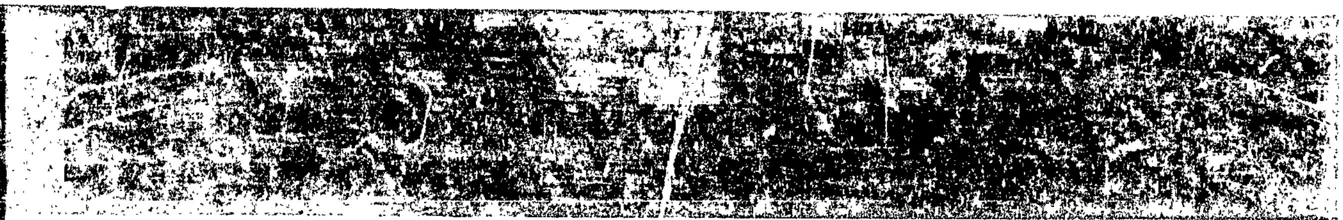

Technical Report 12

SOME ADDITIONAL THEOREMS FOR A NON-STATIONARY STOCHASTIC PROCESS WITH A CONTINUOUS, NON-RANDOM, TIME-DEPENDENT COMPONENT

Prepared for:

ADVANCED RESEARCH PROJECTS AGENCY
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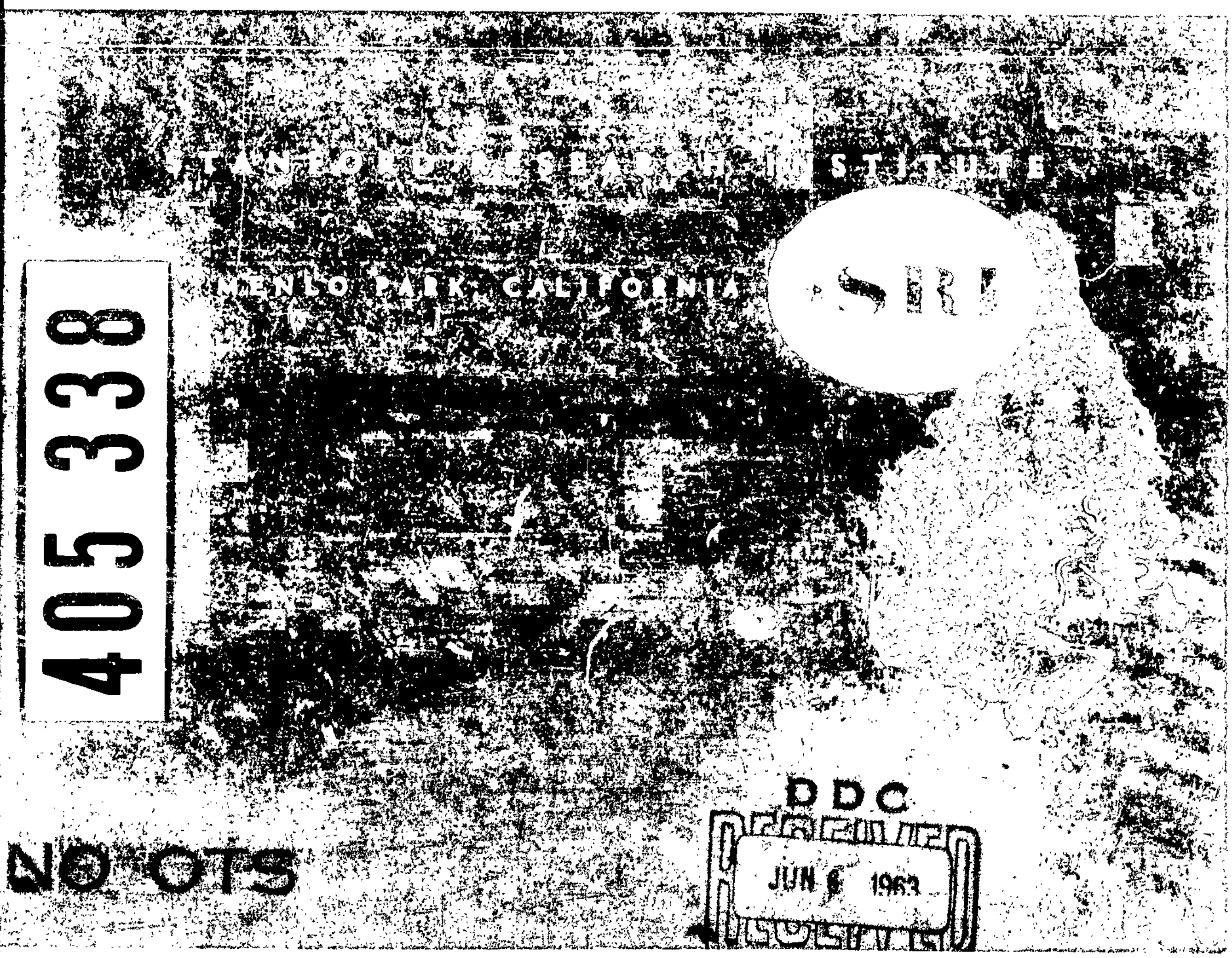
By: *R. C. McCarty* *G. W. Evans II* *G. L. Sutherland* *Z. W. Birnbaum*



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ABSTRACT

In Technical Report 4,[1†] several definitions and theorems were presented to aid in the analysis of a sub-class of non-stationary processes consisting of a random component and a continuous non-random function of time, each component defined over the same finite time interval. This report supplements the results of Technical Report 4 both by extending the sub-class of non-stationary processes under consideration and by including additional theorems.

† References are listed at the end of this report.

CONTENTS

I INTRODUCTION

This report is divided into three sections. Section I summarizes some of the results of Technical Report 4.[1] Section II describes both theorems that are natural extensions to those of Technical Report 4, and extensions to the sub-class of processes considered in the above report. Section III indicates areas for further investigation. The approach followed in the first two sections is to generalize or extend the power spectral density function for stationary stochastic processes so that it may be applied to non-stationary stochastic processes. That is, the approach follows from a desire to obtain, from the observation of a time-dependent non-stationary signal, the energy contributions at one or more average frequencies. The time interval of observation is to be sufficiently large that an energy contribution to an average frequency may be extracted, but not so large that the process changes appreciably during the time of observation. Thus, through successive observations, the change in average frequency may be observed.

Consider a random process $\{Y(t)\}$ consisting of an ensemble of member functions $y_m(t)$, $m = 1, 2, \ldots, M$, defined for the interval $0 \leq t \leq 2T$. For this process the autocovariance $\psi(\tau)$ is redefined as

DEFINITION 1

$$\psi(\tau) = E[A_{y_m(t)y_m(t+\tau)}] - E[A_{y_m(t)}]E[A_{y_m(t+\tau)}]$$

where

$$A_{y_m(t)} = \frac{1}{T}\int_0^T y_m(t)\,dt$$

$$A_{y_m(t+\tau)} = \frac{1}{T}\int_0^T y_m(t+\tau)\,dt \ , \ 0 \leq \tau \leq T$$

and

$$A_{y_m(t)y_m(t+\tau)} = \frac{1}{T}\int_0^T y_m(t)y_m(t+\tau)dt \ , \quad 0 \leq \tau \leq T \quad .$$

The power spectral density function $\Phi(\omega)$ of the process $\{Y(t)\}$ is given with respect to $\psi(\tau)$ in the usual way except that the interval over which the Fourier cosine transform is performed is finite, hence

DEFINITION 2

$$\Phi(n\omega_T) = \frac{1}{T}\int_0^T \psi(\tau) \cos n\omega_T\tau d\tau$$

where

$$\omega_T T = 2\pi \quad .$$

The preceding definitions are applied to a subclass of non-stationary stochastic processes represented by those processes $\{Y(t)\}$ consisting of member functions $y_m(t)$, $m = 1,2, \ldots, M$ and $t \in [0, 2T]$ which may be represented by

$$y_m(t) = \theta_m(t) + f(t)$$

where $\theta_m(t)$ is a member function of a stationary stochastic process $\{\Theta(t)\}$ and $f(t)$ is a deterministic function inducing non-stationarity and is statistically independent of $\theta_m(t)$. In particular, the ensemble $\{Y(t)\}$ is said to satisfy

Condition [*A*]

(1) $\theta_m(t)$ has c.d.f.† $G(\theta_m)$ with $-\infty < \theta_m < +\infty$ for every $m = 1,2, \ldots, M$ and each $t \in [0, 2T]$.

†Cumulative distribution function is abbreviated c.d.f.

(2) $E[\theta_m(t)] = E[\theta_m(t+\tau)] = E[\theta] = \int_{-\infty}^{\infty} \theta dG(\theta) < \infty$ for every $m = 1, 2, \ldots, M$ and each $t \in [0, 2T]$.

(3) $f(t) \in L_2$ where L_2 is the class of square integrable functions.

The following theorems have been proven in Technical Report 4.

<u>THEOREM 1</u>

Given an ensemble of member functions $y_m(t) = \theta_m(t) + f(t)$ satisfying Condition $[A]$, then the autocovariance, $\psi(\tau)$, of the process is separable into two components such that

$$\psi(\tau) = \psi_\theta(\tau) + \psi_f(\tau)$$

where

$$\psi_\theta(\tau) = E[\theta(t)\theta(t+\tau)] - E^2[\theta(t)]$$

and

$$\psi_f(\tau) = A_{f(t)f(t+\tau)} - A_{f(t)}A_{f(t+\tau)} \quad .$$

Theorem 1 shows that if $f(t) \equiv 0$, the autocovariance, $\psi(\tau) = \psi_\theta(\tau)$, is the accepted definition for stationary stochastic process. Similarly, if $\theta_m(t) \equiv 0$ for all m and if $f(t)$ is periodic with fundamental period T, then the autocovariance, $\psi(\tau) = \psi_f(\tau)$, is the accepted definition for periodic functions.

<u>THEOREM 2</u>

The power spectral density function, $\Phi(n\omega_T)$, for an ensemble of member functions $y_m(t)$ satisfying Condition $[A]$ is separable into two components such that

$$\Phi(n\omega_T) = \Phi_\theta(n\omega_T) + \Phi_f(n\omega_T) \quad .$$

where

$$\Phi_\theta(n\omega_T) = \frac{1}{T}\int_0^T \psi_\theta(\tau)\cos n\omega_T\tau d\tau$$

and

$$\Phi_f(n\omega_T) = \frac{1}{T}\int_0^T \psi_f(\tau)\cos n\omega_T\tau d\tau$$

This theorem follows directly from Theorem 1 and from the fact that a Fourier transform is a linear operation.

If in the preceding theorem $\psi_f(\tau) \equiv 0$—*e.g.*, $f(t) \equiv 0$—then

$$\lim_{T\to\infty}\Phi(n\omega_T) = \lim_{T\to\infty}\Phi_\theta(n\omega_T) = \Phi_\theta(\omega) = \frac{1}{\pi}\int_0^\infty \psi_\theta(\tau)\cos\omega\tau d\tau$$

is an acceptable definition for the power spectral density function of a stationary stochastic process; and if $\psi_\theta(\tau) \equiv 0$ and $f(t)$ is periodic with fundamental period T, then $\Phi(n\omega_T) = \Phi_f(n\omega_T)$ represents a definition for the power spectral density function of a periodic function.

II EXTENSIONS

Generally, in applications of power spectral density methods, an additional condition is imposed on ensembles $\{Y(t)\}$ satisfying Condition [A] so that estimates of $\Phi(n\omega_T)$ may be obtained from a single member function $y_m(t)$. The additional condition is one of ergodicity and is stated as follows:

Condition [B]

For the random component $\theta_m(t)$ of Condition [A]

$$E[\theta] = \int_{-\infty}^{\infty} \theta dg(\theta) = E[\theta_m(t)] = \lim_{T\to\infty}\left\{\frac{1}{M}\sum_{m=1}^{M}\theta_m(t)\right\} = \lim_{M\to\infty}\left\{\frac{1}{T}\int_0^T \theta_m(t)dt\right\}$$

$$E[\theta] = E[\theta_m(t+\tau)] = \lim_{M\to\infty}\frac{1}{M}\sum_{m=1}^{M}\theta_m(t+\tau) = \lim_{T\to\infty}\left\{\frac{1}{T}\int_0^T \theta_m(t+\tau)dt\right\}$$

$$E[\theta(t)\theta(t+\tau)] = \int_{-\infty}^{\infty}\theta(t)\theta(t+\tau)dH[\theta(t),\ \theta(t+\tau)] = E[\theta_m(t)\theta_m(t+\tau)]$$

$$= \lim_{M\to\infty}\left\{\frac{1}{M}\sum_{m=1}^{M}\theta_m(t)\theta_m(t+\tau)\right\} = \lim_{T\to\infty}\left\{\frac{1}{T}\int_0^T \theta_m(t)\theta_m(t+\tau)dt\right\}$$

where the time averages hold for each value of $m = 1, 2, \ldots, M$, and $H[\theta(t),\ \theta(t+\tau)]$ is the joint c.d.f. of $\theta(t)$ and $\theta(t+\tau)$

Applications assuming ergodicity imply that the frequency analysis is derived from a single member function, indicating that an estimate for $\psi(\tau)$ must be used in place of Definition 1. Thus the following estimates are defined:

DEFINITION 3

$$\hat{E}[\theta_*(t)] = \frac{1}{T}\int_0^T \theta_*(t)dt$$

$$\hat{E}[\theta_*(t+\tau)] = \frac{1}{T}\int_0^T \theta_*(t+\tau)dt$$

$$\hat{E}[\theta_*(t)\theta_*(t+\tau)] = \frac{1}{T}\int_0^T \theta_*(t)\theta_*(t+\tau)dt \quad .$$

$\hat{E}[\theta_*(t)]$ and $\hat{E}[\theta_*(t+\tau)]$ are estimates for $E[\theta]$, and $\hat{E}[\theta_*(t)\theta_*(t+\tau)]$ is an estimate for $E[\theta_*(t)\theta_*(t+\tau)]$ in the sense of Condition [*B*]. The expressions in Definition 3 permit an estimate for $\psi(\tau)$—namely, $\hat{\psi}(\tau)$—which is given as follows:

DEFINITION 4

$$\hat{\psi}_*(\tau) = A_{y_*(t)y_*(t+\tau)} - A_{y_*(t)}A_{y_*(t+\tau)} \quad .$$

and for the random component

$$\hat{\psi}_{\theta_*}(\tau) = A_{\theta_*(t)\theta_*(t+\tau)} - A_{\theta_*(t)}A_{\theta_*(t+\tau)} \quad .$$

Note that $\hat{\psi}_f(\tau) = \psi_f(\tau)$. Using the expression for $\hat{\psi}_*(\tau)$, an estimate for the power-spectral-density, $\hat{\Phi}(n\omega_T)$, is given by Definition 5.

DEFINITION 5

$$\hat{\Phi}(n\omega_T) = \frac{1}{T}\int_0^T \hat{\psi}_*(\tau)\cos n\omega_T\tau d\tau \quad .$$

Because of Conditions [*A*] and [*B*] and Definitions 1 through 5, $\hat{\Phi}_*(n\omega_T)$ is an estimate for $\Phi(n\omega_T)$ in the sense of the following theorem.

THEOREM 3

Let $\overline{\Phi_M(n\omega_T)} = \frac{1}{M}\sum_{m=1}^{M} \hat{\Phi}_m(n\omega_T)$. Then for a non-stationary process $\{Y(t)\}$ containing member functions $y_m(t)$ satisfying Conditions [A] and [B],

$$\lim_{M\to\infty} \overline{\{\Phi_M(n\omega_T)\}} = \Phi_f(n\omega_T) + \frac{1}{T}\int_0^T \left[\lim_{M\to\infty}\left\{\frac{1}{M}\sum_{m=1}^{M} \hat{\psi}_{\theta_m}(\tau)\right\}\right] \cos n\omega_T\tau d\tau$$

and by an additional limiting process

$$\lim_{M\to\infty}\lim_{T\to\infty} \overline{\{\Phi_M(n\omega_T)\}} = \Phi_f(n\omega_T) + \Phi_\theta(n\omega_T) \quad .$$

Proof:

By Definitions 4 and 5

$$\hat{\Phi}_m(n\omega_T) = \frac{1}{T}\int_0^T \left[A_{y_m(t)y_m(t+\tau)} - A_{y_m(t)}A_{y_m(t+\tau)}\right] \cos n\omega_T\tau d\tau$$

$$= \frac{1}{T}\int_0^T \left[\frac{1}{T}\int_0^T y_m(t)y_m(t+\tau)dt - \frac{1}{T^2}\int_0^T y_m(t)dt \int_0^T y_m(t+\tau)dt\right] \cos n\omega_T\tau d\tau$$

$$= \frac{1}{T}\int_0^T \left[\frac{1}{T}\int_0^T \theta_m(t)\theta_m(t+\tau)dt + \frac{1}{T}\int_0^T f(t)f(t+\tau)dt + \frac{1}{T}\int_0^T \theta_m(t)f(t+\tau)dt\right.$$

$$+ \frac{1}{T}\int_0^T \theta_m(t+\tau)f(t)dt - \frac{1}{T^2}\int_0^T \theta_m(t)dt \int_0^T \theta_m(t+\tau)dt$$

$$- \frac{1}{T^2}\int_0^T f(t)dt \int_0^T f(t+\tau)dt - \frac{1}{T^2}\int_0^T \theta_m(t)dt \int_0^T f(t+\tau)dt$$

$$\left.- \frac{1}{T^2}\int_0^T \theta_m(t+\tau)dt \int_0^T f(t)dt\right] \cos n\omega_T\tau d\tau$$

$$= \frac{1}{T}\int_0^T \Big[A_{\theta_m(t)\theta_m(t+\tau)} - A_{\theta_m(t)}A_{\theta_m(t+\tau)} + A_{f(t)f(t+\tau)}$$

$$- A_{f(t)}A_{f(t+\tau)} + R(y_m)\Big] \cos n\omega_T\tau d\tau$$

$$= \Phi_f(n\omega_T) + \frac{1}{T}\int_0^T [\dot{\hat{\psi}}_{\theta_m}(\tau) + R(y_m)] \cos n\omega_T\tau d\tau$$

where

$$R(y_m) = \frac{1}{T}\int_0^T \theta_m(t)f(t+\tau)dt - \frac{1}{T^2}\int_0^T \theta_m(t)dt \int_0^T f(t+\tau)dt$$

$$+ \frac{1}{T}\int_0^T \theta_m(t+\tau)f(t)dt - \frac{1}{T^2}\int_0^T \theta_m(t+\tau)dt \int_0^T f(t)dt \quad .$$

Now $R(y_m)$ has the following property as a result of Condition $[B]$:

$$\lim_{M\to\infty}\left\{\frac{1}{M}\sum_{m=1}^{M}\left[\frac{1}{T}\int_0^T R(y_m) \cos n\omega_T\tau d\tau\right]\right\}$$

$$= \frac{1}{T}\int_0^T \left\{\lim_{M\to\infty}\left[\frac{1}{M}\sum_{m=1}^{M} R(y_m)\right]\right\} \cos n\omega_T\tau d\tau$$

$$= \frac{1}{T}\int_0^T \Bigg[\frac{E(\theta)}{T}\int_0^T f(t+\tau)dt - \frac{E(\theta)}{T}\int_0^T f(t+\tau)dt + \frac{E(\theta)}{T}\int_0^T f(t)dt$$

$$- \frac{E[\theta]}{T}\int_0^T f(t)dt\Bigg] \cos n\omega_T\tau d\tau = 0 \quad .$$

Summing $\hat{\Phi}_m(n\omega_T)$ for $m = 1, 2, \ldots, M$, and taking the limit as $M \to \infty$, we establish the first part of the theorem—*i.e.*,

$$\lim_{M\to\infty} \overline{\{\Phi_M(n\omega_T)\}} = \lim_{M\to\infty}\left\{\frac{1}{M}\sum_{m=1}^{M} \hat{\Phi}_m(n\omega_T)\right\}$$

$$= \Phi_f(n\omega_T) + \frac{1}{T}\int_0^T \left[\lim_{M\to\infty}\left\{\frac{1}{M}\sum_{m=1}^{M} \hat{\psi}_{\phi_m}(\tau)\right\}\right] \cos n\omega_T\tau d\tau \quad .$$

Now to consider the limit as $T \to \infty$ we note that

$$\lim_{T\to\infty} \{\psi_{\theta_m}(\tau)\} = \lim_{T\to\infty}\left\{\frac{1}{T}\int_0^T \theta_m(t)\theta_m(t+\tau)dt - \frac{1}{T^2}\int_0^T \theta_m(t)dt \int_0^T \theta_m(t+\tau)dt\right\} \quad .$$

From Definition 3 this is equivalent to

$$\lim_{T\to\infty} \{\hat{\psi}_{\theta_m}(\tau)\} = \lim_{T\to\infty} \{E[\theta_m(t)\theta_m(t+\tau)] - \hat{E}[\theta_m(t)]\hat{E}[\theta_m(t+\tau)]\}$$

and by Condition [B] and Theorem 1,

$$\lim_{T\to\infty} \{\hat{\psi}_{\theta_m}(\tau)\} = E[\theta(t)\theta(t+\tau)] - E^2[\theta] = \psi_\theta(\tau) \quad .$$

Now consider the limit as $T \to \infty$ in the term

$$\lim_{M\to\infty}\left\{\frac{1}{M}\sum_{m=1}^{M}\left[\frac{1}{T}\int_0^T \lim_{T\to\infty} \{\hat{\psi}_{\theta_m}(\tau)\} \cos n\omega_T\tau d\tau\right]\right\}$$

$$= \lim_{M\to\infty}\left\{\frac{1}{M}\sum_{m=1}^{M}\left[\frac{1}{T}\int_0^T \psi_\theta(\tau) \cos n\omega_T\tau d\tau\right]\right\}$$

$$= \Phi_\theta(n\omega_T) \text{ as a result of Theorem 2} \quad .$$

Thus $\lim_{M\to\infty} \lim_{T\to\infty} \overline{\{\Phi_M(n\omega_T)\}} = \Phi_f(n\omega_T) + \Phi_\theta(n\omega_T)$ and the proof of Theorem 3 is concluded.

As given by Theorem 2, $\Phi_f(n\omega_T)$ was defined with respect to the time of observation, T; and the calculated frequencies are multiples (harmonics) of the fundamental frequency, ω_T. However, suppose $\{Y(t)\}$ is stationary where $f(t)$ is a periodic function with fundamental period $T_0 = 2\pi/\omega_0$. It is then desirable to know the relation between the calculated frequencies and the harmonics of ω_0. Since

$$\Phi_f(n\omega_0) = \frac{1}{T_0}\int_0^{T_0} \psi_{fN}(\tau) \cos n\omega_0\tau d\tau$$

then

$$\Phi_f\left(\frac{n}{N}\omega_0\right) = \frac{1}{NT_0}\int_0^{NT_0} \psi_{fN}(\tau) \cos \frac{n}{N}\omega_0\tau d\tau$$

will exhibit the fundamental frequency, ω_0, when $n = N$, and will exhibit harmonics or multiples of ω_0 when $n = 2N, 3N, 4N, \ldots.$ The notation ψ_{fN} is shorthand for

$$\psi_{fN}(\tau) = \frac{1}{NT_0}\int_0^{NT_0} f(t)f(t+\tau)dt - f^2$$

where

$$f = \frac{1}{NT_0}\int_0^{NT_0} f(t)dt = \frac{1}{NT_0}\int_0^{NT_0} f(t+\tau)dt \quad .$$

The following theorem establishes a relation between the calculated and actual frequencies.

<u>THEOREM 4</u>

For $f(t) \in L_2$, a periodic function with fundamental angular frequency ω_0 and fundamental period T_0,

$$\lim_{T\to NT_0} \{\Phi_f(n\omega_T)\} = \Phi_f\left(\frac{n}{N}\omega_0\right)$$

and

$$\left|\Phi_f(n\omega_T) - \frac{N^2}{(N+\beta)^2}\,\Phi_f\!\left(\frac{n}{N}\,\omega_0\right)\right| \le f_M^2\,\beta\,\frac{(5N+2\beta)}{(N+\beta)^2}$$

where

$$T = (N+\beta)T_0\,,\ N \ge 1,\ 0 \le \beta < 1,\ \text{and}\ f_M = \max_{0\le t\le T_0} |f(t)| \quad .$$

Proof:

Take $\psi_f(\tau) = \dfrac{1}{T}\displaystyle\int_0^T f(t)f(t+\tau)dt - \dfrac{1}{T^2}\int_0^T f(t)dt\int_0^T f(t+\tau)dt$.

We may write each integral in two parts, as follows:

$$\psi_f(\tau) = \frac{1}{T}\int_0^{NT_0} f(t)f(t+\tau)dt + \frac{1}{T}\int_{NT_0}^{T} f(t)f(t+\tau)dt$$

$$- \frac{1}{T^2}\left\{\left[\int_0^{NT_0} f(t)dt + \int_{NT_0}^{T} f(t)dt\right]\left[\int_0^{NT_0} f(t+\tau)dt + \int_{NT_0}^{T} f(t+\tau)dt\right]\right\}$$

$$= \frac{N}{N+\beta}\,\frac{1}{NT_0}\int_0^{NT_0} f(t)f(t+\tau)dt - \frac{N}{N+\beta}\bar{f}^2 + \frac{N}{N+\beta}\bar{f}^2 + \frac{1}{T}\int_{NT_0}^{T} f(t)f(t+\tau)dt$$

$$- \frac{1}{T^2}\left\{\left[NT_0\bar{f} + \int_{NT_0}^{T} f(t)dt\right]\left[NT_0\bar{f} + \int_{NT_0}^{T} f(t+\tau)dt\right]\right\}$$

$$= \frac{N}{N+\beta}\psi_{fN} + \frac{\beta N}{(N+\beta)^2} f^2 + \frac{1}{T}\int_{NT_0}^{T} f(t)f(t+\tau)dt$$

$$-\frac{1}{T^2}\left\{NT_0 f \int_{NT_0}^{T} [f(t) + f(t+\tau)]dt + \int_{NT_0}^{T} f(t)dt \int_{NT_0}^{T} f(t+\tau)dt\right\} .$$

If this expression is substituted for $\psi_f(\tau)$ in the definition of $\Phi_f(n\omega_T)$ then

$$\Phi_f(n\omega_T) = \frac{N^2}{(N+\beta)^2}\Phi_f\left(\frac{n}{N+\beta}\omega_0\right) + \frac{N}{(N+\beta)T}\int_{NT_0}^{T} \psi_{fN}(\tau) \cos n\omega_T\tau d\tau$$

$$+\frac{1}{T}\int_0^T\left\{\frac{\beta N}{(N+\beta)^2} f^2 + \frac{1}{T}\int_{NT_0}^{T} f(t)f(t+\tau)dt\right.$$

$$\left.-\frac{1}{T^2}\left[NT_0 f\int_{NT_0}^{T} [f(t+\tau) + f(t)]dt + \int_{NT_0}^{T} f(t)dt \int_{NT_0}^{T} f(t+\tau)dt\right]\right\}\cos n\omega_T\tau d\tau .$$

Thus, since $\lim\limits_{T\to NT_0} \beta = 0$, taking the limit as $T \to NT_0$ establishes the first part of the theorem:

$$\lim_{T\to NT_0} \{\Phi_f(n\omega_T)\} = \Phi_f\left(\frac{n}{N}\omega_0\right) .$$

Next, consider upper bounds for each of the three last terms in the expression for $\Phi_f(n\omega_T)$:

$$\left|\frac{N}{(N+\beta)T}\int_{NT_0}^{T} \psi_{fN}(\tau)\cos n\omega_T\tau d\tau\right| = \left|\frac{N}{(N+\beta)T}\int_{NT_0}^{T}\left[\frac{1}{NT_0}\int_0^{NT_0} f(t)f(t+\tau)dt - f^2\right]\cos n\omega_T\tau d\tau\right|$$

$$\leq \frac{N^2}{(N+\beta)^2}\frac{1}{NT_0}\int_{NT_0}^{(N+\beta)T_0} 2f_M^2\left|\cos\frac{n\omega_0}{N+\beta}\tau\right|d\tau \leq \frac{2\beta N}{(N+\beta)^2} f_M^2 ,$$

$$\left|\frac{1}{T}\int_0^T\left\{\frac{\beta N}{(N+\beta)^2}f^2\right\}\cos n\omega_T\tau d\tau\right| = \left|\frac{\beta N}{(N+\beta)^3 T_0}f^2\int_0^{(N+\beta)T_0}\cos\frac{n}{N+\beta}\omega_0\tau d\tau\right| = 0 \quad ,$$

and finally

$$\left|\frac{1}{T}\int_0^T\left\{\frac{1}{T}\int_{NT_0}^T f(t)f(t+\tau)\right.\right.$$

$$\left.\left.-\frac{1}{T^2}\left[NT_0 f\int_{NT_0}^T [f(t+\tau)+f(t)]dt+\int_{NT_0}^T f(t)dt\int_{NT_0}^T f(t+\tau)dt\right]\right\}\cos n\omega_T\tau d\tau\right|$$

$$\leq\frac{1}{(N+\beta)T_0}\int_0^{(N+\beta)T_0}\left[\frac{1}{(N+\beta)T_0}\int_{NT_0}^{(N+\beta)T_0}f_M^2dt+\frac{N}{(N+\beta)^2T_0}f_M\int_{NT_0}^{(N+\beta)T_0}2f_Mdt\right.$$

$$\left.+\frac{1}{(N+\beta)^2T_0^2}\int_{NT_0}^{(N+\beta)T_0}f_Mdt\int_{NT_0}^{(N+\beta)T_0}f_Mdt\right]\left|\cos\frac{n}{N+\beta}\omega_0\tau\right|d\tau$$

$$\leq\left[\frac{\beta}{N+\beta}f_M^2+\frac{2N\beta f_M^2}{(N+\beta)^2}+\frac{\beta^2f_M^2}{(N+\beta)^2}\right] = f_M^2\frac{(3N\beta+\beta^2)}{(N+\beta)^2} \quad .$$

Thus

$$\left|\Phi_f(n\omega_T)-\frac{N^2}{(N+\beta)^2}\Phi_f\left(\frac{n}{N+\beta}\omega_0\right)\right|\leq f_M^2\frac{\beta(5N+2\beta)}{(N+\beta)^2}$$

and Theorem 4 is proved.

A corollary to Theorem 4 gives the relations between $\psi_f(\tau)$ and $\psi_{f1}(\tau)$, $N = 1$, and between $\Phi_f(n\omega_T)$ and $\Phi_f(n\omega_0)$ where $0 < T \leq T_0$. For this investigation set $T = \gamma T_0$ or $\omega_T = \omega_0/\gamma$ where $0 < \gamma \leq 1$. Remembering that

$$\psi_{f1}(\tau) = \frac{1}{T_0}\int_0^{T_0} f(t)f(t+\tau)dt - f^2$$

where

$$f = \frac{1}{T_0}\int_0^{T_0} f(t)dt = \frac{1}{T_0}\int_0^{T_0} f(t+\tau)dt$$

then the corollary may be stated as follows.

Corollary:

For $f(t) \in L_2$, a periodic function with fundamental frequency ω_0 and period T_0, then

$$\lim_{T\to T_0} \Phi_f(n\omega_T) = \Phi_f(n\omega_0)$$

and

$$\left| \Phi_f(n\omega_T) - \frac{1}{\gamma^2}\Phi_f\left(\frac{n}{\gamma}\omega_0\right) \right| \le 5f_M^2(\gamma - 1)/\gamma^2$$

where

$$f_M = \max_{0\le t\le T_0}\left|f(t)\right| = \max_{0\le t\le T_0}\left|f(t+\tau)\right| \quad \text{and} \quad 0 \le \tau \le T_0 .$$

Proof:

$$\psi_f(\tau) = \frac{1}{T}\int_0^T f(t)f(t+\tau)dt - \frac{1}{T^2}\int_0^T f(t)dt\int_0^T f(t+\tau)dt$$

$$= \frac{1}{\gamma T_0}\int_0^{T_0} f(t)f(t+\tau)dt + \frac{1}{\gamma T_0}\int_{T_0}^{\gamma T_0} f(t)f(t+\tau)dt$$

$$- \frac{1}{(\gamma T_0)^2}\left[\int_0^{T_0} f(t)dt + \int_{T_0}^{\gamma T_0} f(t)dt\right]\left[\int_0^{T_0} f(t+\tau)dt + \int_{T_0}^{\gamma T_0} f(t+\tau)dt\right].$$

$$= \frac{1}{\gamma}\psi_{f1}(\tau) + \frac{\gamma - 1}{\gamma^2} f^2 + \frac{1}{\gamma T_0}\int_{T_0}^{\gamma T_0} f(t)f(t+\tau)dt$$

$$- \frac{1}{(\gamma T_0)^2}\left[T_0 f \int_{T_0}^{\gamma T_0} [f(t+\tau) + f(t)]dt + \int_{T_0}^{\gamma T_0} f(t)dt \int_{T_0}^{\gamma T_0} f(t+\tau)dt\right] .$$

Now since

$$\Phi_f(n\omega_T) = \frac{1}{T}\int_0^T \psi_f(\tau)\cos n\omega_T\tau d\tau = \frac{1}{\gamma T_0}\int_0^{\gamma T_0} \psi_f(\tau)\cos\frac{n\omega_0}{\gamma}\tau d\tau ,$$

substituting the preceding expression for $\psi_f(\tau)$ into $\Phi_f(n\omega_T)$ gives

$$\Phi_f(n\omega_T) = \frac{1}{\gamma T_0}\int_0^{\gamma T_0}\left\{\frac{1}{\gamma}\psi_{f1}(\tau) + \frac{\gamma - 1}{\gamma^2} f^2 + \frac{1}{\gamma T_0}\int_{T_0}^{\gamma T_0} f(t)f(t+\tau)dt\right.$$

$$\left. - \frac{1}{\gamma^2 T_0} f \int_{T_0}^{\gamma T_0} [f(t+\tau) + f(t)]dt - \frac{1}{(\gamma T_0)^2}\int_{T_0}^{\gamma T_0} f(t)dt \int_{T_0}^{\gamma T_0} f(t+\tau)dt\right\}\cos\frac{n\omega_0}{\gamma}\tau d\tau$$

$$= \frac{1}{\gamma^2}\Phi_f\left(\frac{n}{\gamma}\omega_0\right) + \frac{1}{\gamma^2 T_0}\int_{T_0}^{\gamma T_0} \psi_{f1}(\tau)\cos\frac{n\omega_0}{\gamma}\tau d\tau$$

$$+ \frac{1}{(\gamma T_0)^2}\int_0^{\gamma T_0}\int_{T_0}^{\gamma T_0} f(t)f(t+\tau)dt\cos\frac{n\omega_0}{\gamma}\tau d\tau$$

$$- \frac{1}{\gamma^3 T_0^2} f \int_0^{\gamma T_0}\int_{T_0}^{\gamma T_0} [f(t+\tau) + f(t)]dt\cos\frac{n\omega_0}{\gamma}\tau d\tau +$$

$$-\frac{1}{(\gamma T_0)^3}\int_0^{\gamma T_0}\left[\int_{T_0}^{\gamma T_0} f(t)dt \int_{T_0}^{\gamma T_0} f(t+\tau)dt\right]\cos\frac{n\omega_0}{\gamma}\tau d\tau$$

and

$$\lim_{T\to T_0} \Phi_f(n\omega_T) = \Phi_f(n\omega_0)$$

since $\lim_{T\to T_0} \gamma = 1$, and the first part of the corollary is established. Next, consider upper bounds for the individual terms in the expression for $\Phi_f(n\omega_T)$

$$\left|\frac{1}{\gamma^2 T_0}\int_{T_0}^{\gamma T_0} \psi_{f1}(\tau)\cos\frac{n\omega_0}{\gamma}\tau d\tau\right| = \left|\frac{1}{\gamma^2 T_0}\int_{T_0}^{\gamma T_0}\left[\frac{1}{T_0}\int_0^{T_0} f(t)f(t+\tau)dt - f^2\right]\cos\frac{n\omega_0}{\gamma}\tau d\tau\right|$$

$$\leq \frac{1}{\gamma^2 T_0}\int_{\gamma T_0}^{T_0} 2f_M^2\left|\cos\frac{n\omega_0}{\gamma}\tau\right| d\tau \leq \frac{(1-\gamma)}{\gamma^2}2f_M^2 \quad ,$$

$$\left|\frac{1}{(\gamma T_0)^2}\int_0^{\gamma T_0}\int_{T_0}^{\gamma T_0} f(t)f(t+\tau)dt\cos\frac{n\omega_0}{\gamma}\tau d\tau\right| \leq \frac{1}{(\gamma T_0)^2}\int_0^{\gamma T_0}(1-\gamma)T_0 f_M^2\left|\cos\frac{n\omega_0}{\gamma}\tau\right| d\tau$$

$$\leq \frac{\gamma(1-\gamma)}{\gamma^2} f_M^2 \quad ,$$

$$\left|\frac{1}{\gamma^3 T_0^2} f \int_0^{\gamma T_0}\int_{T_0}^{\gamma T_0}[f(t+\tau)+f(t)]dt\cos\frac{n\omega_0}{\gamma}\tau d\tau\right| \leq \frac{2f_M^2(1-\gamma)}{\gamma^3 T_0}\int_0^{\gamma T_0}\left|\cos\frac{n\omega_0}{\gamma}\tau\right| d\tau$$

$$\leq \frac{(1-\gamma)}{\gamma^2}2f_M^2 \quad ,$$

and for the final term

$$\left|\frac{1}{(\gamma T_0)^3}\int_0^{\gamma T_0}\left[\int_{T_0}^{\gamma T_0} f(t)dt \int_{T_0}^{\gamma T_0} f(t+\tau)dt\right]\cos\frac{n\omega_0}{\gamma}\tau d\tau\right|$$

$$\leq \frac{(1-\gamma)^2 f_M^2}{\gamma^3 T_0}\int_0^{\gamma T_0}\left|\cos\frac{n\omega_0}{\gamma}\tau\right|d\tau \leq \frac{(1-\gamma)^2}{\gamma^2} f_M^2 \quad .$$

The proof of the corollary is completed since

$$\left|\Phi_f(n\omega_T) - \frac{1}{\gamma^2}\Phi\left(\frac{n}{\gamma}\omega_0\right)\right| \leq \frac{2(1-\gamma)}{\gamma^2} f_M^2 + \frac{\gamma(1-\gamma)}{\gamma^2} f_M^2 + \frac{2(1-\gamma)}{\gamma^2} f_M^2 + \frac{(1-\gamma)^2}{\gamma^2} f_M^2$$

$$\leq f_M^2 \frac{(1-\gamma)}{\gamma^2}(4+\gamma+1-\gamma) = 5f_M^2 \frac{(1-\gamma)}{\gamma^2} \quad .$$

The power-spectral-density techniques are applicable to non-stationary processes consisting of member functions which may be reduced to the form $y_s(t) = \theta_s(t) + f(t)$. For example, consider a non-stationary process $\{X(t)\}$ consisting of member functions

$$x_s(t) = \lambda_s(t)g(t)$$

where $\lambda_s(t)$ satisfies Condition [B] and Parts 1 and 2 of Condition [A] and $g(t)$ satisfies Part 3 of Condition [A]. Then by setting

$$y_s(t) = ln\ x_s(t)$$

$$\theta_s(t) = ln\ \lambda_s(t)$$

and

$$f(t) = ln\ g(t)$$

the frequency analysis of the product $x_*(t) = \lambda_*(t)g(t)$ may be treated as the analysis of the sum

$$\ln \lambda_*(t) + \ln g(t) = \ln x_*(t) = y_*(t) = \theta_*(t) + f(t)$$

providing the dominant frequencies of $\ln x_*(t)$ are the same as those for $x_*(t)$. This restriction is stated mathematically for functions $x_*(t)$, observed for $0 \leq t \leq 2T$, satisfying the following theorem.

<u>THEOREM 5</u>

Given $x_*(t)$ expandable in a Fourier series as

$$x_*(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\omega_T t + b_n \sin n\omega_T t = \frac{a_0}{2}\,[1 + z_*(t)]$$

where the a_n and b_n are random coefficients, $a_0 > 0$, $\omega_T T = 2\pi$, and $|z_*(t)| << 1$, then the dominant frequencies contained in the set $n\omega_T$ for $n = 1,2, \ldots$ are in the same order for $x_*(t)$ and $\ln x_*(t)$.

Proof:

Set $\ln x_*(t) = \ln \dfrac{a_0}{2} + \ln\,[1 + z_*(t)]$

where

$$z_*(t) = \frac{2}{a_0}\left[\sum_{n=1}^{\infty} a_n \cos n\omega_T t + b_n \sin n\omega_T t\right] \quad .$$

Since $|z_*(t)| < 1$ and $a_0 > 0$, then $\ln\,[1 + z_*(t)]$ may be expanded in powers of $z_*(t)$ and therefore

$$\ln x_*(t) = \ln \frac{a_0}{2} + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} z_*^n(t) \quad .$$

The condition $|z_*(t)| << 1$ is used to imply that the first term of the power series is the dominant term:

$$ln\ x_s(t) \doteq ln \frac{a_0}{2} + z_s(t)$$

$$\doteq ln \frac{a_0}{2} + \frac{2}{a_0}[\sum_{n=1}^{\infty} a_n \cos n\omega_T t + b_n \sin n\omega_T t]$$

which provides the same ordering for the magnitudes of the $a_n^2 + b_n^2$, $n \geq 1$, for $ln\ x_s(t)$ as for $x_s(t)$.

Thus the analysis of $x_s(t) = \lambda_s(t)g(t)$ may be performed on $y_s(t) = ln\ x_s(t) = \theta_s(t) + f(t)$ by the usual methods of taking the Fourier cosine transform of the autocovariance of $y_s(t)$. This is true because the choice of $\omega_T = 2\pi/T$ for the fundamental frequency of $g(t)$ carries over into the analysis, since $g(t)$ and $f(t) = ln\ g(t)$ both have the same fundamental frequency.

Note that the condition $|z_s(t)| << 1$ is equivalent to

$$\frac{a_0}{2} >> \sum_{n=1}^{\infty} \left|[a_n^2 + b_n^2]^{1/2}\right|$$

That is, the dc component of the signal must be greater than the maximum amplitude of the sum of the non-dc component of the signal so that the logarithmic transformation produces a real function which is physically interpretable.

III ADDITIONAL THEOREMS

Consider a periodic function $f(t)$ with fundamental frequency ω_0 and period T_0 which is expandable in a Fourier series—*i.e.*,

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\omega_0 t + b_n \sin n\omega_0 t \quad .$$

The frequency spectrum of $f(t)$ may be obtained from the product of the finite Fourier transform with the finite conjugate Fourier transform. That is, as the finite Fourier transform is defined by

$$F(n\omega_0) = \frac{1}{T_0}\int_0^{T_0} f(t)e^{in\omega_0 t}dt$$

and as the finite conjugate Fourier transform is defined by

$$F^*(n\omega_0) = \frac{1}{T_0}\int_0^{T_0} f(t)e^{-in\omega_0 t}dt$$

the frequency spectrum is given by

$$|F(n\omega_0)|^2 = F(n\omega_0)F^*(n\omega_0) = \frac{a_n^2 + b_n^2}{4} \quad .$$

For periodic functions $f(t)\epsilon L_2$, $|F(n\omega_0)|^2 = \Phi_f(n\omega_0)$ when $\Phi_f(n\omega_0)$ is evaluated for $f(t)$ observed over the time interval $0 \leq t \leq T_0$. However, $|F(n\omega_0)|^2$ is functionally different from $\Phi(n\omega_0)$, and other methods of calculating the frequency spectrum of $f(t)$ may be investigated (differing from the power-spectral-density function described in the previous sections). It is hoped that these methods can be extended to non-stationary stochastic processes $\{Y(t)\}$ satisfying Conditions $[A]$ and $[B]$.

Consider a theorem which parallels Theorem 4. That is, let $f(t)$ be observed for $0 \leq t \leq T$ where $T = (N + \beta)T_0$, $N \geq 1$ and $0 \leq \beta < 1$. Again, define $\omega_T T = 2\pi$ and let

$$F(n\omega_T) = \frac{1}{T}\int_0^T f(t)e^{in\omega_T t}dt$$

and

$$F^*(n\omega_T) = \frac{1}{T}\int_0^T f(t)e^{-in\omega_T t}dt \quad .$$

Further, since

$$F(n\omega_0) = \frac{1}{T_0}\int_0^{T_0} f(t)e^{in\omega_0 t}dt$$

then

$$F\left(\frac{n}{N}\,\omega_0\right) = \frac{1}{NT_0}\int_0^{NT_0} f(t)e^{i(n/N)\omega_0 t}dt \quad .$$

Similarly,

$$F^*\left(\frac{n}{N}\,\omega_0\right) = \frac{1}{NT}\int_0^{NT_0} f(t)e^{-i(n/N)\omega_0 t}dt \quad ,$$

and the following theorem is stated and proved.

THEOREM 6

Given $f(t)$ periodic with fundamental frequency ω_0 and period T_0 and given $f(t)\epsilon L_2$ for $0 \leq t \leq T$, then

$$\left|F(n\omega_T)F^*(n\omega_T) - \left(\frac{N}{N+\beta}\right)^2 F\left(\frac{n}{N+\beta}\,\omega_0\right)F^*\left(\frac{n}{N+\beta}\,\omega_0\right)\right| \leq f_M^2\,\frac{\beta(4N+\beta)}{(N+\beta)^2}$$

where

$$f_M = \max_{0<t<T_0} \{|f(t)|\} .$$

Proof:

Substituting

$$e^{in\omega_T t} = \cos n\omega_T t + i \sin n\omega_T t$$

and

$$e^{-in\omega_T t} = \cos n\omega_T t - i \sin n\omega_T t$$

into $F(n\omega_T)F^*(n\omega_T)$ gives

$$F(n\omega_T)F^*(n\omega_T) = \left[\frac{1}{T}\int_0^T f(t)\cos n\omega_T t\,dt\right]^2 + \left[\frac{1}{T}\int_0^T f(t)\sin n\omega_T t\,dt\right]^2$$

$$= \left[\frac{1}{(N+\beta)T_0}\int_0^{NT_0} f(t)\cos\frac{n\omega_0 t}{N+\beta}\,dt\right.$$

$$\left. + \frac{1}{(N+\beta)T_0}\int_{NT_0}^{(N+\beta)T_0} f(t)\cos\frac{n\omega_0 t}{N+\beta}\,dt\right]^2$$

$$+\left[\frac{1}{(N+\beta)T_0}\int_0^{NT_0} f(t)\sin\frac{n\omega_0 t}{N+\beta}\,dt\right.$$

$$\left. + \frac{1}{(N+\beta)T_0}\int_{NT_0}^{(N+\beta)T_0} f(t)\sin\frac{n\omega_0 t}{N+\beta}\,dt\right]^2$$

$$F(n\omega_T)F^*(n\omega_T) = \frac{N^2}{(N+\beta)^2} F\left(\frac{n}{N+\beta}\omega_0\right) F^*\left(\frac{n}{N+\beta}\omega_0\right)$$

$$+ \left[\frac{1}{(N+\beta)T_0} \int_{NT_0}^{(N+\beta)T_0} f(t) \cos \frac{n\omega_0 t}{N+\beta}\, dt\right]^2$$

$$+ \left[\frac{1}{(N+\beta)T_0} \int_{NT_0}^{(N+\beta)T_0} f(t) \sin \frac{n\omega_0 t}{N+\beta}\, dt\right]^2$$

$$+ 2\left[\frac{1}{(N+\beta)T_0} \int_{0}^{NT_0} f(t) \cos \frac{n\omega_0 t}{N+\beta}\, dt\right] \cdot$$

$$\cdot \left[\frac{1}{(N+\beta)T_0} \int_{NT_0}^{(N+\beta)T_0} f(t) \cos \frac{n\omega_0 t}{N+\beta}\, dt\right]$$

$$+ 2\left[\frac{1}{(N+\beta)T_0} \int_{0}^{NT_0} f(t) \sin \frac{n\omega_0 t}{N+\beta}\, dt\right]$$

$$\cdot \left[\frac{1}{(N+\beta)T_0} \int_{NT_0}^{(N+\beta)T_0} f(t) \sin \frac{n\omega_0 t}{N+\beta}\, dt\right] .$$

By Schwartz's inequality

$$(I)^2 = \left[\frac{1}{(N+\beta)T_0}\int_{NT_0}^{(N+\beta)T_0} f(t)\,\cos\frac{n\omega_0 t}{N+\beta}\,dt\right]^2$$

$$\leq \left[\frac{1}{(N+\beta)T_0}\int_{NT_0}^{(N+\beta)T_0} [f(t)]^2 dt\right]\cdot\left[\frac{1}{(N+\beta)T_0}\int_{NT_0}^{(N+\beta)T_0}\left[\cos\frac{n\omega_0 t}{N+\beta}\right]^2 dt\right].$$

and

$$(II)^2 = \left[\frac{1}{(N+\beta)T_0}\int_{NT_0}^{(N+\beta)T_0} f(t)\,\sin\frac{n\omega_0 t}{N+\beta}\,dt\right]^2$$

$$\leq \left[\frac{1}{(N+\beta)T_0}\int_{NT_0}^{(N+\beta)T_0} [f(t)]^2 dt\right]\left[\frac{1}{(N+\beta)T_0}\int_{NT_0}^{(N+\beta)T_0}\left[\cos\frac{n\omega_0 t}{N+\beta}\right]^2 dt\right].$$

Thus,

$$(I)^2 + (II)^2 \leq \left[\frac{1}{(N+\beta)T_0}\int_{NT_0}^{(N+\beta)T_0} f^2(t)dt\right]\cdot\left[\frac{1}{(N+\beta)T_0}\int_{NT_0}^{(N+\beta)T_0} dt\right].$$

$$\leq f_M^2\left(\frac{\beta}{N+\beta}\right)^2 .$$

Since

$$\left|\cos\frac{n\omega_0 t}{N+\beta}\right| \leq 1 , \qquad \left|\sin\frac{n\omega_0 t}{N+\beta}\right| \leq 1 ,$$

and

$$|f(t)| \leq f_M ,$$

then

$$\left| 2\left[\frac{1}{(N+\beta)T_0}\int_0^{NT_0} f(t)\cos\frac{n\omega_0 t}{N+\beta}\,dt\right]\cdot\left[\frac{1}{(N+\beta)T_0}\int_{NT_0}^{(N+\beta)T_0} f(t)\cos\frac{n\omega_0 t}{N+\beta}dt\right]\right.$$

$$+\ 2\left[\frac{1}{(N+\beta)T_0}\int_0^{NT_0} f(t)\sin\frac{n\omega_0 t}{N+\beta}\,dt\right]$$

$$\left.\cdot\left[\frac{1}{(N+\beta)T_0}\int_{NT_0}^{(N+\beta)T_0} f(t)\sin\frac{n\omega_0 t}{N+\beta}\,dt\right]\right| \le 4f_M^2\,\frac{\beta N}{(N+\beta)^2}$$

and

$$\left|F(n\omega_T)F^*(n\omega_T) - \left(\frac{N}{N+\beta}\right)^2 F\left(\frac{n}{N+\beta}\omega_0\right)F^*\left(\frac{n}{N+\beta}\omega_0\right)\right|$$

$$\le f_M^2\,\frac{\beta(4N+\beta)}{(N+\beta)^2} \quad .$$

As in the case of Theorem 4, a corollary to Theorem 6 exists. The statement of the corollary follows without proof since the proof parallels that for the corollary to Theorem 4.

Corollary:

Given $f(t)$ periodic with fundamental frequency ω_0 and period T_0 and given $f(t)\epsilon L_2$ for $0 \le t \le T$, then

$$\left|F(n\omega_T)F^*(n\omega_T) - \frac{1}{\gamma^2}\, f\left(\frac{n}{\gamma}\omega_0\right)F^*\left(\frac{n}{\gamma}\omega_0\right)\right| \le f_M^2\,\frac{(5-\gamma)(1-\gamma)}{\gamma^2}$$

where

$$f_M = \underset{0\le t\le T_0}{\mathrm{Max}}\ \{|f(t)|\}\ ,\ T = \gamma_0\ ,$$

and

$$0 < \gamma \leq 1 \quad .$$

For stochastic processes, consider the process $\{Y(t)\}$ containing member functions $y_m(t) = \theta_m(t) + f(t)$ satisfying conditions [A] and [B] as before; and define a sample function $g_m(t)$ as follows.

DEFINITION 6

Let a sample function $g_m(t)$ be constructed for each member function $y_m(t)$ by

$$g_m(t) = y_m(t) - A_{y_m(t)} = \theta_m(t) + f(t) - [A_{\theta_m(t)} + A_{f(t)}]$$

$$= [\theta_m(t) - A_{\theta_m(t)}] + [f(t) - A_{f(t)}]$$

where

$$A_{y_m(t)} = \frac{1}{T}\int_0^T y_m(t)dt$$

$$A_{\theta_m(t)} = \frac{1}{T}\int_0^T \theta_m(t)dt$$

and

$$A_{f(t)} = \frac{1}{T}\int_0^T f(t)dt \quad .$$

The Fourier transform of $g_m(t)$ for $0 \leq t \leq T$ and for $\theta_m(t)\epsilon L_2$ and $f(t)\epsilon L_2$ is given by

$$F_m(n) = \frac{1}{T}\int_0^T g_m(t)e^{in\omega_T T}dt$$

$$= \frac{1}{T}\int_0^T [\theta_m(t) - A_{\theta_m(t)}]e^{in\omega_T t}dt + \frac{1}{T}\int_0^T [f(t) - A_{f(t)}]e^{in\omega_T t}dt$$

where $\omega_T T = 2\pi$; and, similarly, the conjugate transform is given by

$$F_m^*(n) = \frac{1}{T}\int_0^T g_m(t)e^{-in\omega_T t}dt$$

$$= \frac{1}{T}\int_0^T [\theta_m(t) - A_{\theta_m(t)}]e^{-in\omega_T t}dt + \frac{1}{T}\int_0^T [f(t) - A_{f(t)}]e^{-in\omega_T t}dt \; .$$

The expected value of the power spectrum is given by the mathematical expectation of the ensemble of the products of the Fourier transform of each stochastic member function with its conjugate transform—*i.e.*,

$$\begin{aligned}
E[|F(n)|^2] &= E[F_m(n)F_m^*(n)] \\
&= E\left[\left\{\frac{1}{T}\int_0^T \theta_m(t)\cos n\omega_T t dt\right\}^2 - 2\left\{\frac{1}{T}\int_0^T A_{\theta_m(t)}\cos n\omega_T t d\right\}\right. \\
&\quad \left.\cdot\left\{\frac{1}{T}\int_0^T \theta_m(t)\cos n\omega_T t dt\right\} + \left\{\frac{1}{T}\int_0^T A_{\theta_m(t)}\cos n\omega_T t dt\right\}^2\right] \\
&\quad + E\left[\left\{\frac{1}{T}\int_0^T \theta_m(t)\sin n\omega_T t dt\right\}^2 - 2\left\{\frac{1}{T}\int_0^T A_{\theta_m(t)}\sin n\omega_T t dt\right\}\right. \\
&\quad \left.\cdot\left\{\frac{1}{T}\int_0^T \theta_m(t)\sin n\omega_T t dt\right\} + \left\{\frac{1}{T}\int_0^T A_{\theta_m(t)}\sin n\omega_T t dt\right\}^2\right] \\
&\quad + \left\{\frac{1}{T}\int_0^T [f(t) - A_{f(t)}]\cos n\omega_T t dt\right\}^2 + \left\{\frac{1}{T}\int_0^T [f(t) - A_{f(t)}]\sin n\omega_T t dt\right\}^2 \; .
\end{aligned}$$

Since $E[\theta_n(t)] = E[A_{\theta_n(t)}]$, the preceding expression reduces to

$$E[|F(n)|^2] = E\left[\left\{\frac{1}{T}\int_0^T \theta_n(t)\cos n\omega_T t dt\right\}^2\right] - E^2[\theta_n(t)]\left\{\frac{1}{T}\int_0^T \cos n\omega_T t dt\right\}^2$$

$$+ E\left[\left\{\frac{1}{T}\int_0^T \theta_n(t)\sin n\omega_T t dt\right\}^2\right] - E^2[\theta_n(t)]\left\{\frac{1}{T}\int_0^T \sin n\omega_T t dt\right\}^2$$

$$+ \left\{\frac{1}{T}\int_0^T [f(t) - A_{f(t)}]\cos n\omega_T t dt\right\}^2$$

$$+ \left\{\frac{1}{T}\int_0^T [f(t) - A_{f(t)}]\sin n\omega_T t dt\right\}^2 .$$

Next, consider

$$E\left[\left\{\frac{1}{T}\int_0^T \theta_n(t)\cos n\omega_T t dt\right\}^2\right] = E\left[\frac{1}{T^2}\int_0^T\int_0^T \theta_n(t)\theta_n(\tau)[\cos n\omega_T t][\cos n\omega_T \tau]dt d\tau\right]$$

$$= E[\theta_n^2(t)] \cdot \left\{\frac{1}{T}\int_0^T \cos n\omega_T t dt\right\}^2$$

and, similarly,

$$E\left[\left\{\frac{1}{T}\int_0^T \theta_n(t)\sin n\omega_T t dt\right\}^2\right] = E[\theta_n^2(t)] \cdot \left\{\frac{1}{T}\int_0^T \sin n\omega_T t dt\right\}^2 .$$

Thus,

$$E[|F(n)|^2] = \left[\left\{\frac{1}{T}\int_0^T \cos n\omega_T t dt\right\}^2 + \left\{\frac{1}{T}\int_0^T \sin n\omega_T t dt\right\}^2\right] \cdot \left[E[\theta^2] - E^2[\theta]\right]$$

$$+ \left\{\frac{1}{T}\int_0^T [f(t) - A_{f(t)}] \cos n\omega_T t dt\right\}^2$$

$$+ \left\{\frac{1}{T}\int_0^T [f(t) - A_{f(t)}] \sin n\omega_T t dt\right\}^2 .$$

Finally, since $\omega_T T = 2\pi$

$$\frac{1}{T}\int_0^T \cos n\omega_T t dt = \begin{cases} 1 & \text{for } n = 0 \\ 0 & \text{for } n \geq 1 \end{cases} .$$

and

$$\frac{1}{T}\int_0^T \sin n\omega_T t dt = 0 \quad \text{for all } n$$

and the following theorem has been proved.

THEOREM 7

For the sample functions of Definition 6, the expected value of the power spectrum is separable into a stochastic component and a deterministic component—*i.e.*,

$$|F(n)|^2 = |F_\theta|^2 + |F_f(n)|^2$$

where

$$|F_\theta|^2 = \begin{cases} E[\theta^2] - E^2[\underline{\theta}] & \text{for } n = 0 \\ 0 & \text{for } n \geq 1 \end{cases}$$

and

$$|F_f(n)|^2 = \left\{\frac{1}{T}\int_0^T [f(t) - A_{f(t)}] \cos n\omega_T t dt\right\}^2 + \left\{\frac{1}{T}\int_0^T [f(t) - A_{f(t)}] \sin n\omega_T t dt\right\}^2 .$$

The result stated in this theorem is to be expected since $f(t)$ and the $\theta_k(t)$ are statistically independent.

In the following discussion, possible extensions for the use of transform methods are considered for the analysis of non-stationary deterministic functions $f(t)\epsilon L_2$. For example, let $f(t)$, $0 \leq t \leq T_0$, be defined by

$$f(t) = \alpha_0 \cos [\omega(t)t] = \alpha_0 \cos\left[\omega_0\left(t + \frac{t^2}{T_0}\right)\right]$$

where

$$\omega(t) = \omega_0\left(1 + \frac{t}{T_0}\right) \quad \text{and} \quad \omega_0 T_0 = 2\pi .$$

Define a transform by

$$F(n) = \frac{1}{T_0}\int_0^{T_0} f(t) e^{in\omega_0\left(t + \frac{t^2}{T_0}\right)} dt = \frac{1}{T_0}\int_0^{T_0} \alpha_0\left[\cos\omega_0\left(t + \frac{t^2}{T_0}\right)\right] e^{in\omega_0\left(t + \frac{t^2}{T_0}\right)} dt$$

and a conjugate transform by

$$F^*(n) = \frac{1}{T_0}\int_0^{T_0} f(t) e^{-in\omega_0\left(t + \frac{t^2}{T_0}\right)} dt = \frac{1}{T_0}\int_0^{T_0} \alpha_0\left[\cos\omega_0\left(t + \frac{t^2}{T_0}\right)\right] e^{-in\omega_0\left(t + \frac{t^2}{T_0}\right)} dt .$$

Substituting

$$\cos\omega_0\left(t + \frac{t^2}{T_0}\right) = \frac{1}{2}\left[e^{-i\omega_0\left(t + \frac{t^2}{T_0}\right)} + e^{i\omega_0\left(t + \frac{t^2}{T_0}\right)}\right]$$

in $F(n)$ and $F^*(n)$ gives

$$F(n) = \frac{\alpha_0}{2}\left[\frac{1}{T_0}\int_0^{T_0} e^{i(n+1)\omega_0\left(t+\frac{t^2}{T_0}\right)}dt + \frac{1}{T_0}\int_0^{T_0} e^{i(n-1)\omega_0\left(t+\frac{t^2}{T_0}\right)}dt\right]$$

and

$$F^*(n) = \frac{\alpha_0}{2}\left[\frac{1}{T_0}\int_0^{T_0} e^{-i(n+1)\omega_0\left(t+\frac{t^2}{T_0}\right)}dt + \frac{1}{T_0}\int_0^{T_0} e^{-i(n-1)\omega_0\left(t+\frac{t^2}{T_0}\right)}dt\right].$$

Completing the square in each exponent,

$$\frac{t^2}{T_0} + t = \frac{t^2}{T_0} + t + \frac{T_0}{4} - \frac{T_0}{4} = \left(\frac{t}{\sqrt{T_0}} + \frac{\sqrt{T_0}}{2}\right)^2 - \frac{T_0}{4}$$

then

$$e^{\pm i(n+1)\omega_0\left(t+\frac{t^2}{T_0}\right)} = e^{\pm i(n+1)\omega_0\left(\frac{t}{\sqrt{T_0}}+\frac{\sqrt{T_0}}{2}\right)^2} e^{\pm i(n+1)\omega_0\frac{T_0}{4}}$$

$$e^{\pm i(n-1)\omega_0\left(t+\frac{t^2}{T_0}\right)} = e^{\pm i(n-1)\omega_0\left(\frac{t}{\sqrt{T_0}}+\frac{\sqrt{T_0}}{2}\right)^2} e^{\pm i(n-1)\omega_0\frac{T_0}{4}}$$

$$F(n) = \frac{\alpha_0}{2}\left[\frac{e^{-i(n+1)\pi/2}}{\sqrt{2\pi(n+1)}}\int_a^b e^{i\theta^2}d\theta + \frac{e^{-i(n-1)\pi/2}}{\sqrt{2\pi(n-1)}}\int_c^d e^{i\theta^2}d\theta\right]$$

and

$$F^*(n) = \frac{\alpha_0}{2}\left[\frac{e^{i(n+1)\pi/2}}{\sqrt{2\pi(n+1)}}\int_a^b e^{-i\theta^2}d\theta + \frac{e^{i(n-1)\pi/2}}{\sqrt{2\pi(n-1)}}\int_c^d e^{-i\theta^2}d\theta\right]$$

where

$$a = \frac{1}{2}\sqrt{2\pi(n+1)}$$

$$b = \frac{3}{2}\sqrt{2\pi(n+1)}$$

$$c = \frac{1}{2}\sqrt{2\pi(n-1)}$$

and

$$d = \frac{3}{2}\sqrt{2\pi(n-1)} \quad .$$

Thus,

$$F(n)F^*(n) = \frac{\alpha_0^2}{4}\left\{\frac{1}{2\pi(n+1)}\int_a^b [\cos\theta^2 + i\sin\theta^2]d\theta \int_a^b [\cos\theta^2 - i\sin\theta^2]d\theta\right.$$

$$+ \frac{e^{-i\pi}}{2\pi\sqrt{n^2-1}}\int_a^b e^{i\theta^2}d\theta\int_c^d e^{-i\theta^2}d\theta + \frac{e^{i\pi}}{2\pi\sqrt{n^2-1}}\int_a^b e^{-i\theta^2}d\theta\int_c^d e^{i\theta^2}d\theta$$

$$\left. + \frac{1}{2\pi(n-1)}\int_c^d [\cos\theta^2 + i\sin\theta^2]d\theta\int_c^d [\cos\theta^2 - i\sin\theta^2]d\theta\right\} .$$

Finally, it can be shown that

$$F(n)F^*(n) = \frac{\alpha_0^2}{8\pi}\left\{\left[\frac{1}{\sqrt{n+1}}\int_a^b \cos\theta^2 d\theta - \frac{1}{\sqrt{n-1}}\int_c^d \cos\theta^2 d\theta\right]^2\right.$$

$$\left. + \left[\frac{1}{\sqrt{n+1}}\int_a^b \sin\theta^2 d\theta - \frac{1}{\sqrt{n-1}}\int_c^d \sin\theta^2 d\theta\right]^2\right\}, \quad n \geq 1 \quad .$$

That is, this extension to a transform permits the calculation of $|F(n)|^2$ for the function $f(t) = \alpha_0 \cos \omega_0[(t + t^2/T_0)]$ in terms of Fresnel integrals.†

To see that $F(n)$ may be interpreted in terms of a Fourier transform consider

$$F(\omega) = \frac{1}{T}\int_0^T f(t)e^{in\omega_T\left(t+\frac{\omega_1}{n\omega_T}t^2\right)}$$

where ω_1 is a constant. Then, we may write

$$F(\omega) = \frac{1}{T}\int_0^T f(t)e^{i\omega_1 t^2} e^{in\omega_T t}\,dt = \frac{1}{T}\int_0^T f(t)[\cos \omega_1 t^2 + i \sin \omega_1 t^2]e^{in\omega_T t}dt \ .$$

By setting

$$x(t) = f(t) \cos \omega_1 t^2$$

$$y(t) = f(t) \sin \omega_1 t^2$$

and

$$z(t) = x(t) + iy(t) \quad ,$$

then

$$F(\omega) = \frac{1}{T}\int_0^T z(t)e^{in\omega_T t}\,dt$$

and $F(\omega)$ is the Fourier transform of the complex function[2] $z(t)$.

The preceding analysis suggests considering finite transformations of the form

$$F(\omega) = \frac{1}{T}\int_0^T f(t)e^{in\omega_0 P(t)}\,dt$$

†For $n = 1$, $c = d$ and two of the integrals in $F(1)F^*(1)$ are zero. For $n = 0$, $F(0)F^*(0) = \left(\frac{1}{T_0}\int_0^{T_0} f(t)dt\right)^2$.

where

$$P(t) = t + \sum_{p=2}^{P} \frac{\omega_p}{n\omega_0} t^p .$$

The ω_p are constants to be adjusted to find a "best fit" with non-stationary functions $f(t)$, $0 \leq t \leq T$, for a series similar to the Fourier series. As a first step in this direction, the function $F(n)$ is shown to be the Fourier transform of the complex function $z(t) = x(t) + iy(t)$ where

$$z(t) = f(t) \exp\left[i \sum_{p=2}^{P} \omega_p t^p\right] = f(t)\left[\cos\left(\sum_{p=2}^{P} \omega_p t^p\right) + \sin\left(\sum_{p=2}^{P} \omega_p t^p\right)\right]$$

$$= x(t) + iy(t) .$$

For $f(t) \epsilon L_2$, $0 \leq t \leq T$, the Fourier transform is represented by

$$F(\omega) = \frac{1}{T}\int z(t) e^{in\omega_T t} \, dt$$

and a sufficient condition for the existence[3] of $F(\omega)$ is that

$$\frac{1}{T}\int_0^T |z(t)| dt < \infty .$$

Consider the modulus of $z(t)$,

$$|z(t)| = \left| \left\{ f^2(t) \left[\cos\left(\sum_{p=2}^{P} \omega_p t^p\right) + i \sin\left(\sum_{p=2}^{P} \omega_p t^p\right)\right] \left[\cos\left(\sum_{p=2}^{P} \omega_p t^p\right) - i \sin\left(\sum_{p=2}^{P} \omega_p t^p\right)\right] \right\}^{1/2} \right| = |f(t)| .$$

Thus, since $f(t) \epsilon L_2$, $|z(t)| \epsilon L_2$ and the transform exists. The existence of $F^*(\omega)$ is demonstrated in a similar fashion.[3]

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3. For proof see N. Wiener, *The Fourier Integral and Certain of Its Applications*, Dover Publications, Inc., New York, N.Y. (1933), Chapter 1.

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