63-3-5

MEMORANDUM
RM-3596
MAY 1963

PROPAGATION OF ACOUSTIC-GRAVITY WAVES FROM A SMALL SOURCE ABOVE THE GROUND IN AN ISOTHERMAL ATMOSPHERE

Allan D. Pierce

The RAND Corporation
SANTA MONICA • CALIFORNIA

MEMORANDUM

RM-3596

MAY 1963

PROPAGATION OF ACOUSTIC-GRAVITY WAVES FROM A SMALL SOURCE ABOVE THE GROUND IN AN ISOTHERMAL ATMOSPHERE

Allan D. Pierce

The RAND Corporation

1700 MAIN ST • SANTA MONICA • CALIFORNIA

PREFACE

This report discusses the effects of gravity and of the atmosphere's nonuniform density on sound propagation. It is part of RAND's continuing study of the atmospheric waves generated by nuclear explosions.

This report should be of interest to agencies and contractors concerned with the detection of nuclear explosions. It should also be of interest to other research workers who are studying low frequency sound propagation in the atmosphere.

SUMMARY

A theoretical discussion is presented of the influence of gravity on sound propagation from a small source in an isothermal atmosphere where ambient pressure and density decrease exponentially with height. A solution for the free-space case is derived which indicates that waves with angular frequency ω between $(\gamma-1)^{1/2}g/c$ and $(\gamma/2)g/c$ will not be propagated, while those with ω between 0 and $(\gamma-1)^{1/2}(g/c)\cos\theta$ will not be propagated in a direction making an angle of θ with the vertical axis. A formal solution incorporating appropriate boundary conditions at the ground is derived and discussed. The field along the vertical line passing through the source is found explicitly. A consideration of the energy intensity shows that no energy is propagated within a cone above and below the source if $\omega < (\gamma-1)^{1/2}g/c$. A calculation of the intensity for the case when $(\gamma-1)^{1/2}g/c < \omega < (\gamma/2)g/c$ indicates that the energy flowing from the source tends to concentrate in the lowest layers of the atmosphere. The field for large horizontal distances appears as a sum of a direct wave, a reflected wave, and a surface wave. Reflection coefficients are derived and the criteria for the surface wave to be dominant are discussed.

CONTENTS

I. INTRODUCTION

The effects of the earth's gravitational force on the propagation of infrasonic waves to great distances has been extensively studied by a number of workers.[1-10] Interest in this subject dates back to 1883, when the eruption of the Krakatoa volcano generated a pressure pulse which was detected in widely spaced regions of the world. The detection of waves from the great Siberian meteorite in 1908 and, in recent years, from the detonation of hydrogen bombs created additional interest.

Early theoretical studies[1-5] showed that one of the principal effects of the earth's gravitational attraction was that it permitted the existence of horizontally propagating modes trapped in the lower atmosphere. These modes were found to exist when the assumed temperature-height profile of the atmosphere had no temperature minimum, and when the conventional acoustic theories (i.e., those neglecting the influence of gravity) precluded the existence of such modes.

In this paper we shall study the manner in which these trapped modes (or _gravity_ modes) evolve from a small source located above the ground. In particular, we shall derive expressions for the acoustic field in regions relatively close to the source and shall study the effects of the ground on this field.

The model we shall study is that of an isothermal atmosphere bounded by a flat earth. Although this model is an oversimplification of the real atmosphere and, as is well known, it does not lead to a correct prediction of the form of the pressure pulse which would be observed at large distances, we feel that a study of this model may lead to a better understanding of the origin of the phenomena predicted at great distances by more sophisticated models. In particular, the theory developed here should complement the recent asymptotic theories of Hunt, Palmer, and Penney[7] and of Weston.[8]

The subject of wave propagation in an isothermal atmosphere has been considered previously by Lamb,[1] by Pekeris,[4] and by Sretenskii.[11] Lamb discussed the problem of the vertical and horizontal propagation of plane waves and Pekeris extended the discussion of horizontal propagation to illustrate his theory of the excitation of gravity modes by a small source on the ground in an atmosphere having a constant lapse rate in the troposphere. Sretenskii considered the propagation of waves from a point source in an isothermal atmosphere with the influence of the earth's surface neglected (i.e., the source was sufficiently far above the earth that reflections could be disregarded). It appears, however, that Sretenskii's theory is in error. His initial equation,

$$\frac{\partial^2 \Phi}{\partial t^2} = c^2 \nabla^2 \Phi - g \frac{\partial \Phi}{\partial z} , \qquad (1.1)$$

for the velocity potential is inconsistent with the equations of hydrodynamics for air. Lamb[1] has shown that a velocity potential does not exist for the isothermal atmosphere unless $\gamma = 1$. If Sretenskii's theory were to have any application to the real atmosphere, he would have to take $\gamma = 1.4$. Furthermore, Sretenskii imposed the requirement that

$$\frac{\partial \Phi}{\partial z} = 0 \qquad (1.2)$$

everywhere in the horizontal plane of the source except at the source. The only justification given by Sretenskii for this requirement is that it is applicable when gravity is neglected. The early portion (Sec. III) of this paper will be concerned with the presentation of the correct solution of this problem.

II. MATHEMATICAL FORMULATION OF THE PROBLEM

For an isothermal atmosphere, the characteristic length is the scale height,

$$H = c^2/(\gamma g) \tag{2.1}$$

where c is the (constant) speed of sound, g is the acceleration of gravity, and γ is the specific heat ratio, which for air may be taken as 1.40. The characteristic time is H/c. We shall accordingly develop our theory in a system of units in which distance is in units of H and time is in units of H/c. (Typical values of H and H/c for the real atmosphere are 8 km and 24 sec.)

Let the pressure and density at a point $\mathbf{r}$ at time t be represented by $p_o + p$ and $\rho_o + \rho$, where p_o and ρ_o are their ambient values. For an isothermal atmosphere, p_o and ρ_o are given by the expressions

$$p_o = c^2\rho_o/\gamma = p_{oo}e^{-z} , \tag{2.2}$$

where p_{oo} is the pressure at the ground and z denotes the height above the ground (in units of H). The quantities p, ρ, and the particle velocity $\mathbf{v}$ are presumed to obey the linearized equations of hydrodynamics. For an isothermal atmosphere, with distances in units of H and times in units of H/c, these have the form:

$$c\partial(\rho_o\mathbf{v})/\partial t + \nabla p + \mathbf{e}_z c^2\rho/\gamma = -\{\epsilon c\rho_o\mathbf{v}\} \tag{2.3a}$$

$$c\partial\rho/\partial t + \nabla\cdot(\rho_o\mathbf{v}) = 0 \tag{2.3b}$$

$$\partial p/\partial t - c^2\partial\rho/\partial t + c\gamma A^2\mathbf{e}_z\cdot(\rho_o\mathbf{v}) = 0 \tag{2.3c}$$

where $\underset{\sim}{e}_z$ denotes the unit vector in the z-direction and

$$A^2 = (\gamma - 1)/\gamma^2 \approx .204 . \qquad (2.4)$$

These equations may be derived in the manner outlined by Pridmore-Brown.[12] The term in braces in (2.3a) is included in order that we may qualitatively take frictional forces into account. The parameter ϵ is a positive constant which we shall assume to be very small. The inclusion of this term is a well known device[13] which is generally attributed to Rayleigh.

The Fourier transforms $\hat{p}$, $\hat{\rho}$, and $\hat{\underset{\sim}{v}}$ of p, ρ, and $\underset{\sim}{v}$, where

$$\hat{p}(\omega) = (2\pi)^{-1} \int_0^\infty p(t)\, e^{i\omega t}\, dt , \qquad (2.5)$$

etc., will satisfy three simultaneous differential equations which may be formally obtained by replacing $\partial/\partial t$ by $-i\omega$ in equations (2.3). (Here ω represents the angular frequency in units of c/H.) From these equations one may derive a single differential equation for the Fourier transform of any one of a number of quantities associated with the acoustic field. (Lamb,[1] for example, derived a differential equation for the vorticity.) We choose to concentrate on the pressure. If we define

$$P = \hat{p}e^{z/2} , \qquad (2.6)$$

then this quantity P will satisfy the following differential equation

$$\underset{\sim}{\nabla}\cdot(\hat{\underset{\sim}{\nabla}}P) + (\beta^2/\mu^2)P = 0 \qquad (2.7)$$

where

$$\beta^2 = \Omega^2 - \tfrac{1}{4} , \qquad (2.8)$$

$$\mu^2 = \Omega^2 - A^2, \qquad (2.9)$$

and

$$\Omega^2 = \omega(\omega + i\epsilon) . \tag{2.10}$$

The operator $\hat{\nabla}$ has components $(1/\Omega^2)\partial/\partial x$, $(1/\Omega^2)\partial/\partial y$, $(1/\mu^2)\partial/\partial z$ in Cartesian coordinates. Thus

$$\nabla\cdot\hat{\nabla} = \frac{1}{\Omega^2}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + \frac{1}{\mu^2}\frac{\partial^2}{\partial z^2} . \tag{2.11}$$

The quantity P plays the role of a potential. Once it is known, $\hat{\rho}$ and $\hat{v}$ may be obtained by use of the following formulas:

$$c^2\hat{\rho} = e^{-z/2}\left\{\Omega^2 - (\gamma/2)A^2 + \gamma A^2(\partial/\partial z)\right\} P/\mu^2 , \tag{2.12}$$

$$i(\omega + i\epsilon)c\rho_o\hat{v} = e^{-z/2}\left\{\Omega^2\hat{\nabla}P + (\Omega^2/\mu^2)e_z B^2 P\right\} , \tag{2.13}$$

where

$$B^2 = (1/\gamma) - (1/2) . \tag{2.14}$$

Since the vertical component of the particle velocity must vanish at the ground, P must satisfy the boundary condition

$$\partial P/\partial z + B^2 P = 0 \tag{2.15}$$

at $z=0$. This follows directly from (2.13).

In addition, P must conform to causality. In conventional acoustic theories, this would be interpreted as requiring that P behave as an outgoing wave in regions remote from the source (radiation condition). A general formulation of this requirement is that, for all r such that $z > 0$, the quantity $Pe^{-i\omega t_o}$, where t_o corresponds to any time before the source is initially excited, should be analytic everywhere in the upper half of the complex ω-plane and should vanish as $\mathrm{Im}(\omega)$ tends to infinity. This implies, in particular, that P should have no poles or branch lines in the region for which $\mathrm{Im}(\omega) > 0$. That this requirement guarantees causality is clear, since the inverse transform

$$P(\underset{\sim}{r}, t) = e^{-z/2} \int_{-\infty}^{\infty} P(\underset{\sim}{r}, \omega) e^{-i\omega t} d\omega \tag{2.16}$$

will then vanish for all $t < t_o$.

III. THE FREE-SPACE SOLUTION FOR A SMALL SOURCE

To examine the nature of the field close to the source, we shall solve the free-space problem. The presence of the ground is neglected and the boundary condition (2.15) is discarded. The causality condition is required for all $\underset{\sim}{r}$ rather than for just those for which $z > 0$.

A Green's function $G(\underset{\sim}{r}-\underset{\sim}{r}_o,\omega)$ may readily be found which is finite everywhere except at a point $\underset{\sim}{r}_o$ (assuming $\epsilon > 0$), and which conforms to causality in the sense described above. We may choose the normalization of the Green's function such that it satisfies the equation.

$$\underset{\sim}{\nabla}\cdot\hat{\underset{\sim}{\nabla}}G + (\beta^2/\mu^2)G = -4\pi\delta(\underset{\sim}{r}-\underset{\sim}{r}_o)/(\Omega^2\mu). \tag{3.1}$$

It is clear that this equation is formally equivalent to that for the Green's function e^{ikR}/R for the scalar Helmhltz equation. One need only replace Ωx, Ωy, and μz by x', y', and z' to cast it in a form identical to that given by Morse and Feshbach.[14] Thus we may write

$$G(\underset{\sim}{r}-\underset{\sim}{r}_o,\omega) = R'^{-1}\, e^{i(\beta/\mu)R'} \tag{3.2}$$

where

$$R' = \left[\Omega^2(x-x_o)^2 + \Omega^2(y-y_o)^2 + \mu^2(z-z_o)^2\right]^{\frac{1}{2}}. \tag{3.3}$$

The requirement of causality may be satisfied by a suitable definition of the phases of β, μ, and R' in the complex ω-plane. The branch lines for these quantities are taken as extending vertically downwards from their respective branch points--all of which lie slightly below the real axis. One requires that the phases of β, μ, and R' be continuous everywhere except at their branch lines and that, for $Im(\omega) > 0$, their phases approach the phase of ω as $|\omega|$ approaches infinity. Thus, in the limit of $\epsilon = 0$, the phase of β, μ, or R' is 0 for ω lying on the real axis to the right of the branch point lying on the positive real axis. Between the two branch points the phase is $\pi/2$ and, to the left of the branch point on the negative real axis, the phase is π. Thus the phase of the expression $\beta R'/\mu$ which appears in the exponent in (3.2) has the following behavior on the real axis:

$$\begin{aligned}
\mathrm{ph}(\beta R'/\mu) &= 0 && \omega > \tfrac{1}{2} && (3.4a)\\
&= \pi/2 && A < \omega < \tfrac{1}{2} && (3.4b)\\
&= 0 && A|\cos\theta| < \omega < A && (3.4c)\\
&= \pi/2 && |\omega| < A\,|\cos\theta| && (3.4d)\\
&= \pi && -A < \omega < -A|\cos\theta| && (3.4e)\\
&= \pi/2 && -\tfrac{1}{2} < \omega < -A && (3.4f)\\
&= \pi && \omega < -\tfrac{1}{2} && (3.4g)
\end{aligned}$$

where

$$\cos\theta = (z-z_o)/|r-r_o| \,. \qquad (3.5)$$

That the Green's function $G(r-r_o,\omega)$ may be considered as describing the spatial dependence of P for a small source located at r_o may be readily seen if one makes use of Green's theorem. One may show in general that

$$P(\underset{\sim}{r},\omega) = \frac{\mu\Omega^2}{4\pi} \iint [P(\underset{\sim}{r}_o)\, \hat{\underset{\sim}{\nabla}}_o\, G(\underset{\sim}{r}-\underset{\sim}{r}_o)$$

$$-G(\underset{\sim}{r}-\underset{\sim}{r}_o)\, \hat{\underset{\sim}{\nabla}}_o\, P(\underset{\sim}{r}_o)]\cdot \underset{\sim}{n}_o\, dS_o \,, \qquad (3.6)$$

where the integration is carried over the surface of a small sphere enclosing the source ($\underset{\sim}{r}_o$ denoting the position vector of points on the surface and $\underset{\sim}{n}_o$ denoting the outward-pointing normal) and $\underset{\sim}{r}$ is outside the surface. Then one may show, in a manner similar to that outlined by Weston,[15] that if the radius of the source is sufficiently small, the expression (3.6) becomes approximately

$$P(\underset{\sim}{r},\omega) = -\frac{\mu\Omega^2}{4\pi} G(\underset{\sim}{r}-\underset{\sim}{r}_o) \iint \hat{\underset{\sim}{\nabla}}_o\, P(\underset{\sim}{r}_o,\omega)\cdot \underset{\sim}{n}_o\, dS_o \,. \qquad (3.7)$$

The approximation (3.7) is invalid if either R' or μ is zero. In the limit of $\epsilon = 0$ this will be the case if $\omega = A$ or $\omega = A \cos \theta$. However, we expect that the deviation of the actual pressure field $p(\underset{\sim}{r},t)$ from the approximate field which would be calculated by taking the inverse transform of (3.7) should be small if the source is small and if its spectrum is not sharply peaked at these frequencies. With these reservations, we shall accept the product $G(\underset{\sim}{r}-\underset{\sim}{r}_o,\omega)e^{-z/2}$ as describing the spatial dependence of the Fourier transform of the pressure field.

The acceptance of (3.7) makes it possible to describe the field if the gradient of the pressure is known over the surface of a small sphere enclosing the source. If one were instead to consider his input data as consisting of the recording $p_1(t)$ of the pressure vs. time at some point $\underset{\sim}{r}_1$ close to the source, the quantity $P(\underset{\sim}{r},\omega)$ would be given by

$$P(r,\omega) = \frac{G(\underset{\sim}{r}-\underset{\sim}{r}_o,\omega)}{G(\underset{\sim}{r}_1-\underset{\sim}{r}_o,\omega)} \hat{p}_1(\omega) \quad e^{z_1/2} \tag{3.8}$$

where $\hat{p}_1(\omega)$ is the transform of $p_1(t)$.

The Green's function $G(\underset{\sim}{r}-\underset{\sim}{r}_o,\omega)$ has some interesting features which should be noted. As indicated in equations (3.4), it corresponds to a propagating wave only if $|\omega| > \frac{1}{2}$ or $A|\cos \theta|<|\omega|< A$. Thus there are two distinct pass bands, the width of the lower band depending on the angle θ. If one is directly above or below the source, θ will be zero or π and the lower band will have zero width--in accordance with the results derived by Lamb[1] for the vertical propagation of plane waves in the atmosphere.

One may derive a phase velocity and a group velocity for frequencies lying in the two pass bands. These velocities (in units of c and letting $\epsilon = 0$) are given by

$$v_p = \omega\mu/(\beta\zeta) \tag{3.9}$$

and

$$v_g = \left[\frac{d(\beta\zeta/\mu)}{d\omega}\right]^{-1} = \frac{\beta\zeta\mu^3/\omega}{[\mu^4+A^2(\frac{1}{4}-A^2)\sin^2\theta]} , \qquad (3.10)$$

where

$$\zeta = R'/|\underset{\sim}{r}-\underset{\sim}{r}_o| = [\Omega^2 - A^2\cos^2\theta]^{\frac{1}{2}} . \qquad (3.11)$$

Thus the phase and group velocities will depend on the angle θ in addition to the frequency ω. The phase velocity represents the apparent direction with which wave crests move in the radial direction from the source, while the group velocity corresponds to the average velocity with which a signal of frequency ω will have apparently traveled from the source to an observer along a radial line from the source. One may readily show from (3.9) that the phase velocity decreases monotonically from ∞ to 0 as ω increases from $A|\cos\theta|$ to A and that it decreases monotonically from ∞ to 1 as ω increases from $\frac{1}{2}$ to ∞. Similarly, one may show that the group velocity is zero at $\omega = A|\cos\theta|$ and at $\omega = A$ and that it is positive and less than 1 for intermediate frequencies. The group velocity increases monotonically from 0 to 1 as the frequency increases from $\frac{1}{2}$ to ∞. In Fig. 1 we plot v_p and v_g vs. ω for $\theta = 60^o$.

In the frequency bands for which $0 < |\omega| < A|\cos\theta|$ or $A < |\omega| < \frac{1}{2}$, there is no propagation and the Green's function decreases exponentially with increasing radial distance. The coefficient of attenuation may be taken as

$$\alpha = -i\beta\zeta/\mu . \qquad (3.12)$$

The parameter α is $(\frac{1}{2})|\cos\theta|$ at $\omega = 0$ and decreases monotonically to zero as ω increases from 0 to $A|\cos\theta|$. As ω increases from A to $\frac{1}{2}$, α decreases monotonically from ∞ to 0. In Fig. 2 we plot α vs. ω for $\theta = 60^o$.

The factor of $1/R'$ in the Green's function (3.2) will cause the Green's function to be singular in the limit of $\epsilon = 0$ if $|\omega| < A$ and $|\cos\theta| = |\omega|/A$. It is at such angles, however, that the approximation (3.7) ceases to be valid. Nevertheless, we may expect that the Fourier transform of the pressure field computed via (3.6) for a small source will be large at such frequencies, if not singular.

IV. FORMAL SOLUTION WHEN SOURCE IS ABOVE A FLAT GROUND

Let us assume that the source is located at a distance h (in units of H) above the ground. The coordinate system is chosen such that the origin lies on the ground directly below the source.

To find the spatial dependence of the field corresponding to a given frequency, one looks for a Green's function $G_h(\underset{\sim}{r},\omega)$ which satisfies (3.1) with $\underset{\sim}{r}_o = \underset{\sim}{e}_z h$, which satisfies the boundary condition (2.15) at $z = 0$, and which satisfies the causality requirement for all $\underset{\sim}{r}$ for which $z > 0$.

The method of finding such a Green's function is well-known. One expresses G_h as a Fourier-Bessel transform,

$$G_h(\underset{\sim}{r},\omega) = \int_{-\infty}^{\infty} H_o(kr)\, Z(k,\omega,z)\, kdk\ , \qquad (4.1)$$

where $H_o(kr)$ is the Hankel function of the first kind (defined such that its branch line extends vertically downwards in the complex k-plane), and r represents the radial distance in cylindrical coordinates. A differential equation for the quantity $Z(k,\omega,z)$ together with sufficient boundary conditions may be obtained from the required properties of G_h. In this manner, one may obtain the following expression for $Z(k,\omega,z)$:

$$Z(k,\omega,z) = \frac{i\mu}{2\Omega^2}\left[e^{i\phi|z-h|} + e^{i\phi|z+h|} - \frac{2B^2}{B^2+i\phi}\, e^{i\phi|z+h|}\right], \qquad (4.2)$$

where

$$\phi = \left[\beta^2 - \mu^2 k^2/\Omega^2\right]^{\frac{1}{2}}. \qquad (4.3)$$

The phase of ψ is chosen such that, if k is real, ψ has no branch line in the upper half of the complex ω-plane and such that the phase of ψ approaches the phase of ω as $|\omega|$ approaches infinity if $\mathrm{Im}(\omega) > 0$. (This will insure that G_h conforms to causality in the sense previously described.)

The phase requirements on ψ in the ω-plane enable one to derive the phase of ψ for all real k for any given real ω. In general, for ϵ not identically zero, one may specify the phase of ψ uniquely for all real k and real ω by requiring the phase of ψ to be between 0 and π. The specification may be extended to include complex values of k by placing branch lines in the k-plane and requiring that the phase of ψ be continuous everywhere except at these branch lines. For $\epsilon > 0$, the placing of these branch lines is restricted by the fact that no branch line may cross the real axis, since, in this case, the phase must be continuous along the real axis. The simplest placing is to take the branch lines as extending vertically upwards from branch points lying above the real axis and vertically downwards from those lying below the real axis. One may readily show that the branch points of ψ in the k-plane lie in the first and third quadrants if $\omega > 0$ and in the second and fourth quadrants if $\omega < 0$. Thus, in the limit of $\epsilon = 0$, branch lines extend upwards from branch points lying on the positive real axis and downwards from branch points lying on the negative real axis if $\omega > 0$, while the converse is true if $\omega < 0$.

The specification of the phase of ψ as outlined above reduces to the following for $\epsilon = 0$ and real k:

1. For $|\omega| > \frac{1}{2}$,

$$\mathrm{Ph}(\psi) = \mathrm{Ph}(\omega) \qquad |k| < |\beta\omega/\mu| \tag{4.4a}$$

$$= \pi/2 \qquad |k| > |\beta\omega/\mu| \tag{4.4b}$$

2. For $\frac{1}{2} > |\omega| > A$,

$$\mathrm{Ph}(\psi) = \pi/2 \qquad \text{all } k \tag{4.5}$$

3. For $|\omega| < A$,

$$\mathrm{Ph}(\psi) = \pi/2 \qquad |k| < |\beta\omega/\mu| \tag{4.6a}$$

$$= \pi + \mathrm{Ph}(\omega) \qquad |k| > |\beta\omega/\mu| \tag{4.6b}$$

The relation given by (4.6b) should particularly be noted. More naive considerations might lead one to believe that the phase of ψ is either equal to the phase of ω or to $\pi/2$ for real k.

For $\epsilon > 0$, the integration in (4.1) may be taken as proceeding along the real axis. The poles of $Z(k,\omega,z)$ will lie off the axis--specifically, at the points

$$k = \pm (\omega + i\epsilon/2) \tag{4.7}$$

for sufficiently small ϵ. If $\epsilon = 0$, the poles will lie on the real axis, and the contour must pass below the pole on the positive real axis and above that on the negative real axis if $\omega > 0$. If $\omega < 0$, the converse is true.

Let us now note that the free-space Green's function $G(\underset{\sim}{r}-\underset{\sim}{e}_z h,\omega)$ given by (3.2) may also be expanded in the form (4.1). The appropriate expression for $Z(k,\omega,z)$ may be readily shown to be

$$\frac{i\mu}{2\Omega^2 \psi} e^{i\psi|z-h|} \tag{4.8}$$

in a manner similar to that outlined above. Thus, with the decomposition of Z indicated by (4.2), our Green's function G_h appears as a sum of three terms:

$$G_h(\underset{\sim}{r},\omega) = G(\underset{\sim}{r}-\underset{\sim}{e}_z h,\omega) + G(\underset{\sim}{r}+\underset{\sim}{e}_z h,\omega) + I(r,z+h,\omega). \tag{4.9}$$

The first term represents the free-space Green's function for a source located at $\underset{\sim}{r}_o = \underset{\sim}{e}_z h$, while the second term represents the free-space Green's function for the image source. The third term is given by the integral

$$I(r,z+h,\omega) = \frac{-i\mu}{\Omega^2} \int_{-\infty}^{\infty} \frac{B^2}{\psi(B^2+i\psi)} e^{i\psi(z+h)} H_o(kr)\, kdk \,. \tag{4.10}$$

This term is present as a direct result of the fact that $\partial P/\partial z$ is not identically zero at the surface of the earth.

We may regard the product

$$e^{z/2}\, G_h(\underset{\sim}{r},\omega)$$

as describing the spatial dependence of the pressure field from a harmonic source or as describing the spatial dependence of the Fourier transform of the field. This follows from Eqn. (3.8).

V. THE FIELD BELOW THE SOURCE

The fictional parameter ϵ is of no further use to us and will henceforth be set to zero. Also, since

$$G_h(\underset{\sim}{r}, -\omega) = - G_h^*(\underset{\sim}{r}, \omega) \tag{5.1}$$

for real ω, we shall limit ourselves to positive ω.

Since ψ is even in k, the integral in (4.10) may also be taken in the form

$$I(r,z+h,\omega) = \frac{-i\mu}{\omega^2} \int_0^\infty \frac{2B^2}{\psi(B^2+i\psi)} e^{i\psi(z+h)} J_o(kr)\, kdk \tag{5.2}$$

with the contour passing below the pole at $k = \omega$. This form is most convenient for studying the field at small r. If we take $r = 0$, the Bessel function becomes 1 and the variable of integration may be changed to

$$s = -(B^2+i\psi)(z+h)$$

to give

$$I(0,z+h,\omega) = (2B^2/\mu)\, e^{-B^2(z+h)} \int_{-s_o}^\infty (e^{-s}/s)\, ds \,, \tag{5.3}$$

with

$$s_o = (B^2 + i\beta)(z+h) \,, \tag{5.4}$$

and the contour passing below the pole at $s = 0$. The integral in (5.3) is the exponential integral of complex argument and has been tabulated by the National Bureau of Standards.[16] If $|s_o|$ is very large, the integral is very nearly $-\exp(s_o)/s_o$ and we have

$$I(0,z+h,\omega) \approx \frac{-2B^2}{\mu(B^2+i\beta)} \frac{e^{i\beta(z+h)}}{(z+h)} . \tag{5.5}$$

If, on the other hand, $|s_o|$ is small,

$$I(0,z+h,\omega) \approx \frac{2B^2}{\mu} e^{-B^2(z+h)} \log_e \left\{-(B^2+i\beta)(z+h)e^{.5772}\right\} , \tag{5.6}$$

with the branch cut for the logarithm on the positive imaginary axis.

We note that when (5.4) is applicable the ratio of I to $G(\underset{\sim}{r}+\underset{\sim}{e}_z h,\omega)$ at $r = 0$ is

$$\frac{-2B^2}{B^2 + i\beta} \tag{5.7}$$

This ratio is vanishingly small for large frequencies, but may be very large if ω is close to A. If (5.6) is applicable, the ratio is

$$2B^2(z+h) \log_e \left\{-(B^2+i\beta)(z+h)e^{.5772}\right\} , \tag{5.8}$$

which is singular for $\omega = A$.

Let us now consider (5.2) when r is not identically zero. If $\omega < A$ and

$$r < (z+h)\tan\theta_c , \tag{5.9}$$

where

$$\tan\theta_c = \left[(A/\omega)^2 - 1\right]^{\frac{1}{2}}, \tag{5.10}$$

the contour in (5.2) may be deformed to the negative imaginary axis. With an appropriate change of variable, (5.2) then becomes

$$I(r,z+h,\omega) = \frac{-i|\mu|}{\omega^2} \int_0^\infty f(\xi)\, I_0(\xi r)\, e^{-|\psi|(z+h)}\, \xi d\xi \qquad (5.11)$$

where $I_0(\xi r)$ is the Bessel function of imaginary argument,

$$|\psi| = \left[|\beta|^2 + (|\mu|/\omega)^2\right]^{\frac{1}{2}} ,$$

and

$$f(\xi) = \frac{2B^2}{|\psi|(|\psi|-B^2)}$$

is a positive real function of ξ. It is immediately clear that the magnitude of (5.11) increases with increasing r and decreasing z+h. It is also clear that the phase of I will not depend on r or z.

Let us note that the latter property is shared by $G(\underset{\sim}{r}+\underset{\sim}{e}_z h)$ if $\omega < A$ and (5.9) is satisfied, and that it also holds for $G(\underset{\sim}{r}-\underset{\sim}{e}_z h)$ if $r < |z-h|\tan\theta_c$. We may accordingly conclude that

$$\underset{\sim}{S} = i(G_h \hat{\underset{\sim}{\nabla}} G_h^* - G_h^* \hat{\underset{\sim}{\nabla}} G_h) \qquad (5.12)$$

is zero everywhere above and below the source within a cone of apex angle θ_c with the apex at the source.

The quantity $\underset{\sim}{S}$ introduced above gives the spatial dependence of the acoustic intensity for a harmonic source. (This may be seen by taking the product of the pressure, as given by (2.16), and the complex conjugate of the particle velocity, whose transform is given by (2.13), and then integrating over time.) Thus we may conclude that, for $\omega < A$, there is no energy propagation in the cone of apex angle θ_c.

Let us now consider (5.2) in the case where ω is between A and $\frac{1}{2}$. In this case we may consider the integral as being a sum of its principal value plus an integral around a small semicircle passing below the pole. The latter may be evaluated by the method of residues. The contribution to I from the principal value of the integral will be a real number as ψ has a phase of $\pi/2$ along the real axis. Thus the imaginary part of I will be given by the contribution from the pole:

$$\mathrm{Im}(I) = \frac{2\pi B^2}{\mu} e^{-B^2(z+h)} J_0(\omega r). \qquad (5.13)$$

This also gives the imaginary part of $G_h(\underset{\sim}{r},\omega)$.

The quantity $I(r,z+h,\omega)$ is related to the Green's function $G(\underset{\sim}{r}+\underset{\sim}{e}_z h,\omega)$ by the equation

$$(\partial/\partial z + B^2)\, I = -\, 2B^2 G \qquad (5.14)$$

as may be seen from an examination of (4.10). This equation is useful if one seeks to derive an expression for the acoustic intensity $\underset{\sim}{S}$. The z-component of $\underset{\sim}{S}$ may be found directly from (5.12) if one uses the above relation and the fact that the two Green's functions are real. In this manner, one obtains

$$S_z = -(2/\mu^2)\, \mathrm{Im}(I) \left\{(\frac{\partial}{\partial z} - B^2)G(\underset{\sim}{r}+\underset{\sim}{e}_z h,\omega) + (\frac{\partial}{\partial z} + B^2)G(\underset{\sim}{r}+\underset{\sim}{e}_z h,\omega)\right\} . \qquad (5.15)$$

The r-component may be found if one uses the fact that the divergence of $\underset{\sim}{S}$ is zero. Thus

$$S_r = -(1/r) \int_0^r (\partial S_z/\partial z)\, r dr. \qquad (5.16)$$

After some algebraic manipulations and an integration by parts, this becomes

$$S_r = -(2/\omega^2)\left\{\mathrm{Im}(I)\frac{\partial}{\partial r}\left[G(\underset{\sim}{r}+\underset{\sim}{e}_z h) + G(\underset{\sim}{r}-\underset{\sim}{e}_z h\right]\right.$$

$$\left. - \left[G(\underset{\sim}{r}+\underset{\sim}{e}_z h) + G(\underset{\sim}{r}-\underset{\sim}{e}_z h)\right]\frac{\partial}{\partial r}\mathrm{Im}(I)\right\}$$

$$+(4B^2\mu^{-2}/r)\int_0^r \mathrm{Im}(I)\,(B^2 - \frac{\partial}{\partial z})G(\underset{\sim}{r}+\underset{\sim}{e}_z h)\,r dr\ . \qquad (5.17)$$

From (5.15) and (5.17) it is clear that the intensity near the source is of the form

$$\underset{\sim}{S} = \frac{4\pi B^2}{\mu}\ \frac{(\underset{\sim}{r}-\underset{\sim}{e}_z h)\ e^{-2B^2 h}}{\left[\omega^2 r^2 + \mu^2(z-h)^2\right]^{3/2}} \qquad (5.18)$$

which represents a flow of energy radially outward from the source. However, since $\mu^2 < \omega^2$, the flow is not spherically symmetric. More energy flows out vertically than horizontally. We also note that as much energy flows out in the upward direction as flows out in the downward direction.

One interesting feature of our expressions for $\underset{\sim}{S}$ is that S_z is proportional to the Bessel function $J_o(\omega r)$. Whenever r is such that ωr is a root of the Bessel function, S_z will be zero and the energy flow will be horizontal for all points on the cylinder of radius r.

In the limit of large r the intensity is entirely in the horizontal direction and is given by the last term of (5.17):

$$S_r = (8\pi B^4/\mu^3)\ e^{-B^2(z+h)} r^{-1}\int_0^\infty J_o(\omega r)(B^2 - \partial/\partial z)G(\underset{\sim}{r}+\underset{\sim}{e}_z h,\omega)\ r dr\ . \qquad (5.19)$$

This indicates that the intensity decreases inversely with r at large distances.

The nature of the transition from (5.18) to (5.19) is best illustrated by means of a numerical example. We choose ω = .48 and h = 5.0. (In conventional units h would be approximately 40 km.) The results of the calculation are given in Fig. 3. There we show the typical paths along which energy would flow away from the source. (These lines may be considered as being the sound rays.) The figure also shows the variation of the magnitude of the intensity with distance. The downward bending of the rays in the region above the source is reminiscent of a water fountain and reminds us that we are considering a problem in which the earth's gravity is important.

VI. APPLICATION OF THE METHOD OF STEEPEST DESCENTS

The case when ωr is large is best studied within the context of the method of steepest descents. If one follows the rationale of this method, the Hankel function in (4.10) is replaced by its asymptotic expression and the contour is deformed to one which passes through the saddle point of

$$e^{ikr+i\psi(z+h)} \tag{6.1}$$

and which goes along the path for which this quantity decreases most rapidly with increasing distance from the saddle point.

The saddle point is located at

$$k = k_s = \frac{\omega^2\beta}{\mu\zeta}\sin\theta \tag{6.2}$$

where

$$\tan\theta = r/(z+h)\ , \tag{6.3}$$

$$\zeta = (\omega^2 - A^2\cos^2\theta)^{\frac{1}{2}} \tag{6.4}$$

and θ is between 0 and $\pi/2$. The path of steepest descents is given by

$$k_I = (\omega/\mu)\ \frac{(k_R-k_E)(k_R-|k_s|)}{\left[(k_R-k_s)^2 + Q^2\right]^{\frac{1}{2}}}\tan\theta,\quad \omega > \tfrac{1}{2} \tag{6.5a}$$

$$= (\omega/\mu)(k_R^2 + Q^2)^{\frac{1}{2}},\qquad \tfrac{1}{2} > \omega > A \tag{6.5b}$$

$$= (\omega/|\mu|)\ \frac{(k_R-|k_s|)(k_R-k_E)}{(k_M-k_R)^{\frac{1}{2}}(k_R-k_N)^{\frac{1}{2}}}\tan\theta,\ A > \omega > A\cos\theta \tag{6.5c}$$

and

$$k_R = 0 , \qquad 0 < \omega < A \cos\theta , \tag{6.5d}$$

where

$$k_E = |\beta\zeta/(\mu \sin\theta)| , \tag{6.6a}$$

$$Q = |(\omega\beta \cos\theta)/\zeta| , \tag{6.6b}$$

$$k_N = |k_s| - Q , \tag{6.6c}$$

$$k_M = |k_s| + Q , \tag{6.6d}$$

and k_R and k_I denote the real and imaginary parts of k, and, in (6.5c), k_R is understood to be between k_N and k_M. If we let

$$k_B = |\omega\beta/\mu|$$

represent the distance of the branch points from the origin, then it is readily verified that $k_E > k_B > |k_s|$ for $\omega > \frac{1}{2}$, $|k_s| > k_B$ for $\frac{1}{2} > \omega > A$, and $k_M > |k_s| > k_B > k_E > k_N > 0$ for $A > \omega > A \cos\theta$. In Fig. 4 we show sketches of the path of steepest descents for the various ranges of ω.

The deformation of the contour in (4.10) to the path of steepest descents is valid for $\omega > \frac{1}{2}$ only if $k_E > \omega$, and is valid for $A \cos\theta < \omega < A$ only if $k_E < \omega$. If this is not true, one must add the residue from the pole at $k = \omega$ to the integral I_{SD} along the path of steepest descents to obtain (4.10). This residue is

$$I_{Res} = \frac{2\pi i}{\mu\beta^2} e^{-B^2(z+h)} H_o(\omega r) \qquad (6.7a)$$

$$\approx e^{i\pi/4}(2B^{-2}/\mu)[2\pi/(\omega r)]^{\frac{1}{2}} e^{-B^2(z+h)} e^{i\omega r} . \qquad (6.7b)$$

The condition $k_E > \omega$ for $\omega > \frac{1}{2}$ is equivalent to $\omega > \omega_1(\theta)$, while the condition $k_E < \omega$ for $A \cos\theta < \omega < A$ is equivalent to $\omega < \omega_1(\theta)$, where

$$(2\cos^2\theta)\omega_{1,2}^2(\theta) = \tfrac{1}{4} + A^2\cos^2\theta \pm \left[(\tfrac{1}{4} - A^2)^2 + A^2(\tfrac{1}{4} - A^2)\sin^2 2\theta\right]^{\frac{1}{2}} . \qquad (6.8)$$

(It is readily seen that ω_1 increases monotonically from $\frac{1}{2}$ to ∞ while ω_2 decreases monotonically from A to 0 as θ goes from 0 to $\pi/2$.) The integral (4.10) is thus

$$I(r, z+h, \omega) = I_{SD} + U(\omega,\theta)\, I_{Res} \qquad (6.9)$$

where $U(\omega,\theta)$ is 1 for $\omega_1(\theta) > \omega > \omega_2(\theta)$ and is otherwise zero.

The integral along the path of steepest descents may be evaluated approximately in the usual manner, giving

$$I_{SD} \approx \left[\tau(\omega,\theta) - 1\right] G(\underset{\sim}{r} + \underset{\sim}{e}_z h, \omega) , \qquad (6.10)$$

where

$$\tau(\omega,\theta) = \frac{i(\mu\beta/\zeta)\cos\theta - B^2}{i(\mu\beta/\zeta)\cos\theta + B^2} . \qquad (6.11)$$

It is clear that $|\tau|$ is 1 if $\omega > \frac{1}{2}$ or $A\cos\theta < \omega < A$ and that τ is real and less than -1 if $\frac{1}{2} > \omega > A$. Also, τ is real and greater than 1 if $\omega < A\cos\theta$. In Fig. 5, we plot the phase and magnitude of τ versus ω for several representative values of θ.

With the approximation (6.10), the field $G_h(r,z,\omega)$ becomes

$$G_h(r,z,\omega) = G(\underset{\sim}{r}-\underset{\sim}{e}_z h,\omega) + \tau G(\underset{\sim}{r}+\underset{\sim}{e}_z h,\omega) + U\, I_{Res} \,. \qquad (6.12)$$

The three terms in this equation may be conveniently labelled as a direct wave, a reflected wave, and a surface wave, respectively. The parameter τ thus corresponds to a reflection coefficient.

Let us now study the behavior of (6.12) for given ω and h vs. r for $z = 0$. In this event the two Green's functions are equal. Also, if $\omega > \frac{1}{2}$ or $\omega < A$, the presence of the factor U in (6.12) implies that the surface wave will not be present unless

$$r > r_o = |\beta\mu/(B^2\omega)|h \,. \qquad (6.13)$$

It need scarcely be pointed out that this parameter r_o has no physical significance. It is unlikely that the field should change abruptly at $r = r_o$. Furthermore, at this value of r, the surface wave may be of insignificant value compared to the sum of the direct and reflected waves if the conditions for the validity of the approximation (6.10) are satisfied. Nevertheless, the theory does give us a natural small distance cut-off for the surface wave. This indicates that Eqn. (6.12) will be qualitatively correct for all r for frequencies in these two bands if the height of the source is sufficiently large.

As r increases from r_o, the relative contribution of the surface wave to G_h will increase. This follows for two reasons. First, the surface wave falls off more slowly with r than the direct and reflected waves. Secondly, the direct and reflected waves will tend to cancel each other at large r.

A characteristic distance r_1 may be defined as that value of r for which

$$|I_{Res}| = |1 + \tau|\ |G(\underset{\sim}{r}+\underset{\sim}{e}_z h,\omega)| \,.$$

For values of r greater than r_1 the surface wave will be dominant, while the converse will be true if $r < r_1$. For $\omega > \frac{1}{2}$ or $\omega < A$ this parameter may be shown to be approximately

$$r_1 \approx \left[\left| \frac{\beta^2 \mu^4}{2\pi\omega^3} \right| h^2 \, e^{2B^2 h} \right]^{1/3} , \tag{6.14}$$

provided this expression is much greater than $|\mu|^2 h/(B^2\omega)$ or $|\mu| h/\omega$. (These criteria will be met if h is large.) In the high frequency limit, r_1 is proportional to ω, while, in the low frequency limit, it is proportional to ω^{-1}.

If $A < \omega < \frac{1}{2}$, Eqn. (6.12) gives no small distance cut-off to the surface wave. Although the expression is not valid for r identically zero, we expect it to be approximately valid for any r for which $\omega r > 1$, assuming h is large. This is confirmed by Fig. 4, which shows a surface wave present for almost all points on the ground except directly below the source. For r = 0, the field may be found by use of (5.3). In the limit of large h, (5.5) is applicable and we find

$$G(0,0,\omega) = \frac{-2|\beta|}{B^2 - |\beta|} \, \frac{e^{-|\beta| h}}{\mu h} . \tag{6.15}$$

It is interesting that this has sign opposite to that of the free space Green's function.

VII. DISCUSSION AND CONCLUSIONS

The explicit solution, Eqn. (3.2), for the case when the presence of the ground is ignored, represents the correct solution of the problem originally considered by Sretenskii.[11] The method of deriving this solution (i.e., of proceeding from the equations of hydrodynamics in a linearized form, then seeking a Green's function conforming to causality, and finally using Green's theorem to find the field in the limit of an infinitesimal source) appears to be the most consistent approach, and it is gratifying that our solution has such a relatively simple form. The prediction of the two non-propagating frequency bands is perhaps the most interesting consequence of our theory. It appears difficult to give a simple qualitative explanation for the presence of these bands. We may mention, however, that the frequency $\omega = A$ is Brunt's[17] resonant frequency for the isothermal atmosphere. The frequency $\omega = \frac{1}{2}$ is also a characteristic frequency for the isothermal atmosphere and appears in Lamb's[1] theory as the dividing point between propagation and attenuation of vertically propagating plane waves.

When the presence of the ground was incorporated into the theory, the solution to the problem became more complicated. Although it could be given in integral form, the term $I(r,z+h,\omega)$ could not be evaluated exactly. We were able, however, to derive an explicit expression for the case $r = 0$, enabling one to compute the field anywhere on the vertical line passing through the source. We were also able to give the solution in the limit when the method of steepest descents was applicable. This led to a representation of the solution as a sum of a direct wave, a reflected wave (with reflection coefficient (6.11)), and a surface wave. The surface wave is, of course, the counterpart of the gravity modes discovered in other models of the atmosphere.

For an understanding of the nature of the field when $\omega < \frac{1}{2}$, the study of the acoustic intensity $\underset{\sim}{S}$ was useful. We found that $\underset{\sim}{S}$ was zero if $\omega < A$ everywhere above and below the source within a cone of apex angle θ_c. This was just what might have been expected after considering the free-space problem.

The field for frequencies in the band $A < \omega < \frac{1}{2}$ has some interesting properties which were brought out by our theory. In the free-space problem, no propagation would take place and $\underset{\sim}{S}$ would be everywhere zero. However, the presence of the ground permitted the existence of a propagating surface wave. We were able to derive relatively simple expressions for the intensity above the ground. A numerical example (Fig. 3) showed that the energy flowing out of the source had an overwhelming tendency to concentrate in the lower layers of the atmosphere.

For arbitrary frequencies and at sufficiently large distances, the field near the ground would be entirely given by the surface wave. However, for high frequencies or very low frequencies, the surface wave would not predominate unless the distance were very large.

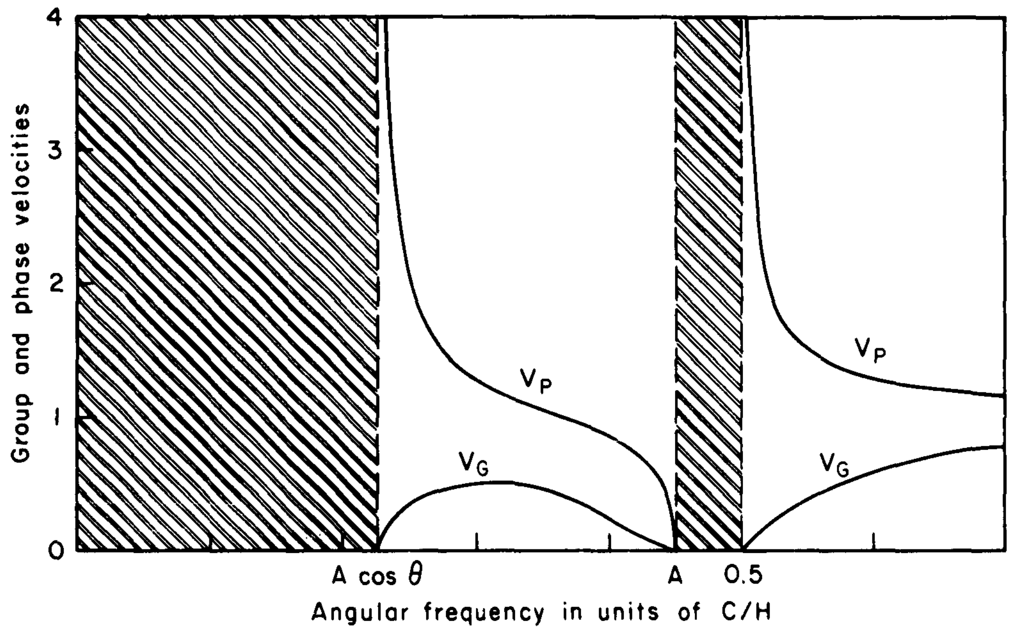

Fig. 1. -- Phase velocity V_P and group velocity V_G (units of C) versus ω (units of C/H)for free-space propagation from a point source along a line making an angle of 60° with the vertical.

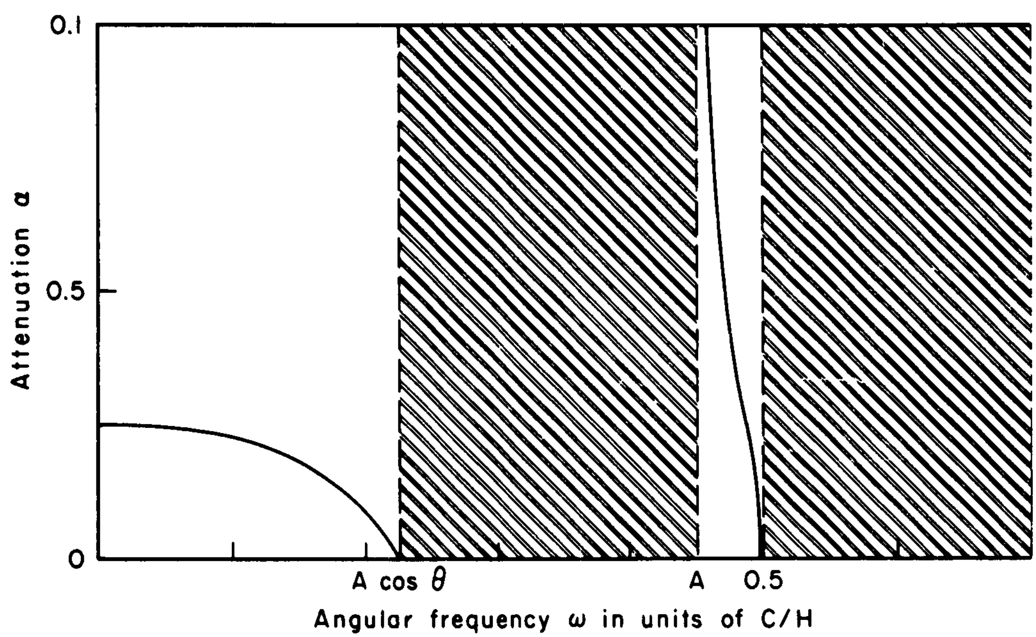

Fig. 2. -- Attenuation coefficient α (units of H^{-1}) versus ω (units of C/H) for free-space propagation from a point source along a line making an angle of 60° with the vertical.

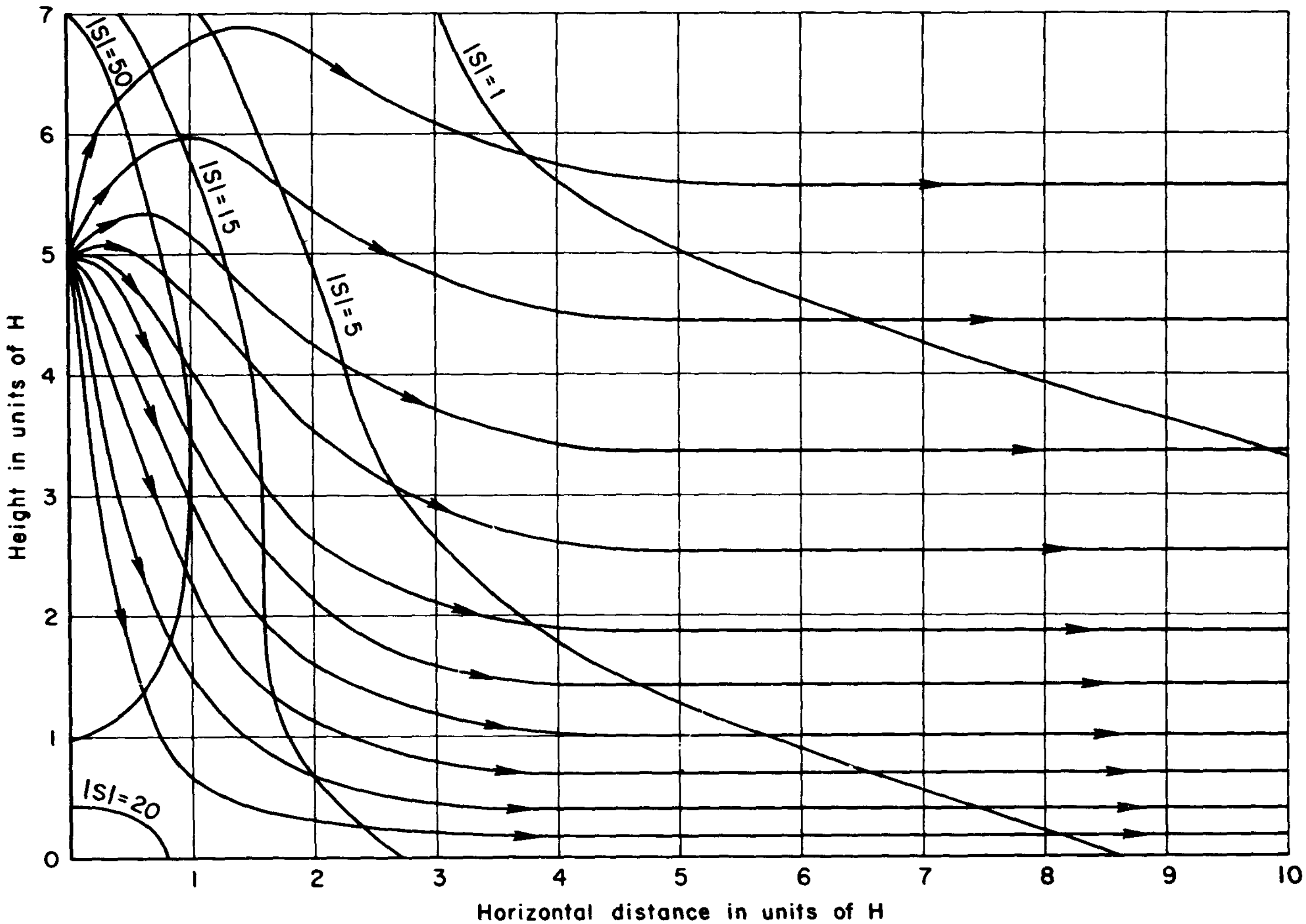

Fig. 3. -- Direction and relative magnitude of acoustic intensity $\underset{\sim}{S}$ as functions of height and horizontal distance for a source of frequency $\omega = 0.48$ (in units of H/C) located at a height of 5H.

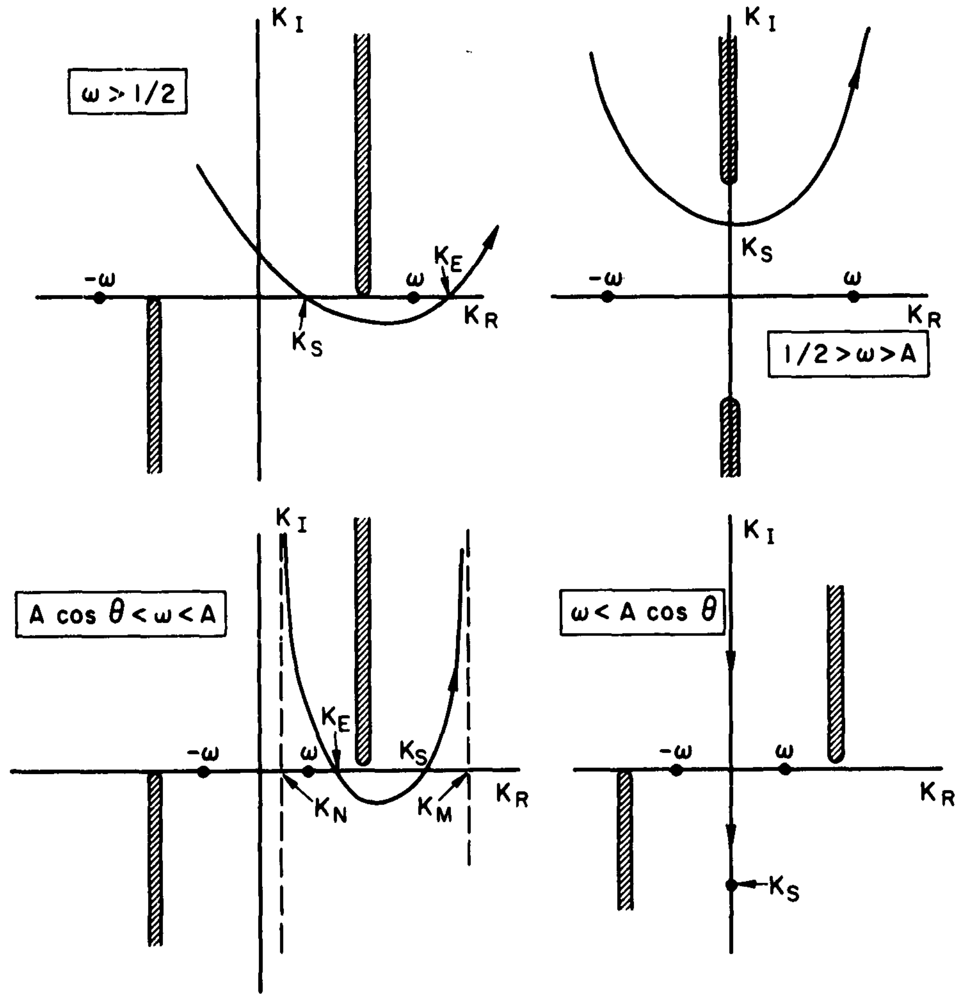

Fig. 4. -- Sketches of the complex K-plane showing relative location of branch lines, poles, and path of steepest descents for various ranges of the angular frequency ω. (The saddle point is denoted by K_S.)

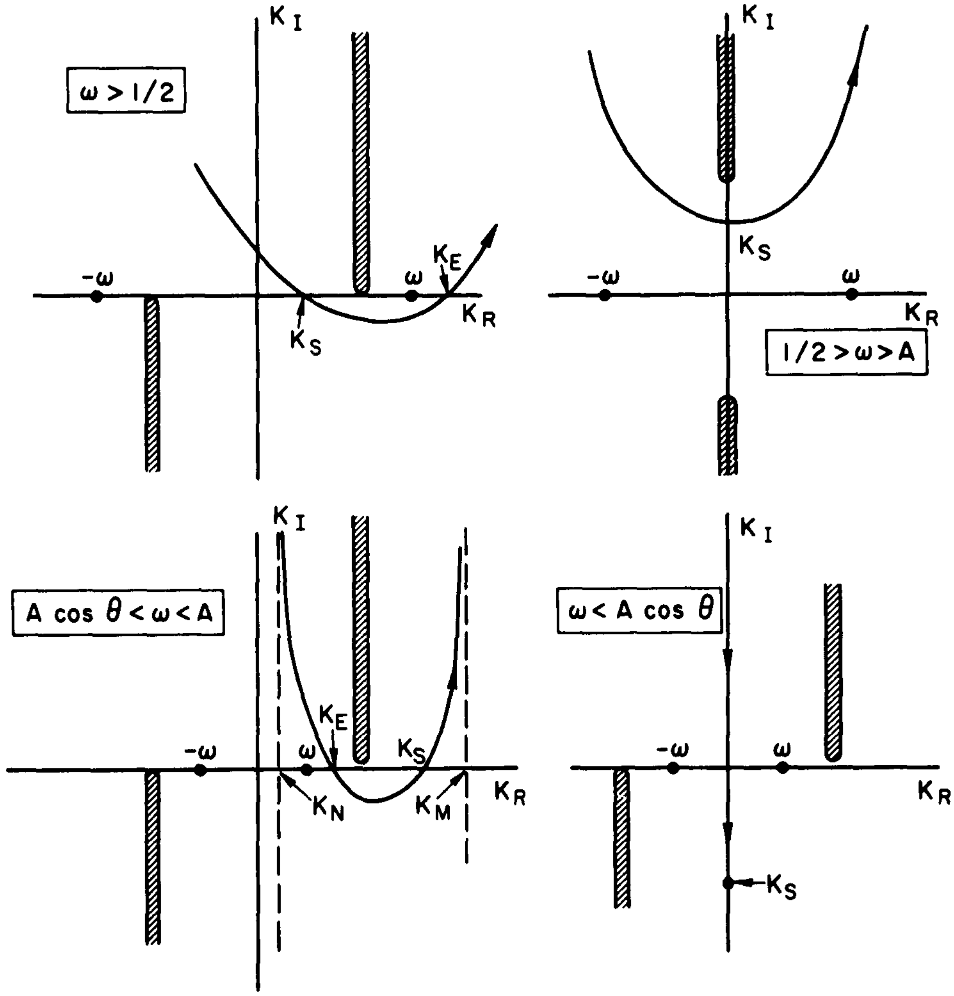

Fig. 4. -- Sketches of the complex K-plane showing relative location of branch lines, poles, and path of steepest descents for various ranges of the angular frequency ω. (The saddle point is denoted by K_S.)

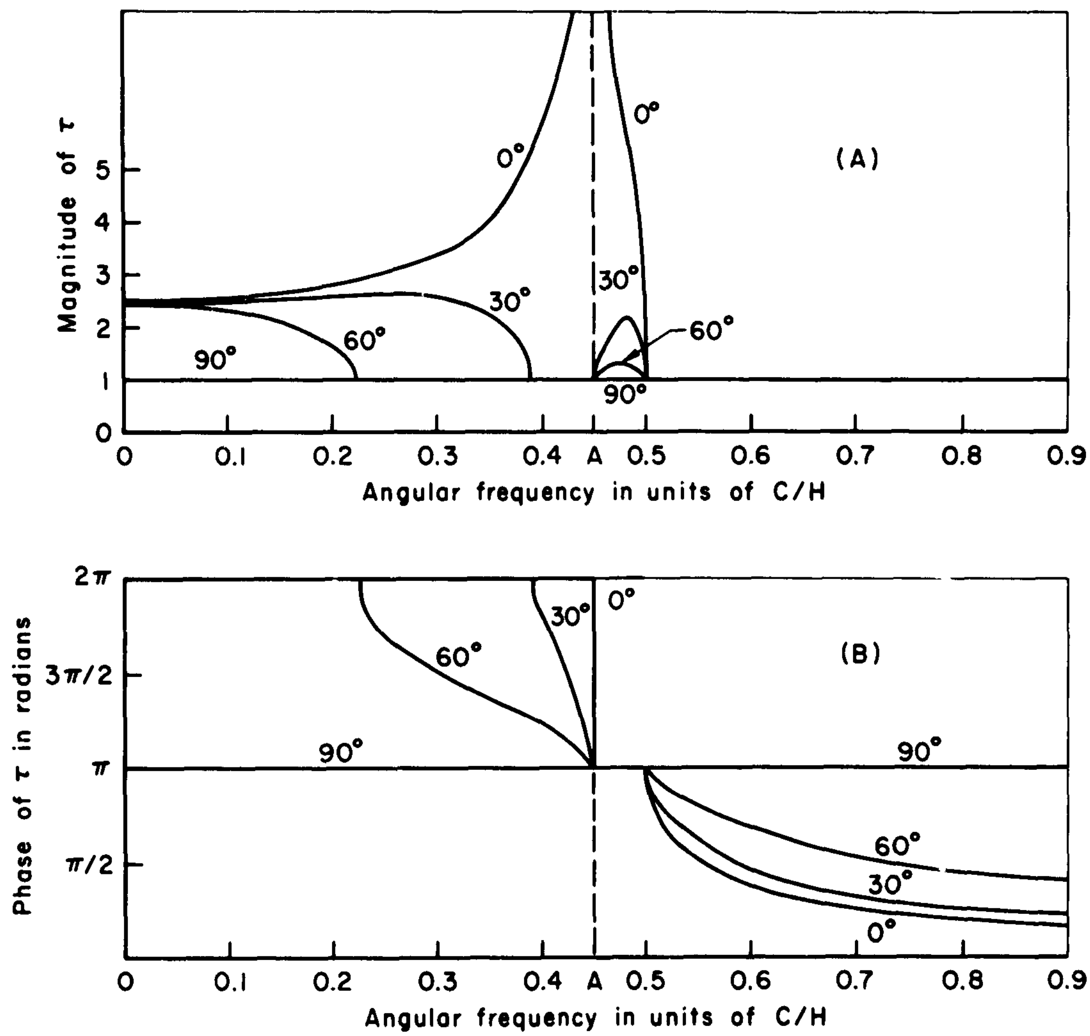

Fig. 5. -- Magnitude and phase of the reflection coefficient τ versus frequency for $\theta = 0^\circ$, 30°, 60°, and 90°.

REFERENCES

1. Lamb, H., Hydrodynamics, Dover Publications, New York, 1945, pp. 541-554.
2. Taylor, G. I., Proc. Roy. Soc. (London), Vol. A126, 1929, pp. 169-183.
3. Pekeris, C. L., Proc. Roy. Soc. (London), Vol. A171, 1939, pp. 434-449.
4. Pekeris, C. L., Phys. Rev., Vol. 73, 1948, pp. 145-154.
5. Scorer, R. S., Proc. Roy. Soc. (London), Vol. A201, 1950, pp. 137-157.
6. Yamamoto, R., J. Meteorol. Soc. Japan, Vol. 35, 1957, pp. 288-296.
7. Hunt, J. N., R. Palmer, and W. Penney, Phil. Trans. Roy. Soc. London, Vol. A252, 1960, pp. 275-315.
8. Weston, V. H., Can. J. Phys., Vol. 39, 1961, pp. 993-1009; Vol. 40, 1962, pp. 431-445, pp. 446-453.
9. Pfeffer, R. L., and J. Zarichny, J. Atmos. Sci., Vol. 19, 1962, pp. 256-263.
10. Gazaryan, Yu. L., Soviet Phys. - Acoustics, Vol. 7, 1961, pp. 17-22.
11. Sretenskii, L. N., Izvestiia Akad. Nauk S.S.S.R., Ser. Geofiz., No. 2, 1954, pp. 134-142.
12. Pridmore-Brown, D. C., J. Acoust. Soc. Am., Vol. 34, 1962, pp. 438-443.
13. Lamb, H., Hydrodynamics, Dover Publications, New York, 1945, p. 399.
14. Morse, P. M., and H. Feshbach, Methods of Theoretical Physics, Vol. 1, McGraw-Hill Book Company, Inc., New York, 1953, pp. 808-811.
15. Weston, V. H., Univ. of Mich. Radiation Lab. Report No. 2886-1-T, 1960. (See also Can. J. Phys., Vol. 39, 1961, pp. 993-1009.)
16. National Bureau of Standards, Tables of the Exponential Integral for Complex Arguments, N.B.S. Applied Math. Series 51, 1958.
17. Brunt, D., Quart. J. Roy. Meteorol Soc., Vol. 53, 1927, p. 30.