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STEP LOAD MOVING WITH LOW SUBSEISMIC VELOCITY ON THE SURFACE OF A HALF-SPACE OF GRANULAR MATERIAL

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ABSTRACT

The plane strain problem of a step load, p, moving on the surface of a half-space of granular material governed by internal friction and cohesion is being considered because of its importance to Air Force research in protective construction. As a preliminary step towards a more complete solution, in this paper it is assumed that in regions of slip the elastic deformations may be neglected in favor of those resulting from slip. The latter assumption limits the application to Mach numbers of less than about 0.20.

Two possible types of behavior of the material are considered. During slip, one material exhibits dilatancy, while the other does not change in volume. Because of neglect of the elastic deformations in the slip region, only degenerate results are obtained for the case of dilatancy; while stresses can be determined, deformations, velocities and accelerations vanish. For the other material, all desired quantities are obtained.

A significant finding is that for values of p above a certain limit, granular particles will be expelled at the surface ahead of the pressure front. The applied pulse will be preceded by a precursor of expelled grains.

PUBLICATION REVIEW

This report has been reviewed and is approved.

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Note re: Numbering of Equations.

Throughout this report, equations are numbered by hyphenated numbers, such as Eq. (2-7), (3-18), or (B-14). The first number, 2, 3, B, respectively, indicates that the equation occurs in Section 2, 3, or in Appendix B. When an equation is referred to, the full number, say (2-7), is quoted if the reference occurs in a different section. However, if an equation is mentioned in the section in which it originally appears, only the second half of the number is quoted: In Section 2, Eq. (2-7) would, therefore, be quoted simply as Eq. (7), etc.

1. INTRODUCTION

The present paper is the first step toward the determination of the effects of a pressure distribution p (x-Vt) moving with a velocity V on the surface of a half-space consisting of a granular material, (Fig. 1). Because of the complexity of the matter only the steady-state situation is being studied, i. e., the effects of initial (starting) conditions are excluded.

The equivalent steady-state problem for an elastic half-space has been treated by Cole and Huth, [Ref. 1], for the case of a moving line load $p\ (x-Vt) = \delta\ (x-Vt)$, where δ is Dirac's delta function. Using the principle of superposition, the effect of an arbitrary load p (x-Vt) can be found by integrations. Considering the behavior of a granular material to be governed by internal friction of the Coulomb type and possibly cohesion, the elastic solution can be applied up to a certain intensity of the loads p; beyond this critical intensity, which depends on the velocity V and the distribution of p, internal slip must occur in the material requiring a different analysis which is to be developed in this paper.

For a complete understanding of all possible situations, the elastic properties of the material and the slipping effects must be taken into account simultaneously. However, the problem becomes very involved, and it appears advantageous to treat at first the simple case of a step load, (Fig. 2), for an elastically extremely rigid material, such that the elastic deformations can be assumed to vanish. In spite of the serious implications of this assumption, which limits the application of the theory to low Mach numbers, the results obtained permit a physical understanding of a peculiarity of the Coulomb type material in dynamic situations: When the pressure wave exceeds a certain intensity, a precursor occurs, consisting of particles expelled from the surface ahead of the applied pressure wave.

The equivalent of the dynamic problem treated here is the static one of determining the limiting load p which may act on one half of the surface of a half-space, (Fig. 3a). This problem is a special case of one treated by Prandtl, [Ref. 2], who considers a load p of finite length l, (Fig. 3b). The present treatment for the limiting case $V \to o$ overlaps Prandtl's problem for $l \to \infty$.

In order to formulate equations of motion, the "yield" condition between the stresses defining the occurrence of slip must be supplemented by a flow rule defining the strain rates. Assuming the material to be isotropic, the principal axes of the stress tensor and the strain-rate tensor must necessarily coincide everywhere. However, this statement does not suffice and further relations are required. In accordance with the theory of plasticity, Drucker and Prager, [Ref. 3], have formulated such relations by postulating the existence of a plastic potential. The condition

$$f = aJ_1 + \sqrt{J_2} - c = 0 \qquad \text{(I-1)}$$

in terms of the invariants $J_1 = \sigma_1 + \sigma_2 + \sigma_3$, $J_2 = \frac{1}{6}[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2]$ is the generalization of the Coulomb yield condition for a three-dimensional state of stress, where σ_1, σ_2, σ_3 are the principal stresses and where a and c are appropriate constants connected with the values of the slip angle and cohesion, [Ref. 3]. The concept of the plastic potential then defines the plastic strain rates $\dot{\epsilon}_{ij}$ as derivatives of the yield function f with respect to the stresses σ_{ij}:

$$\dot{\epsilon}_{ij} = \lambda \frac{\partial f}{\partial \sigma_{ij}} \qquad \text{(I-2)}$$

where λ is an arbitrary positive quantity which may be a function of the location. For the case of plane strain, of interest here, the resulting stress-strain rate relations are given in [Ref. 4]. It is noted that the use of the yield condition f as plastic potential requires that slip be accompanied by a volume expansion.

As an alternative to the ideal isotropic material defined by [3], one can define the behavior of a material in plane strain completely by supplementing the Coulomb yield condition, and the requirement that the axes of the stress tensor and strain rate tensors agree, by the statement that the plastic deformations are incompressible. This assumption has been utilized by Ishlinski, [Ref. 5], to formulate equations of motion for plane problems and for problems of spherical symmetry. The assumption of incompressibility which appears not unreasonable - or physically impossible - is incompatible with the formulation by Drucker and Prager. To clarify the fundamental difference between the two materials, one can generalize the "assumed" incompressibility in plane strain to a three dimensional statement. For an isotropic material any condition may be only in terms of invariants; the appropriate flow rule defines the strain rates again as derivatives of a plastic potential, but the function to be used is $f_1 = F(J_2)$, where F may be an arbitrary function. As the result does not depend on the form of the function F, its choice only affects the value of the arbitrary factor λ, one may, without loss of generality, use $f_1 = J_2$. The combination of Eq. (1) and

$$\dot{\epsilon}_{ij} = \lambda \frac{\partial f_1}{\partial \sigma_{ij}} \equiv \lambda \frac{\partial J_2}{\partial \sigma_{ij}} \qquad \text{(I-3)}$$

defines the three-dimensional behavior of the material postulated in [5].

The difference in the formulations of [3] and [5] lies in the fact that [3] assumed the function f in the yield condition to be the plastic potential. It is conventional to use the yield function as plastic potential, but the use of another function of the invariants will not be ruled out here, in spite of certain thermodynamic difficulties resulting from the use of different functions*) in the yield condition and in the flow rule.

Without attempting to decide on the physical applicability of the two concepts, the following paper will pursue both possibilities. It will be seen that the solutions of dynamical problems for the dilating material (Drucker-Prager) become degenerate if the elastic deformations are ignored, such that the alternative case of the "incompressible" material gives more interesting results and its analysis will be emphasized.

A comment on steady-state solutions of the type considered here is in order. Due to the fact that the manner in which the loads at large distances from the front have been applied at early times is not included in the statement of the problem, there can be no uniqueness or existence theorems, and there may be more than one solution for a given case. In the elastic problem [1], there is no steady-state solution if the velocity V equals that of Rayleigh waves; for other velocities

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*) An elementary consequence of the two different assumptions may be of interest. In the non-dilating material, Eq. (3), in the case of plane strain the transverse stress σ_3 is equal to the mean of the other two principal stresses

$$\sigma_3 = \frac{\sigma_1 + \sigma_2}{2}$$

For the dilating material, on the other hand, the absolute value of the transverse stress is necessarily larger than the mean,

$$\left|\sigma_3\right| > \left|\frac{\sigma_1 + \sigma_2}{2}\right|$$

the excess depending on the state of stress and on the material constants.

the solutions still contain arbitrary constants. The horizontal stress σ_x contains, e.g., an open constant, equivalent to the superposition of a uniform state of horizontal stress; this indeterminacy can not be removed. In some cases, solutions can be excluded by additional general considerations of the starting situation one has in mind. This is best seen on the elementary example of a half-space of an inviscid compressible fluid, loaded by a pressure pulse p progressing with supersonic velocity V. There is an obvious solution, (Fig. 4a), where the load produces a plane wave of intensity p, progressing with a front inclined at an appropriate angle α. However, this is not the only steady-state solution; an alternative is a plane wave whose front is inclined at an angle of $180^\circ - \alpha$. Combinations of the two are also correct steady-state solutions. If one is interested in states generated by the application of pressures on the surface only, one can reason that solutions including the wave front shown in (Fig. 4b) can not occur, such that a unique solution is obtained.

Similar simple considerations do not exist in subsonic situations, leaving one even in the relatively simple elastic case with non-unique solutions. It is, therefore, to be expected that non-uniqueness may also occur in the case of the material considered in this paper.

2. BEHAVIOR OF MATERIAL

It will be found convenient to formulate the equations describing the behavior of the material in an unusual manner, using the major principal stress and its direction as independent variables.

In accordance with the usual conventions, positive stresses and positive strain rates indicate tension and elongation, respectively. The inequalities required to describe the material behavior will be on the quantities $-\sigma_1$, $-\sigma_2$, etc. As the material to be considered requires that at least the major principal stress σ_1 is compression, it follows that the quantities at each side of the inequalities are positive numbers.

Two types of material will be considered. Both are assumed to permit no elastic deformations at all; any deformations which occur are caused by slip. Whenever such slip occurs, the Coulomb condition between the stresses must be satisfied.

CASE A. Incompressible Material.

This material agrees with the one assumed in [5] and is described by the following assumptions:

1. The material is incompressible, i. e., in addition to assuming that the elastic deformations vanish, there is no volume change during slip.
2. The state of stress which may exist in any element is restricted, such that at least one of the two principal stresses, (Fig. 5), must be compressive; let

$$-\sigma_1 \geq -\sigma_2 \qquad (2\text{-}1)$$

and the restriction requires

$$-\sigma_1 > 0; \tag{2-2}$$

σ_1 will be referred to as the "major" principal stress.

3. The permissible states of stress are further restricted by the condition:

$$-\sigma_2 \geq k\sigma \tag{2-3}$$

where k is a positive number, k < 1, which defines the angle of interior friction and the variable σ is defined by

$$\sigma = -(\sigma_1 + s) \tag{2-4}$$

In this relation s $\geq$ 0 is a material constant defining the amount of cohesion. The requirement $\sigma_1 \leq 0$ leads to the condition $\sigma \geq -s$; if, on physical grounds, one wishes to insure that σ_2 is also compressive, $\sigma > 0$ could be specified.

4. If

$$-\sigma_2 > k\sigma \tag{2-5}$$

the element will be rigid, and all strains vanish. If, however,

$$-\sigma_2 = k\sigma \tag{2-6}$$

the element, while incompressible, may deform but only in such fashion that the principal axes of the strain rate tensor $\dot{\varepsilon}_1$, $\dot{\varepsilon}_2$, are parallel to σ_1, σ_2, respectively. In addition, the strain rate $\dot{\varepsilon}_1$ must be negative or vanish:

$$\dot{\varepsilon}_1 \geq 0 \tag{2-7}$$

Due to the assumed incompressibility

$$\dot{\varepsilon}_1 + \dot{\varepsilon}_2 = 0 \tag{2-8}$$

such that Eq. (7)* assures dissipation of energy during deformation.

*) See note on page ii.

The statements 1. to 4. express the situation in a material with a coefficient f of internal Coulomb friction and cohesion. The relation between k and f can be found in the following manner: Consider a rectangular element, (Fig. 5), under the action of the principal stresses σ_1 and σ_2. If a plane making an arbitrary angle α with the direction of σ_1 is drawn, the normal and tangential stresses, σ_α and τ_α, respectively, are:

$$\sigma_\alpha = \sigma_1 \cos^2\alpha + \sigma_2 \sin^2\alpha$$

$$\tau_\alpha = (\sigma_1 - \sigma_2)\sin\alpha\cos\alpha \qquad (2\text{-}9)$$

No slip can occur if for all values of α the direct pressure multiplied by the friction coefficient, $-f\sigma_\alpha$, is larger than the excess of the shear stress $|\tau_\alpha|$ over a value $|\tau_s|$ which defines cohesion, or

$$\frac{|\tau_\alpha| - |\tau_s|}{-\sigma_\alpha} < f$$

The condition of slip will be reached if the largest value of the above ratio as function of α just reaches f, i.e., when

$$\max \frac{|\tau_\alpha| - |\tau_s|}{-\sigma_\alpha} = f \qquad (2\text{-}10)$$

Substitution of σ_1 and σ_2 from Eqs. (4) and (6) into Eqs. (9) gives after rearrangement

$$\sigma_\alpha = -\sigma\frac{1+k}{2} + \sigma\frac{1-k}{2}\cos 2\alpha - s\cos^2\alpha$$

$$|\tau_\alpha| = \sigma\frac{1-k}{2}\sin 2\alpha + \frac{s}{2}\sin 2\alpha \qquad (2\text{-}11)$$

To find the angle α for which the maximum of the fraction in Eq. (10) occurs, Eqs. (11) are substituted, and differentiation with respect to α is performed. After cancellation of common factors, one finds a condition for α:

$$\left[\sigma \frac{1+k}{2} + \sigma \frac{1-k}{2} \cos 2\alpha + s \cos^2\alpha\right] \cos 2\alpha +$$

$$+ \left[\sigma \frac{1-k}{2} \sin 2\alpha + \frac{s}{2} \sin 2\alpha - |\tau_s|\right] \sin 2\alpha = 0 \qquad (2\text{-}12)$$

Noting from Eq. (11) that the expressions in parentheses are, respectively, the numerator and denominator of Eq. (10), one obtains f in terms of the as yet unknown angle α:

$$f = -\frac{\cos 2\alpha}{\sin 2\alpha} \qquad (2\text{-}13)$$

Rearranging Eq. (12) to separate terms which contain σ and those which do not, gives after simplification

$$\sigma\left[\frac{1+k}{2} \cos 2\alpha + \frac{1-k}{2}\right] + \cos \alpha \left[s \cos \alpha - 2|\tau_s| \sin \alpha\right] = 0 \qquad (2\text{-}14)$$

This relation must hold regardless of the value of σ, requiring that the coefficient of σ and the second term each vanish individually. The vanishing of the coefficient of σ gives a condition on α:

$$\cos 2\alpha = -\frac{1-k}{1+k} \qquad (2\text{-}15)$$

From Eq. (13)

$$f = -\frac{\cos 2\alpha}{\sqrt{1-\cos^2 2\alpha}} = \frac{1-k}{2\sqrt{k}} \qquad (2\text{-}16)$$

The vanishing of the second term in Eq. (14) relates finally

$|\tau_s|$ and s:

$$|\tau_s| = s\frac{\cot\alpha}{2} = \frac{s\sqrt{k}}{2} \qquad (2\text{-}17)$$

where the identity $\cot^2\alpha = \frac{1 + \cos 2\alpha}{1 - \cos 2\alpha}$ was used.

CASE B. Material with Dilatancy

It is again assumed that the elastic deformations of the material vanish, but in accordance with the assumptions in [3] there will be an increase in volume associated with slip.

Retaining Eqs. (3) and (4), the material will remain rigid if the inequality (5) applies, and may slip if Eq. (6) is satisfied. In this case, the direction of the principal strain rates $\dot{\varepsilon}_1$ and $\dot{\varepsilon}_2$ coincide with those of the principal stresses σ_1, σ_2, respectively. The values of $\dot{\varepsilon}_1$, $\dot{\varepsilon}_2$ required by the assumptions in [3] for the case of plane strain can be found conveniently from [Ref. 4, Eq. 6] by letting the x, y-axes coincide with the principal ones, and noting that sin ϕ in the reference has the value $\frac{1 - k}{1 + k}$ in the symbols employed here:

$$\dot{\varepsilon}_1 = \frac{\lambda}{2}\left[\frac{1-k}{1+k} - 1\right]$$

$$\dot{\varepsilon}_2 = \frac{\lambda}{2}\left[\frac{1-k}{1+k} + 1\right] \qquad (2\text{-}18)$$

where, similarly to Eq. (I-2), λ denotes an arbitrary, location dependent, positive quantity.

3. SOLUTIONS IN REGIONS WITH, AND WITHOUT SLIP, RESPECTIVELY.

As a preliminary to the construction of solutions satisfying the boundary conditions on the surface, it is convenient to consider, respectively, solutions in regions where either Eq. (2-5) or Eq. (2-6) is satisfied, i.e., regions without slip, and with slip, respectively.

In areas where Eq. (2-6) is satisfied, the equations of motion in Cartesian coordinates x, y with respect to the fixed origin O, (Fig. 6), are:

$$\begin{aligned} \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau}{\partial y} &= \rho\left[\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right] \\ \frac{\partial \tau}{\partial x} + \frac{\partial \sigma_y}{\partial y} &= \rho\left[\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right] \end{aligned} \qquad (3\text{-}1)$$

where u, v are, respectively, the x and y components of the velocity, and σ_x, σ_y and τ are the stresses.

Consider an element, (Fig. 5), in which the major principal stress is inclined at an angle θ to the horizontal. In the condition of slip, Eqs. (2-4) and (2-6) give

$$\begin{aligned} \sigma_1 &= -\sigma - s \\ \sigma_2 &= -k\sigma \end{aligned} \qquad (3\text{-}2)$$

where k and s are positive quantities. Expressing the stress components in terms of the principal stresses and of the angle θ, one obtains after substitution of (2)

$$\begin{aligned} \sigma_x &= -\sigma\left[\frac{1+k}{2} + \frac{1-k}{2}\cos 2\theta\right] - s\cos^2\theta \\ \sigma_y &= -\sigma\left[\frac{1+k}{2} - \frac{1-k}{2}\cos 2\theta\right] - s\sin^2\theta \\ \tau &= -\sigma\left[\frac{1-k}{2}\right]\sin 2\theta - \frac{s}{2}\sin 2\theta \end{aligned} \qquad (3\text{-}3)$$

From this point on, the two types of material described in Section 2 require separate treatment.

A. Region with Slip in the Incompressible Material

As stated in Section 2, the principal strain rates $\dot{\varepsilon}_1$ and $\dot{\varepsilon}_2$ are parallel to σ_1 and σ_2; further, from Eq. (2-7), $\dot{\varepsilon}_1 \leq 0$. Introducing a new variable, the positive quantity $\dot{e}$, defined by

$$\dot{e} \equiv -\dot{\varepsilon}_1 \tag{3-4}$$

the condition (2-8), expressing incompressibility, requires

$$\dot{\varepsilon}_2 = \dot{e} \tag{3-5}$$

The components of the strain rate tensor, $\dot{\varepsilon}_{xx}$, $\dot{\varepsilon}_{yy}$, $\dot{\varepsilon}_{xy}$, can now be expressed, alternatively, in terms of $\dot{e}$ and θ, or in terms of u and v. This leads to the following relations:

$$\dot{\varepsilon}_{xx} = -\dot{e} \cos 2\theta = \frac{\partial u}{\partial x} \tag{3-6a}$$

$$\dot{\varepsilon}_{yy} = +\dot{e} \cos 2\theta = \frac{\partial v}{\partial y} \tag{3-6b}$$

$$\dot{\varepsilon}_{xy} = -\dot{e} \sin 2\theta = \frac{1}{2}\left[\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right] \tag{3-6c}$$

At this point, use is made of dimensional considerations and of the fact that only steady-state solutions are to be determined. Stresses, velocities, strain rates, etc., can be expressed in the general form:

$$p^{n_1} \quad y^{n_2} \quad \rho^{n_3} \quad f\left[\frac{x-Vt}{y}, \frac{p}{s}, \frac{p}{\rho V^2}\right] \tag{3-7}$$

where f means a function of the variables stated.

If the ratios $\frac{p}{s}$ and $\frac{p}{\rho V^2}$ are considered as parameters defining the particular physical problem, the exponents n_k are uniquely defined by the dimension of the expressed quantity. Therefore:

$$\begin{aligned} \sigma,\ \sigma_x,\ \sigma_y,\ \tau \ldots &= p\ f(\xi) \\ Vu,\ Vv \ldots &= \frac{p}{\rho}\ f(\xi) \\ V\dot{\theta} &= \frac{p}{\rho}\,\frac{1}{y}\ f(\xi) \\ \theta &= f(\xi) \end{aligned} \tag{3-8}$$

where

$$\xi = \frac{x - Vt}{y} \tag{3-8a}$$

and the functions f differ, of course, for different quantities. The derivatives of $f \equiv f(\frac{x-Vt}{y}) \equiv f(\xi)$ with respect to x, y and t are

$$\begin{aligned} \frac{\partial f}{\partial x} &= \frac{1}{y}\frac{df}{d\xi} \\ \frac{\partial f}{\partial y} &= -\frac{\xi}{y}\frac{df}{d\xi} \\ \frac{\partial f}{\partial t} &= -\frac{V}{y}\frac{df}{d\xi} \end{aligned} \tag{3-9}$$

Applying these relations to the derivatives in Eqs. (1) and (6) gives

$$\begin{aligned} \frac{d\sigma_x}{d\xi} - \xi\frac{d\tau}{d\xi} &= -\rho V\frac{du}{d\xi}\left[1 - \frac{u}{V} + \xi\frac{v}{V}\right] \\ \frac{d\tau}{d\xi} - \xi\frac{d\sigma_y}{d\xi} &= -\rho V\frac{dv}{d\xi}\left[1 - \frac{u}{V} + \xi\frac{v}{V}\right] \end{aligned} \tag{3-10}$$

and

$$\frac{du}{d\xi} = -y\dot{e} \cos 2\theta$$

$$\xi \frac{dv}{d\xi} = -y\dot{e} \cos 2\theta \qquad (3\text{-}11)$$

$$\frac{dv}{d\xi} - \xi \frac{du}{d\xi} = -2y\dot{e} \sin 2\theta$$

The term $\dot{e}$ being a function of ξ which might vanish identically, the two situations $\dot{e} \neq 0$ and $\dot{e} = 0$ require separate consideration.

Case 1, $\dot{e} \neq 0$

Elimination of $\frac{du}{d\xi}$ and $\frac{dv}{d\xi}$ in Eqs. (11) furnishes the relation:

$$(\xi - \frac{1}{\xi}) \cos 2\theta + 2 \sin 2\theta = 0 \qquad (3\text{-}12)$$

The variable ξ, defined by Eq. (8a), can be expressed by the angle ϕ, (Fig. 6), defining the position of the element with respect to the moving front:

$$\xi = \cot \phi \qquad (3\text{-}13)$$

Substitution into Eq. (12) gives two possible values for the direction of the major principal axis

$$\theta = \phi \pm \frac{\pi}{4} \qquad (3\text{-}14)$$

To obtain the stresses, one finds from the first two Eqs. (11)

$$\frac{du}{d\xi} - \xi \frac{dv}{d\xi} = 0 \qquad (3\text{-}15)$$

which permits elimination of all velocity terms from Eqs. (10).

Changing the independent variable in the resulting differential equations from ξ to ϕ, one obtains

$$\left[\frac{d\sigma_x}{d\phi} - 2\cot\phi \frac{d\tau}{d\phi} + \cot^2\phi \frac{d\sigma_y}{d\phi}\right] \frac{d\phi}{d\xi} = 0 \qquad (3\text{-}16)$$

Excluding the points $\phi = 0, \pi$ where $\frac{d\phi}{d\xi}$ vanishes, substitution of Eqs. (3) furnishes a first order differential equation for the quantity σ:

$$\frac{d\sigma}{d\phi} \pm 2\,\frac{1-k}{1+k}\,\sigma \pm \frac{2s}{1+k} = 0 \qquad [\phi \neq 0, \pi] \qquad (3\text{-}17)$$

Its solution is

$$\sigma = Ce^{\mp 2\frac{1-k}{1+k}\phi} - \frac{s}{1-k} \qquad [\phi \neq 0, \pi] \qquad (3\text{-}18)$$

The upper or lower signs are to be selected in the same manner as in Eq. (14), while C is an arbitrary constant.

The accelerations a_x and a_y, and velocities u and v can be obtained from Eqs.(1) and (10). The right hand sides of Eqs. (1) are ρa_x and ρa_y, respectively, and one recognizes therefore that the right hand sides of Eqs. (10) are $\rho y a_x$ and $\rho y a_y$, respectively. Substitution of Eqs. (3) and (14) into Eqs. (10) gives therefore

$$\begin{aligned} a_x &= \frac{1-k}{2\rho y}\cos\phi\,\sin\phi\,\frac{d\sigma}{d\phi} = \frac{1-k}{2\rho\sqrt{x^2+y^2}}\cos\phi\,\frac{d\sigma}{d\phi} \\ a_y &= \frac{1-k}{2\rho y}\sin^2\phi\,\frac{d\sigma}{d\phi} = \frac{1-k}{2\rho\sqrt{x^2+y^2}}\sin\phi\,\frac{d\sigma}{d\phi} \end{aligned} \qquad (3\text{-}19)$$

which are the components of a radial acceleration

$$a = \frac{1-k}{2\rho\sqrt{x^2+y^2}}\,\frac{d\sigma}{d\phi} \qquad (3\text{-}20)$$

Eqs. (10) also furnish simple differential equations for u and v

$$\begin{aligned} \rho V\,\frac{du}{d\phi}\left(1 - \frac{u}{V} + \frac{v}{V}\cot\phi\right) &= \frac{1-k}{2}\cot\phi\,\frac{d\sigma}{d\phi} \\ \rho V\,\frac{dv}{d\phi}\left(1 - \frac{u}{V} + \frac{v}{V}\cot\phi\right) &= \frac{1-k}{2}\,\frac{d\sigma}{d\phi} \end{aligned} \qquad (3\text{-}21)$$

For any wedge shaped region, ϕ_o to ϕ_1, these equations define the velocities in terms of those at one side, say ϕ_o. The fact that the acceleration is purely radial suggests introduction of polar velocities u_r and u_ϕ. One then finds two separate equations

$$u_r = - \frac{du_\phi}{d\phi}$$

$$\left[\frac{d^2 u_\phi}{d\phi^2} + u_\phi\right] (V \sin \phi + u_\phi) = - \frac{1-k}{2\rho} \frac{d\tau}{d\phi} \tag{3-22}$$

The fact that the term $(V \sin \phi + u_\phi)$ may vanish indicates two distinct ranges of solutions. If the velocities u and v are small, $\frac{u}{V} \ll 1$ and $\frac{v}{V} \ll 1$, Eqs. (21) may be solved directly:

$$u(\phi) = \frac{1-k}{2\rho V} \int_{\phi_o}^{\phi} \cot \phi \; \frac{d\sigma}{d\phi} \; d\phi + u(\phi_o)$$

$$v(\phi) = \frac{1-k}{2\rho V} [\sigma(\phi) - \sigma(\phi_o)] + v(\phi_o) \tag{3-23}$$

Case 2, $\dot{e} = 0$.

If the conditions (2) concerning the stresses are satisfied, but $\dot{e} = 0$, the region is on the verge of slip. In this case, Eqs. (11) give $\frac{du}{d\xi} = \frac{dv}{d\xi} = 0$, and the right hand sides of Eqs. (10) vanish, leaving the equations of static equilibrium

$$\frac{d\sigma_x}{d\xi} - \xi \frac{d\tau}{d\xi} = 0$$

$$\frac{d\tau}{d\xi} - \xi \frac{d\sigma_y}{d\xi} = 0 \tag{3-24}$$

Substitution of Eqs. (3) furnishes two non-linear simultaneous differential equations for σ and θ

$$\left[\frac{1-k}{2}(\xi \sin 2\theta - \cos 2\theta) - \frac{1+k}{2}\right] \frac{d\sigma}{d\xi} + [\xi \cos 2\theta + \sin 2\theta][\sigma(1-k) + s] \frac{d\theta}{d\xi} = 0$$

$$\left[\frac{1-k}{2}(\sin 2\theta + \xi \cos 2\theta) - \xi \frac{1+k}{2}\right]\frac{d\sigma}{d\xi} + \left[\cos 2\theta - \xi \sin 2\theta\right]\left[\sigma(1-k) + s\right] \frac{d\theta}{d\xi} = 0 \tag{3-25}$$

These equations can be separated, giving

$$\frac{d\sigma}{d\xi} = 0$$

$$[\sigma(1-k) + s] \frac{d\theta}{d\xi} = 0 \tag{3-26}$$

unless the determinant

$$\begin{vmatrix} \left[\frac{1-k}{2}(\xi \sin 2\theta - \cos 2\theta) - \frac{1+k}{2}\right] & \left[\xi \cos 2\theta + \sin 2\theta\right] \\ \left[\frac{1-k}{2}(\sin 2\theta + \xi \cos 2\theta) - \xi \frac{1+k}{2}\right] & \left[\cos 2\theta - \xi \sin 2\theta\right] \end{vmatrix} \tag{3-27}$$

vanishes.

If the determinant does not vanish, the second Eq. (26) gives $\frac{d\theta}{d\xi} = 0$, because by definition the term $[\sigma(1-k) + s] > 0$. One obtains therefore the obvious, yet important, solution:*)

$$\sigma = \text{const.} \qquad \theta = \text{const.} \tag{3-28}$$

Consider next the singular case when the determinant (27) vanishes. Expanding and noting Eq. (13), one finds:

$$\theta = \phi \pm \gamma \tag{3-29}$$

*) Eqs. (28) describe a region of uniform stress, the stresses satisfying the slip condition (2).

where the angle γ is defined by

$$\cos 2\gamma = \frac{1-k}{1+k} \tag{3-30}$$

Substitution of Eq. (29) into Eq. (25) finally gives

$$\frac{d\sigma}{d\phi} + \frac{1-k}{\sqrt{k}}\sigma \pm \frac{s}{\sqrt{k}} = 0 \tag{3-31}$$

and

$$\sigma = Ce^{\mp\frac{1-k}{\sqrt{k}}\phi} - \frac{s}{1-k} \tag{3-32}$$

Except for the value of the exponent, this result and Eq. (18) are similar. The solution (32) satisfies static equilibrium and is the one obtained in the classical static problem [Ref. 2].

B. Material with Dilatancy.

The components of the strain rate tensor, $\dot{\epsilon}_{xx}$, $\dot{\epsilon}_{yy}$ and $\dot{\epsilon}_{xy}$ can be expressed in terms of the principal strains, and therefore from Eq.(2-18) by the arbitrary function λ. Proceeding as in Section A., one finds that Eqs. (10) remain valid, but in lieu of Eqs. (11) one finds

$$\frac{du}{d\xi} = -\frac{\lambda y}{2}\cos 2\theta + \frac{\lambda y}{2}\,\frac{1-k}{1+k}$$

$$\xi\frac{dv}{d\xi} = -\frac{\lambda y}{2}\cos 2\theta - \frac{\lambda y}{2}\,\frac{1-k}{1+k}$$

$$\frac{dv}{d\xi} - \xi\frac{du}{d\xi} = -\lambda y\sin 2\theta \tag{3-33}$$

Elimination of (λy) furnishes the relation

$$\left(\frac{1}{\xi} - \xi\right)\cos 2\theta - 2\sin 2\theta + \frac{1-k}{1+k}\left(\xi + \frac{1}{\xi}\right) = 0$$

and after substitution of Eq. (13) one finds

$$\theta = \phi + \gamma \tag{3-34}$$

where γ has the value defined in Eq. (30). Proceeding by substitution of Eqs. (3) into Eqs. (10), one obtains a set of two simultaneous non-homogeneous linear equations in $\frac{d\sigma}{d\xi}$ and $\frac{d\theta}{d\xi}$:

$$\left[\frac{1-k}{2}(\xi \sin 2\theta - \cos 2\theta) - \frac{1+k}{2}\right]\frac{d\sigma}{d\xi} + \left[\xi \cos 2\theta + \sin 2\theta\right]\left[\sigma(1-k) + s\right]\frac{d\theta}{d\xi} = -\rho V \frac{du}{d\xi}\left(1 - \frac{u}{V} + \xi \frac{v}{V}\right)$$

$$\left[\frac{1-k}{2}(\sin 2\theta + \xi \cos 2\theta) - \xi \frac{1+k}{2}\right]\frac{d\sigma}{d\xi} + \left[\cos 2\theta - \xi \sin 2\theta\right]\left[\sigma(1-k) + s\right]\frac{d\theta}{d\xi} = -\rho V \frac{dv}{d\xi}\left(1 - \frac{u}{V} + \xi \frac{v}{V}\right) \tag{3-35}$$

Observing that the value θ given by Eq. (34) is just the one for which the determinant (27) of the coefficients of the above equations vanishes, one must conclude that these equations have solutions only if a determinant formed by the coefficients of $\frac{d\theta}{d\xi}$ and by the right hand sides vanishes. Ignoring trivial factors, it is necessary that

$$(V - u + \xi v)\begin{vmatrix} \xi \cos 2\theta + \sin 2\theta & \frac{du}{d\xi} \\ \cos 2\theta - \xi \sin 2\theta & \frac{dv}{d\xi} \end{vmatrix} = 0 \tag{3-36}$$

for $\theta = \phi \pm \gamma$. The determinant vanishes if $\phi = 0$ or π (which are trivial roots) or if $\lambda = 0$. The factor $V - u + \xi v$ can also vanish, giving a velocity distribution $u = V$, $v = 0$, which in turn again leads to $\lambda = 0$. One finds therefore that the only possible motion occurs without slip. The velocities must be constant

$$u = \text{constant} \qquad v = \text{constant} \tag{3-37}$$

while the stresses satisfy the equations of equilibrium, and are therefore the stresses for the static problems treated in [2] and derived above in Eq. (32).

C. Regions without Slip.

In locations without slip, when the inequality (2-5) is satisfied, the two materials considered may be treated jointly. If no slip occurs, the velocities u and v in Eqs. (1,10) are necessarily constants, such that their derivatives vanish, and one is left with the equation of equilibrium

$$\frac{d\sigma_x}{d\xi} - \xi \frac{d\tau}{d\xi} = 0$$

$$\frac{d\tau}{d\xi} - \xi \frac{d\sigma_y}{d\xi} = 0 \qquad (3\text{-}38)$$

In analogy with the solution in [2] for the static case, the construction of solutions in the present problem will involve the matching of sectors with and without slip, (Fig. 7). Prior to undertaking the matching of sectors, solutions applicable to the sectors without slip must therefore be obtained. The problem to be studied concerns the possible states of equilibrium in wedges of arbitrary angles β,(Fig. 8a or b), loaded on the horizontal surface by a normal load p; on the inclined face, matching with the solution in the region with slip leads to two conditions:

a.) The major principal stress must have the direction γ required by the solution in A or B above.

b.) The principal stresses at the boundary must satisfy the slip conditions Eq. (2-6).

However, there is one additional condition.

c.) Among the solutions of Eqs. (38) only those are acceptable which satisfy, in addition, the inequality (2-3), which limits the possible state of stress in the Coulomb material.

Eqs. (38) can be satisfied by expressing the stresses in the usual manner by an Airy stress-function F. In the absence of a compatibility equation, there is no further condition on $F(\xi)$, and it is left arbitrary. In these circumstances, families of solutions satisfying all requirements exist, at least for certain ranges of β.

The multitude of solutions can be restricted by noting that the assumption of an elastically undeformable material was introduced because, in case of slip, for sufficiently large values of E and G, there will be important situations where the elastic deformations can be expected to be small versus the slip deformations. However, in regions without slip, the elastic deformations ought not to be ignored because they are not small versus the (non-existent) slip deformations. Retaining the elastic deformations, the acceleration terms on the right hand side of Eqs. (10) remain, and an additional condition expressing elastic compatibility applies. The appropriate differential equations which contain the velocity V as parameter, would be those of reference [1], to be applied to a wedge, (Fig. 8a, b), in lieu of the half space. However, having ignored the elastic terms required for wave propagation in the slip region, the analysis here is restricted to values

$$V \ll c_s < c_p \qquad (3\text{-}39)$$

For the present purpose, it suffices therefore to study the solutions of the elastic problem for the limiting case $V \to 0$. In this case, the acceleration terms vanish again, such that the equations of equilibrium remain valid; however, the compatibility equation has still to be satisfied. All stresses being functions of $\xi = \frac{x}{y}$, the latter becomes

$$\left[\frac{d^2}{d\xi^2} + \frac{d}{d\xi}(\xi^2 \frac{d}{d\xi})\right](\sigma_x + \sigma_y) = 0 \qquad (3\text{-}40)$$

The solution of Eqs. (38 and 40) subject to the boundary conditions a.) and b.) above is routine, but contains three parameters β, k, and s. Having obtained the solution, it remains to be determined for what range of the parameters condition c.) above is satisfied. This complicates the presentation appreciably and the entire matter is therefore relegated to Appendix A; only pertinent conclusions are listed below:

1. For the "incompressible" material, when at the inclined face of the wedge $\gamma = \pm \frac{\pi}{4}$, it was found that solutions satisfying all conditions (a, b, and c) exist in the range

$$\frac{\pi}{4} \leq \beta \leq \frac{3}{4}\pi \qquad (3\text{-}41)$$

provided the directions of the major principal stress is as shown in (Fig. 9a or b), i. e., when σ_1 makes the angle $\frac{\pi}{4} + \beta$ with the horizontal side of the wedge. For this orientation of the major principal stress, no solution exists for other angles, just one solution exists for each combination of k and s. The value of σ on the inclined face of the wedge is given in Eq. (A-35). For the wedge angles $\beta = \frac{\pi}{4}$ and $\frac{3\pi}{4}$, the solutions represent uniform states of stress, and are of the type previously found, Eq. (28), for the special case $\dot{e} = 0$.

2. For the "incompressible" material, similar solutions exist if the direction of the major principal stress with the horizontal is $(\beta - \frac{\pi}{4})$, (Fig. 10a, b), but only if

$$\beta = \frac{\pi}{4} \quad \text{or} \quad \beta = \frac{3\pi}{4} \qquad (3\text{-}42)$$

For these two values of β, a solution exists for any combination of k and s. These solutions represent states of uniform stress, and are of the type previously found in Eq. (28).

3. In part A of this section, under Case 2, a solution without slip with a non-uniform state of stress, Eqs. (29 to 32), has been developed for the "incompressible" material. In view of our present reasoning, the compatibility condition (40) should be satisfied in a region without slip. The solution Eqs. (29 to 32) must, therefore, be checked in this respect. Not unexpectedly, one finds that compatibility is violated, such that this solution will not be considered further.

4. In the case of the material with dilatancy when the angle $\gamma(\beta)$ satisfies the equation $\cos 2\gamma = \frac{1-k}{1+k}$, solutions also exist for ranges of wedge angles β. The details are not presented, for reasons discussed in Section 5 where the material with dilatancy is discussed.

5. For completeness sake, the possibility of regions of no slip between two regions with slip has been considered, because this possibility would permit geometries different from (Fig. 7). Such regions exist, provided $\beta = \frac{\pi}{2}$. The solutions are uniform states of stress, covered by Eq. (28). (The situation, however, can not be utilized in the construction of solutions.)

6. It is of interest to know for what values of p, and up to what values of p, elastic solutions for the half space exist without slip, i. e., solutions where the inequality (2-3) is satisfied everywhere. Eqs. (38, 40) apply again with appropriate boundary conditions at $\phi = 0, \pi$, (Fig. 12), and are considered in Appendix A.

If there is no cohesion, $s = 0$, no solution satisfying Eq. (2-3) exists for $p \neq 0$.

However, if $s \neq 0$, an unusual situation occurs. The solutions of the differential equations for the stresses satisfying the boundary condition on the surface are not unique, as a state of uniform horizontal stress σ_x = constant can be added to any solution one may find. This arbitrary horizontal stress enters the condition for slip, Eq. (2-3), and the value of p up to which no slip occurs depends on the assumed intensity of the horizontal state of stress. There is, however, a most "beneficial" state for which the load p becomes a maximum. This "distinguished solution" defines the value of p above which solutions without slip can not possibly exist. A plot of these critical values of p vs. k is given in Appendix A, (Fig. A.4).

4. SOLUTIONS FOR THE "INCOMPRESSIBLE" MATERIAL.

a. Construction of Solutions.

In this section, the solutions applying for the "incompressible" material, Case A in Section 2, will be obtained. The available ingredients are:

1. A solution in a region with slip, treated under A, Case 1, in the previous section. It was found that in such a region, the major principal stress makes an angle $\pm \frac{\pi}{4}$ with the position vector ϕ, Eq. (3-14), while σ is given by Eq. (3-18). It will be seen that the case of the lower sign, (Fig. 13), will be used. In this case

$$\sigma = Ce^{+2\frac{1-k}{1+k}\phi} - \frac{s}{1-k} \tag{4-1}$$

where C is an open but positive constant. The equation indicates that σ increases with increasing values of ϕ.

2. Of solutions without slip obtained under A, Case 2 in Section 3, only the ones defined by Eq. (3-28)

$$\sigma = \text{const.} \qquad \theta = \text{const.} \tag{4-2}$$

representing a state of uniform stress can be considered. The other solution, Eqs. (3-29 to 32) is not to be used, for reasons stated in Section 3 under C, item 3. Solutions of the type of Eq. (2) will be used for 45° wedges as shown in (Fig. 14a, b). In (Fig. 14a), the major principal stress is inclined at $\theta = \frac{\pi}{2}$, while in (Fig. 14b) the inclination is $\theta = 0$.

3. Additional solutions without slip in wedge shaped regions are described in Section 3 under C, item 1. Of interest are solutions for wedges $\frac{\pi}{4} \leq \beta < \frac{3\pi}{4}$, (Fig. 15).

Let us now consider the problem of a step load progressing with velocity V, (Fig. 2), for material properties k and s, for various values of the load, starting with small values of $\frac{p}{s}$. For sufficiently small values of this ratio, it has been shown in Appendix A that (elastic) solutions without slip exist, provided that p is less than a critical value p_s, given in (Fig. A-4). As pointed out in Section 3 under C, item 6, the solutions are not even unique, i. e., there are families of solutions if $p < p_s$, but there is just one if $p = p_s$. The critical value p_s vanishes if $s = 0$. If cohesion is present, assuming likely values of k, the limit p_s is 2s to 3s, i. e., at an uninterestingly low level of stress.

When constructing solutions with slip, it is clear that solution 1. above is insufficient to satisfy the condition at either the loaded or the free surface, because the directions of the principal stresses at the surface cannot be at angles $\pm \frac{\pi}{4}$ to a surface. Transition regions without slip are necessary as indicated in (Fig. 7).

The solutions 1, 2 and 3 above can be combined, (Fig. 16), to form a continuous solution for the half space, consisting of a non-slip wedge of angle $\frac{\pi}{4} \leq \beta < \frac{3\pi}{4}$ on the loaded side, a non-slip wedge of angle $\frac{\pi}{4}$ on the unloaded side, and an interior wedge of angle $(\frac{3\pi}{4} - \beta)$ in which slip occurs.

Consider first the case $\beta = \frac{\pi}{4}$, such that slip occurs in a 90° wedge. Under the load, the state of stress, (Fig. 14a), is uniform, the major principal stress being vertical; In this range

$$\frac{3\pi}{4} \leq \phi \leq \pi \qquad \sigma = p - s \qquad \theta = \frac{\pi}{2} \tag{4-3}$$

In the adjoining sector with slip, the major principal stress makes therefore the angle $-\frac{\pi}{4}$ with the position vector; obtaining the value of C in Eq. (1) by matching the values of σ at $\phi = \frac{3\pi}{4}$, one finds

$$\frac{\pi}{4} \le \phi \le \frac{3\pi}{4}: \quad \sigma = \left(p + \frac{ks}{1-k}\right)e^{-2\frac{1-k}{1+k}\left(\frac{3\pi}{4} - \phi\right)} - \frac{s}{1-k} \tag{4-4}$$

As the direction of the major principal stress in the region $\phi \le \frac{\pi}{4}$ must match the direction in the slip region, σ_1 must be horizontal, such that (Fig. 14b) applies, and one finds that the solution described requires a vertical pressure at $\phi = 0$:

$$p_o = k\left(p + \frac{ks}{1-k}\right)e^{-\frac{1-k}{1+k}\pi} - \frac{ks}{1-k} \tag{4-5}$$

The surface at $\phi = 0$ is, however, a free surface, where the pressure should vanish, $p_o = 0$. Solving Eq. (5) for the value $p = p_L$ for which $p_o = 0$, one obtains

$$p_L = \frac{s}{1-k}\left[e^{\frac{1-k}{1+k}\pi} - k\right] \tag{4-6}$$

such that a steady state solution for the specific value $p = p_L$ has been obtained. The subscript L has been selected because this value represents a limiting situation, as will be recognized later.

One could now proceed to vary the angle β in (Fig. 16) and try to obtain solutions for values $p \neq p_L$. However, the presence of cohesion masks the true state of affairs, and as a first step only the case $s = 0$ will be considered. In this case, Eqs. (1, 4, 5) retain the exponential terms on the right hand side only, and Eq. (6) gives $p_L = 0$, i. e., only a trivial solution, for $p = 0$, has been determined.

If one uses different angles β in (Fig. 16) on the loaded side, one again finds only trivial solutions $p = 0$, as is easily confirmed. The relation between p and σ on the inclined face for the wedge, (Fig. 15), is

$$\sigma = \frac{2p\ (1-\cos 2\beta) - s(1-\cos 2\beta + \sin 2\beta - 2\beta \cos 2\beta)}{(1+k)\ (1-\cos 2\beta) + (1-k)\ (\sin 2\beta - 2\beta \cos 2\beta)} \qquad (4-7)$$

When $s = 0$, the value of σ at the inclined face is therefore proportional to p; matching of the solution (1) gives C, and the stress σ at $\phi = \frac{\pi}{4}$, both proportional to p. This is to be matched with the 45° wedge shown in (Fig. 14b). If $s = 0$, and the stress at $\phi = 0$ vanishes, the stress σ in this wedge necessarily vanishes, and p must also vanish.

There is, therefore, for $s = 0$, no solution for the boundary value problem formulated, i. e., for a finite value of p behind the moving front, and no pressure ahead of the front. From a purely mathematical point of view, this is connected with the fact that the equations are hyperbolic, and having prescribed conditions on one side, one cannot simultaneously prescribe conditions on the other. However, the physical problem must have a solution - there must be a response - if a pressure wave is applied to the half-space. The mathematical and physical concepts can be reconciled, by allowing for the possibility of particles of the material being expelled from the unloaded surface ahead of the pressure front. It is reasoned, that the non-existence of solutions of the problem for certain applied loads in the formulation used so far indicates that the body cannot exist under these loads, and will disintegrate. The process of disintegration has been studied for a simple equivalent situation in Appendix B on a mechanical model exhibiting Coulomb properties. A mechanism is demonstrated which expels particles on an unloaded surface, the momentum of the particles providing a reactive pressure required to obtain a solution in the interior of the body.

In this revised situation, regardless of the value of s, Eqs. (3, 4) represent a solution of the steady state problem for $p \geq p_L$, but a reactive pressure according to Eq. (5) is exerted by particles expelled from the surface ahead of the pressure front p.

Continuing the discussion for s = 0, the limit $p_L = 0$, and the solution Eqs. (4, 5) applies for all values of p. However, it is not the only solution. Varying the angle β of the non-slip region in (Fig. 16), other solutions can be obtained. The value of σ at $\phi = \pi - \beta$ follows from Eq. (7) for s = 0:

$$\sigma(\pi - \beta) = \frac{2p\,(1-\cos 2\beta)}{(1+k)\,(1-\cos 2\beta) + (1-k)\,(\sin 2\beta - 2\beta \cos 2\beta)} \qquad (4-8)$$

In the adjoining slip region σ is given by Eq. (1), where C is obtained by matching values at $\phi = \pi - \beta$. In the non-slip region, $0 < \phi \leq \frac{\pi}{4}$, the value of σ is constant, and one finds

$$\sigma(0) = \sigma(\pi - \beta)\, e^{-2\,\frac{1-k}{1+k}\left(\frac{3\pi}{4} - \beta\right)} \qquad (4-9)$$

The required reactive pressure to be exerted by expelled particles at $\phi = 0$ becomes

$$p_o = k\sigma(0) = k\sigma(\pi - \beta)\, e^{-2\,\frac{1-k}{1+k}\left(\frac{3\pi}{4} - \beta\right)} \qquad (4-10)$$

In addition to the solution for $\beta = 45^\circ$ found first, there exists therefore a whole family of solutions for $\frac{\pi}{4} \leq \beta \leq \frac{3\pi}{4}$. All these solutions are mathematically unobjectionable; this lack of uniqueness in steady-state problems was discussed in the introduction and is nothing unusual.

Comparing the possible solutions for various angles β, (Fig. 16), it is seen that the one for $\beta = 45^\circ$ where slip occurs in a 90° wedge has distinctive properties, and it will be called the "principal solution". Among all solutions for a given value of the applied pressure p, it requires the smallest pressure p_o exerted by the particles expelled on the free surface. To demonstrate this consider (Fig. 17). It shows the values of σ for the "principal solution" as function of ϕ plotted as Curve I; in the range $\frac{\pi}{4} < \phi < \frac{3\pi}{4}$ the curve decays exponentially as given by Eq. (4), while the remainder is horizontal. Curve II in (Fig. 17) is the locus of the values $\sigma(\pi - \beta)$ according to Eq. (8), i. e., the value of σ at the interface of the slip and non-slip regions. The end points of the curve are unity and $\frac{1}{k}$ respectively, and it has a minimum at $\beta = \frac{\pi}{2}$, as can be found by differentiation of Eq. (8). One can easily demonstrate that between $\frac{\pi}{4}$ and $\frac{3\pi}{4}$ Curve II is always above Curve I, and between $\frac{\pi}{4}$ and $\frac{\pi}{2}$ the slope of Curve I is always steeper (downward) than that of Curve II.

Curve III indicates the situation for an intermediate value of β. At the interface $\phi = (\pi - \beta)$ the value of σ is defined by point C; the value σ then decreases exponentially, i. e., the decay curve C-D is a portion C^1-D^1 of the curve A B displaced horizontally, such that the final value at the surface, $\sigma(0)$, and $p_o = k\sigma(0)$ is necessarily larger than for the "principal solution".

In the general case, when cohesion does not vanish, the situation is more complicated. For low pressures, $p < p_s$, there are elastic solutions without slip anywhere; for $p \geq p_L$ there is the "principal solution", requiring disintegration at the free surface if $p > p_L$, while for $p = p_L$ no disintegration occurs. (Fig. 18) shows a plot of p_s and p_L versus k, indicating that everywhere $p_L > p_s$. There is, therefore, a gap between the levels of p_s and p_L where neither the elastic nor the "principal solution" applies. For such values of p solutions can be constructed by utilizing wedge angles $\beta > \frac{\pi}{4}$, (Fig. 16). If $s \neq 0$ the Curve II, (Fig. 17), describing σ at the inclined face is modified, but the general situation remains similar, such that for any value $p_L > p > p_s$, a value of β exists for which p_o given by Eq. (10) will vanish. In other words, in the range

$$p_L > p > p_s \tag{4-11}$$

the Eqs. (8, 9, and 10) and the condition $p_o = 0$ define a "principal solution" in which slip occurs in the range $\frac{\pi}{4} < \phi < \pi - \beta$, where $\frac{\pi}{4} < \beta < \frac{3\pi}{4}$, (Fig. 16). For this solution the pressure on the surface ahead of the wave vanishes such that no disintegration occurs. (It would also be possible to find solutions in this range where $p_o > 0$, requiring disintegration. Such solutions are not considered, because a "principal solution" with $p_o = 0$ exists.)

The method of selecting the "principal solution" is not very satisfactory as intuitive reasoning has to be employed. More definite conclusions showing that these solutions are the ones approached in transient cases after a long time, would require consideration of the starting conditions, which would pose a major yet unsolved problem. As an alternative, one might study the stability of the various solutions against perturbations. This might permit elimination of some excess solutions, but such an approach might not be successful. Stability considerations are unable to resolve the lack of uniqueness in the elastic solutions for $p < p_s$.

An additional point is worth discussing and illustrates the inherent complexities of the type of problem considered. (Fig. 18) shows plots of p_s and p_L, and also of the value of a static load p_p which can be supported by a half space according to the classical analysis by Prandtl [2]. It is seen that $p_p > p_L$, such that the present paper contains in the limit $V \to 0$, solutions where steady-state motion occurs at pressure levels below p_p (where equilibrium is possible). It is outside the province of this paper to investigate if this indicates an instability of the static solution [2] for the material considered here or not. It is important, however, to point out that the fact that the limit $V \to 0$ of the present solution does not give the static situation is not due to a mathematical error. In the static case, when the load $p = p_p$ is applied, slip must occur in some locations before the state of equilibrium is reached. If the load is moved with a velocity V, however small, slip has to occur continuously, and a slipless solution at the level $p = p_p$ is therefore not possible. The dynamic solution for $V \to 0$ does not approach the static solution.

b. Discussion of Strains and Accelerations due to the "Principal Solution"

In the preceding subsection (a) solutions with slip have been obtained covering the range of pressures, $p > p_s$, where p_s defines the limit up to which elastic solutions without slip exist. If cohesion is present, $s \neq 0$, then there is a narrow range $p_s < p \leq p_L$ where the "principal solution" selected involves slip but does not require surface disintegration ahead of the pressure pulse. For larger pressures, $p > p_L$, solutions exist only with slip and surface disintegration, the "principal solution" being the one with the least reactive pressure due to expelled particles.

It is naturally of interest to discuss the magnitude of accelerations and strains associated with the principal solutions. For these quantities, there being only small, quantitative differences between the ranges $p < p_L$ and $p > p_L$, only the latter will be considered.

A quantity of interest is the total permanent strain produced at any point due to the passing of the shock wave on the surface. This strain gives an indication of the deformation to which a target, say a cylindrical shell, (Fig. 19), would be subjected if the presence of the target would not affect the force field*). To obtain the total strain, the strain rates at the target at depth y must be obtained, and integrated with respect to time. Substituting Eq. (3-23) into Eq. (3-11) and noting that the angle θ in the solution employed is defined by Eq. (3-14) with the lower sign, one finds

$$\dot{e} = \frac{1-k}{4\rho V y} \frac{d\sigma}{d\phi} \tag{4-12}$$

This expression applies in the slip region $\frac{\pi}{4} \leq \phi \leq \frac{3\pi}{4}$. The value of σ is

*) Whether or not the target affects the free field substantially or not depends on relative stiffness, but little is known about the matter at present.

given by Eq. (4),

$$\frac{d\sigma}{d\phi} = 2\left(p\,\frac{1-k}{1+k} + \frac{sk}{1+k}\right) e^{-2\,\frac{1-k}{1+k}\left(\frac{3\pi}{4} - \phi\right)} \tag{4-13}$$

which permits the determination of the strain rates $\dot{\epsilon}_{xx}$, $\dot{\epsilon}_{yy}$ and $\dot{\epsilon}_{xy}$ from Eqs. (3-6).

To integrate the resulting expression with respect to time, it is noted that

$$dt = \frac{y}{V\,\sin^2\phi}\,d\phi \tag{4-14}$$

Hence

$$\begin{aligned} -\epsilon_x = \epsilon_y &= \int \dot{\epsilon}_y\,dt = 2A\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} e^{2\,\frac{1-k}{1+k}\,\phi}\cot\phi\,d\phi \\ \epsilon_{xy} &= \int \dot{\epsilon}_{xy}\,dt = -A\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} (1-\cot^2\phi)\,e^{2\,\frac{1-k}{1+k}\,\phi}\,d\phi \end{aligned} \tag{4-15}$$

where

$$A = \frac{1-k}{2\rho V^2}\left(p\,\frac{1-k}{1+k} + \frac{sk}{1+k}\right) e^{-2\,\frac{3\pi(1-k)}{(1+k)}} \tag{4-16}$$

The limits in the above integrals are those of the slip region $\frac{\pi}{4} < \phi < \frac{3\pi}{4}$. The nature of the integrals requires numerical integration. Another quantity of interest is the acceleration history. The accelerations in the non-slip regions, (Fig. 19), vanish; in the slip region, $\frac{\pi}{4} \leq \phi \leq \frac{3\pi}{4}$, Eq. (3-20) defines the radial acceleration. Using Eq. (13)

$$a = \frac{1-k}{\rho\sqrt{x^2+y^2}}\left(p\,\frac{1-k}{1+k} + \frac{sk}{1+k}\right)\exp\left\{-\frac{1-k}{1+k}\left[\frac{3\pi}{2} - 2\cot^{-1}\left(\frac{x}{y}\right)\right]\right\} \tag{4-17}$$

<u>Example</u>: The physical meaning of these results is best discussed for a typical case. We select $k = \frac{1}{3}$ and no cohesion, $s = 0$. Using Eqs. (15) to compute

the permanent strains, one finds the principal permanent strains

$$\epsilon_{1,2} \simeq \pm 0.111 \frac{p}{\rho V^2} \qquad \text{(a)}$$

the direction being defined by $\theta \simeq 0.5$. To interpret the result it will be compared with the magnitude of elastic strains which are of the order $\frac{p}{E}$. Noting the expression for the velocity of P-waves in an elastic medium $c_p^2 = \frac{E(1-\nu)}{\rho(1+\nu)(1-2\nu)}$, Eq. (a) may be written

$$\epsilon_{1,2} \simeq \pm \frac{p}{E}\left[\frac{c_p^2}{10V^2}\right] \qquad \text{(b)}$$

Due to the basic assumptions on which this paper is based, V must be small versus c_p, such that the term in parentheses is much larger than unity. The permanent strain is, therefore, much larger than elastic strains at this stress level.

Eq. (b) also permits an estimate of the range of applicability of the present theory, which ignores elastic strains: It should be applicable as long as the slip strains, Eq. (b), are appreciably larger than elastic ones, $\simeq \frac{p}{E}$. This condition is reasonably satisfied up to Mach numbers $\frac{V}{c_p} < 0.2$, and the theory presented will therefore be applicable up to such values of $\frac{V}{c_p}$.

The value of the radial acceleration a, given by Eq. (17) is plotted in (Fig. 20). The acceleration a vanishes if $\left|\frac{x}{y}\right| > 1$; its maximum value in the range $\left|\frac{x}{y}\right| \leq 1$ is

$$\max a \simeq 0.24 \frac{p}{\rho y} \qquad \text{(c)}$$

For comparison purposes one can express ρ in terms of the velocity of elastic shear waves c_s and of the modulus of rigidity, G,

$$\max a \simeq 0.24 \frac{c_s^2 p}{G y} \qquad \text{(d)}$$

This value may be compared with the acceleration due to a step wave in the elastic case, which can be obtained from the results in Ref. [1]. Using the curve in Fig. 5 of this paper, the effect of a step wave can be found by integration. It is seen that the acceleration given by Eq. (d) is of equal magnitude as the elastic one for $M_T = 0.5$ which corresponds to $\frac{V}{c_p} = 0.25$. As the accelerations in the elastic case decrease with the fourth power of V, the present solution gives again substantially larger values than [Ref. 1] provided $\frac{V}{c_p} < 0.2$.

It is important to note that the accelerations do not depend on the velocity V, but that the total strains are proportional to V^{-2}. The independence of a is due to the fact that the accelerations are solely determined by the stress field, by virtue of Eqs. (3-1), and the velocity V does not explicitly appear in these equations. To find strain rates and strains, one and two integrations, respectively, with respect to time are required. Because of Eq. (14), each such integration introduces a factor $\frac{1}{V}$, explaining the fact that the strains are proportional to V^{-2}.

The above comparison of effects permits an interesting prediction for the range of larger Mach numbers, $\frac{V}{c_p} > 0.2$, for a material having combined elastic and Coulomb-slip properties. While for low Mach numbers, the slip effects are much larger than the elastic ones, this is not the case for larger values of $\frac{V}{c_p}$. For such values, the occurrence of slip will not change the order of magnitude of acceleration and strains found by a purely elastic analysis ignoring slip. One can understand this conclusion by noting that for large values of V, the slip stage lasts only a short time such that displacements due to slip have only little time to develop, and will remain of the same magnitude as the elastic ones.

Another prediction for results at larger Mach numbers concerns the disintegration of the surface ahead of the pressure wave. This effect will occur also at larger Mach numbers, but of course only for sub-seismic velocities.

5. THE MATERIAL WITH DILATANCY.

For this material a solution has been obtained in Section 3 under B, where the direction of the major principal stress makes an angle $\pm\gamma$ with the radial, (Fig. 21), where

$$\cos 2\gamma = \frac{1-k}{1+k} \qquad (5\text{-}1)$$

This solution could be matched with regions of uniform stress if the angles are selected as shown in (Fig. 22). One can then find a value $p = p_P$ where the surface ahead is a free surface, the solution agreeing with the static one found by Prandtl [2]. For larger values of the load, $p > p_P$, one obtains a solution requiring a pressure p_o ahead of the pressure front, p_o being supplied again by the disintegration of the surface.

One could further investigate if solutions exist for wedges without slip at angles other than $90^\circ - \gamma$ and γ, utilised in (Fig. 22). This investigation has not been made for reasons explained hereafter.

In Section 3 under C, it was concluded that in regions without slip the equations of elastic compatibility should be satisfied, such that the solutions found in this paper would apply for an actual elastic-Coulomb material in the limit when E and G are large. The fact that elastic deformation in areas where slip occurs are neglected in the present paper is defensible as long as the slip deformations are large versus the elastic ones. This argument, unfortunately, does not apply for the solutions found in Section 3 under B, because it was found that the solution requires $\lambda = 0$, and according to Eq. (3-33) all strains and deformations vanish. In these circumstances ignoring of the elastic deformations is not justified*) if one desires, as we do, to obtain a solution which applies to a material having elastic-Coulomb properties.

In these circumstances, it was decided not to pursue all the details of the possible solutions, as was done in Section 4 for the alternative material. It would be unreasonable to waste effort on a solution because all deformations vanish and one obtains in the end rather trivial results: All accelerations and the free field strains (corresponding to Eq. (4-15) for the other material) vanish, the only non-vanishing results concern stresses.

For materials with Dilatancy an analysis including elastic deformations is necessary, and is planned for the future. Incidentally, one can easily understand that the coupling of slip and volume expansion in Eqs. (2-9) requires that a meaningful analysis must retain the effects of elastic volume changes.

*) In order to avoid misunderstandings, it should be stressed that the above objection to ignoring elastic effects only applies in the dynamic problem studied here. The objection does not apply in a static case, treated by Prandtl [2], and the above reasoning should not be misconstrued into an objection to the latter solution.

6. SUMMARY OF CONCLUSIONS

The possibility of steady-state solutions with slip has been investigated when the half-space is loaded by a moving step load, (Fig. 2). Restricting the study to low Mach numbers, the elastic deformations were neglected in the slip regions. Two materials were considered; both follow the Coulomb rule, defining the stresses at slip, but have different flow rules.

One material is assumed to remain constant in volume during slip, while the other, using the concepts of Drucker and Prager [2] exhibits Dilatancy. For the latter material only unsatisfactory results are obtained, because elastic deformations were neglected. For the other material, the results obtained are expected to be valid up to Mach numbers $\frac{V}{c_p} \sim 0.2$.

A major finding of the present study is the recognition that for pressures p above a critical value, disintegration of the surface ahead of the pressure wave must occur, producing a precursor of dust particles. The mechanism of the disintegration has been studied on a mechanical model, and it has been shown that elements in a boundary layer near the free surface will be accelerated upward and finally expelled, (Fig. 23). This manner of disintegration occurs for both types of materials considered, and is expected to occur beyond the range $\frac{V}{c_p} < 0.2$ of the present study, but only for subseismic velocities V.

Detailed results for the material without volume change during slip are presented in Section 4, obtaining stresses, accelerations and permanent strains. It is demonstrated (Section 4.b) that accelerations and permanent strains, in the range $\frac{V}{C_p} \not> 0.2$, are substantially larger than the corresponding quantities in an elastic medium when no slip occurs.

The results can be utilized to study the equivalent problem for a locking medium having Coulomb properties behind the locking front. Such an investigation is under way.

For the material with Dilatancy only stresses can be obtained as the analysis degenerates and accelerations and strains are found to vanish. To obtain meaningful results elastic effects must be retained.

APPENDIX A

STATIC, ELASTIC SOLUTIONS FOR WEDGE SHAPED REGIONS AND FOR THE HALF-SPACE WITHOUT SLIP.

As background for Section 3C, the elastic stresses are obtained for wedge shaped regions, (Fig. A-1). Using for convenience polar coordinates and stress components σ_r , σ_ϕ and τ , the following conditions on the surfaces are to be satisfied.

On the surface $\phi = 0$:

$$\sigma_\phi(0) = -p, \quad \tau = 0 , \tag{A-1}$$

while on the surface $\phi = \beta$, the direction of the major principal stress shall be in a specified direction, defined by the angle $\gamma(\beta)$, (Fig. A-1). (Of interest will be the angles $\gamma(\beta) = \pm \frac{\pi}{4}$). Further, at $\phi = \beta$ the two principal stresses σ_1 and σ_2 are required to satisfy the slip condition, such that

$$\begin{aligned} \sigma_1 &= -(\sigma + s) \\ \sigma_2 &= -k\sigma . \end{aligned} \tag{A-2}$$

Having obtained solutions for these boundary conditions, it is to be investigated which of the solutions, if any, satisfy the Coulomb rule

$$-\sigma_2 \geq k\sigma = -k(\sigma_1 + s) \tag{A-3}$$

everywhere.

The equations of equilibrium, (3-38), in polar coordinates for stresses which are independent of r, become

$$\begin{aligned} \frac{d\tau}{d\phi} + \sigma_r - \sigma_\phi &= 0 \\ \frac{d\sigma_\phi}{d\phi} + 2\tau &= 0 \end{aligned} \tag{A-4}$$

while the compatibility equation (3-40) becomes

$$\frac{d^2}{d\phi^2}[\sigma_r + \sigma_\phi] = 0 \tag{A-5}$$

The general solution of the differential equations (4,5) may be written

$$\begin{aligned} \sigma_r &= A + B\phi - C \sin 2\phi - D \cos 2\phi \\ \sigma_\phi &= A + B\phi + C \sin 2\phi + D \cos 2\phi \\ \tau &= -\frac{B}{2} - C \cos 2\phi + D \sin 2\phi \end{aligned} \tag{A-6}$$

where A, B, C and D are arbitrary constants. Determining A and B from the boundary conditions (1) gives the stresses in terms of C and D:

$$\begin{aligned} \sigma_r &= -p - D(1 + \cos 2\phi) - C(2\phi + \sin 2\phi) \\ \sigma_\phi &= -p - D(1 - \cos 2\phi) - C(2\phi - \sin 2\phi) \\ \tau &= D \sin 2\phi + C(1 - \cos 2\phi) \end{aligned} \tag{A-7}$$

It is now convenient to express the σ_r , σ_ϕ and τ in terms of principal stresses σ_1 , σ_2 and of the angle $\gamma = \gamma(\phi)$ which the direction of σ_1 makes with the radial, (Fig. A-1). Further, the stresses σ_1 and σ_2 may be expressed by two other functions of ϕ, namely $\sigma \equiv \sigma(\phi)$ and $\bar{k} \equiv \bar{k}(\phi)$,

$$\begin{aligned} \sigma_1 &= -(\sigma + s) \\ \sigma_2 &= -\bar{k}\sigma \end{aligned} \tag{A-8}$$

Using the conventional expression for the stresses in terms of the principal stresses, one finds

$$\begin{aligned} \sigma_r &= \sigma_1 \cos^2\gamma + \sigma_2 \sin^2\gamma = -[(1-\bar{k})\sigma + s] \cos^2\gamma - \bar{k}\sigma \\ \sigma_\phi &= \sigma_1 \sin^2\gamma + \sigma_2 \cos^2\gamma = -[(1-\bar{k})\sigma + s] \sin^2\gamma - \bar{k}\sigma \\ \tau &= (\sigma_1 - \sigma_2)\sin\gamma \cos\gamma = -[(1-\bar{k})\sigma + s] \sin\gamma \cos\gamma \end{aligned} \tag{A-9}$$

Elimination of σ from these equations gives two relations between σ_r , σ_ϕ , τ, $\bar{k}$ and γ

$$\tau - \frac{\sigma_r - \sigma_\phi}{2} \tan 2\gamma = 0$$

$$\tau - \frac{\sigma_r + \sigma_\phi}{2} \left(\frac{1-\bar{k}}{1+\bar{k}}\right) \sin 2\gamma + \frac{\bar{k}s}{1+\bar{k}} \sin 2\gamma = 0 \quad . \tag{A-10}$$

a. SOLUTIONS FOR WEDGE SHAPED REGIONSWHEN $\gamma(\beta) = \pm \frac{\pi}{4}$.

Applying these relations at the boundary $\phi = \beta$, using the values

$$\bar{k}(\beta) = k \quad , \quad \gamma(\beta) = \pm \frac{\pi}{4} \tag{A-11}$$

one substitutes the values of σ_r , σ_ϕ and τ for $\phi = \beta$ from Eqs. (7), and obtains two simultaneous equations for the constants C and D:

$$C \sin 2\beta + D \cos 2\beta = 0$$

$$C \left[(1 - \cos 2\beta) \pm 2\beta \frac{1-k}{1+k}\right] + D \left[\sin 2\beta \pm \frac{1-k}{1+k}\right] = \mp \left[p \frac{1-k}{1+k} + \frac{ks}{1+k}\right] . \tag{A-12}$$

The upper and lower sign is to be used in this and subsequent equations as in Eq. (11). The values of the constant C and D are

$$C = \quad N \cos 2\beta$$

$$D = - N \sin 2\beta \tag{A-13}$$

where

$$N = \pm \frac{p \frac{1-k}{1+k} + \frac{ks}{1+k}}{1-\cos 2\beta \pm \frac{1-k}{1+k} (\sin 2\beta - 2\beta \cos 2\beta)} \tag{A-14}$$

It is noted that the denominator of this expression may vanish for certain combinations of β and k, if the lower sign applies. In such cases, no solutions exist.

It is nov necessary to investigate if any of the solutions obtained satisfy the Coulomb rule Eq. (3). It is easy to see that for the special values $\beta = \frac{\pi}{4}$ and $\beta = \frac{3}{4}\pi$, Eq. (3) is satisfied everywhere. In these cases $C = 0$, and the state of stress defined by Eqs. (7) is uniform, σ_x and σ_y being constant, while τ vanishes. Having prescribed that the slip condition (2) is satisfied at $\phi = \beta$, it is obvious that Eq. (3) is satisfied everywhere. This gives the following simple solutions

$$\left.\begin{array}{ll} \beta = \frac{\pi}{4}, & \gamma(\beta) = +\frac{\pi}{4} \\ \beta = \frac{3\pi}{4}, & \gamma(\beta) = -\frac{\pi}{4} \end{array}\right\} \quad \sigma_y = -p\,,\ \sigma_x = -k(p-s),\ \tau = 0$$

$$\left.\begin{array}{ll} \beta = \frac{3\pi}{4}, & \gamma(\beta) = +\frac{\pi}{4} \\ \beta = \frac{\pi}{4}, & \gamma(\beta) = -\frac{\pi}{4} \end{array}\right\} \quad \sigma_y = -p\,,\ \sigma_x = -(\frac{p}{k}+s),\ \tau = 0 \qquad \text{(A-15)}$$

The discussion of the solutions for other angles β becomes much more involved. As a first step, the angle γ can be determined from Eq. (10) by substitution of Eqs. (7) and (13):

$$\cot(2\gamma) = \frac{\sin 2(\beta-\phi)}{\cos 2\beta - \cos 2(\beta-\phi)} \quad . \qquad \text{(A-16)}$$

The fact that the denominator vanishes for $\phi = 0$ and $\phi = 2\beta - \pi$ (if $\beta > \frac{\pi}{2}$) presents no difficulty. One can now determine the derivative $\gamma' \equiv \frac{d\gamma}{d\phi}$. As will be seen shortly, the behavior of the quantity $1+\gamma'$ is of interest, and one finds

$$1+\gamma' = \frac{\cos 2\beta\,[\cos 2\beta - \cos 2(\beta-\phi)]}{1 + \cos^2(2\beta) - 2\cos(2\beta)\cos 2(\beta-\phi)} \quad . \qquad \text{(A-17)}$$

Applying this equation for the boundary $\phi = \beta$ one finds

$$1+\gamma'(\beta) = -\frac{\cos 2\beta}{1 - \cos 2\beta} \qquad \text{(A-18)}$$

In the range $0 < \beta < \pi$ the right hand side has the sign of $(-\cos 2\beta)$ and one finds the inequalities

$$\left.\begin{array}{l} 0 < \beta < \frac{\pi}{4} \\ \frac{3}{4}\pi < \beta < \pi \end{array}\right\} \quad 1+\gamma'(\beta) < 0 \tag{A-19}$$

$$\frac{\pi}{4} < \beta < \frac{3\pi}{4} \quad , \quad 1+\gamma'(\beta) > 0 \quad . \tag{A-20}$$

As a next step, an expression for the derivative $\bar{k}' \equiv \frac{d\bar{k}}{d\phi}$ will be obtained. For this purpose, Eqs. (9) are substituted into Eqs. (4) giving after manipulations

$$(1+\gamma')[\sigma(1-\bar{k}) + s] \sin \gamma + \bar{k}\sigma' \cos \gamma + \sigma\bar{k}' \cos \gamma = 0 \tag{A-21}$$

$$(1+\gamma')[\sigma(1-\bar{k}) + s] \cos \gamma + \sigma' \sin \gamma = 0 \quad . \tag{A-22}$$

Excluding locations where $\sin 2\gamma = 0$, one can eliminate σ' and obtain a relation

$$\bar{k}' = \frac{(1+\bar{k})\,[\sigma(1-\bar{k}) + s]\,(1+\gamma')}{\sigma \sin 2\gamma}\left[\cos 2\gamma - \frac{1-\bar{k}}{1+\bar{k}}\right] \tag{A-23}$$

Applying this result to the determination of the sign of $\bar{k}'$ at $\phi = \beta$, one has, in addition to Eqs. (19,20), the following information

$$\begin{array}{l} \bar{k}(\beta) = k \quad , \quad \gamma(\beta) = \pm \frac{\pi}{4} \\ 1+\bar{k} > 0 \quad , \quad \sigma(1-\bar{k}) + s > 0 \quad , \quad \sigma > 0 \\ \cos 2\gamma - \frac{1-\bar{k}}{1+\bar{k}} = -\frac{1-k}{1+k} < 0 \quad . \end{array} \tag{A-24}$$

Using Eqs. (19), (20), (23) and (24) one finds that $\bar{k}' < 0$ provided:

$$\begin{array}{l} \gamma(\beta) = \frac{\pi}{4} \quad \text{and} \quad \frac{\pi}{4} < \beta < \frac{3\pi}{4} \\ \text{or} \\ \gamma(\beta) = -\frac{\pi}{4} \quad \text{and} \quad 0 < \beta < \frac{\pi}{4} \quad \text{or} \quad \frac{3\pi}{4} < \beta < \pi \quad . \end{array} \tag{A-25}$$

For the complementary ranges of β one finds $\tilde{k}' > 0$. The important point is that the material at the boundary $\phi = \beta$ is just at the verge of the slip, $\tilde{k} = k$, such that the condition $\tilde{k}' \leq 0$ must be satisfied, otherwise the Coulomb rule is violated at small distances from $\phi = \beta$. This restricts the ranges to be studied further to those defined in Eq. (25).

In the geometries defined by Eq. (25), the Coulomb rule is satisfied near $\phi = \beta$; to investigate other locations, Eq. (16) is used to obtain the range of γ as ϕ varies from 0 to β. Equation (A-16) is simple enough to draw general, qualitative conclusions. The three cases in Eq. (25) separate into a total of four situations, listed in the following table depending on the prescribed value of $\gamma(\beta) = \pm \frac{\pi}{4}$ and on the wedge angle β. The function $\gamma(\phi)$ decreases in all cases monotonically from $\gamma(0)$ to $\gamma(\beta)$:

	$\gamma(\beta)$	$\gamma(0)$
$0 < \beta \leq \frac{\pi}{4}$	$-\frac{\pi}{4}$	0
$\frac{\pi}{4} < \beta \leq \frac{\pi}{2}$	$+\frac{\pi}{4}$	$\frac{\pi}{2}$
$\frac{\pi}{2} < \beta < \frac{3\pi}{4}$	$+\frac{\pi}{4}$	π
$\frac{3\pi}{4} < \beta < \pi$	$-\frac{\pi}{4}$	$\frac{\pi}{2}$

(A-26)

To proceed, a general expression for $\tilde{k}(\phi)$ is required, and may be obtained from Eqs. (10). One could solve the second of these equations for $\tilde{k}$, but the result would break down whenever $\sin 2\gamma = 0$, i.e., according to Eq. (26) on the surface $\phi = 0$. To circumvent this difficulty, one forms the difference of the two equations (10) and finds the alternative relation

$$\frac{\sigma_r - \sigma_\phi}{2} - \frac{1-\tilde{k}}{1+\tilde{k}} \frac{\sigma_r + \sigma_\phi}{2} \cos 2\gamma + \frac{\tilde{k}s}{1+\tilde{k}} \cos 2\gamma = 0 \qquad \text{(A-27)}$$

Using Eqs. (7) and (13), one obtains

$$\bar{k}(\phi) = \frac{p+N\left[\dfrac{\sin 2(\beta-\phi)}{\cos 2\gamma} - \sin 2\beta + 2\phi \cos 2\beta\right]}{p-s-N\left[\dfrac{\sin 2(\beta-\phi)}{\cos 2\gamma} + \sin 2\beta - 2\phi \cos 2\beta\right]} \tag{A-28}$$

This result can be used to check on the values of $\bar{k}(0)$. Taking the value of $\gamma(0)$ from Eq. (26), and considering values β permitted by Eq. (25) one finds:

If $0 < \beta < \frac{\pi}{4}$, or if $\frac{\pi}{2} < \beta < \frac{3\pi}{4}$, the value of the cosine is $\cos[2\gamma(0)] = +1$, and

$$\bar{k}(0) = \frac{p}{p-s-2N \sin 2\beta} \tag{A-29}$$

If, however, $\frac{\pi}{4} < \beta < \frac{\pi}{2}$, or if $\frac{3\pi}{4} < \beta < \pi$, then $\cos[2\gamma(0)] = -1$, and

$$\bar{k}(0) = \frac{p - 2N \sin 2\beta}{p-s} \tag{A-30}$$

Before evaluating Eqs. (29,30) knowledge of the permissible range of $\bar{k}$ is required. From the definition, Eq. (8), for the principal stresses it is clear that $\bar{k} > k$ is a sufficient condition for the satisfaction of the Coulomb rule if σ_1 is the major stress. But there will be locations where σ_2 is the major stress, and in such locations $\bar{k}$ must not exceed an upper limit, which can be easily computed. If $s = 0$, the upper limit is simply $\frac{1}{k}$, but in general

$$k \leq \bar{k} \leq \frac{1}{k} + \frac{s(1+k)}{\sigma k} \tag{A-31}$$

Figure (A-2) gives the typical appearance of the plot of the four branches of $\bar{k}(0)$. Regardless of the values of s, the value of $\bar{k}(0)$ is always at the limit of slip if $\beta = \frac{\pi}{4}$ or $\frac{3\pi}{4}$, because the solution Eq. (15), (a constant state of stress) applies there. For $\beta < \frac{\pi}{4}$ or $\beta > \frac{3\pi}{4}$ (Fig. A-2) shows that the Coulomb rule is always violated, and no solutions exist for such angles.

In the range $\frac{\pi}{4} < \beta < \frac{3\pi}{4}$ the value $\bar{k}(0)$ has a maximum for $\beta = \frac{\pi}{2}$. To verify that Eq. (31) is satisfied at this point, Eqs. (13) are substituted in Eqs. (7) and one finds for $\phi = 0$, $\sigma_r = \sigma_\phi = -p$, $\tau = 0$, a state of stress which certainly satisfies the Coulomb rule and therefore Eq. (31). If $s = 0$, the peak value of $\bar{k}$ at $\beta = \frac{\pi}{2}$ is unity.

We have now arrived at the conclusion that in addition to Eqs. (15) suitable solutions may exist only if $\gamma(\beta) = +\frac{\pi}{4}$ and $\frac{\pi}{4} < \beta < \frac{3\pi}{4}$. It is still not known if the solutions in this range satisfy the Coulomb rule expressed by Eq. (31) everywhere or not. To confirm that (31) is satisfied it is necessary to consider the situations for $\beta \gtrless \frac{\pi}{2}$ separately. For simplicity, the argument will be demonstrated for $s = 0$ only.

For the case $\frac{\pi}{4} < \beta \leq \frac{\pi}{2}$ the value of γ ranges from $\frac{\pi}{4}$ to $\frac{\pi}{2}$ (for $\phi = 0$), such that, with the exclusion of $\phi = 0$,

$$\sin 2\gamma > 0 \quad \text{and} \quad \cos 2\gamma \leq 0 \ .$$

In the above range for β the sign of $1 + \gamma'$ given by Eq. (17) does not change, and for $\phi \neq 0$

$$1 + \gamma' > 0 \ .$$

The ratio

$$\frac{1 + \gamma'}{\sin 2\gamma} > 0$$

is therefore positive; this statement holds even in the limit $\phi = 0$.

Equation (23) gives the sign of $\bar{k}'$ for $s = 0$:

$$\text{sign } (\bar{k}') = \text{sign} \left[\left(1-\bar{k}\right)\left(\cos 2\gamma - \frac{1-\bar{k}}{1+\bar{k}}\right)\right] \ . \qquad \text{(A-32)}$$

It is known already from Fig. (A-2) that $1 \geq \bar{k}(0) > \bar{k}(\beta) \equiv k$, and the above equation indicates that, beginning at $\phi = 0$, the value of $\bar{k}'$ is negative.

As ϕ increases this could only change if $\bar{k}$ became equal to unity, which is impossible as $\bar{k}(0) \leq 1$. The value of $\bar{k}$ will therefore, decrease smoothly from $\phi = 0$ to $\phi = \beta$, such that the Coulomb rule for $\frac{\pi}{4} < \beta < \frac{\pi}{2}$ is satisfied everywhere.

For values $\frac{\pi}{2} < \beta < \frac{3\pi}{4}$ the proof is a little more complicated, because $\bar{k}(\phi)$ increases from $\phi = 0$ to a maximum value, and subsequently decreases to $\bar{k}(\beta) \equiv k$.

According to Eq. (26), γ ranges from $\gamma(0) = \pi$ to $\gamma(\beta) = \frac{\pi}{4}$. Equation (16) yields easily the locations where $\gamma = \frac{3\pi}{4}$ or $\frac{\pi}{2}$, respectively:

$$\gamma(\beta - \frac{\pi}{2}) = \frac{3\pi}{4} \quad \text{and} \quad \gamma(2\beta - \pi) = \frac{\pi}{2} \quad .$$

Using this knowledge one can show again that

$$\frac{1 + \gamma'}{\sin 2\gamma} > 0$$

for the entire range $0 \leq \phi \leq \beta$, and the sign of $\bar{k}'$ is therefore again given by Eq. (32). Substituting the appropriate values of γ at $\phi = 0$, β and noting that in these locations $\bar{k} < 1$, one finds

$$\bar{k}'(0) > 0 \quad , \quad \bar{k}'(\beta) < 0$$

indicating a maximum of $\bar{k}$ in the interior. To locate the maximum, examine the value of $\bar{k}$ at $\phi = \beta - \frac{\pi}{2}$, where $\gamma = \frac{3\pi}{4}$ and $\cos 2\gamma = 0$, $\sin 2\gamma = -1$.

In general, the value of $\bar{k}(\phi)$ is given by Eq. (28), but this equation breaks down if $\cos 2\gamma = 0$. However, using Eq. (16) one finds

$$\frac{\sin 2(\beta-\phi)}{\cos 2\gamma} = \frac{\cos 2\beta - \cos 2(\beta-\phi)}{\sin 2\gamma} \tag{A-33}$$

Substituting this expression in Eq. (28), one obtains a valid expression for $\phi = \beta - \frac{\pi}{2}$. Forming the combination $\frac{1-\bar{k}}{1+\bar{k}}$, one obtains (for $s = 0$)

$$\left.\frac{1-\bar{k}}{1+\bar{k}}\right|_{\phi = \beta - \frac{\pi}{2}} = \frac{(1 + \cos 2\beta)\,\frac{1-k}{1+k}}{1 - \cos 2\beta - \pi\,\frac{1-k}{1+k}\cos 2\beta} > 0 \quad . \tag{A-34}$$

The inequality is easily proved, because in the range considered we have $0 > \cos 2\beta > -1$.

The above inequality implies $\bar{k} < 1$, and Eq. (32), with $\cos 2\gamma = 0$, gives

$$\left.\bar{k}'\right|_{\phi = \beta - \frac{\pi}{2}} < 0$$

indicating that the maximum value of k occurs between $\phi = 0$ and $\phi = \beta - \frac{\pi}{2}$. In this range the angle γ is limited by $\frac{3\pi}{4} < \gamma < \pi$ or $0 < \cos 2\gamma < 1$.
For the maximum value of $\bar{k}$ the derivative (32) must vanish. Approaching from $\phi = 0$, having started with $\bar{k} < 1$, and noting the above inequality on $\cos 2\gamma$, it is clear that the second factor in Eq. (32) will be the one to vanish when $\bar{k} = \max.\ \bar{k}$; or

$$\cos 2\gamma - \frac{1 - \max \bar{k}}{1 + \max \bar{k}} = 0 \quad .$$

In this equation $\cos 2\gamma \geq 0$, such that $\max \bar{k} \leq 1$ completing the proof.

In the final stage, only the case $s = 0$ has been considered. However, the values of $\bar{k}$ being a continuous function of s, one reasons that the conclusions must also hold for small values of the ratio $\frac{s}{p}$ (small vs. unity). This region suffices for the present purpose, as values of p comparable to s are not of interest anyway*).

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*) It is demonstrated in subsection b that elastic solutions exist up to some value $\frac{p}{s} > 1$, the critical ratio being a function of k. If solutions for wedges without slip, satisfying the Coulomb rule everywhere did exist for all values of $\frac{\pi}{4} < \beta < \frac{3\pi}{4}$, even for values where $\frac{s}{p}$ is not small vs. unity, they would ultimately lead to solutions (for the half-space) of the steady-state problem with slip, for values p where elastic solutions also exist. In such a case solutions with slip would not be expected to have physical significance, even if they do exist.

Summarizing, it has been found that for $\gamma(\beta) = \frac{\pi}{4}$ solutions exist only for wedges $\frac{\pi}{4} \leq \beta \leq \frac{3\pi}{4}$, while for $\gamma(\beta) = -\frac{\pi}{4}$ solutions (Eq. 15) exist only for $\beta = \frac{\pi}{4}$ or $\beta = \frac{3\pi}{4}$.

For the case $\gamma(\beta) = \frac{\pi}{4}$, the following expression for the value of the stress parameter σ on the inclined face $\phi = \beta$ can be obtained:

$$\sigma(\beta) = \frac{2p(1 - \cos 2\beta) - s(1 - \cos 2\beta - 2\beta \cos 2\beta + \sin 2\beta)}{(1+k)(1 - \cos 2\beta) + (1-k)(\sin 2\beta - 2\beta \cos 2\beta)} \qquad \text{(A-35)}$$

b. SOLUTIONS WITHOUT SLIP FOR THE HALF-SPACE.

In order to determine up to what level of the load p, the elastic stresses satisfy the Coulomb rule for a half-space, the left side of which is loaded, (Fig. A-3), the solution is obtained from Eq. (6). Surprisingly, the four boundary conditions, $\sigma_\phi(0) = 0$, $\sigma_\phi(\pi) = -p$, $\tau(0) = \tau(\pi) = 0$ permit the determination of only three of the four constants. The last one, D, remains open, such that the stresses become, with $\bar{D} = -\frac{\pi}{p} D$

$$\frac{\pi\sigma_r}{p} = -\phi - \tfrac{1}{2}\sin 2\phi + \bar{D}(1 + \cos 2\phi)$$

$$\frac{\pi\sigma_\phi}{p} = -\phi + \tfrac{1}{2}\sin 2\phi + \bar{D}(1 - \cos 2\phi) \qquad \text{(A-36)}$$

$$\frac{\pi\tau}{p} = \tfrac{1}{2}(1 - \cos 2\phi) + \bar{D}\sin 2\phi \quad .$$

A restriction on the constant $\bar{D}$ follows immediately by considering the stresses at the free surface $\phi = 0$; $\sigma_\phi = \tau = 0$ and $\sigma_r = \frac{2\bar{D}p}{\pi}$. Requiring one principal stress to be compressive, the Coulomb rule gives

$$0 \geq \bar{D} \geq -\frac{s\pi}{2p} \quad . \qquad \text{(A-37)}$$

The free constant is physically equivalent to an additional uniform horizontal state of stress $\sigma_x = \frac{2\bar{D}p}{\pi}$, $\sigma_y = \tau = 0$. The lack of uniqueness of the solution is a consequence of the lack of a prescribed state of stress at infinity.

To determine if slip occurs at any point, the value of

$$\bar{k} = \frac{\sigma_2}{\sigma_1 + s}$$

is required. Using the usual expressions for the principal stresses one obtains

$$\bar{k} = \frac{\phi - \bar{D} - \sqrt{\Delta}}{\phi - \bar{D} + \sqrt{\Delta} - \frac{\pi s}{p}} \qquad \text{(A-38)}$$

where

$$\Delta = \sin^2\phi + \bar{D}^2 - 2\bar{D}\sin\phi\cos\phi \tag{A-39}$$

If the value of $\bar{D}$ is given, one could find the minimum value of $\bar{k}$ for any ratio $\frac{p}{s}$, and the solution would satisfy the Coulomb rule when $k \leq \min \bar{k}$. A change of $\bar{D}$ will, of course, effect min $\bar{k}$; it is of interest to know the most beneficial value of $\bar{D}$ which raises min $\bar{k}$ to its highest value, because it defines values $\frac{p}{s}$ above which elastic solutions without slip cannot exist.

To find extremal values of $\bar{k}$ with respect to ϕ and $\bar{D}$, the somewhat simpler expression $1 + \bar{k}$ is formed,

$$1+\bar{k} = \frac{2(\bar{D}-\phi) + \frac{\pi s}{p}}{\bar{D} - \phi - \sqrt{\Delta} + \frac{\pi s}{p}} \tag{A-40}$$

and differentiation with respect to ϕ and $\bar{D}$ gives two equations

$$\left[2(\bar{D}-\phi) + \frac{\pi s}{p}\right]\left[-1 - \frac{1}{2\sqrt{\Delta}}\frac{\partial\Delta}{\partial\phi}\right] = -2\left(\bar{D} - \phi - \sqrt{\Delta} + \frac{\pi s}{p}\right)$$

$$\left[2(\bar{D}-\phi) + \frac{\pi s}{p}\right]\left[1 - \frac{1}{2\sqrt{\Delta}}\frac{\partial\Delta}{\partial\bar{D}}\right] = 2\left(\bar{D} - \phi - \sqrt{\Delta} + \frac{\pi s}{p}\right) . \tag{A-41}$$

Addition of these equations indicates that either

$$2(\bar{D}-\phi) + \frac{\pi s}{p} = 0 \tag{A-42}$$

or

$$\frac{\partial\Delta}{\partial\phi} + \frac{\partial\Delta}{\partial\bar{D}} = 0 . \tag{A-43}$$

If Eq. (42) applies, the right hand side of (41) must also vanish, giving $\phi - \bar{D} = \sqrt{\Delta}$, and according to Eq. (38), $\bar{k} = 0$. This situation is trivial. The alternative, Eq. (43), becomes after substitution

$$4\bar{D}\sin^2\phi = 0 \tag{A-44}$$

yielding the conditions

$$\bar{D} = 0, \text{ or } \phi = 0, \text{ or } \phi = \pi . \tag{A-45}$$

The last two possibilities, $\phi = 0$ or π, lead again to trivial cases.

This leaves the root $\bar{D} = 0$. Substitution of this value in either of Eqs. (41) gives a relation between $\frac{s}{p}$ and ϕ :

$$\frac{p}{s} = \frac{\pi(1 - \cos\phi)}{2(\sin\phi - \phi\cos\phi)} \quad . \tag{A-46}$$

It can be shown that the solution minimizes $\bar{k}$ with respect to ϕ , and that $\bar{D} = 0$ gives the highest values of the minimum as desired. Computing min $\bar{k}$ and equating it to the value k of the material to be considered, one finds

$$k = \min \bar{k} = \frac{1 - \cos\phi}{1 + \cos\phi} \quad . \tag{A-47}$$

This is a parametric representation of $\frac{p}{s}$ as function of k. It defines the values of p above which no elastic solution without slip exist. Figure (A-4) shows a plot of $\frac{p}{s}$ versus k. It is noted that the location ϕ at which the minimum occurs varies with k; for small values of k the angle ϕ is also quite small; ϕ increases with k, and reaches the value $\phi = \frac{\pi}{2}$ for k = 1.

APPENDIX B

ON THE DISINTEGRATION OF BODIES OF MATERIALS GOVERNED BY THE COULOMB RULE.

In the materials considered the premissible states of stress are limited by the inequality (2-3)

$$- \sigma_2 \geq k\sigma.$$

If one subjects a body to loads where this condition cannot be satisfied, the body being unable to support the loads must disintegrate entirely or at least in part. To establish the manner in which the disintegration occurs requires a physical description of the behavior of the elements of the body beyond the mathematical statements in Section I. In the following a simple situation will be considered in which the material can be represented by a mechanical model which has properties in agreement with relations Eq. (2-2 to 2-8) when the cohesion s vanishes. (An alternative model with cohesion can also be formed).

Consider the plane structure covering a rectangular area, (Fig. B1); the structure is built of identical square elements and carries loads P_1 in one direction, and P_2 in the other. Each element consists of four masses connected by inextensible linkage bars under 45 degrees to the vertical, (Fig. B2). Let the elements be small and of width 2a and height 2a. The hinges are assumed to have friction, the friction moments being proportional to the load transferred to the hinge by the bar. If the unit is to be in equilibrium under forces $P_1 > P_2$ acting on the element, (Fig. B2), the forces must satisfy the inequality

$$P_1 \geq P_2 > kP_1 \qquad \text{(B-1)}$$

If however

$$P_2 = kP_1 \qquad \text{(B-2)}$$

the unit can be in steady motion or at rest. The coefficient k can be obtained by equilibrium considerations from the friction in the hinges, where by reason of symmetry the forces in one link only need be considered.

The friction in a hinge may be defined by the maximum normal distance f at which the resultant force S in the linkage bar will bypass the center of the hinge. This applies to the hinges at both ends of the link and due to the type of linkage motion, the resultant must pass through the mid-point of the linkage, (Fig. B-3). The resultant will therefore make an angle α with the link, such that

$$f = \frac{a \sin \alpha}{\sqrt{2}} \tag{B-3}$$

Later, it will be convenient to know the connection between α and k. Equilibrium at the four hinges requires the following two relations between the compressive resultant S and the forces P_1 and $P_2 = kP_1$, acting on the element in the limiting state,

$$P_1 = 2S \cos\left(\frac{\pi}{4} - \alpha\right)$$

$$P_2 = kP_1 = 2S \sin\left(\frac{\pi}{4} - \alpha\right)$$

or

$$k = \tan\left(\frac{\pi}{4} - \alpha\right) \quad . \tag{B-4}$$

The changes in vertical and horizontal "strain" $\frac{\epsilon}{a}$, (Fig. B-2), will be of equal magnitude in similarity with Eq. (2-8). It is further assumed that the angular motion of the links is restricted to a small angle, such that the motion ϵ cannot exceed a set value $\bar{\epsilon}$. When this situation is reached, the unit will be deemed to become rigid.

The element described behaves like the material considered under A in Section 2, provided the vertical axis of the element, (Fig. B-2), is in the direction of the major principal stress. The only difference lies in the limitation on the strain resulting from the condition $\epsilon \leq \bar{\epsilon}$.

A number (2N) of such elements arranged in a horizontal line, (Fig. B-4), will be in incipient motion if each element is vertically loaded by the force P, while horizontal forces kP are applied at the ends and transmitted from element to element. The arrangement simulates a rectangular region of a Coulomb material at the verge of slip. Let us determine the response of the model if the forces kP on the sides are suddenly removed. The behavior of the model ought to give an insight on what will happen in case of a granular material.

It has already been stated that the linkage bars are inextensible; for simplicity it is further assumed that they are massless, while each of the four masses at the hinges has one quarter of the total mass M of each unit. Also, for simplicity, it is assumed that rotation of the masses is prevented (e.g. on top and bottom by the stiffness of plates in a testing machine by means of which the loads P are applied, while the other masses will not rotate by reason of symmetry).

Consider the motion of the i-th element, (Fig. B-5). It is acted on by the external forces P on top and bottom; by as yet unknown horizontal forces P_{i-1} and P_i from the two adjoining elements; in addition, at each end of each linkage bar there will be friction moments acting on these bars, fS_L and fS_R, where S_L and S_R are the resultants transferred by the left and right linkage, respectively. At the top and bottom hinges, the friction moments do not balance, the difference being supplied by the external support preventing rotation.

The motion of the element is fully described by two generalized coordinates: the horizontal displacement x_i of the center O_i of the unit, and the relative motions ϵ_i , horizontal and vertical, respectively, of the mass points with respect to the center O_i . For reasons of symmetry in the arrangement, (Fig. B-4), no vertical motion of the center O_i need be considered. For small motions, the rotation of the linkages is $\frac{\epsilon}{a}$. To write Lagrange's equations, one obtains the kinetic energy

$$2T = M(\dot{x}_i^2 + \dot{\epsilon}_i^2) \tag{B-5}$$

while the potential energy vanishes. The generalized forces Q_{x_i} and Q_{ϵ_i} due to the physical forces shown in (Fig. B-5) are

$$\begin{aligned} Q_{x_i} &= P_{i-1} - P_i \\ Q_{\epsilon_i} &= 2P - P_{i-1} - P_i - \frac{4f}{a}(S_L + S_R) . \end{aligned} \tag{B-6}$$

Noting Eq. (3), the equations of motion become

$$M\ddot{x}_i = Q_{x_i} = P_{i-1} - P_i \tag{B-7}$$

$$M\ddot{\epsilon}_i = Q_{\epsilon_i} = 2P - P_{i-1} - P_i - 2\sqrt{2}\sin\alpha\,(S_L + S_R). \tag{B-8}$$

The second equation still contains the resultants S_L and S_R in the linkage bars, which can be expressed in terms of $\ddot{x}_i$ and $\ddot{\epsilon}_i$. The forces acting on the two lateral masses are shown in (Fig. B-6); their accelerations are $\ddot{x}_i \pm \ddot{\epsilon}_i$, respectively, such that

$$\frac{M}{4}(\ddot{x}_i - \ddot{\epsilon}_i) = P_{i-1} - 2\sin\left(\frac{\pi}{4} - \alpha\right) S_L$$

$$\frac{M}{4}(\ddot{x}_i + \ddot{\epsilon}_i) = 2\sin\left(\frac{\pi}{4} - \alpha\right) S_R - P_i$$

or

$$2\sin\left(\frac{\pi}{4} - \alpha\right)(S_R + S_L) = \frac{M}{2}\ddot{\epsilon}_i + (P_i + P_{i-1}). \tag{B-9}$$

Substitution into Eq. (8) gives

$$\frac{1+k}{2k} M\ddot{\epsilon}_i = 2P - \frac{1}{k}(P_i + P_{i-1}) . \tag{B-10}$$

where the relation

$$\frac{\sin \alpha}{\sqrt{2}\,\sin(\frac{\pi}{4} - \alpha)} \equiv \frac{1-k}{2k} \tag{B-11}$$

was obtained from Eq. (4).

Eq. (10) was derived on the tacit understanding that $\dot{\varepsilon}_i > 0$, indicating that the element deforms in the manner shown in (Fig. B-5). An alternative relation valid if $\dot{\varepsilon}_i < 0$ could easily be derived, but is not required for the present purpose. Further, it is possible that the external and inertial forces are such that the friction is sufficient to prevent any deformation of the unit, such that $\dot{\varepsilon}_i = \ddot{\varepsilon}_i = 0$. If, in this case the unit is not subject to acceleration, $\ddot{x}_i = 0$, the forces P_i and P_{i-1} will be equal,

$$P_i = P_{i-1} \tag{B-12}$$

and must satisfy the Coulomb-inequality

$$\frac{1}{k} P > P_i > kP \tag{B-13}$$

Equivalent relations for $\ddot{x}_i \neq 0$ could be derived, but will not be required for the present purpose.

The differential equations (7) and (10) and the relations (12, 13) permit determination of the response of the group of elements shown in (Fig. B-4) when the lateral loads kP are suddenly removed at a time t = 0. Considering x_i, ε_i and P_i as functions of t, the forces P_N and P_{-N} must vanish for $t \geq 0$:

$$P_N \equiv P_{-N} = 0 \tag{B-14}$$

Further, when adjoining elements are in contact, the geometric relation

$$x_{i+1} - x_i = \varepsilon_{i+1} + \varepsilon_i \tag{B-15}$$

must be satisfied. At $t = 0$, Eq. (15) will apply for all values $N-1 > i \geq -N$, but thereafter separation might occur if $P_i \leq 0$, or due to the limitation on the motion of the linkages, $\varepsilon_i \leq \bar{\varepsilon}$. These possibilities, and the question whether the strain rate $\dot{\varepsilon}_i$ is positive, negative or zero, must be carefully considered.

To start, the simple case of two elements, $N = 1$, will be solved, (Fig. B-7), where for reasons of symmetry only unit (1) need be studied. Subject to later check, it is assumed that $\dot{\varepsilon}_1 > 0$, such that Eqs. (7) and (10) apply.

$$M \ddot{x}_1 = P_o$$

$$\frac{1+k}{2k} M \ddot{\varepsilon}_1 = 2 P - \frac{1}{k} P_o \tag{B-16}$$

Eq. (15) becomes

$$x_1 = \varepsilon_1 \tag{B-17}$$

and elimination of $\ddot{x}_1$ and $\ddot{\varepsilon}_1$ gives the simple relation

$$P_o = \frac{4k}{3+k} P \tag{B-18}$$

P_o is positive as required. Further, using the initial conditions $\varepsilon_1 = \dot{\varepsilon}_1 = x_1 = \dot{x}_1 = 0$, one finds

$$\dot{\varepsilon}_1 > 0, \ \dot{x}_1 > 0, \ \varepsilon_1 > 0, \ x_1 > 0$$

For sufficiently small values of t all conditions for the validity of Eqs. (16) are therefore satisfied. The centroids 0_i of the elements move away from each other, while the "strain rate" $\dot{\varepsilon}_1$ is positive. However, at an instant $t = \bar{T}$, the strain ε_i reaches the limit $\bar{\varepsilon}$, and the elements freeze, $\dot{\varepsilon}_1 = 0$.

At this instant, the centroids of both elements still have outward velocities, $\dot{x}_1 > 0$, such that the elements will separate and fly apart.

The value of the pressure between the two elements prior to separation is given by Eq. (18). It is noted that the value P_0, regardless of the exact value of k, is always slightly larger than the value kP required to prevent slip under static conditions. This means that the solution for a larger number of elements, N > 1, can be stated without further analysis. For $t < \bar{t}$, the outside elements N, -N, in Fig. (B-4) behave exactly as the element just analyzed in the case N = 1. The interior elements, N-1 to -(N-1) do not move or change shape, because the force, Eq. (18), exerted by the outside elements does not produce slip. At $t = \bar{t}$, the outside elements separate, and a new situation arises. The elements (N-1) and -(N-1) begin to move, while no motion occurs in the other elements, until separation occurs again, etc.

We find, therefore, that in case of a large number of elements, the disintegration of the model structure occurs in a layer of elements at the boundary. The accelerations of the elements in the boundary layer produce a reactive pressure which produces in the interior a state of stress satisfying the Coulomb-rule?

In the above derivations, linearity was assumed, and enforced by permitting only a finite small deformation $\bar{\epsilon}$ of the elements. If the restriction had been dropped, the element, after experiencing large strains, could still not exceed a certain finite strain where the linkage is fully extended; separation would therefore occur, and is not just due to the assumption $\epsilon_1 \leq \epsilon$.

Visualizing the mechanism of acceleration and subsequent separation in a granular medium, one can see, (Fig. B-8), how a grain 3 might be expelled by appropriate wedge-like surfaces of two other grains, 1 and 2, if the latter are pressed together.

In analogy to the model, we expect that on a free surface in a Coulomb-type granular material, disintegration will occur if the problem has no solution for which the pressure acting on the surface vanishes. The process of disintegration and acceleration will occur in a thin boundary layer, and will restore a pressure sufficient to satisfy the Coulomb inequality below the surface.

One might be tempted to critise the use of the mechanical model because it cannot represent a Coulomb material if the principal stresses should change direction during loading. However, this criticism is not pertinent because the process of disintegration occurs in the immediate vicinity of a surface where such directions necessarily cannot change.

The model could be adapted to permit representation of a material with dilatancy, or with cohesion. In the former case, linkages making angles other than 90° would be used. A model for the second case is obtained by specifying the friction in the hinges to be a constant plus fS. Such a model could also be given tensile strength by providing attraction between the elements. The basic conclusions are not affected by the use of the above modifications.

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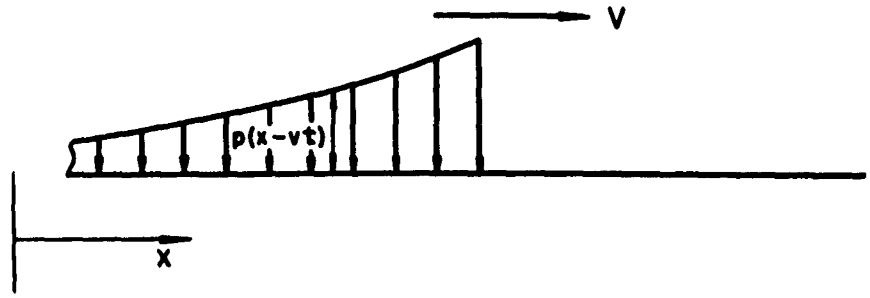


FIG. 1

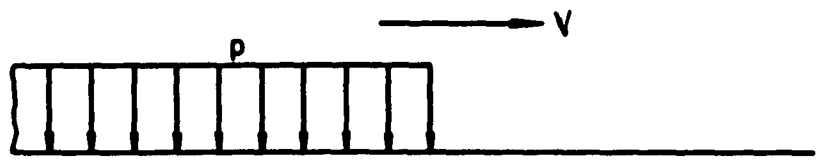


FIG. 2

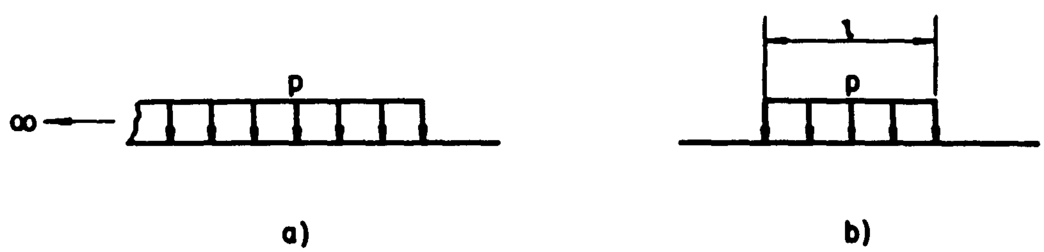


FIG. 3

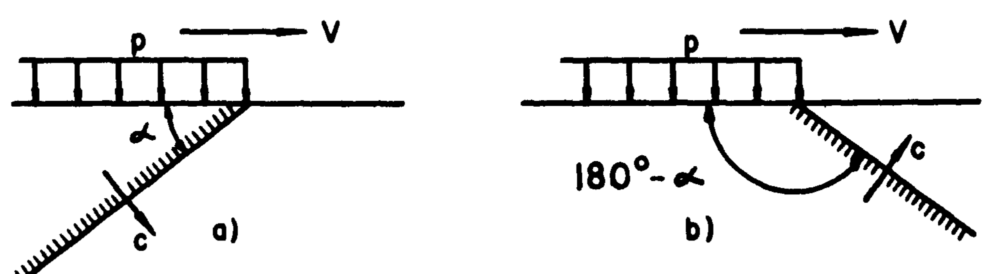
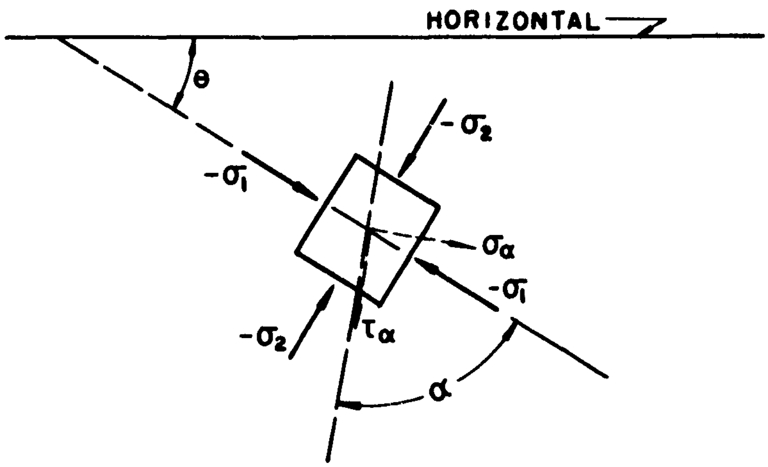
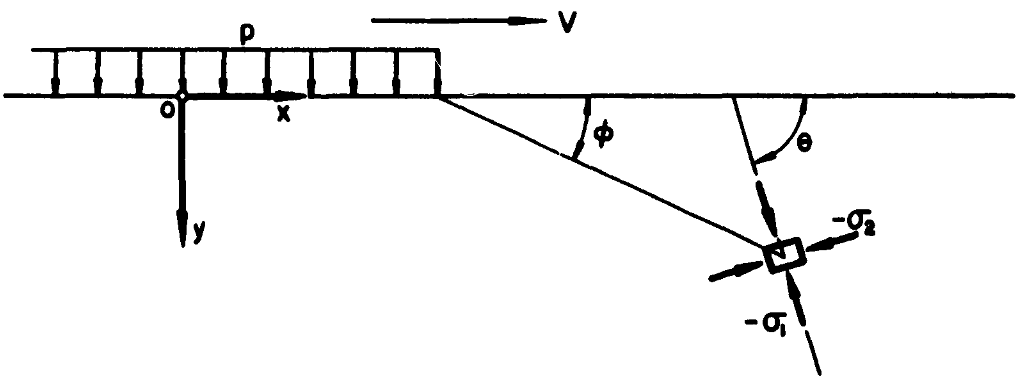


FIG. 4


HORIZONTAL
θ
$-\sigma_2$
$-\sigma_1$
σ_α
$-\sigma_1$
$-\sigma_2$
τ_α
α

F I G . 5


p
V
O
x
y
φ
θ
$-\sigma_2$
$-\sigma_1$

F I G . 6

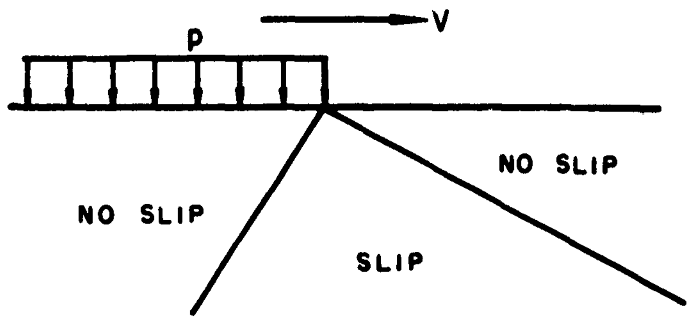


FIG. 7

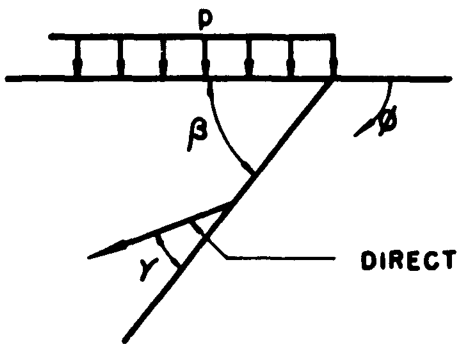


FIG. 8a

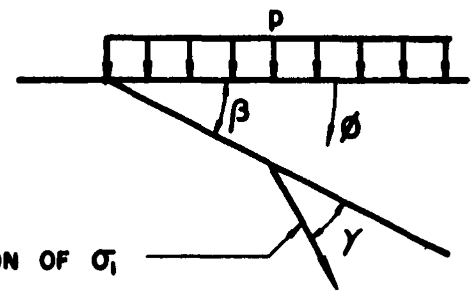


FIG. 8b

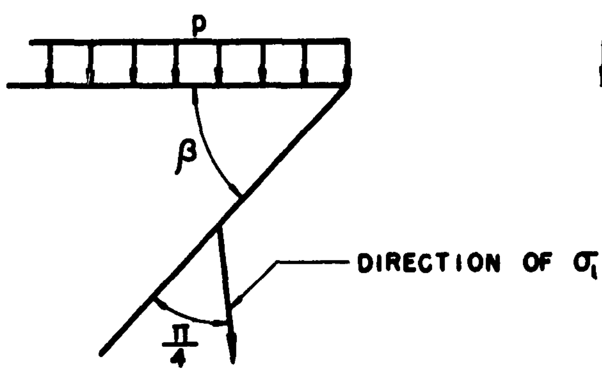


FIG. 9a

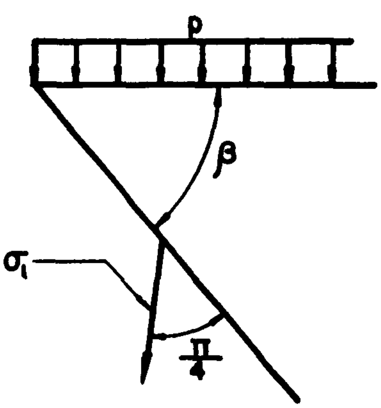


FIG. 9b

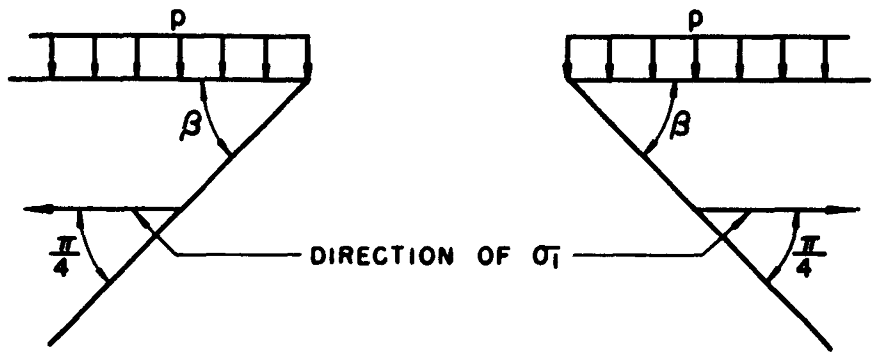


FIG 10a

FIG 10b

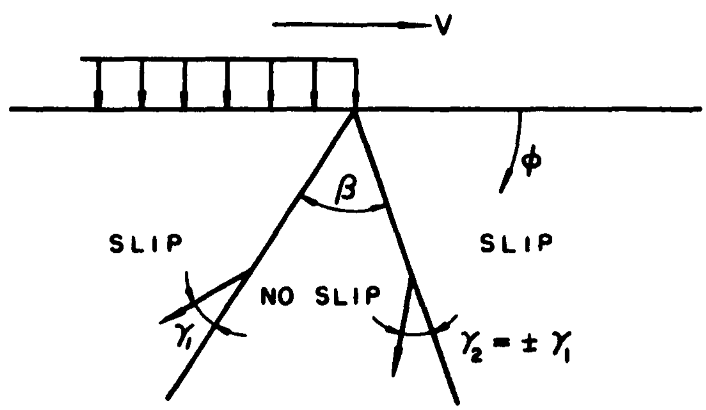


FIG. 11

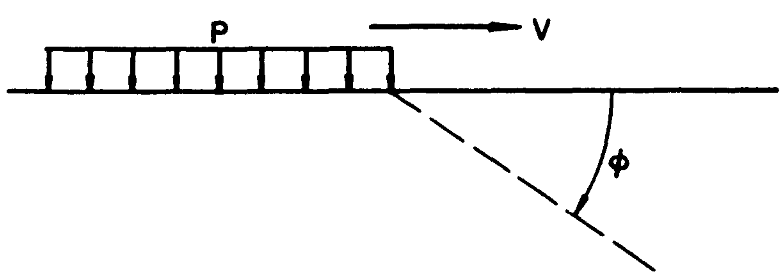


FIG. 12

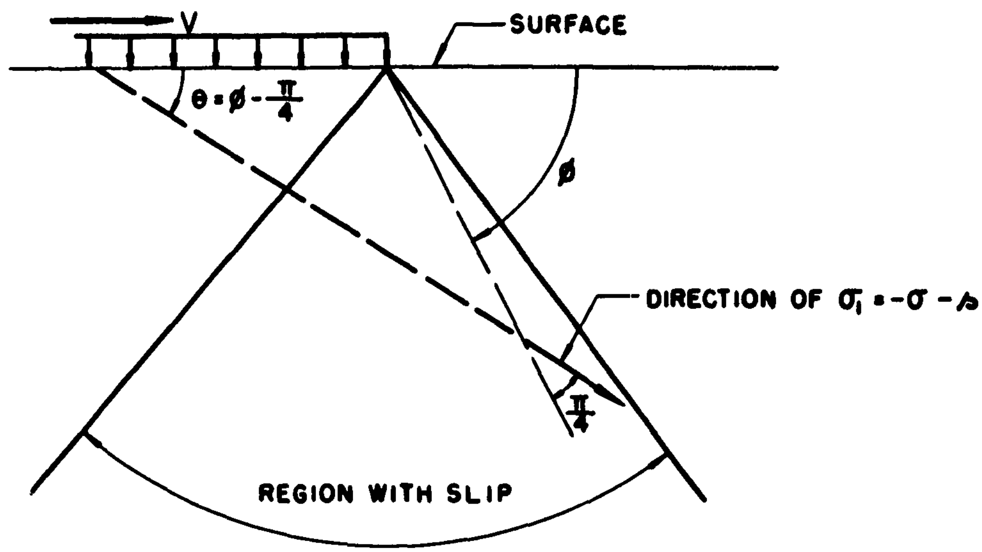


FIG. 13

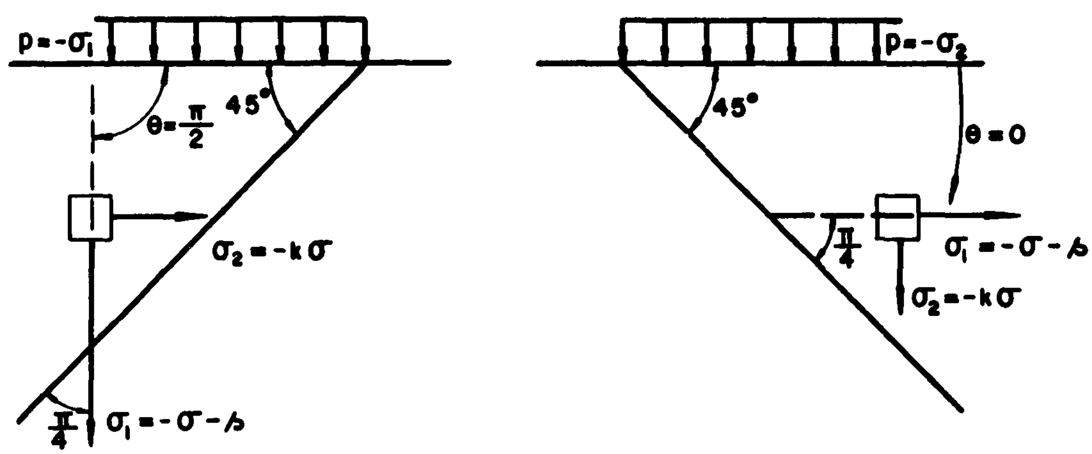


FIG. 14a

FIG. 14b

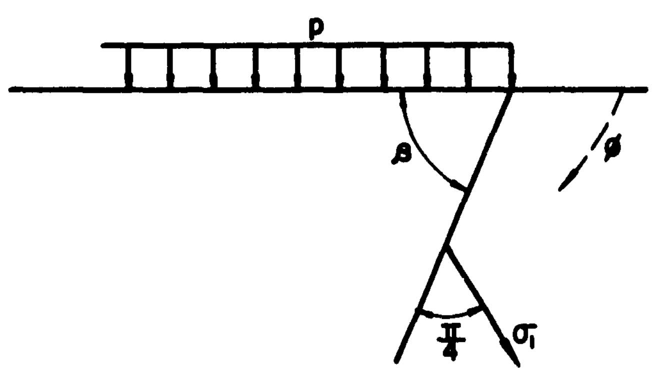


FIG. 15

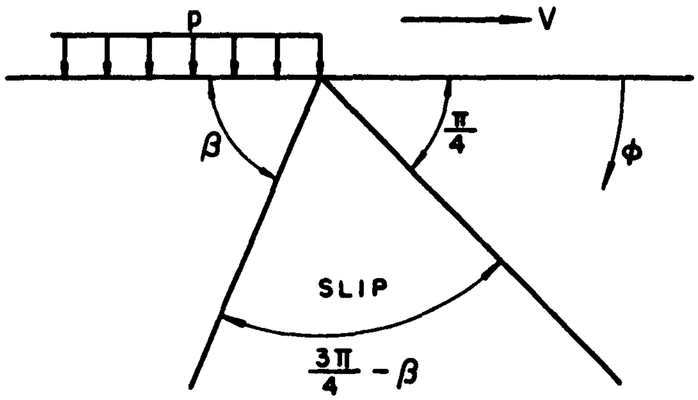

p
V
β
$\frac{\pi}{4}$
ϕ
SLIP
$\frac{3\pi}{4} - \beta$

FIG. 16

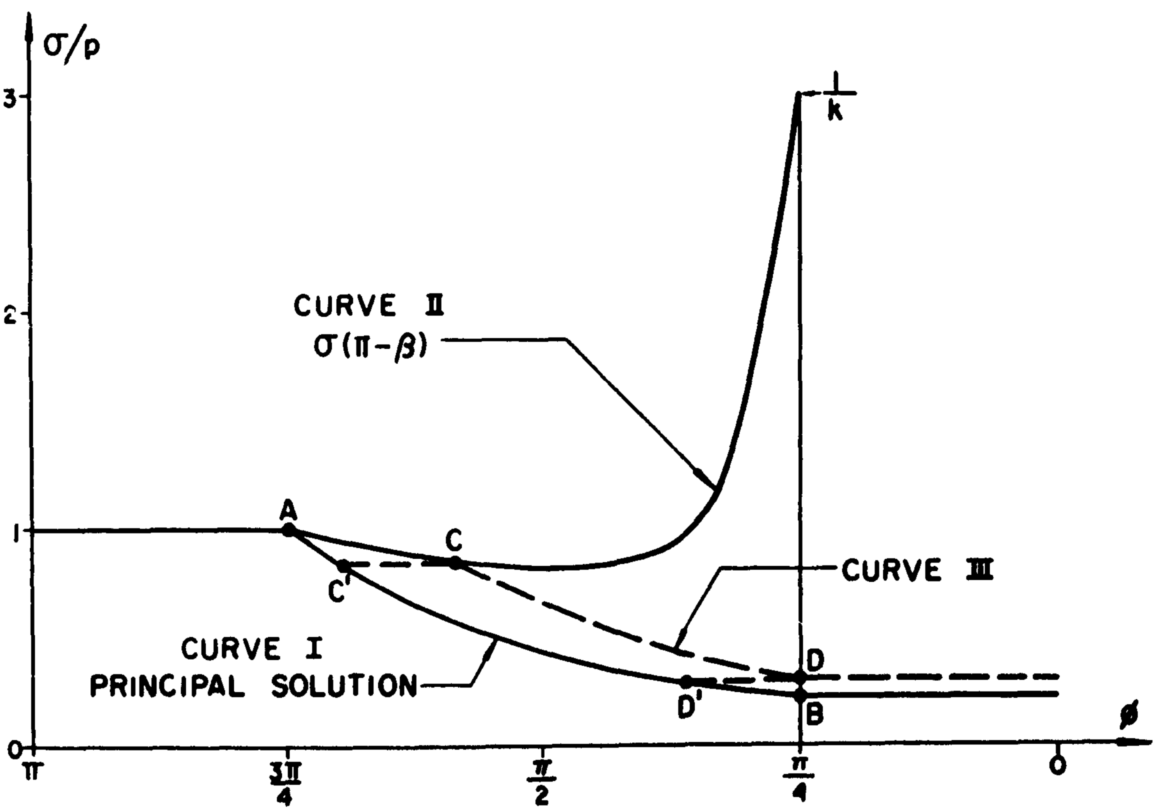

σ/p
3
2
1
0
$\frac{1}{k}$
CURVE II
$\sigma(\pi-\beta)$
CURVE III
CURVE I
PRINCIPAL SOLUTION
A
C
C'
D
D'
B
ϕ
π
$\frac{3\pi}{4}$
$\frac{\pi}{2}$
$\frac{\pi}{4}$
0

FIG. 17

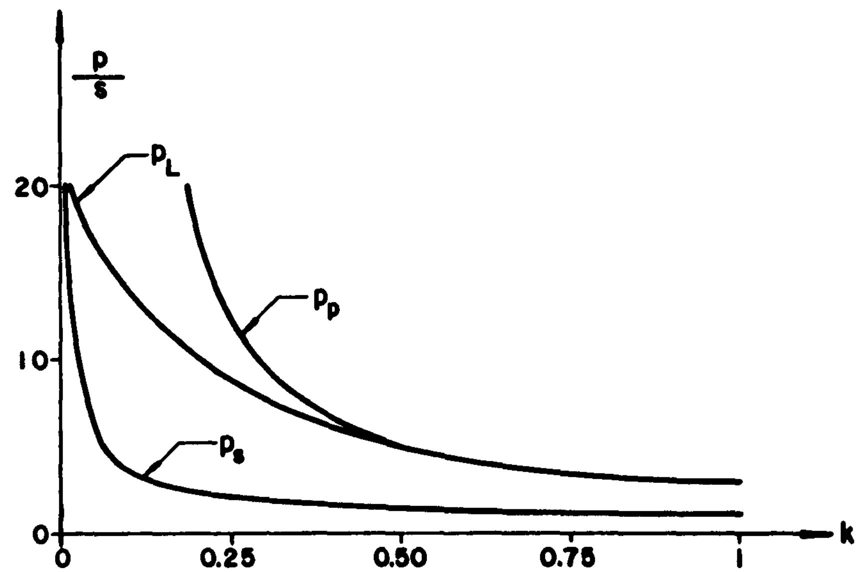


FIG.18

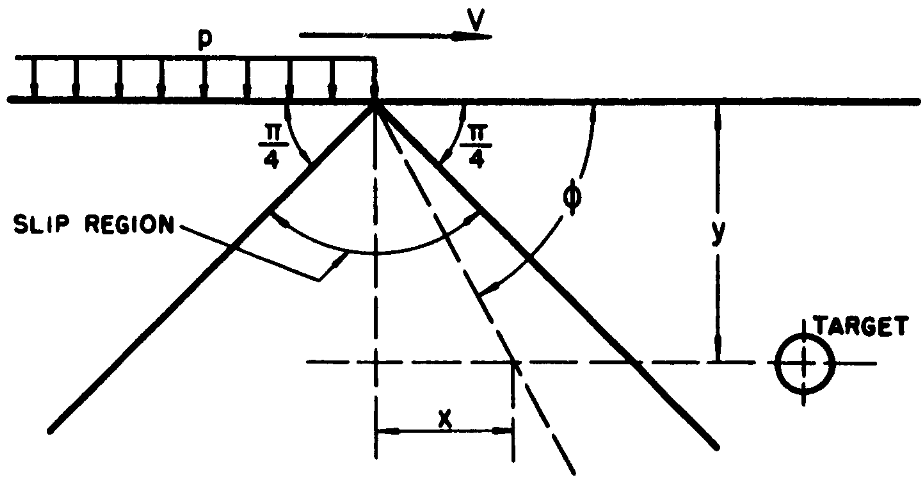
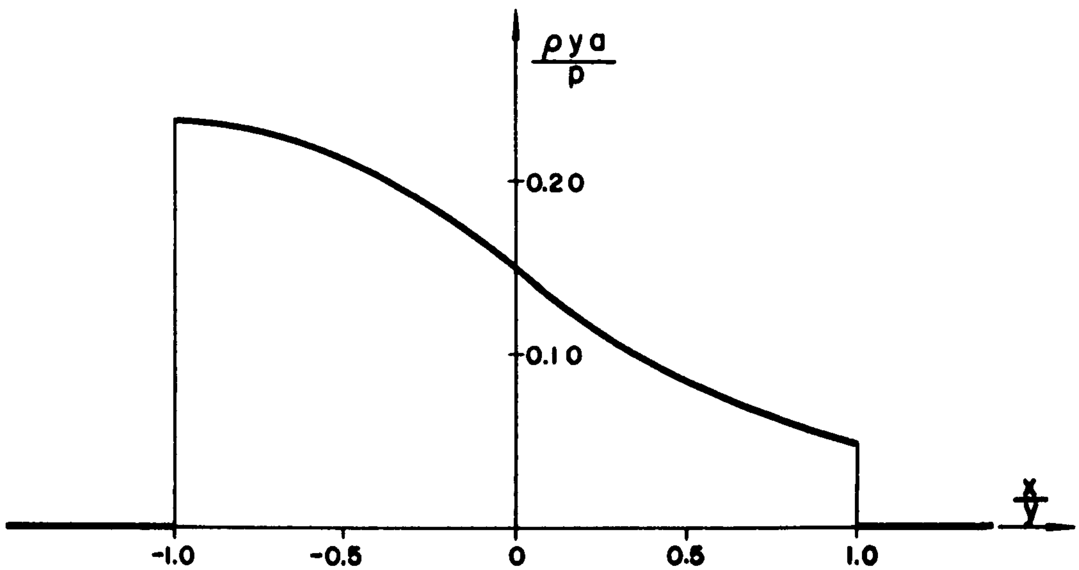


FIG. 19



RADIAL ACCELERATION a

FIG. 20

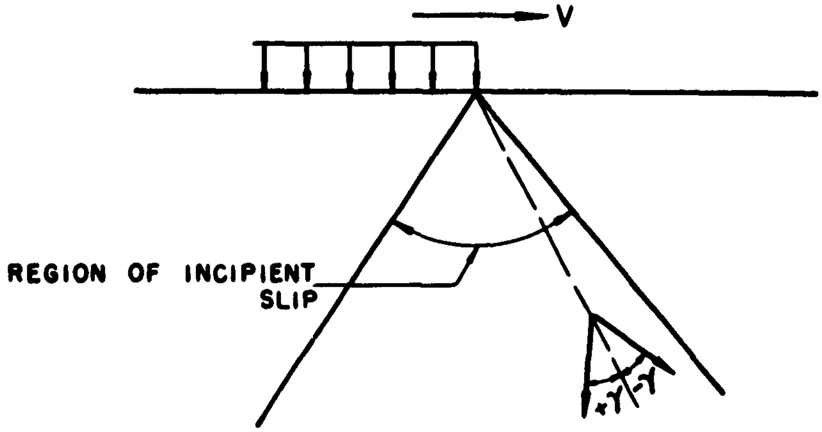

V
REGION OF INCIPIENT SLIP
+γ
-γ

FIG. 21

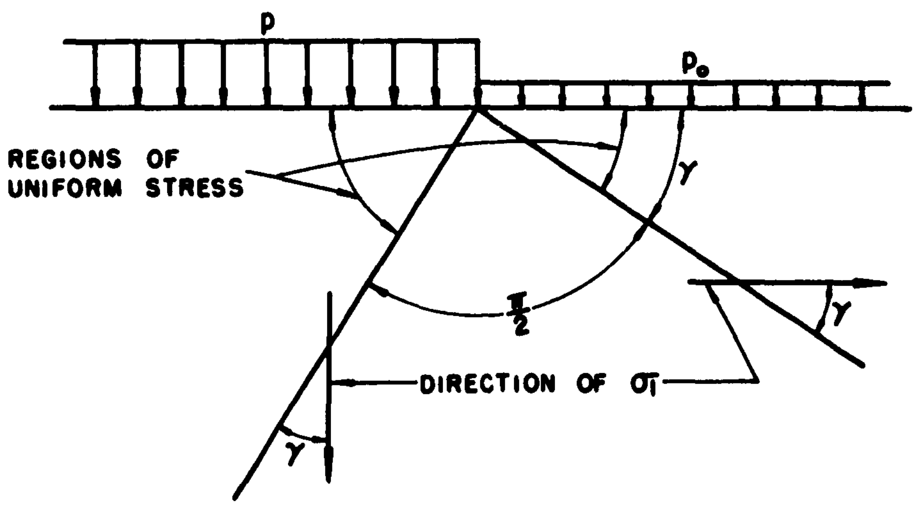
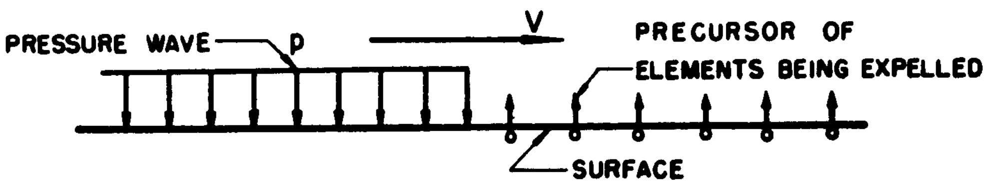

p
p_o
REGIONS OF UNIFORM STRESS
γ
$\frac{\pi}{2}$
γ
DIRECTION OF σ_1
γ

FIG. 22



PRESSURE WAVE
p
V
PRECURSOR OF
ELEMENTS BEING EXPELLED
SURFACE

FIG. 23

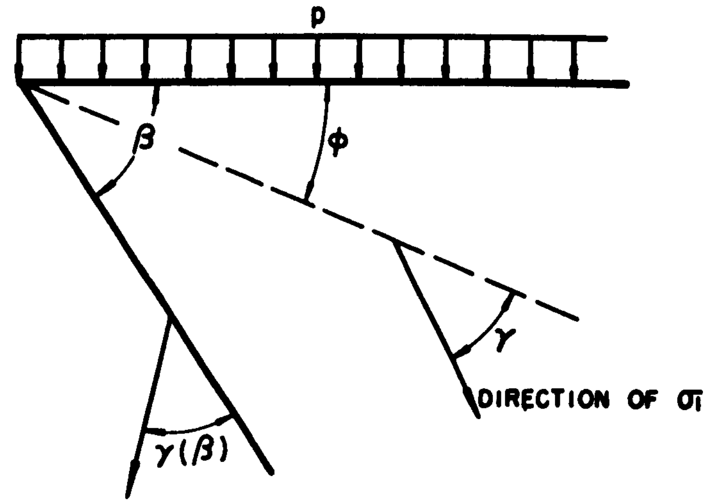


FIG. A-1

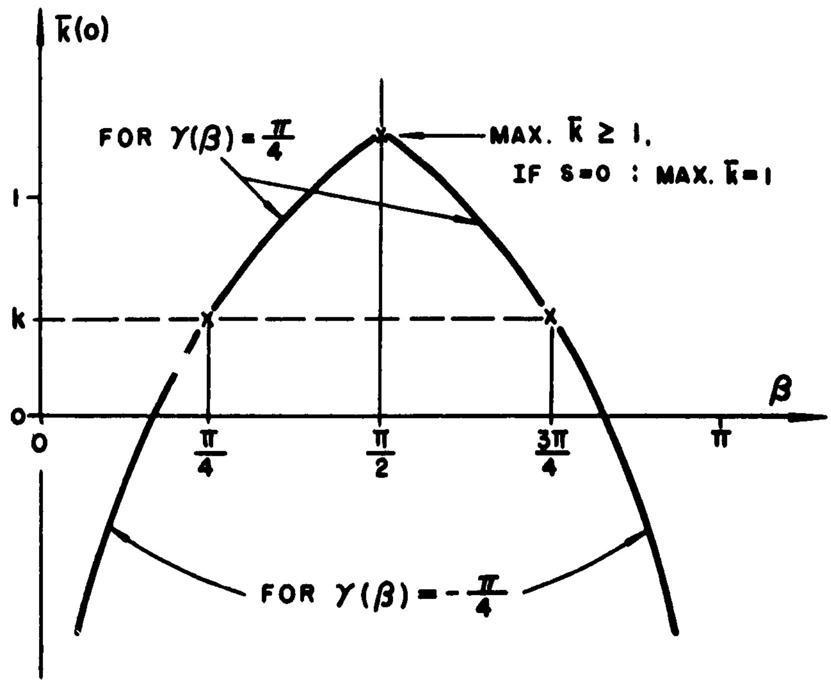


FIG A-2 TYPICAL PLOT OF $\bar{k}(o)$ VS. β

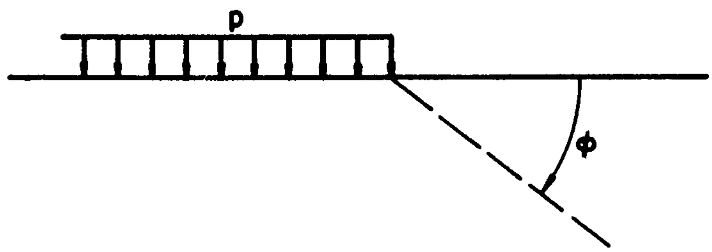


FIG. A-3

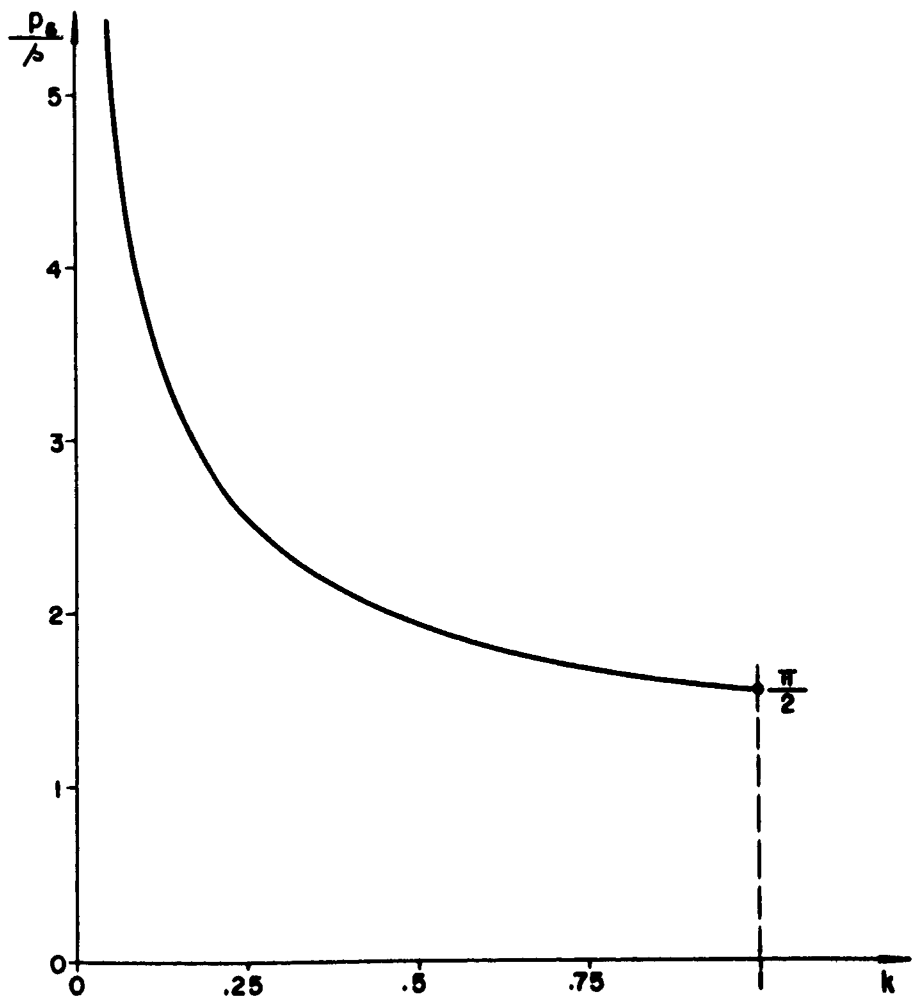
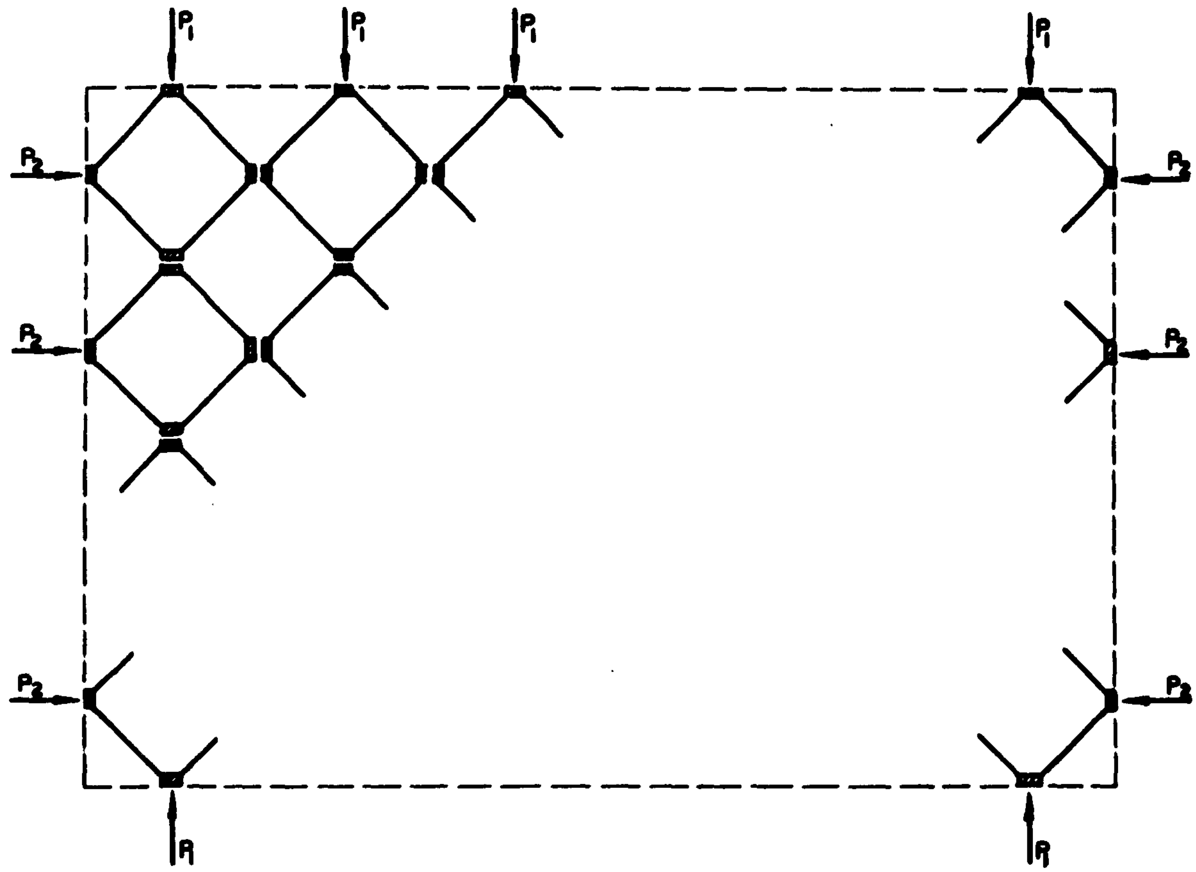
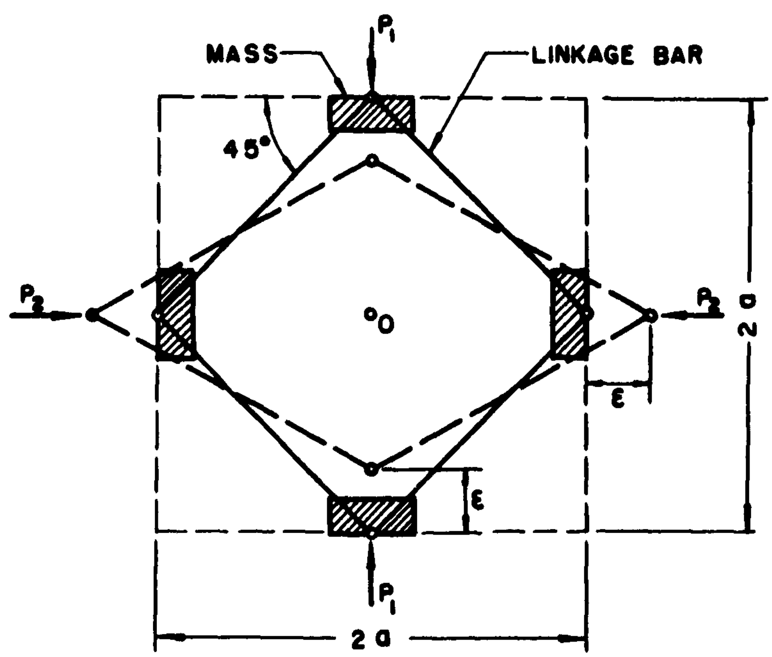


FIG. A-4



P1
P1
P1
P1
P2
P2
P2
P2
P2
P2
P1
P1

FIG. B-1



P1
MASS
LINKAGE BAR
45°
P2
O
P2
2a
ε
ε
P1
2a

FIG. B-2

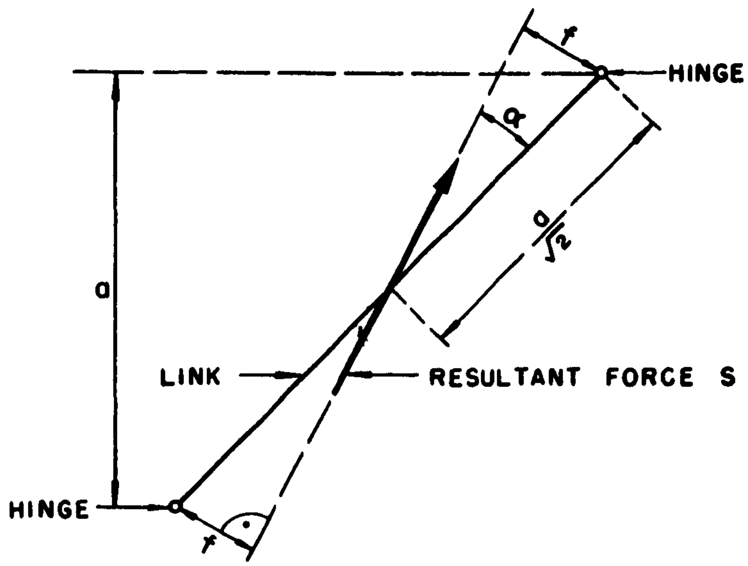


FIG B-3

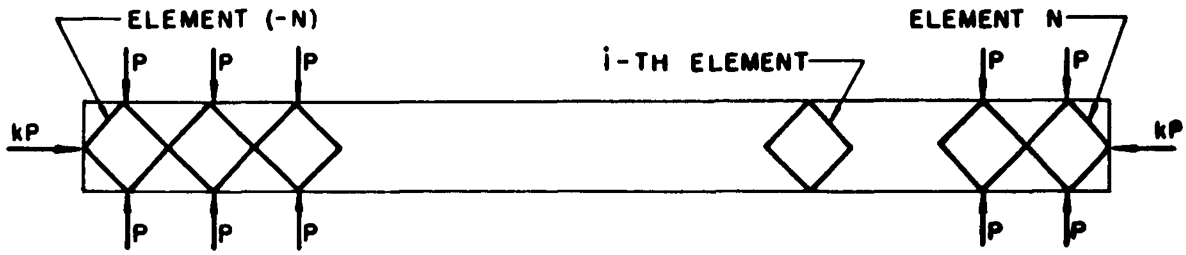


FIG B-4

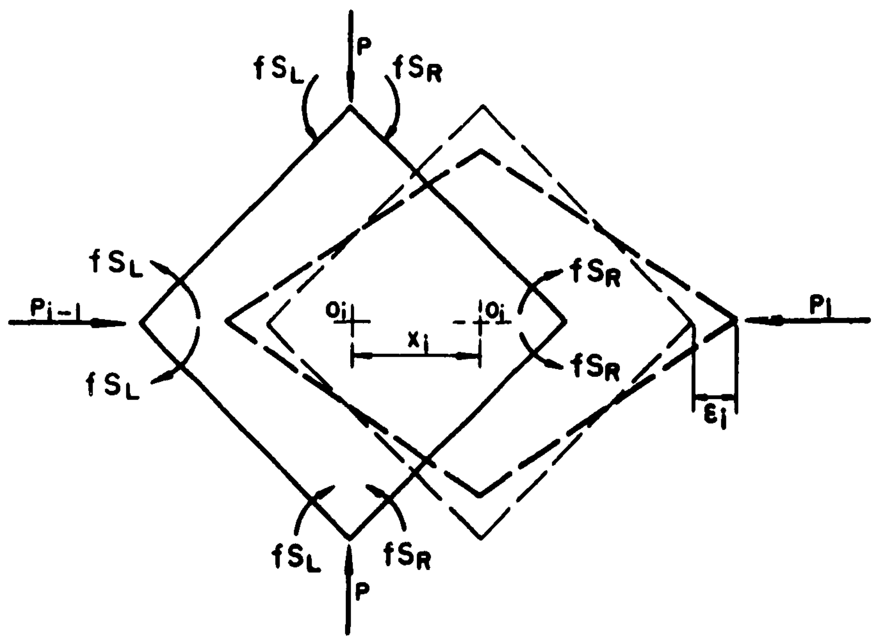


FIG. B-5

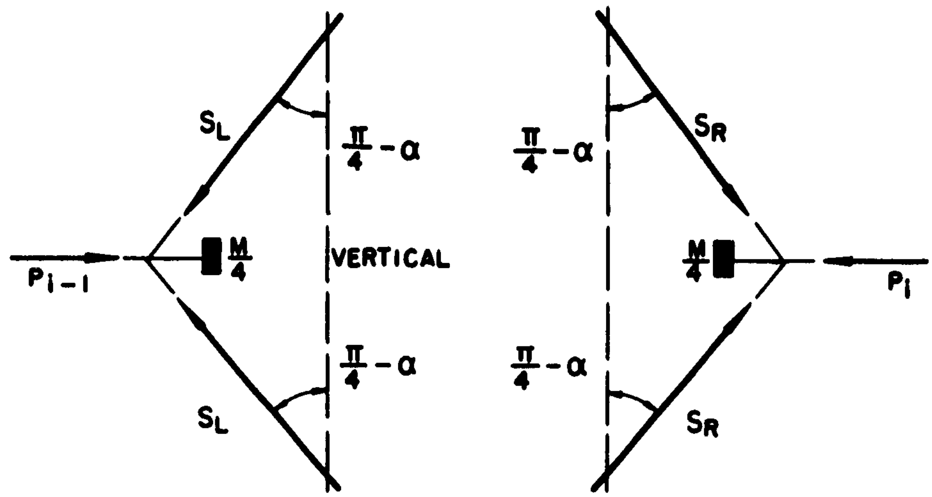


FIG. B-6

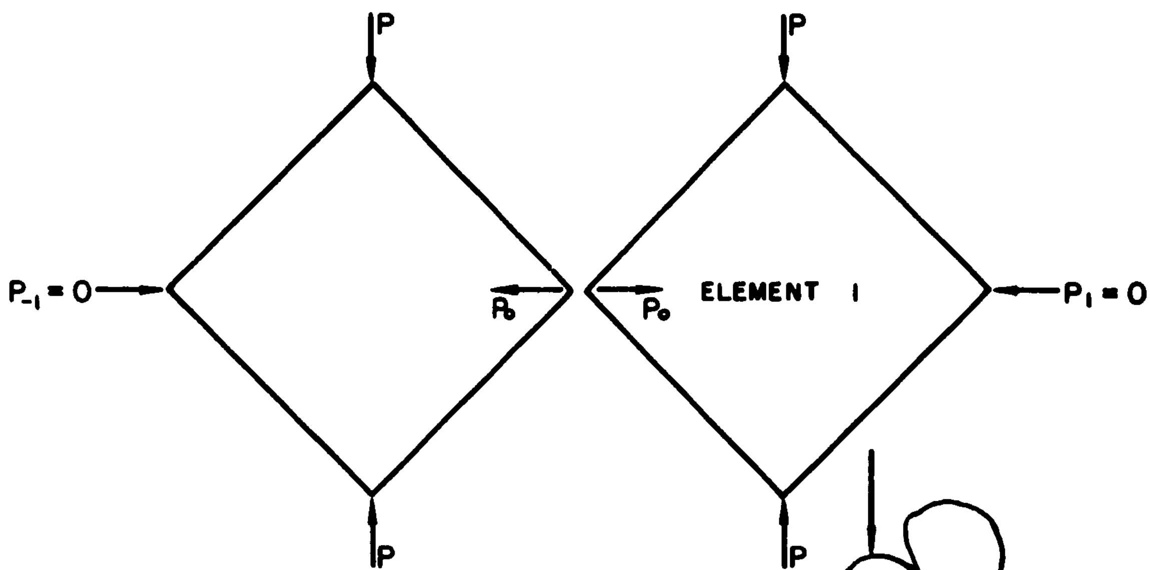


FIG. B-7

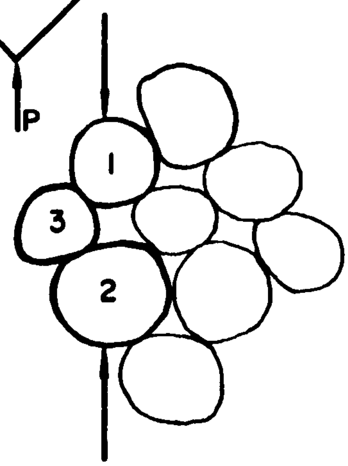


FIG B-8

LIST OF SYMBOLS.*)

$A, B, C, D, \bar{D}$	In Appendix A: arbitrary constants of differential equation.
a, a_x, a_y	Acceleration in radial, x- and y- direction, respectively.
a	In Appendix B: one half of height of elements.
c_p, c_s	Velocities of compressive and shear waves, respectively, in an elastic material.
c	In Section 1: a constant in Eq. (1-1).
e	Basis of natural logarithms.
$\dot{e}$	Positive quantity, defining the principal strain rates.
E	Young's modulus.
f, f_1	In Section 1: yield function, plastic potential, respectively.
f	In Section 3: any function.
f	In Appendix B: distance shown in (Fig. B-3).
F	In Section 1: an arbitrary function.
G	Modulus of rigidity.
i,j	Subscripts, $i,j = 1,2, \ldots$
J_1, J_2	Invariants of strain deviator.
k	Constant defining the stress at which slip occurs in Coulomb material, Eq. (2-6).
$\bar{k}$	In Appendix A: a parameter expressing the principal stresses in a non-slip elastic region, Eq. (A-8).
l	Length of load in Fig. 3.
n_1, n_2, n_3	Exponents in Eq. (3-7).
N	In Appendix A: expression defined by Eq. (A-14).

*) Other symbols which are locally used are defined where they occur.

O_i	In Appendix B: centroids of elements.
P, P_1, P_2, P_i	In Appendix B: forces acting on the elements.
p	Intensity of applied step pressure.
p_o	Surface pressure at $\phi = 0$ (ahead of applied load).
p_L	Limiting value of p, Eq. (4-6).
p_P	Maximum static pressure according to Prandtl [Ref. 2].
p_s	Limiting value of p below which solutions without slip exist.
Q_{x_i}, Q_{e_i}	Generalized forces.
$s > 0$	Parameter defining cohesion.
S	In Appendix B: force in links.
t	Time.
T	Kinetic energy.
u, u_ϕ, u_r, v	Components of velocity, horizontal, tangential, radial and vertical, respectively.
V	Velocity of applied pressure pulse.
x, y	Cartesian coordinates, (Fig. 1).
x_i	In Appendix B: displacement of i-th elements.
$\cdot$ '	Derivative with respect to t, ξ, respectively.
α	In Section 1: Coefficient in Drucker and Prager's (Eq. 1-1), also inclination of pressure front, (Fig. 4). In Section 2: Angle defining position of slip plane, (Fig. 5). In Appendix B: angle between resultant force and link, (Fig. B-3).
β	Opening angles of wedge shaped regions in which no slip occurs.
γ	Angle, $\cos 2\gamma = \frac{1-k}{1+k}$.

δ	Dirac's Delta Function.
Δ	In Appendix A: expression defined by Eq. (A-39).
ϵ, $\dot{\epsilon}$	Strain and strain rate, respectively, with appropriate subscripts, $\dot{\epsilon}_{ij}$, etc.
ϵ	In Appendix B: Deformation of element.
$\dot{\epsilon}_1$, $\dot{\epsilon}_2$	Principal strain rates in direction of principal stresses, σ_1 , σ_2 , respectively.
θ	Angle defining direction of major principal stress, (Fig. 5).
λ	Location dependent quantity when deriving strain rates from potential functions, Eqs. (1-2), (1-3), (2-9).
ν	Poisson's ratio.
$\xi = \frac{x - Vt}{y}$	Non-dimensional variable.
$\pi = 3.1415...$	
ρ	Mass density of material.
σ	Independent, necessarily positive variable, defining the principal stresses, Eqs. (3-2).
σ_1 , σ_2	Major and minor principal stresses, respectively.
σ_x , σ_y , σ_r , σ_ϕ , σ_{ij}	Stress components.
τ	Shear stress.
ϕ	Position angle of an element, (Fig. 6), measured clockwise from horizontal.

Note re Numbering of Equations.

Throughout this report, equations are numbered by hyphenated numbers, such as Eq. (2-7), (3-18), or (B-14). The first number, 2, 3, B, respectively, indicates that the equation occurs in Section 2, 3, or in Appendix B.
When an equation is referred to, the full number, say (2-7), is quoted if the reference occurs in a different section. However, if an equation is mentioned in the section in which it originally appears, only the second half of the number is quoted: In Section 2, Eq. (2-7) would therefore, be quoted simply as Eq. (7), etc.

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than about 0.20
Two possible types of behavior of the material are considered. During slip, one material exhibits dilatancy, while the other does not change in volume. Because of neglect of the elastic deformations in the slip region, only degenerate results are obtained for the case of dilatancy; while stresses can be determined, deformations, velocities and accelerations vanish. For the other material, all desired quantities are obtained.

A significant finding is that for values of p above a certain limit, granular particles will be expelled at the surface ahead of the pressure front. The applied pulse will be preceded by a precursor of expelled grains.

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