication will be required for the calculations.

From the point of view of the rate of calculation of this function for $\frac{3}{4} \leq |x| < 1$, it is more advantageous to use the following formula:

$$\arcsin x = \operatorname{arctg} \frac{x}{\sqrt{1 - x^2}}, \qquad (54)$$

from which the calculation of the values of arcsinx requires the performance of 8 addition, 7 multiplication, 2 division and 3 comparison operations.

Although formula (54) holds for all $|x|<1$, for $|x| < \frac{3}{4}$ it is less useful than formulas (51), (52) and (53).

7. Calculation of the function $f(x) = \frac{1}{x}$

Some computers do not have the division operation; hence an iteration process is used to obtain inverse quantities.

We shall examine the possibility of obtaining inverse quantities using best polynomials constructed for different segments into which the entire interval of definition of the independent variable is divided.

We shall assume that the inverse quantity is calculated for a normalized argument, i.e. that $\frac{1}{2} \leq x_1 < 1$.

We shall divide the entire interval of definition of the independent variable into 32 partial segments, and at the

midpoint of each of these segments we shall calculate the values of the function and its first 6 derivatives.

Through estimations we establish that function $f(x)=\frac{1}{x}$ can be approximated accurately to 10 decimal places by fifth-degree polynomials of the form (13) on the 16 left segments, and by fourth-degree polynomials of the form (12) on the 16 right intervals. The coefficients of the polynomials are calculated just as they were calculated for polynomials (12) and (13). Calculation of $f(x) = \frac{1}{x}$ with these polynomials will require 6 addition and 5 multiplication, or 5 addition and 4 multiplication operations. Additionally, 5 conditional branches will be necessary to search the required interval. In the computer's memory unit 208 constants must be stored.

In conclusion we shall determine the values A_1 and A_2 for the methods examined above of calculating the elementary functions. This will permit an estimation of the increse in the rate of calculation of these functions. In cases where the degree of the approximating polynomial does not increase from one sub-interval to the next (in calculating e^x by the second method, $\sqrt{x}$, arctgx), the value of A_1 will be determined relative to the old formulas for calculating elementary functions.

In finding A_2 it should be noted that τ_0 (the time for searching in the table) usually does not exceed two τ of addition, since in the majority of cases we have a division into four sub-intervals, and thus two comparisons must be performed to find the necessary interval. In the case where the number of

sub-intervals is greater than four, the number of comparisons will be $\ln_2 l$, where l is the number of sub-intervals [7]. Typically, in all cases the boundaries of the sub-intervals can be defined by two or three decimal places. This will permit the insertion into the computer code of an instruction for shortened comparison which can effect a comparison over half of a word in the computer. Additionally, if automatic search of the table is employed, as in the IBM-650 computer [7], or the arithmetic unit is specially adapted for such problems, the search time can be cut at least in half. Hence we shall estimate A_2 from two sides. The right estimate will be suitable for computers not specially adapted for the problem of searching in the table, while the left estimate will be suitable for computers which are thus adapted. Here all estimations will be based on the assumption that the ratio of the time required to perform a multiplication to the time required to perform an addition operation will equal two, and the ratio of the time required to perform a division operation to the time required to perform an addition operation will equal four. These ratios hold in the best present-day computers. If these ratios are larger, the values of A_1 and A_2 will diminish significantly.

The application of subroutines which use the methods discussed above to calculate elementary functions requires an additional volume of memory units. This increase in volume is caused by:

1) the additional storage of constants for polynomials which approximate the value of a function in various sub-inter-

No.	Calculated function	Ratio of calculating times of new and old formulas	Ratio of table search time to calculation time with old formula	Increase in rate of calculation of formulas according to new method compared to old method
		A_1	A_2	A
1.	sinx, cosx, tgx	0,58	$0,04 \leq A_2 \leq 0,07$	$35 \leq A \leq 38$
2.	lnx	0,42	$0,06 \leq A_2 \leq 0,12$	$46 \leq A \leq 52$
3.	e^x 1st method	0,88	$0,04 \leq A_2 \leq 0,08$	$4 \leq A \leq 8$
	e^x 2nd method	0,54	$0,08 \leq A_2 \leq 0,17$	$29 \leq A \leq 38$
4.	$\sqrt{x}$ 1st method	0,85	$0,12 \leq A_2 \leq 0,38$	$-23 \leq A \leq 3$
	$\sqrt{x}$ 2nd method	0,71	$0,06 \leq A_2 \leq 0,18$	$11 \leq A \leq 23$
	$\sqrt{x}$ 3rd method	0,73	$0,06 \leq A_2 \leq 0,18$	$9 \leq A \leq 21$
5.	arctgx	0,51	$0,06 \leq A_2 \leq 0,14$	$10 \leq A \leq 49$
6.	arcsinx	0,57	$0,02 \leq A_2 \leq 0,04$	$39 \leq A \leq 41$

vals, and of constants which define the boundaries of these sub-intervals;

2) instructions necessary for the calculation of the polynomials in these sub-intervals;

3) instructions which effect a branch to a sub-routine for calculating the polynomial in the appropriate sub-interval (in the simplest case these will be instructions for comparing values of the argument with the table of sub-interval boundaries).

An analysis of the sub-routines in the BESM-1 code indicates that the sub-routines are increased as follows:

No.	Calculated function	BSEM-1 sub-routines			New sub-routine		
		No. of constants	No. of inst.	Cells occupied	No. of constants	No. of inst.	Cells occupied
1	$\sin x, \cos x, \operatorname{tg} x$	12	38	50	33	40	73
2	$\ln x$	8	19	27	28	22	50
3	e^x 1st method	14	25	39	33	28	61
	e^x 2nd method	-	-	-	96	21	117
	$\sqrt{x}$ 1st method	10	27	37	102	28	130
4	$\sqrt{x}$ 2nd method	-	-	-	64	24	88
	$\sqrt{x}$ 3rd method	-	-	-	80	25	105
5	$\operatorname{arctg} x$	13	23	41	18	25	43
6	$\arcsin x$	24	57	81	27	67	94
7	$10 \rightarrow 2$	12	25	37	12	25	37
8	$2 \rightarrow 10$	13	26	39	13	26	39
	Total		≈	351		≈	540

CONCLUSIONS

We have described a method which, at the expense of a certain additional loading of the computer's memory unit, permits an increase in the rate of calculation of elementary functions.

The formulas suggested here are not the best for every electronic computer. For any specific computer, appropriate formulas should be selected which allow for the properties of the computer relative to the time required for the performance of arithmetic operations, which insure calculation with the maximum speed, and whose programming does not severely load the computer's memory unit.

The formulas presented here can be quite suitable for specialized computers, where the occurrence of values of the function's argument is not uniform in the reduced interval. In this case, instead of using all of the divisions, we may use only certain ones and thus increase the rate of operation of the computer with only slight additional loading of the memory unit.

If the methods discussed in 3 and 6 are used, the number of multiplications in the formulas presented here can be reduced still further because of the increase in the number addition operations.

All formulas derived here are designed for calculation of functions accurate to 10 decimal places.

If the values of functions are to be calculated with fewer digits, the formulas may be simplified considerably.

One of the indicated variants for calculating elementary functions will require the use of approximately 200 memory units to store necessary constants. The maximum number of constants required for the accomplishment of all of the calculation sub-routines for the elementary functions examined here is around 300. However, around 200 of these are used to formulate the sub-routines of the function $\sqrt{x}$ with 4th- and 5th-degree polynomials, and the function e^x using 4th-degree polynomials.

The use of the suggested formulas for the function arctgx significantly increases the rate of calculation of the arctangent.

The methods described here apparently can serve as one way of achieving a significant reduction of the operating time of the computer during the solution of various problems.

These methods can be transferred in their entirety to the case of rational approximation of elementary functions. In this case their application permits a significant economy of computer operating time.

FOOTNOTES

1. Throughout the article we use the phrase "best polynomials to denote polynomials obtained from the Taylor expansion of a function by multiple rotation with Chebyshev polynomials. These polynomials are close to polynomials which are best in the generally accepted sense. The error in approximating polynomials used in the article consists of the remainder term of the Taylor series and the errors which arise with each rotation.

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