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This technical report covers work performed on the Acoustic Signal Processing Study with the Office of Naval Research on Contract No. Nonr 3320(00).

April 30, 1963

Technical Report TR-63-2-BF
on
Linear Signal Processing Theory and Measurements

by

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ABSTRACT

A time-varying random linear system model is hypothesized for the communication medium. The fundamental parameters and analysis techniques for the linear system are described, and applied in a multiple alternative communication mode with Rayleigh fading. A comparison of several receiver types is then effected. Finally some measurement limitations and shortcomings are indicated.

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LIST OF ILLUSTRATIONS

LIST OF SYMBOLS

t	time
f	frequency
$h(\tau, t)$, $h(\tau)$	medium impulse response
$d(\tau, t)$, $d(\tau)$	deterministic component of impulse response
$r(\tau, t)$, $r(\tau)$	random component of impulse response
$s(t)$, $s_k(t)$	deterministic transmitted signals
$a(t)$	random component of transmitted waveform
T	duration of signal; observation interval
E	energy of received signal in time T
S_p	peak signal power
S, S_a	average signal power (= E/T)
$n(t)$	additive noise
N	average noise power
N_d	double-sided noise spectrum level
N_o	single-sided noise spectrum level (= $2N_d$)
W	signal bandwidth
$w(t)$	received waveform
$w_k(t)$	component of received waveform
$H(f, t)$, $H(f)$	transfer function
$A(\tau, f)$	spreading function
$\mathcal{H}(\eta, f)$	bi-frequency function
B	bandwidth of $A(\tau, f)$
L	delay spread of $h(\tau, t)$
F	bandwidth of $H(\eta, t)$
$\Psi(f)$	positive frequency spectrum of input signal
$H_+(f, t)$	positive frequency spectrum of transfer function
$H_{LF}(f, t)$	complex low-frequency transfer function

LIST OF SYMBOLS (Cont.)

$h_{LF}(\tau, t)$	complex low-frequency impulse response
$Z(f)$	complex low-frequency signal spectrum
$z(t)$, $\xi(t)$	complex low-frequency signals
$E(t)$	complex low-frequency system output envelope
$\delta(t)$	Dirac delta function
$G(t)$	medium fading; gain
$\phi(\tau)$	correlation function of medium fading
τ_o	coherence interval
M	number of communication alternatives
P_c	probability of correct decision
P_e, P_2	probability of error
Δ, τ	signal timing parameters
H	source information rate
ρ	signal-to-noise ratio ($= S_a/N_o H$)
W_n	non-stationary character bandwidth
σ^2	variance
Re	real part of
*	conjugate
i	$\sqrt{-1}$
Superscript bars	ensemble average
lg	$\log_2$

1. Introduction

The detection and correct identification of weak signals in the presence of strong noise backgrounds, intentional and otherwise, is an important problem in many fields. In particular, it has held the rapt attention of communications engineers for some time now. Many important situations and problems have been analyzed and solved, and used as guides for dealing with other more complicated, but similar, cases, and much of this wealth of information and analysis can be applied to the problems of underwater detection, classification, and communication.

In order to determine the applicability of existing techniques and solutions to improvement of ASW system performance, as well as to assess the needs for new theory and techniques, it is of first importance to know whether it is valid to consider that source, medium (channel), and receiver can be distinguished and separately characterized in a one-way system, as it can in the usual communications application. This separation is desired in such a way that the receiver output due to an arbitrary source signal can be predicted from knowledge of its output due to a previously applied source signal of different characteristics. In other words, can a model of the system be derived which will permit source and channel to be independently varied in analysis, or are the two so inextricably tied together, that results for one signal and set of environmental conditions are useless for predicting results for other signals and other channel conditions? We doubt that the latter situation is the one given to us by nature; in fact, this is our basic working assumption in the following.

Even when separability is possible, there are still substantial questions to be answered concerning the channel; e.g., whether it is stationary, deterministic, etc; what is the detailed structure and effect on transmitted signals;

and last, but far from least, what measurements should be made in order to answer these questions? In the following we shall discuss a general linear model which seems suitable for a first try at characterizing the channel and answering these questions. There are several reasons for choosing a linear model of the channel. First, and perhaps most important, the actual medium is believed to behave linearly for limited excitations. Second, the analysis of a linear model, although not simple, is fairly tractable for certain quantities of interest. And last, optimization of the performance of the linear model is often possible; this is not always so for other models which include nonlinearities. The importance of the last reason can not be overstressed, for it often provides methods and suggests guidelines to be considered in the physical application to improve performance.

In this report we shall present some of the fundamental techniques available for studying and analyzing random time-varying linear systems with both qualitative and quantitative results, and also indicate some important limitations on measurements. It is felt that this work will furnish an adequate background with which to delve deeper into additional topics of particular interest.

2. Linear Model; Generalities

One important theme running through virtually all current communication work is the recognition and acceptance of the transmission medium as being a linear system, although perhaps random and time-varying. Since the transmission medium has been space itself, it is easily observed and measured, and tested for specific applications. Even when communication is over long ranges, through ionospheric disturbances, or via moving reflectors, the linear model of the transmission medium is an excellent one.

Furthermore, the degree of randomness of the linear medium is usually relatively small in comparison with the deterministic component. That is, in most cases of interest, the impulse response of the medium $h(t)$ (taken time invariant for the moment), expressed as a sum of a deterministic and a random part,

$$h(t) = d(t) + r(t),$$

has negligible energy in the random part compared with that in the deterministic part. Mathematically this is expressed by

$$\int r^2(t)\, dt \ll \int d^2(t)\, dt,^*$$

with probability near unity. Even in those cases where the impulse response varies with time, as for a moving transmitter or receiver, or for communication through the ionosphere, the deterministic part of the impulse response (that part which can be predicted from recent measurements) dominates the random part.

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*Integrals without limits are taken over the range of non-zero integrand.

The communications engineer makes great use of the deterministic components of both radiated signal and medium impulse response by utilizing matched filters (time varying in some cases) at the receiver. These filters add the signal portion of the received waveform in direct proportion to the processing time, while building up the noise only as the square root of the processing time. Hence, for long processing times, the output signal-to-noise ratio of the matched filter can be made quite large, provided that the receiving filter can be kept matched all this time. Mathematically, if s(t) is the received signal in the absence of additive noise (known quite accurately to the receiver), and n(t) is the additive (white) noise from other sources, such as receiver noise, environmental noise, or jamming, the output of the optimum receiver is

$$r(T) = \int_T s(t)\,[\,s(t) + n(t)]\,dt,$$

where T is the processing time or duration of the signal. This quantity has a mean value of E, and a variance $N_d E$, giving a (power) signal-to-noise ratio of

$$\frac{E}{N_d} = 2TW\,\frac{S}{N}\,.$$

E is the received signal energy, N_d is the double-sided spectrum level of the noise, W is the bandwidth of the signal, S is the average received signal power over T, and N is the average received noise power in the band W. The possibilities of improved performance are well evident from this formula; a large TW product signal can combat noise very well.

The situation for underwater transmissions is nowhere near as amenable. The validity and utility of the linear model of the transmission medium has apparently not been verified to any high degree of certainty. And the percentage of useful deterministic component (much less than the corresponding figure for space communication), may even be less than the percentage of random component.

To further complicate the situation, the radiated "signal" does not maintain a high degree of consistency from trial to trial, whether the source be friend or foe. Even for a friendly submarine repeating a run at the same range, speed, bearing, etc., the movement of water over the transducers and hull, and the vibrations of the auxiliary machinery and the submarine itself, add substantial random effects to the intentionally radiated communication signal. The percentage of total radiated signal which is deterministic is apparently not known with any great degree of accuracy.

In order to be able to combat the underwater environment, and corroborate or disprove the linear model of the medium, some quantitative measurements should be undertaken. The guide as to which measurements may be furnished by considering the model of the complete communication system given in Figure 1 below. This model is felt to be a fairly reasonable characterization of the actual situation. s(t) is that portion of the transmitted

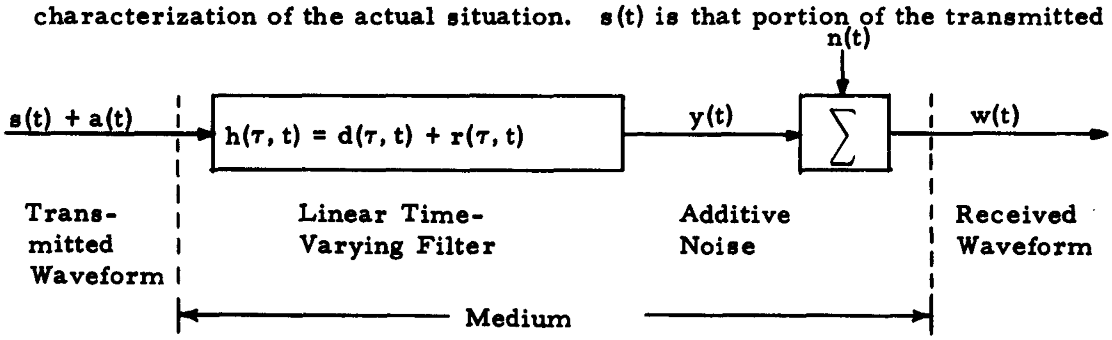

Figure 1. Model of the Communication System

waveform which is identical from trial to trial, under precisely the same environmental conditions, and is called the transmitted signal. a(t) is the random component of the transmitted waveform and is unpredictable.*

The medium is characterized by a linear time varying filter with impulse response $h(\tau, t)$. That is, $h(\tau, t)$ is the response (with external noise $n(t) = 0$) at time t due to an impulse excitation time τ ago. The response is measured at the receiving point of interest. (If the medium is time invariant, $h(\tau, t) = h(\tau)$.) $d(\tau, t)$ represents the deterministic portion of the impulse response (predictable from recent past measurements) and $r(\tau, t)$ the random component. The output y(t) of the filter (in the absence of external noise) for an input x(t) is given by

$$
\begin{aligned}
y(t) &= \int x(t-\tau)\, h(\tau, t)\, d\tau \\
&= \int x(t-\tau)\, d(\tau, t)\, d\tau + \int x(t-\tau)\, r(\tau, t)\, d\tau,
\end{aligned}
$$

and is composed, likewise, of a deterministic and a random part (if x(t) is known). Notice that the random component, $\int x(t-\tau)\, r(\tau, t)\, d\tau$, consists of a linear operation on the input waveform x(t), and represents therefore not merely an additive noise component, but more like an averaged multiplicative noise effect on the input x(t). If x(t) were zero, this random term disappears--- quite different from the case where an additive noise effect is still present when the signal is absent.

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*This portion of the waveform is not useless. Random signals for communication and detection have recently attracted the attention of several investigators [1], [2] and have possibilities. In fact, this is the only waveform on which detection and classification can be attempted when the submarine is not transmitting a known communication signal, as for example, while hovering. However, for the present, in order to retain simplicity, we shall attempt to analyze only the non-random component s(t), and save the generalizations for the future.

Since the ocean is not an all-pass linear-phase network, the received signal for a single path will not be an impulse at the appropriate delay even if the excitation were an impulse. Rather it will be a smeared out pulse centered at the appropriate range delay. For example, if the (time invariant) linear filter has a frequency domain representation approximated by

$$H(f) = e^{-af - i2\pi bf}, \quad f \geq 0,$$

the impulse response would be approximately*

$$\frac{2a}{a^2 + (2\pi)^2 (t-b)^2},$$

which is peaked at b, with a width of a/π at the half amplitude points. (The bandwidth of the filter is $2 \ln 2/a$ at the half amplitude points.)

Depending on the particular geometry, range, water temperature, etc., there may be one or more major contributions to the medium impulse response. Thus, for a situation where there is one bottom bounce, the impulse response on one trial may appear as in Figure 2.

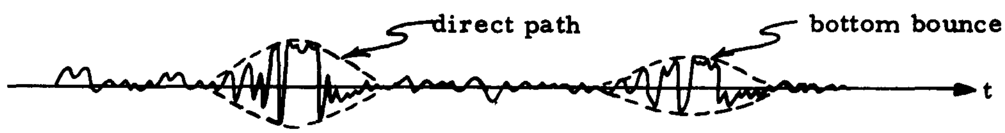

Figure 2. Possible Impulse Response

And if the difference in any two paths is small enough, their pulse outputs will overlap. But there is still no need for an excitation which is narrower than the inverse bandwidth of the medium, because the two pulse outputs can never be resolved any finer than this quantity in any event.

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*Extremely short duration excitations are not necessary to measure the impulse response; rather, any excitation with an effective duration less than approximately a/π in the example above is acceptable for measuring the impulse response.

An equivalent way of characterizing the medium, if linear, is to measure the amplitude and phase characteristics in response to pure single tone excitations. The function giving this information is called the instantaneous transfer function and is defined by

$$H(\eta, t) = \int h(\tau, t) \exp(-i2\pi\eta\tau)\, d\tau$$

$$= D(\eta, t) + R(\eta, t)$$

and contains both deterministic and random components. If a sinewave

$$A \sin 2\pi f_1 t$$

is transmitted, the received waveform (without noise) would be approximately*

$$A\,|D(f_1, t)|\,\sin[2\pi f_1 t + \angle D(f_1, t)],$$

provided $D(f_1, t)$ changes very slowly over times approximately $1/f_1$ seconds in duration.

The final response of the system is given by

$$\begin{aligned} w(t) &= \int [s(t-\tau) + a(t-\tau)]\,[d(\tau, t) + r(\tau, t)]\, d\tau + n(t) \\ &= \int s(t-\tau)\, d(\tau, t)\, d\tau \\ &+ \int s(t-\tau)\, r(\tau, t)\, d\tau \\ &+ \int a(t-\tau)\, d(\tau, t)\, d\tau \\ &+ \int a(t-\tau)\, r(\tau, t)\, d\tau + n(t) \\ &\equiv w_1(t) + w_2(t) + w_3(t) + w_4(t). \end{aligned}$$

---

*With noise present, this expression represents the mean of the received waveform.

The first component of the response, $w_1(t)$, is the (desired) well behaved deterministic component of the received waveform, and is the one upon which normal reception is attempted.

Component $w_2(t)$ is a multiplicative noise term resulting from the passage of a deterministic signal through a random filter. Component $w_3(t)$ is deterministically filtered random noise. These two noise components may be used in some cases to improve performance above that of $w_1(t)$ alone. Whether or not this will be fruitful depends on the relative magnitudes, compared with the first component.

Component $w_4(t)$ is noise which is useless and should be suppressed as much as possible. Whether or not this goal can be adequately reached again depends on the relative magnitudes involved.

It is impossible to measure separately any of the components $\{w_k(t)\}$ or their averages. Rather, they come in an inseparable bundle, and statements about each type can be obtained only through measurements of s, a, d and r. Once these measurements have been made, however, we can obtain many statistical quantities of interest in terms of them. For example,

$$\overline{w_k(t)} = 0, \quad k = 2, 3, 4.$$

$$\overline{w_2^2(t)} = \iint s(t-\tau_1)\, s(t-\tau_2)\, \overline{r(\tau_1, t)\, r(\tau_2, t)}\, d\tau_1\, d\tau_2$$

If r(t) is stationary such that

$$\overline{r(\tau_1, t)\, r(\tau_2, t)} = R(t-\tau_1)\, R(t-\tau_2),$$

we have

$$\begin{aligned} \overline{w_2^2(t)} &= \left[\int s(t)\, R(t)\, dt\right]^2 \\ &= \left[\int G(f)\, S^*(f)\, df\right]^2 \end{aligned}$$

where S(f) and G(f) are the Fourier transforms of s(t) and R(t) respectively.

Also

$$\overline{w_3^2(t)} = \iint \overline{a(t-\tau_1)\, a(t-\tau_2)}\, d(\tau_1, t)\, d(\tau_2, t)\, d\tau_1\, d\tau_2$$

If a(t) is stationary,

$$\overline{a(t-\tau_1)\, a(t-\tau_2)} = A(\tau_1 - \tau_2),$$

and

$$\overline{w_3^2(t)} = \iint A(\tau_1 - \tau_2)\, d(\tau_1, t)\, d(\tau_2, t)\, d\tau_1\, d\tau_2$$

$$= \int \alpha(f)\, |D(f, t)|^2\, df$$

where $\alpha(f)$ is the Fourier transform of A(t).

Lastly, using the independence of the three noise terms $r(\tau, t)$, a(t), and n(t), we have

$$\overline{w_4^2(t)} = \iint \overline{a(t-\tau_1)\, a(t-\tau_2)}\;\overline{r(\tau_1, t)\, r(\tau_2, t)}\; d\tau_1 d\tau_2 + \overline{n^2(t)}$$

$$\longrightarrow \iint A(\tau_1 - \tau_2)\, R(t-\tau_1)\, R(t-\tau_2)\, d\tau_1 d\tau_2 + \overline{n^2}$$

$$= \int \alpha(f)\, |G(f)|^2\, df + \overline{n^2}$$

in the case of stationary statistics.

Combining all the results above, we have

$$\overline{w(t)} = \int s(t-\tau)\, d(\tau, t)\, d\tau$$

$$\sigma^2 [w(t)] = \left[\int G(f)\, S^*(f)\, df\right]^2 + \int \alpha(f)\, |D(f, t)|^2\, df$$

$$+ \int \alpha(f)\, |G(f)|^2\, df + \overline{n^2}$$

from which the receiver input signal-to-noise ratio can be determined. The major contributions to the receiver input noise can be readily evaluated from this equation. The areas of investigation and possible improvement are then more easily ascertained. Other moments of interest can also be derived, provided the higher order correlation functions of the random components are known.

The statistics described above are function of time in general, due to the non-stationary character of the random fluctuations. However, it is possible to perform time averages on these statistics and relate their long term averages. Depending on the duration of the tests, this procedure, a quasi-stationary one, may be the only alternative open for computation.

A more complete statistical description of w(t) is afforded by its p. d. f. 's and not just its moments. However, the relation of the p. d. f. 's of w(t) to those of a, d, etc. is extremely difficult except in special cases. One of those special cases is when the random component $r(\tau, t)$ of $h(\tau, t)$ is negligible in comparison with the deterministic component $d(\tau, t)$. Then, if no signal is transmitted

$$w(t) = \int a(t-\tau)\, d(\tau, t)\, d\tau + n(t)$$

where we recall that a(t) is the random emission of the submarine. If furthermore, $d(\tau, t)$ arises due to just two paths, such as

$$d(\tau, t) = P_1\ \delta(\tau-\Delta_1) + P_2\ \delta(\tau-\Delta_2),$$

where P_1, P_2, Δ_1, Δ_2 may be varying slowly with time, then

$$w(t) = P_1\ a(t-\Delta_1) + P_2\ a(t-\Delta_2) + n(t).$$

The first order p. d. f. of w(t) is then given by

$$p(w, t) = \iint p_a(x_1,\ t-\Delta_1;\ x_2,\ t-\Delta_2)\ p_n(w-P_1x_1 - P_2x_2,\ t)\ dx_1dx_2$$

where the second-order p. d. f. of a and the first order p. d. f. of n are necessary. Therefore if the first order p. d. f's of w and n and the second order p. d. f. of a are measured, they may be substituted above and checked. For second order statistics, or higher, of w, the analytical approach is fraught with integrals of still higher order p. d. f. 's of a and n. Furthermore, if $d(\tau, t)$ does not contain just impulses, the problem is almost completely intractable except in very special cases.

In the above, we have tried to briefly indicate some of the statistical quantities of interest and importance, and how they may be interrelated. In the following sections, we shall delve much deeper into particular aspects of the communication model and obtain quantitative results.

3. Fundamental Parameters of a Time Varying Linear System

In section 2 we defined the filter impulse response of a general time-varying (random) linear system as $h(\tau, t)$, the response of the system at time t to a unit impulse applied τ ago. Also, the instantaneous transfer function of the system was defined as

$$H(\eta, t) = \int h(\tau, t) \exp(-i2\pi\eta\tau)\, d\tau$$

and was indicated to measure the response of a system to pure tone excitations. It will prove convenient to define two more equivalent system functions (characterizations) for later use. They are the conditioned spreading function of the filter

$$A(\tau, f) = \int h(\tau, t) \exp(-i2\pi ft)\, dt$$

and the bi-frequency function

$$\begin{aligned}
\mathcal{H}(\eta, f) &= \int H(\eta, t) \exp(-i2\pi ft)\, dt \\
&= \int A(\tau, f) \exp(-i2\pi\eta\tau)\, d\tau \\
&= \iint h(\tau, t) \exp[-i2\pi(\eta\tau + ft)]\, d\tau\, dt.
\end{aligned}$$

Suppose $h(\tau, t)$ is substantially confined within a rectangle L by D in the τ, t plane; that is, $h(\tau, t)$ is approximately zero outside the rectangle. L is called the "memory time" or delay spread" of the filter, while D is the "duration" or "existence time" of the filter. For an RC filter, for example, $D = \infty$ and $L \cong 3RC$.

Since $H(\eta, t)$ is the Fourier transform on τ of $h(\tau, t)$, $H(\eta, t)$ is confined to a strip of width D on the t-axis, just as $h(\tau, t)$ is. Furthermore, for t in this strip, suppose $H(\eta, t)$ is substantially confined within a band of width F in η. F is called the "bandwidth" of the filter. Then $H(\eta, t)$ is significantly non-zero only within a rectangle F by D in the η, t plane. For the RC filter mentioned above, $F \cong 3/RC$.

Also, since $A(\tau, f)$ is the Fourier transform in t of $h(\tau, t)$, $A(\tau, f)$ is confined to a strip of width L on the τ-axis, just as $h(\tau, t)$ is. Furthermore, for τ in this strip, suppose $A(\tau, f)$ is substantially confined within a band of width B in f. B is called the "frequency spread" of the filter. Then $A(\tau, f)$ is significantly non-zero only within a rectangle L by B in the τ, f plane. Again for the RC filter, $B = 0$ (no spreading).

Lastly, since $\mathcal{H}(\eta, f)$ is the Fourier transform in t of $H(\eta, t)$, and the Fourier transform in τ of $A(\tau, f)$, $\mathcal{H}(\eta, f)$ is confined within a rectangle F by B in the η, f plane.

The four diagrams in Figure 3 depict "slices" of the functions in the various domains. A fundamental property of Fourier transforms has been used extensively here; namely if

$$G(y) = \int g(x)\, e^{-i2\pi xy}\, dx$$

and g(x) has extent X on the x-axis, then G(y) can change significantly in a distance no smaller than $\frac{1}{X}$ on the y-axis, provided g does not have large alternating values; i.e., we do not consider "superdirectivity".

A couple of comments should be made about Figure 3. If the filter is physically realizable, $h(\tau, t) = 0$ for $\tau < 0$, and the rectangle will not extend to the left of the t-axis. Also, for a time invariant network (and many others), $D = \infty$; for this case, the wiggles in f of $A(\tau, f)$ and $\mathcal{H}(\eta, f)$ occur infinitely fast. Figure 3 is drawn for the most general case.

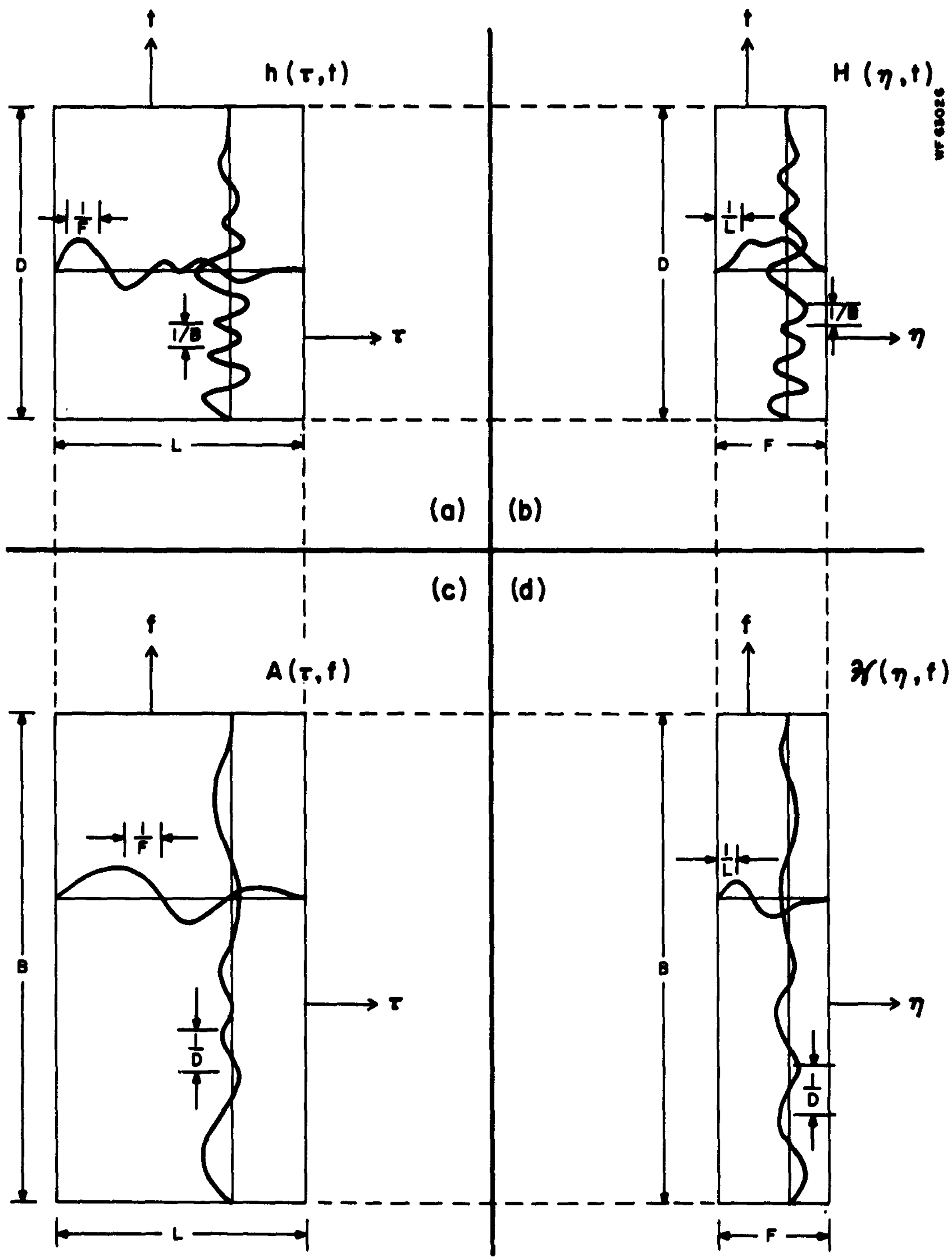

Figure 3. FILTER FUNCTIONS

By way of illustration of the parameters defined above, consider that a sine wave of frequency η_o is fed into the filter described by Figure 3. From 3(a), the output would be significantly non-zero only for approximately the D seconds indicated on the t-scale (D + L seconds, more accurately.) From 3(b), there would be no output at any time unless η_o lies in the pass band F of the filter. And even when η_o lies in this region, the amplitude of the output sine wave is scaled by the factor $|H(\eta_o, t)|$, and therefore varies with the input frequency η_o. And from 3(c), the spectrum of the output signal would not be an impulse, as for the input, but rather, spread out on the f-scale over a range B wide.

To summarize,

D = existence time

L = delay spread

F = bandwidth

B = frequency spread

4. Narrow-Band Linear Systems

In Section 3, general definitions of linear system characterizations were given. However, these characterizations are not convenient for using in situations where the systems are narrow-band, i.e., pass only a small band of frequencies about some high center frequency. (In terms of Figure 3(b) or 3(d), this means the rectangle would be displaced far to the right on the η-scale.) Such a situation can arise naturally when a narrow band excitation passes through a broad-band medium, for reception would then reasonably consist of filtering out only that portion of the spectrum where the signal is expected to lie. Alternately, a piece of reception equipment may be inherently narrow-band and able only to accept a restricted region of input signal spectrum, even though the input is broad-band. In this section, therefore, we specialize the results of Section 3 to narrow-band systems, and indicate some of the more usual types. This material is used extensively in Section 5 for deriving performance in the presence of noise.

We had*

$$H(f, t) = \int h(\tau, t) \exp(-i2\pi f\tau)\, d\tau$$

as the instantaneous transfer function of a linear system. For an input i(t), the output r(t) may be obtained as

$$r(t) = \int h(\tau, t)\, i(t - \tau)\, d\tau$$
$$= \int H(f, t)\, I(f) \exp(i2\pi ft)\, df$$

where I(f) is the voltage density spectrum (Fourier transform) of the input. The latter expression in the frequency domain is a neat generalization of the usual time-invariant result.

---

*We use the more conventional symbol f in this section, rather than η, because there is no chance of confusion as there was in Section 3, where a bi-frequency function was defined. f is only a labeling index, and is not a fundamental parameter.

Since $h(\tau, t)$ and $i(t)$ are real, we can easily show that

$$r(t) = \mathrm{Re}\left\{\int_0^\infty H(f, t)\, I(f)\, \exp(i2\pi ft)\, df\right\}$$

(We drop all irrelevant scale factors in order to facilitate computations. Our result will be proportional to the actual output, and will be sufficient.) Defining

$$\Psi(f) = I(f)\,[1 + \mathrm{sgn}(f)]$$
$$H_+(f, t) = H(f, t)\,[1 + \mathrm{sgn}(f)]$$

we find

$$r(t) = \mathrm{Re}\left\{\int H_+(f, t)\, \Psi(f)\, \exp(i2\pi ft)\, df\right\} \tag{1}$$

Now for a narrow-band system, $H_+(f, t)$ will be significantly non-zero only near one frequency f_h (for all t), and we express it as

$$H_+(f, t) = H_{LF}(f - f_h, t)$$

where $H_{LF}(f, t)$ has a filter characteristic centered about zero frequency--a low-pass filter. Then letting*

$$\Psi(f) = Z(f - f_h),$$

we find

$$r(t) = \mathrm{Re}\left\{\exp(i2\pi f_h t)\int H_{LF}(f, t)\, Z(f)\, \exp(i2\pi ft)\, df\right\}$$

$$= \mathrm{Re}\left\{\exp(i2\pi f_h t)\int h_{LF}(\tau, t)\, z(t - \tau)\, d\tau\right\}$$

where

$$h_{LF}(\tau, t) = \int H_{LF}(f, t)\, \exp(i2\pi f\tau)\, df$$

---

*The "center frequency" of $\Psi(f)$ need not be f_h. Thus $Z(f)$ need not be centered at zero frequency. In fact, $Z(f) = \Psi_{LF}(f - f_s + f_h)$, where f_s is the input signal center frequency.

and

$$z(t) = \int Z(f) \exp(i2\pi ft)\, df$$

$$= \xi(t) \exp[i2\pi(f_s - f_h)t]$$

where $\xi(t)$ is the "complex low frequency envelope"* of the input (Fourier transform of $\Psi_{LF}(f)$). Therefore the complex low frequency envelope of the response r(t) is proportional to

$$E(t) = \int h_{LF}(\tau, t)\, z(t - \tau)\, d\tau. \tag{2}$$

$h_{LF}(\tau, t)$ is the complex low frequency impulse response. A block diagram relating the complex envelopes is shown in Figure 4.

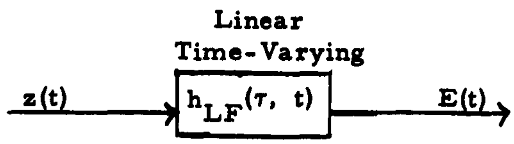

Figure 4. Equivalent Complex Envelope System

Summarizing, we have

$$r(t) = \mathrm{Re}\ \{E(t) \exp(i2\pi f_h t)\}$$

$$E(t) = \int h_{LF}(\tau, t)\, z(t - \tau)\, d\tau$$

$$z(t) = \xi(t) \exp[i2\pi(f_s - f_h)\, t]$$

$$i(t) = \mathrm{Re}\ \{\xi\ (t) \exp(i2\pi f_s t)\} \tag{3}$$

The only quantity not completely tied down here is $h_{LF}(\tau, t)$. We now proceed to relate h_{LF} to h: let $i(t) = \delta(t - t_o)$; then, using (1),

$$r(t) = h(t - t_o, t)$$

$$= \mathrm{Re}\left\{\int H_+(f, t) \exp[i2\pi f(t - t_o)]\, df\right\}.$$

*See [2], Ch. 2.

$$= \text{Re}\left\{\int H_{LF}(f - f_h, t) \exp[i2\pi f(t - t_o)]\, df\right\}$$

$$= \text{Re}\left\{\exp[i2\pi f_h(t - t_o)] \int H_{LF}(\eta, t) \exp[i2\pi\eta(t - t_o)]\, d\eta\right\}$$

$$= \text{Re}\left\{\exp[i2\pi f_h(t - t_o)]\, h_{LF}(t - t_o, t)\right\}$$

Letting $t - t_o = \tau$ (to eliminate t_o), we obtain finally

$$h(\tau, t) = \text{Re}\left\{h_{LF}(\tau, t) \exp(i2\pi f_h \tau)\right\} \tag{4}$$

(which checks with intuition). This relation allows us to obtain h_{LF} easily from h. (It is interesting to note that we cannot derive (4) directly from (2) because $f_s = \infty$ for the impulse, and we get an uninterpretable result.)

Special Cases

In (2), $h_{LF}(\tau, t)$ is a complex function, and furthermore, can be random or deterministic. For example if

$$h_{LF}(\tau, t) = G(t)\, \delta(\tau)$$

where G(t) is a complex random process, then

$$H_{LF}(f, t) = G(t)$$

which is independent of f! This describes a frequency non-selective medium with random gain G(t). The complex envelope of the received signal is, from (2),

$$E(t) = G(t)\, z(t).$$

The block diagram of this system (a special case of Figure 4) is depicted in Figure 5.

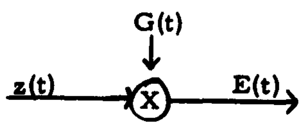

Figure 5. Frequency Non-Selective System

From (4),

$$h(\tau, t) = \delta(\tau) \operatorname{Re} \{G(t)\}$$

and the impulse response of the system is merely a time varying impulse with no delay--a variable gain system. For a time-invariant system, $G(t) = 1$; and the usual simpler results follow.

An alternate form of (2) is given by

$$E(t) = \int H_{LF}(f, t)\, Z(f) \exp(i2\pi ft)\, df$$

Now if, for all t, $H_{LF}(f, t)$ is roughly independent of f in the range where $Z(f)$ is non-zero, then

$$E(t) \cong H_{LF}(f_s - f_h, t)\, z(t)$$

and we have frequency non-selective fading. Thus we obtain the reasonable result that $H_{LF}(f, t)$ need not be independent of f for all f, but only over the signal bandwidth, in order to realize frequency non-selective fading.

As a second example, suppose

$$h_{LF}(\tau, t) = G(t)\, a(\tau).$$

Then

$$H_{LF}(f, t) = G(t)\, \mathcal{A}(f)$$

where $\mathcal{A}(f)$ is the Fourier transform of $a(\tau)$. This is a <u>frequency selective</u> (random) system. There follows that the complex output envelope is given by

$$E(t) = G(t) \int a(\tau)\, z(t - \tau)\, d\tau$$

$$= G(t) \int \mathcal{A}(f)\, Z(f) \exp(i2\pi ft)\, df$$

The block diagram of Figure 6 depicts this system, in terms of the complex envelopes.

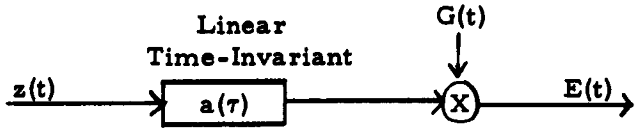

Figure 6. Frequency Selective System

The input signal passes through a filter of characteristic $\mathcal{Q}(f)$ (which may be random) followed by a "gain control" G(t). For G(t)=1, we have the time-dispersive medium. For more general G(t), we have a time- and frequency-dispersive medium (L, B > 0). If $a(\tau)$ is not random, an important statistical quantity of $h_{LF}(\tau, t)$ is the correlation function of the medium fading G(t):

$$\overline{G(t_1)\, G^*(t_2)} \equiv \phi(t_1 - t_2)$$

for stationary fading.

As a last example, consider

$$h_{LF}(\tau, t) = \sum_k G_k(t)\, a_k(\tau)$$

for which

$$H_{LF}(f, t) = \sum_k G_k(t)\, \mathcal{a}_k(f)$$

and

$$E(t) = \sum_k G_k(t) \int \mathcal{a}_k(f)\, Z(f) \exp(i2\pi f t)\, df$$

which is a parallel combination of the building block depicted in Figure 6. For $a_k(\tau) = \delta(\tau - \tau_k)$,

$$E(t) = \sum_k G_k(t)\, z(t - \tau_k)$$

which is the multipath spreading model often met in practice.

It is thus seen that Figure 4 depicts a very general situation for narrow-band communication; we shall analyze a special case of it in the next section.

5. Analysis of a Fading Medium

In this section, results of an early study of the performance of several different receiving structures are summarized. The pertinent derivations are presented in Appendices A, B, and C.

The model to be studied is still the one depicted in Figure 1 of Section 2. However, several specializations have been made. First, there is no random component in the transmitted waveform: $a(t) = 0$. Furthermore, $s(t)$ should more properly be replaced by $s_k(t)$ where $1 \leq k \leq M$; that is, the transmitted signal is one of M (equi-probable) alternatives. Second, $h(\tau, t)$ will be of the form described in Section 4, i.e., narrow-band. Specifically, the first special case described there, Figure 5, will be utilized, and $G(t)$ will be assumed to be a complex Gaussian process.* Thus the fading is assumed to be Rayleigh distributed and non-frequency selective. Third, the additive noise $n(t)$ is assumed to be white Gaussian. These restrictions have been introduced mainly to simplify the analysis; however, they do represent a very reasonable state of affairs.

The three receiving systems to be compared here are the M-ary Weighted Radiometer,** the so-called M-M-ary System, and the M-ary Linear Filtering System. The first system is the optimum system (from the standpoint of minimizing bit error probability using an M-ary alphabet) when the signal energy-to-noise power density ratio is small over a coherence interval, i.e., when $\frac{S_a \tau_o}{N_o} << 1$. τ_o is the length of time over which it is possible to reasonably predict the amplitude and phase of the medium output given the input. In terms of the correlation function of the medium fading introduced in Section 4, we define τ_o, the coherence interval, such that

---

*[3]

**[4]

$$|\phi(\tau_o)| = \frac{1}{2}\,\phi(o).$$

The M-M-ary method of reception is related to the Weighted Radiometer, and produces only slightly higher bit error probabilities. The third technique consists of the optimum system for use in a non-fading channel, and has been included in this early investigation to illustrate the fact that it is extremely hazardous to presume that a system optimized for combatting one type of disturbance in the transmission medium will provide near-optimum, or even good performance, against another type of disturbance.

It has been demonstrated* for a non-fading medium that as $M \longrightarrow \infty$, for both phase-coherent and phase-incoherent receivers, the probability of error $P_e \longrightarrow 0$ if the source information rate H is less than the capacity C_∞ of the infinite bandwidth medium:

$$C_\infty = \lim_{W \to \infty} W \lg \left(1 + \frac{S_a}{N_o W}\right) = \lg e \quad \frac{S_a}{N_o} \text{ bits/sec.}$$

where N_o is the single-sided noise power density level, and S_a is the average signal power. Since such systems are theoretically almost 100 % efficient in the usage of the channel, at least in the non-fading environment, their widespread usage for other environments is suggested. However, some method of combatting or tolerating fading, fast or slow, must be incorporated, for present designs perform miserably in the presence of fading, as will be shortly demonstrated.

Basically, it is anticipated that M-ary systems designed for fading should be near optimum in channel usage. This feeling is based on the fact that adequate communication through a fading channel is possible only if some averaging of instantaneous medium fluctuations is effected, thereby deriving

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*[3, 5, 6, 7]

the benefits of the average medium characteristics. M-ary signals, by their very nature, tend to occupy sizeable portions of the time axis, and experience multiple fades during transmission of a single message of lg M bits. Therefore we have concentrated in this section on modifications of the basic coherent M-ary receiver in an attempt to combat fading and achieve reliable efficient performance; it is believed this has been accomplished.

We now embark on a detailed description of the M-ary systems considered in this report. If the usual M-ary system is used in a fading environment, and orthogonal signals are used, a simple generalization of some past work* shows that the probability of correct decision per (M-ary) word is given by

$$P_c = \prod_{n=1}^{M-1} \left[1 + \frac{1}{n\,(1+\beta)}\right]^{-1}$$

where

$$\beta = \frac{S_a T}{N_o} \int_{-T}^{T} \left(1 - \frac{|u|}{T}\right) |\phi(u)|^2 \, du$$

and ϕ is the correlation function of the medium fluctuations. For a linear correlation function

$$\phi(u) = 1 - \frac{|u|}{2\,\tau_o}, \quad |u| < 2\tau_o$$

it is readily demonstrated that

$$\beta = \frac{S_a T}{N_o} \left\{ \begin{array}{ll} 1 - \frac{1}{6}\frac{T}{\tau_o}, & T < 2\tau_o \\ \frac{2\tau_o}{T}\left(1 - \frac{2}{3}\frac{\tau_o}{T}\right), & T > 2\tau_o \end{array} \right\}$$

---

*[8]

Thus even as $T \longrightarrow \infty$, β saturates at $2 \frac{S_a \tau_o}{N_o}$, and the error probability does not tend to zero. This is the severe limitation of the standard M-ary technique.

An M-ary system of great generality and potentiality, not suffering the above limitations is realized by transmitting a sequence of one of q($\leq$ M) waveforms n times, and employing linear weighting to realize M different possibilities. For example, with q = 2, and Hadamard matrix weighting of size M, M orthogonal signals are realized; yet only two possible signals need be generated. Another special case is realized when q = M; this case is called M-M-ary transmission. Here, one of M waveforms is repeated several times, and the outputs of corresponding matched filters appropriately combined to yield the set of M decision variables. A timing diagram of the waveforms in such a system is indicated below in Figure 7.

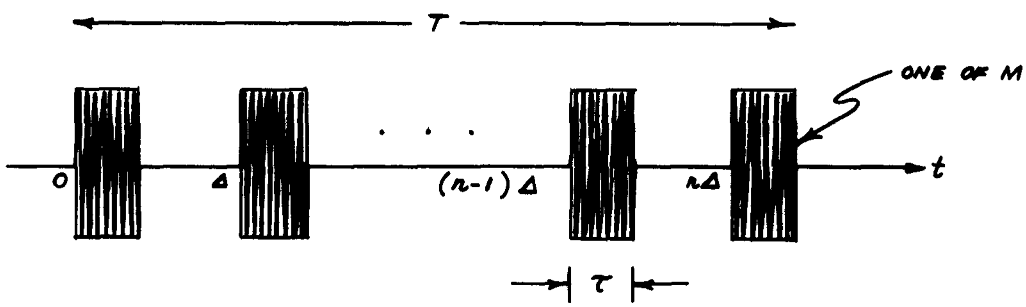

Figure 7. Timing Diagram

τ is the pulse width, Δ is the pulse spacing, and T is the message duration. The probability of correct decision (per M-ary word) is derived in Appendix B and is, for ν-th envelope combining, and for $\Delta >> \tau_o$, $\tau << \tau_o$,

$$P_c \cong \int \phi(x)\, \Phi^{M-1}\, [\alpha x + \beta]\, dx^{*}$$

where

$$\alpha = \left(1 + \frac{S_a \Delta}{N_o}\right)^{\nu/2}$$

$$\beta = n^{1/2} \left[\left(1 + \frac{S_a \Delta}{N_o}\right)^{\nu/2} - 1 \right] \gamma_\nu$$

$$\gamma_\nu = \frac{\Gamma\left(\frac{\nu}{2} + 1\right)}{\sqrt{\Gamma(\nu + 1) - \Gamma^2\left(\frac{\nu}{2} + 1\right)}}$$

$$\phi(x) = (2\pi)^{-1/2} \exp(-x^2/2)$$

$$\Phi(x) = \int_{-\infty}^{x} \phi(y)\, dy.$$

The limitations on Δ and τ above ensure that independent fading occur for each pulse, and negligible fading occur within a pulse. These assumptions allow averaging of medium characteristics over many coherence intervals.

For a peak power limitation on the transmitter, it may be impossible to realize the large pulses required by the M-M-ary system. In such a case, the space between pulses should be filled with signal power, and efficient use made of such waveforms. For the limiting case of peak-equal-average power, $S_p = S_a$, a continuous transmission of signal power over the message interval T is dictated. If also

$$\frac{S_a \tau_o}{N_o} << 1,$$

the optimum receiver is known**, and is called the weighted radiometer. One

*This integral is tabulated for a few selected values of α and β in TR-63-3-BF, "A Multiple Alternative Error Integral", by Albert H. Nuttall, April 30, 1963.

**[4]

of M possible signals is generated at the transmitter; at the receiver, M branches, of which the j-th is depicted below in Figure 8, are constructed and the M decision variables $\{D_j\}$ compared, a decision being made in favor of the largest variable.

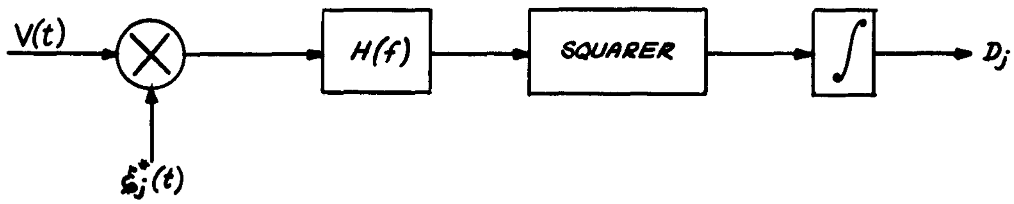

Figure 8. Weighted Radiometer

The bandwidth of H(f) is chosen so as to match the spreading of the medium. The probability of correct decision is derived in Appendix A and is

$$P_c \cong \int \phi(x)\, \Phi^{M-1}(x+\beta)\, dx$$

where

$$\beta \cong \frac{S_a}{N_o}\left[T\int_{-\infty}^{\infty} |\phi(u)|^2\, du\right]^{1/2}$$

For a linear correlation function,

$$\beta \cong \sqrt{\frac{4}{3}\,\frac{S_a T}{N_o}\,\frac{S_a \tau_o}{N_o}}$$

Now since $\frac{S_a \tau_o}{N_o} << 1$ by assumption, we see that $\frac{S_a T}{N_o} >> 1$ is required for good performance. Thus extremely large message durations are required for this system to achieve a high degree of reliability.

To illustrate the degree to which these M-ary systems (with M > 2) are superior to binary systems, four binary systems have also been evaluated:

the binary Weighted Radiometer, the 2-2-ary system, a binary linear filtering system, and a delay autocorrelation technique. The first three of these systems are derived from the M-ary systems with M = 2. The weighted radiometer system is the optimum binary system for low signal-to-noise energy density ratios. The fourth binary system represents one of the many different ways that fairly good performance can be attained with somewhat less complex equipment than is required for an optimum system; it is analyzed in Appendix C.

Because of the widespread reliance on binary digital representations of information, performance is measured by the bit error probability, P_2, for each system, i.e., the long term average probability of a single binary information digit being assigned the wrong value.

The bit error probability for each of the four binary systems is plotted in Figure 9 as a function of the quantity $\rho = \frac{S_a\left(\frac{1}{H}\right)}{N_o}$, where H is the information rate in bits per second. These curves are applicable to the case where the short-term (peak) power limitation, S_p, is equal to the long-term average power capability of the transmitter, S_a. Also it has been assumed that the fading is such that $\tau_o H = \frac{1}{10}$ for each system. The most striking characteristic of Figure 9 is the extremely poor performance of the linear filtering system; although optimum for use in a non-fading medium (under phase incoherent operating conditions), its performance in the presence of fading is quite unsatisfactory. Another feature of interest is that the weighted radiometer, which is the optimum binary system for small values of ρ, is the best system of the four for all values of ρ.

Turning to the multiple alternative systems, the performance of the M-ary Weighted Radiometer, M-M-ary and M-ary Linear Filtering systems is

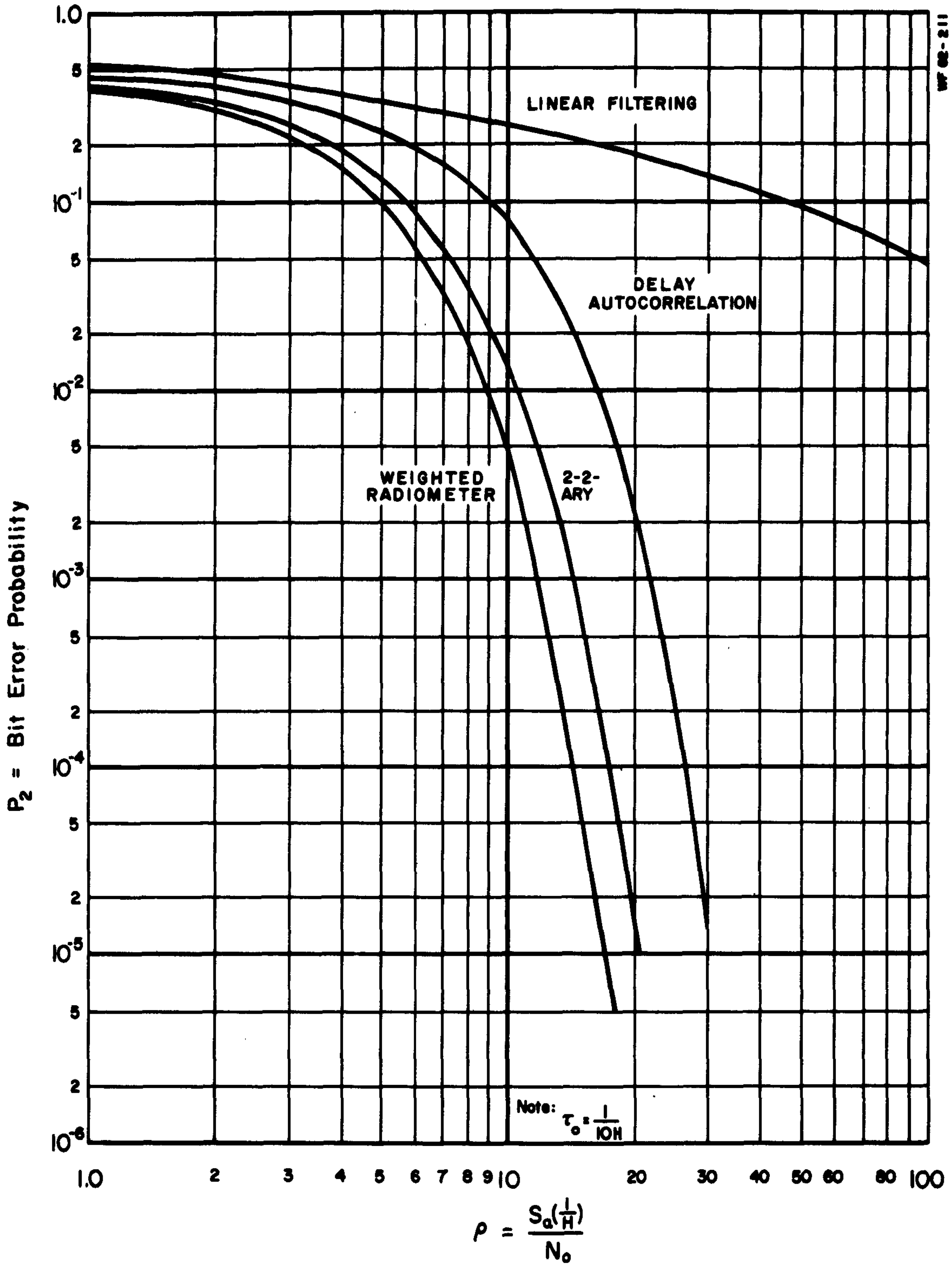

Figure 9. Performance of Binary Systems With Peak-Equal-Average Power Limitations ($S_p = S_a$)

indicated by the curves in Figures 10, 11, 12, and 13. The weighted radiometer has been evaluated for any value of M, under the constraint of peak-equal-average power. Curves are drawn in Figure 10 relating the bit error probability to ρ, for M = 2, 4, 8, and 16, assuming that the fading is such that $\tau_o H = \frac{1}{10}$. As expected, the bit error probability is reduced significantly as the value of M, the alphabet size is increased. As M tends to infinity, if the M-th order Gaussian approximation involved in the evaluation of error probability is valid,* then the limiting performance of this system is indicated by the dashed curve in Figure 10. This suggests (but doesn't rigorously prove) that the limiting performance of an uncoded M-ary weighted radiometer achieves approximately 13 percent of the medium capacity, C_∞, under the $S_p = S_a$ constraint.

Analogous curves are plotted in Figure 11 for the M-M-ary system, under the same constraints. These curves indicate that the M-M-ary system is approximately 0.6 db poorer than the M-ary radiometer for any value of M. The curves in Figure 12 represent the performance of the M-M-ary system when essentially no constraint on peak power exists, i.e., $S_p \geq 10\ S_a$. Under these conditions a method of averaging over many independent fading coherence intervals can be implemented with the M-M-ary system to improve its performance significantly, as indicated by the curves. Again, if the M-th order Gaussian approximation is valid in the limit, the limiting performance of the M-M-ary system (indicated by the dashed line) achieves approximately 56 percent of the medium capacity, C_∞. Although time has not permitted the evaluation of the M-ary weighted radiometer under the no peak power constraint, its slightly better performance than the M-M-ary system under the $S_p = S_a$ constraint suggests that the former system may be capable of achieving nearly 100 percent of the medium capacity, even in the presence of fading.

*Although this approximation is believed to be valid, time has not permitted the completion of a rigorous proof.

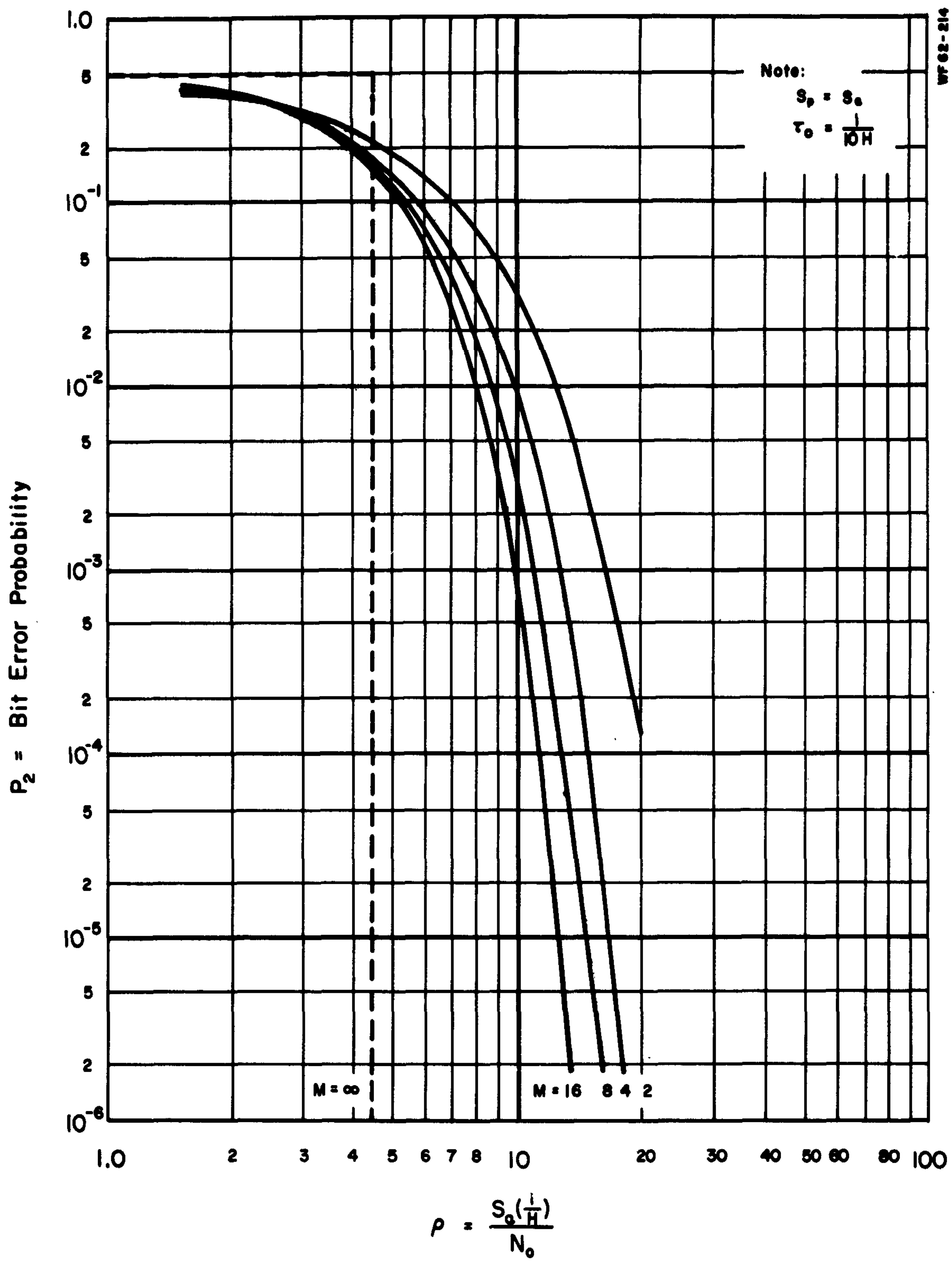

Figure 10. Performance of M-ary Weighted Radiometer With a Peak-Equal-Average Power Limitation

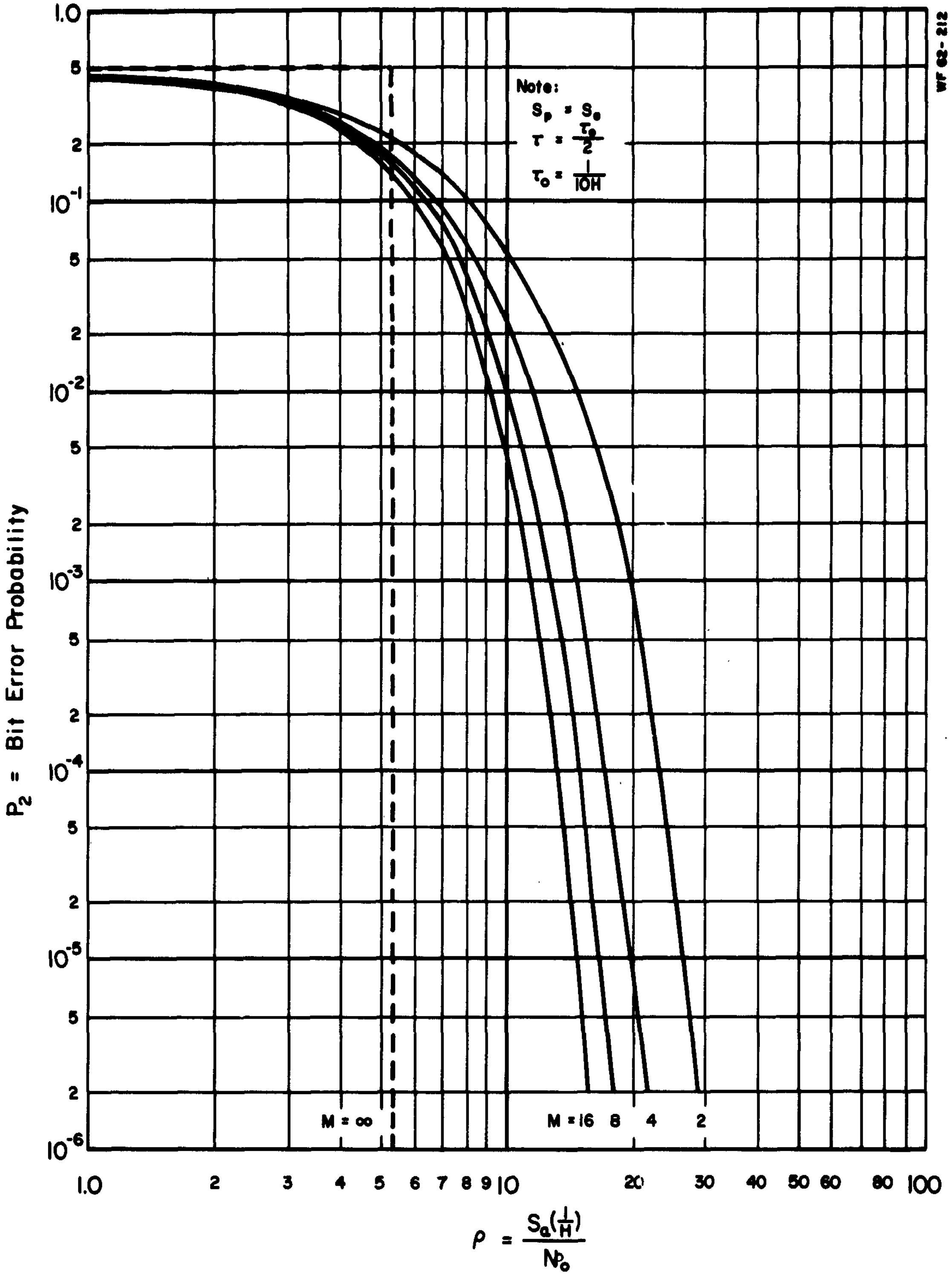

Figure 11. Performance of M-M-ary System With Peak-Equal-Average Power Limitation

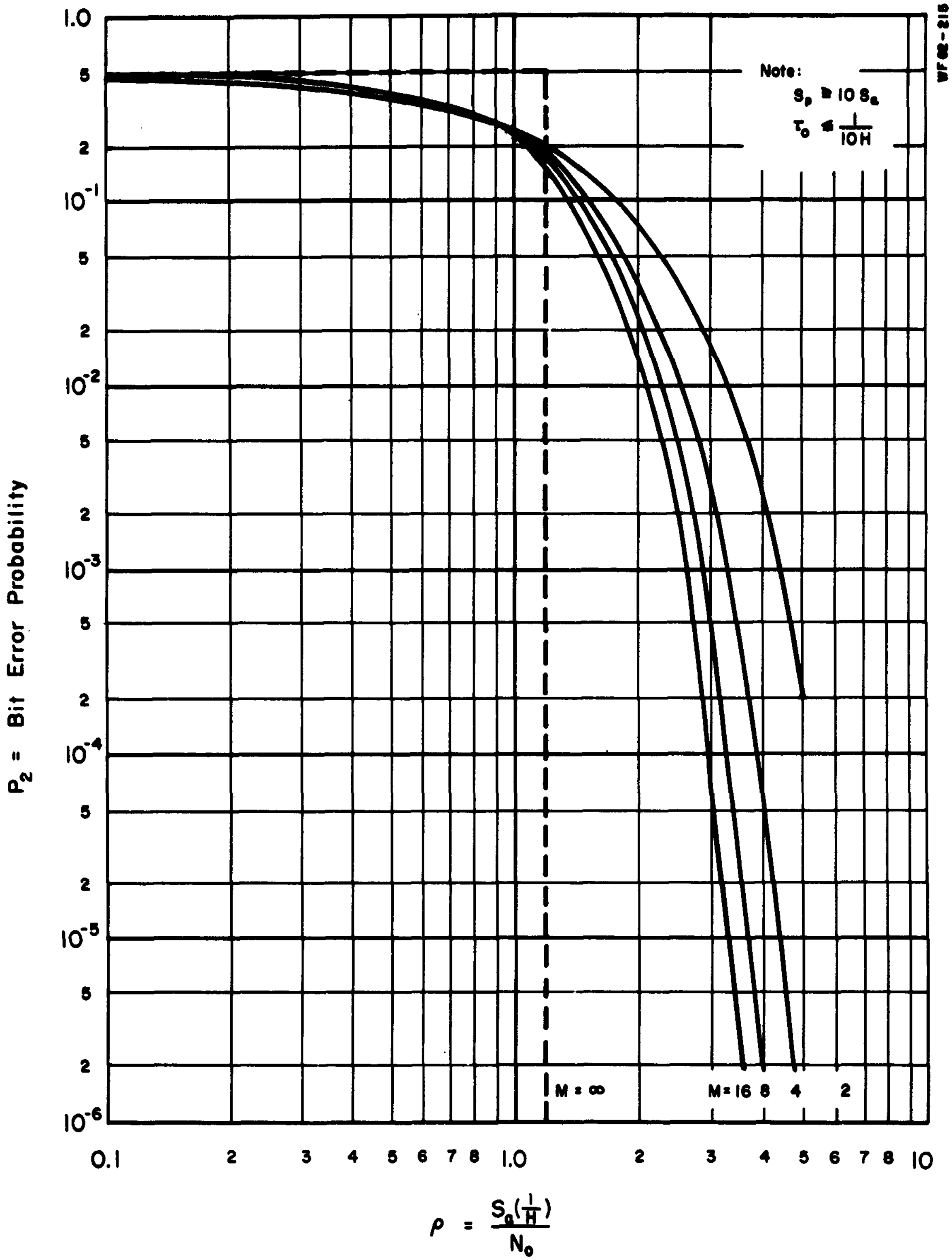

Figure 12. Performance of M-M-ary System With No Peak Power Limitation

The M-ary Linear Filtering Performance curves* are plotted in Figure 13 for both peak power constraints, i.e., $S_p = S_a$, and $S_p \geq 10\, S_a$. The dashed curves correspond to the constraint $S_p = S_a$, and the solid curves apply when $S_p \geq 10\, S_a$. As before, the fading is characterized by $\tau_o H = \frac{1}{10}$. Although increasing the alphabet size does provide some improvement in the system, basically it is unable to cope with a fading medium.

The performance of each of these three M-ary systems is plotted in Figure 14 for M = 16, under the peak-equal-average power constraint, and assuming $\tau_o H = \frac{1}{10}$.

The curves presented in Figures 9-14 suggest strongly that higher order alphabets hold the key to satisfactory performance at high information rates. Further, the use of systems optimized for a non-fading medium can produce extremely poor performance in a fading medium. Therefore, it is of prime importance to place major emphasis on the determination of optimum system structures for fading media, and the development of implementable block diagrams which exploit these structures with a large alphabet.

*These are based on the work reported in [8].

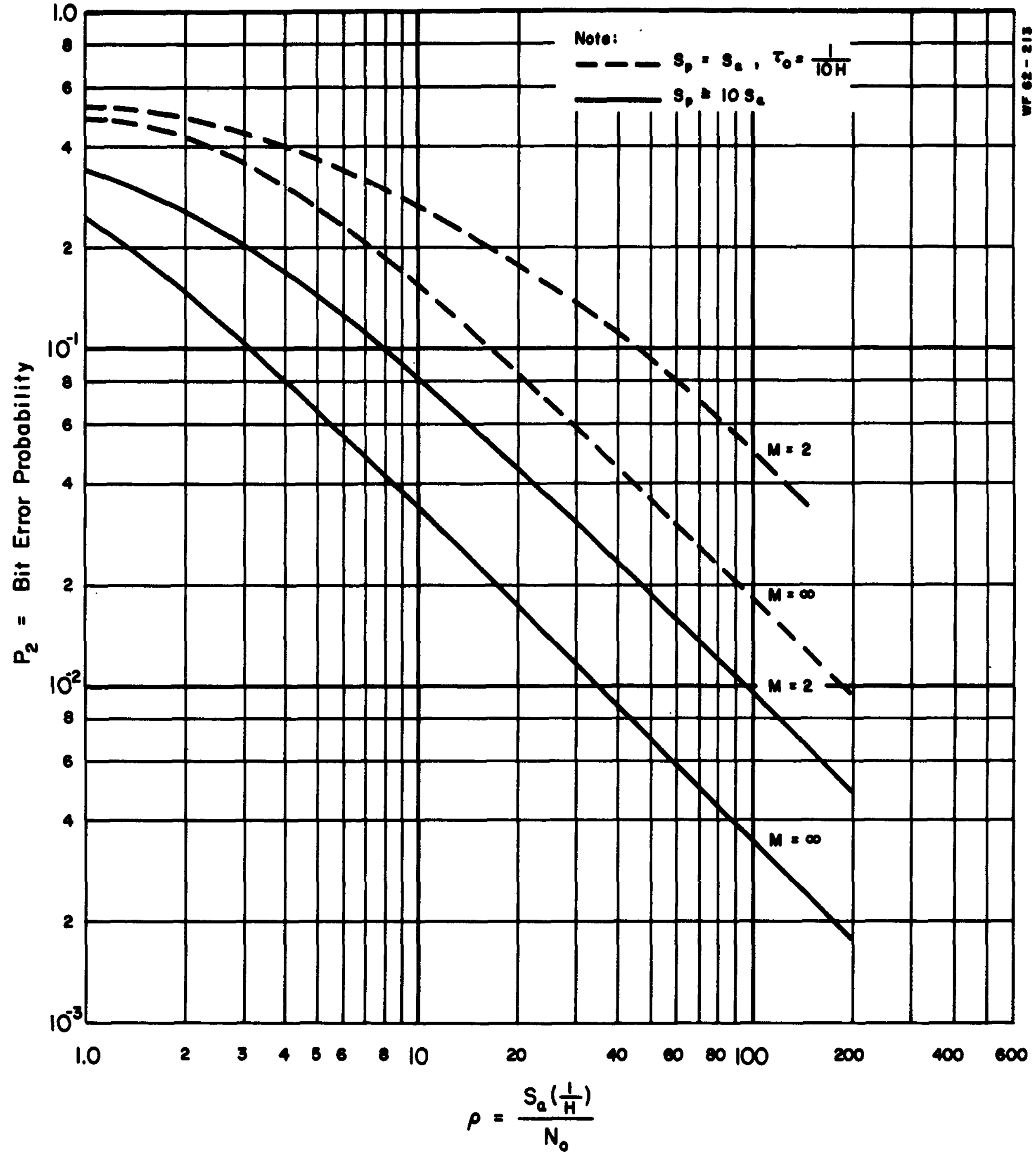

Figure 13. Performance of M-ary Linear Filtering System, With and Without a Peak Power Limitation

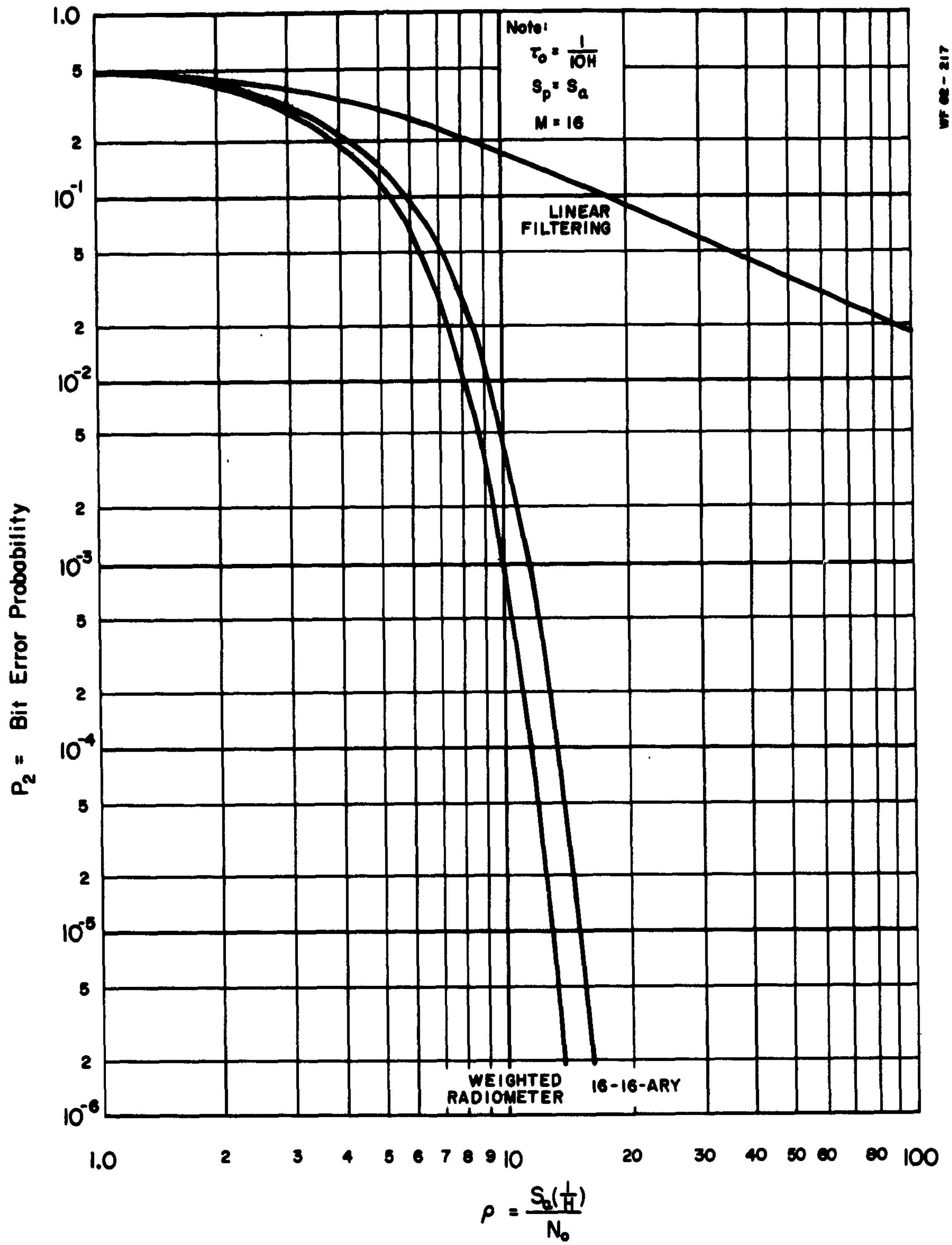

Figure 14. Performance of Weighted Radiometer M-M-ary, and Linear Filtering Systems With M =16, $S_p = S_a$, and $\tau_o H = \frac{1}{10}$.

6. Some Measurement Considerations

Many of the results presented earlier have presumed knowledge of the pertinent medium responses and average behavior. In this section, we shall consider some of the problems in attempting to determine these quantities from measurements.

6.1 Direct Measurement of a Non-Random Time-Variant System

Let us consider the direct measurement of $h(\tau, t)$. If we excite this filter with an impulse at time t_o, the response is $h(t - t_o, t)$ and we have a "slice" of h along a 45^o line in the τ, t plane. Since this function has duration L seconds on the t-axis (see Figure 3(a)), we cannot re-excite the filter with an impulse to get another "slice" until L seconds later than the first excitation; otherwise, the outputswould overlap, thereby generating ambiguities. Thus we can measure, in succession, $h(t - t_o, t)$, $h(t - t_o - L, t)$, $h(t - t_o - 2L, t)$, etc. In order that we not be missing any significant changes in $h(\tau, t)$ by this technique, there should be little change in the sequence of h-functions so obtained. That is, $h(t - t_o, t)$ delayed by L seconds should look like $h(t - t_o - L, t)$ for all t, t_o where the h-functions are not small. Thus we want

$$h(t - t_o - L, t - L) \cong h(t - t_o - L, t)$$

for all t, t_o giving h arguments in the L by D rectangle of Figure 3(a). Alternately this can be written

$$h(\tau, t - L) \cong h(\tau, t)$$

in the L by D rectangle. Since for fixed τ, $h(\tau, t)$ can change in a distance $\frac{1}{B}$ seconds on the t-axis (see Figure 3(a)), this requirement of slow change is satisfied only if

$$L < \frac{1}{B}$$

or

$$BL < 1.$$

When this condition is satisfied, interpolation from the 45° $\{h(t - t_o - k L, t)\}$ slices to the entire $h(\tau, t)$ function for all τ, t is then possible.

If BL > 1, it is still possible to get slices of $h(\tau, t)$ as above. However, the slices may differ significantly, and it is no longer possible to interpolate, and obtain the entire function $h(\tau, t)$*. This is the limitation of overspread (BL > 1) filters.

Notice, however, that if the filter could be "reset" to an earlier time (as for example with a network with a switch), we could interleave additional slices of the $h(\tau, t)$ function, each set of slices necessarily being separated by L seconds to avoid overlap, of course. This resetting procedure could be continued until the density of slices was sufficient to interpolate to the entire function. Thus if, for example, the impulse response were desired known over the interval of D seconds on the t-axis, a processing time of BLD seconds would be necessary to evaluate the entire $h(\tau, t)$ function.

It might be supposed at first that the limitation BL < 1 is due to the particular method of measurement of the system, namely impulse excitation. However, we run up to the same identical relation when we consider the direct measurement of $H(\eta, t)$. If we excite this filter with a pure tone $e^{i2\pi\eta_o t}$ starting at time t_o, the output is, for $t \geq t_o$,

$$e^{i2\pi\eta_o t} \int_{-\infty}^{t-t_o} h(\tau, t)\, e^{-i2\pi\eta_o \tau}\, d\tau$$

$$\cong e^{i2\pi\eta_o t} H(\eta_o, t)$$

*This situation is similar to sampling a function band-limited to W cps, slower than the rate of 2W per second. It is not possible to interpolate to the entire function from the "sparse" samples.

for $t - t_o$ sufficiently large, i.e., outside the region of L seconds on the τ-axis. Thus by exciting the filter sufficiently long ago, we can evaluate H on any line η_o, t in the η, t plane. However, in order to be able to evaluate H at other values of η, we must excite the filter simultaneously with a set of pure tones. (This procedure is the exact dual of the measurement of $h(\tau, t)$. There, we excited with a set of impulses in the time domain. Here, we excite with a set of impulses in the frequency domain.) In order not to miss significant changes of $H(\eta, t)$ in η, we must space the pure tones by no more than $\frac{1}{L}$ cps (see Figure 3(b)). If the pure tones are at frequencies $\{\eta_k\}$, the response, after sufficient time, is approximately

$$\sum e^{i2\pi\eta_k t} H(\eta_k, t).$$

In order to be able to evaluate each $H(\eta_k, t)$ separately, significant changes in $H(\eta_k, t)$ must not occur in less than L seconds; otherwise it would be impossible to separate the individual filter output tones. From Figure 3(b), we see this is possible only if

$$L < \frac{1}{B}$$

or

$$BL < 1.$$

If this condition is satisfied, $H(\eta, t)$ may be interpolated from the slices $\{H(\eta_k, t)\}$. From the complete H-function, $h(\tau, t)$ could of course be obtained by Fourier transformation.

Both measurement methods above require $BL < 1$ in order that complete measurement of the filter be possible. Undoubtedly if other test procedures for the direct evaluation of the A- and $\mathcal{H}$-functions were proposed, they too would require $BL < 1$ for unambiguous filter evaluation. The restriction $BL < 1$ is a fundamental restriction on ability to measure completely a filter's characteristics. Furthermore, we repeat that this filter had no random variation whatsoever; random variations can only generate tighter restrictions.

6.2 Measurement of a Random System

Measurements of the two components $d(\tau, t)$ and $r(\tau, t)$ of a general system $h(\tau, t)$ could be accomplished (for a particular geometry, thermocline, known range, etc.) by using an explosive source, and measuring the response at the desired receiving site, provided the external noise $n(t)$ is small in comparison with the random component $r(\tau, t)$. The signal source (in the submarine) must be turned off for this measurement. In fact, the source should probably be removed from the area because its random output affects the measurements.

Since an explosive source is itself random, the response of the medium will contain a random component due to the source. However, if Δ is the effective duration of the "impulse", estimates of d and r up to frequencies less than $1/\Delta$ will be okay. Equivalently, estimates of D and R will be valid for $f < 1/\Delta$.

These measurements should be repeated as rapidly as possible in succession a number of times, before any important parameters have changed, such as temperature of the water. Of course, a new explosion and measurement must not be undertaken before all the multiple echoes from the different possible bottom bounce reflections (multipaths) have been received from the past explosion, as discussed in Section 6.1. These measurements should be made at a great enough distance from the source of the explosion in order to avoid near field effects, and yet not so far away that the external noise $n(t)$ dominates $r(t)$. The tests should be monitored by measuring the intensity of the explosion as close to the source as reasonable and practical.

In order to ascertain the degree of non-stationarity of $h(\tau, t)$ (more particularly $d(\tau, t)$), the series of measurements of $h(\tau, t)$ should be divided into groups, the members of each of which have been determined under substantially the same conditions. The size of each group should be large enough so as to be able to determine $d(\tau, t)$ accurately (by averaging the measurements), and yet not so large that time varying items, such as water temperature, have made any significant change during that time. For example, suppose time is separated into intervals $\{T_j\}$ such that during each interval T_j, $h(\tau, t) \cong h_j(\tau)$ for $t \in T_j$. (Also, of course, $d(\tau, t) \cong d_j(\tau)$ for $t \in T_j$. This is the mathematical way of saying that the time variant parameters change little during T_j.) Then suppose a series of measurements $\{w_{jk}(t)\}$ of the impulse response were recorded, where, for j fixed, $t_{jk} \in T_j$, $k = 1, 2, \ldots, n$. That is, n measurements of the impulse response were made during the time interval T_j. Then an approximation to the deterministic component of the impulse response is realized according to

$$d_j(\tau) \cong \frac{1}{n} \sum_{k=1}^{n} w_{jk}(\tau + t_{jk}).$$

The shift t_{jk} aligns all the impulse responses at "zero" time, which is chosen for convenience. As j progresses through its values, local averages of $d(\tau, t)$ in intervals of length T_j on the t-scale are obtained. That is, the quantity

$$\frac{1}{T_j} \int_{T_j} d(\tau, t)\, dt$$

is approximated in this fashion. The difference quantities

$$r_{jk}(\tau + t_k) - d_j(\tau), \; k = 1, 2, \ldots, n,$$

for j fixed, are then available for evaluating parameters of the random component of the impulse response, such as stationarity, average power, etc.

Independent measurements of $d(\tau, t)$ and $D(f, t)$ would yield information as to the validity of the linear model, for only in this case are these expressions related by a Fourier transform. Also, if corroborated, the easier of the two measurement techniques can be selected and recommended for future measurement programs. (Notice that we can only relate d and D, and not h and H; the latter are noisy, including $n(t)$, and would be impossible to relate. d and D are approximately related by a Fourier transform because they are the means, respectively, of h and H.)

The only remaining item of importance in Figure 1 is the additive noise $n(t)$. This can be observed with no signal transmitted, either deterministic or random, its average behavior and power determined, its low order p. d. f's approximated, etc. This noise arises from water and wave movement, fish, and other ships.

Although measurement of the transmitted signal is not directly related to measurement of the medium characteristics, a few words about it are appropriate at this point, in view of the background developed above. In order to evaluate the amount of deterministic component $s(t)$ present in the transmitted waveform, and least contaminate the results, and obtain the most consistency, initial measurements of this quantity should probably be done with a stationary submarine using an absolute minimum of auxiliary equipment in a quiet environment. A long duration "close aboard" measurement should be conducted, repeating a basic signal ($\sim$ 100 times). (The "close aboard" feature eliminates medium effects; however the measurement should be conducted far enough away that near-field effects are negligible.) The total duration of the run should not be so long, however, as to allow any important parameters to change. An estimate of $s(t)$ is then realized by averaging the 100 runs, properly aligned in time. The difference between the transmitted waveform and the estimate of $s(t)$ is the random component $a(t)$, and can itself be analyzed to find the average power, degree of stationarity, and first, second, or higher order p. d. f. 's (probability density functions). Evaluation of low-order p. d. f. 's may indicate what type of process the random component is, such as Gaussian. From the above, among other things, quantitative estimates of the percentage of deterministic component in the transmitted waveform will be available, and decisions as to whether to attempt to use the random component in detection and communication can be made.

6.3 Effects of Non-Stationarity

Measurements of the statistical parameters of a random process are virtually always determined through time averages, and the ergodic theorem employed to relate these to corresponding ensemble average. However, for a non-stationary process, these two averages are not equal. It is therefore of interest to see what effect non-stationarity has on some of the typical time averages often computed.

Consider first a process in which the non-stationary character has a bandwidth W_n; i.e., significant changes in statistical parameters occur in a time $1/W_n$. Also, suppose the bandwidth of the process itself is W. Now if

$$\frac{1}{W} << \frac{1}{W_n},$$

many independent samples of the process are available in a time constant $(1/W_n)$ of the non-stationary fluctuations. In this case, estimation of local process parameters is possible from a single member function of the ensemble. For example, an estimate of the mean, from the j-th member function of an ensemble, is

$$\frac{1}{T}\int_T x^{(j)}(t)\,dt \quad (T = 1/W_n)$$

$$\cong \frac{1}{2WT}\sum_{n=1}^{2WT} x^{(j)}(t_n)$$

(This choice of T guarantees that $x^{(j)}(t)$ is quasi-stationary in the averaging interval T.) If $2WT >> 1$, $(W >> W_n)$, we have a reasonably good estimate of the local mean.

If, on the other hand, W_n and W are comparable, good estimates of local parameters are not possible from a single member function. However, good estimates of average local parameters may or may not be, depending on the particular parameter. Let us illustrate this contention. Consider a random discrete voltage, which at time k seconds, can take on voltage values k, k + 1, . . . , k + 5, with equal probability (a non-stationary die). Suppose we attempt to determine the average mean* of the voltage over an interval of N seconds according to the rule

$$y = \frac{1}{N}\sum_{k=1}^{N} x_k$$

where x_k is a voltage sample at time k. Now

$$\overline{x_k} = k + \frac{5}{2}$$

giving

$$\overline{y} = \frac{N}{2} + 3$$

Therefore

$$y - \overline{y} = \frac{1}{N}\sum_{k=1}^{N}(x_k - k - \frac{5}{2}) \equiv \frac{1}{N}\sum_{k=1}^{N} z_k$$

*The word "mean" indicates an ensemble average; the word "average" indicates a time average.

But the distribution of z_k is uniform over the values $-\frac{5}{2}, -\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{5}{2}$, and is independent of k. Therefore $\sigma^2(z_k) = \sigma_z^2$, and

$$\sigma^2(y) = \frac{1}{N}\sigma_z^2$$

Therefore since the variance of y tends to zero as N increases, y tends to the average mean of the random process over any (large) interval. Thus the presence of non-stationarity does not preclude obtaining some useful average local parameters from a single member function. (Of course, $\overline{x_k}$ could not be evaluated in this manner.)

On the other hand, if we attempt to estimate the average mean-square of the voltage according to the rule

$$W = \frac{1}{N}\sum_{k=1}^{N} x_k^2$$

we find a different result altogether. We have

$$\overline{x_k^2} = \frac{1}{6}\left[k^2 + (k+1)^2 + \ldots + (k+5)^2\right]$$

$$= k^2 + 5k + \frac{55}{6}$$

Therefore

$$W - \overline{W} = \frac{1}{N}\sum_{k=1}^{N}\left(x_k^2 - k^2 - 5k - \frac{55}{6}\right) = \frac{1}{N}\sum_{k=1}^{N}\alpha_k$$

where

$$\alpha_k \equiv x_k^2 - \overline{x_k^2}\,.$$

Therefore

$$\sigma^2(\alpha_k) = \overline{x_k^4} - \overline{x_k^2}^{\,2}$$

But

$$\overline{x_k^4} = \frac{1}{6}\ [k^4 + (k+1)^4 + \ldots + (k+5)^4]$$

Combining this result with the above one for $\overline{x_k^2}$, we obtain

$$\sigma^2(\alpha_k) = \frac{35}{3}k^2 + \frac{175}{3}k + 79\frac{5}{36}$$

and

$$\sigma^2(W) = \frac{1}{N}\sum_{k=1}^{N}\left(\frac{35}{3}k^2 + \frac{175}{3}k + 79\frac{5}{36}\right)$$

which tends to infinity as N does. Thus the variance of the estimate gets arbitrarily large as the averaging interval (N seconds) increases.* Here we would require many member functions to obtain a good estimate of the average mean-square voltage over a long interval.

In general, when obtaining an average of a random variable according to

$$y = \frac{1}{N}\sum_{k=1}^{N} x_k,$$

then

$$\sigma^2(y) = \frac{1}{N^2}\sum_{k=1}^{N}\sigma^2(x_k)$$

if $\{x_k\}$ are uncorrelated. Then if

1) $\sigma^2(x_k) = c, \quad \sigma^2(y) \longrightarrow 0$

2) $\sigma^2(x_k) = ck, \quad \sigma^2(y) \sim c$

3) $\sigma^2(x_k) = ck^{\alpha}, \ \alpha < 1, \ \sigma^2(y) \longrightarrow 0$

as N increases. Thus the rate of increase of variance of $\{x_k\}$ is of paramount importance in the adequacy of the end result.

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*It should be noted, however, that $\overline{W}/\sigma(W)$ approaches zero as N increases. Thus the relative error in estimation of $\overline{W}$ decreases.

In considering an estimate of an average local parameter of a process, the merit of the estimate depends heavily on its variance. If the variance is small (in some sense), there will be little difference in estimates obtained with different member functions. And, indeed, this observation allows for a relatively simple check on the adequacy of a particular estimate: two estimates of the desired parameter are calculated from two different member functions. If the estimates are reasonably close (in some sense), either estimate is adequate. If they are widely different, the method of obtaining the estimate is totally inadequate.

We have said nothing about stationarity in the above paragraph. The estimate will be an approximation to a time average of some statistical parameter. This parameter may or may not be appropriate, depending on the problem under investigation. For example,

$$y = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} x(t)\, dt$$

is an unbiased estimate of the average mean of x(t) over the interval (t_1, t_2). (This follows because the ensemble mean of y is

$$\bar{y} = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} m(t)\, dt$$

where m(t) is the ensemble mean of x(t).) If $\sigma^2(y)$ is small, any two member functions will yi eld essentially the same value for y, with probability near unity.

6.4 Recommended Measurements

In view of the relative simplicity of the linear model approach to underwater communications, it is recommended that a concerted measurement program be conducted to verify the model and ascertain its utility in typical applications. It may turn out that although the model is correct, the random fluctuations in the transmitted waveform, the medium, and the environment are so large that satisfactory communication and detection beyond certain ranges and in certain geometries is impossible, even with the so-called optimum receiver. This fact of life, if true, could be determined by this program.

The end products of the measurements should be the following items:

1. Non-random signal component of transmitted waveform when communication is taking place.
2. Random component of transmitted waveform: mean, mean square, correlation function, second-order correlation function, first, second, and third order p. d. f. 's. (These quantities may vary with time.)
3. Non-random component of impulse response of medium.
4. Random component of impulse response of medium: same statistics as under 2.
5. Additive noise: same statistics as under 2.
6. Non-random component of received waveform.
7. Random component of received waveform: same statistics as under 2.
8. Non-random component of medium transfer function, amplitude and phase.
9. Crosscorrelation of random radiated signal with received signal (no deterministic signal transmitted).

In order to determine the quantities above, continuous recordings of the waveforms "close aboard" and at the receiving site should be obtained. The pertinent processing can be done later. If other statistics are deemed important later, they can be computed from the tape records.

7. Discussion

The model of the communication system suggested here is probably the simplest one bearing any reasonable resemblance to the actual situation. How accurate it can be made will have to be borne out by measurements. If measurements indicate that the model is a poor one, other possibilities will have to be investigated. One simple alternative is a combination of a linear filter with memory in cascade with a nonlinear filter without memory (although perhaps time varying). However the analysis and optimization problems will probably be unwieldly except by lengthy computer effort. Very little useful analytical work on nonlinear networks has been accomplished thus far.

A method of circumventing the whole model problem is suggested by likelihood ratio detection and communication. A high order density function of a set of samples is obtained for each communication signal and for noise alone. The decision as to which signal, if any, was transmitted is reached by computing the likelihood ratios and choosing the largest as corresponding to the actual situation. This approach requires a great amount of data acquisition and reduction, mainly because the number of samples necessary to accurately estimate high order p. d. f. 's is enormous. However it benefits from the fact that source, medium, and receiver need not be separable, as they were considered above, but may be treated as an entity with merely one input and one output.

A model of a communication channel, similar to that suggested in Section 2 has recently been considered* and solved for the optimum receiver. However, the relative amount of improvement afforded by the optimum receiver, as contrasted with the usual simpler matched filter for the deterministic components, has not been determined, due to mathematical intractabilities. Further analytical work along this line is necessary for the underwater application, where the random components of signal and medium are very substantial in comparison with the deterministic components.

*[11, 12]

The program of study underway presently is now outlined. For the linear time-varying stochastic model:

a) Relate the statistical properties of components $\{w_k(t)\}$ of the received waveform to those of the medium and transmitted signal. These properties will include ensemble mean, power, correlation function, structure function, higher order moments, and first order probability density function. Calculate the signal-to-noise ratio of the received wave. Investigate the validity and effect of any approximations used.

b) On the basis of the results of a), decide which components are useful and what form of receiver is desirable for detection and/or communication. This approach will be limited to linear filters and simple nonlinear operations.

c) For a linear receiver, relate the statistical properties of the output to those of the transmitted signal, medium, and receiver impulse response. The properties to be investigated are the same as listed under a).

d) Maximize the linear receiver output signal-to-noise ratio by choice of impulse response.

e) Evaluate the detection probabilities at the output of the receiver for several impulse responses; choose those cases that are most easily realized first.

f) For known transmitted signal and receiver response, determine what target characteristics can be deduced from the processed echo. Deduce what form of sounding signal is optimum.

g) For assumed statistical forms of medium and noise, deduce the optimum receiver. Evaluate the performance of this receiver and/or approximations to it in terms of signal-to-noise ratio, detection probabilities, and error probabilities.

h) In all the above, determine how the receiver should adapt to time-varying changes in the medium and noise. Determine what additional measurements are necessary to track the changes, and whether the improvement is significant compared to an average design.

i) Investigate generalizations of the model and results to include spatial parameters.

Almost all the tasks cited above deal with the analysis of a cascade of two linear (time-varying) filters on a transmitted stochastic signal. The second linear operation need not be a strict match to the transmitted signal, however; as such, it could be called a generalized correlation receiver. That is, a more general "multiplier-integrator" form is allowed than conventionally. Results on this configuration will constitute a very worthwhile addition to the signal processing problems of several fields.

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APPENDIX A

M-ARY WEIGHTED RADIOMETER

The optimum detection of small (fading) signals in the presence of noise has been considered extensively* in terms of structure and signal-to-noise ratio. Here we wish to extend the analysis to communication and accompanying error probabilities. Furthermore, the communication mode will not be limited to binary, but rather, to general multiple alternative (M-ary) detection.

To be specific, one of M equiprobable alternatives is transmitted over the time interval (t_a, t_b). Letting $T = t_b - t_a$, the source information rate is

$$H = \frac{\lg M}{T}$$

where $\lg x \equiv \log_2 x$. The transmitted signal, without loss of generality, will be denoted by $s_1(t)$:

$$s_1(t) = \mathrm{Re}\,\{\xi_1(t)\, e^{i\omega_o t}\}$$

where $\xi_1(t)$ is the complex envelope of the signal, and is non-zero only over (t_a, t_b). At the receiver, the "signal" component of the received waveform is

$$\mathrm{Re}\,\{\xi_1(t)\, G(t)\, e^{i\omega_o t}\}$$

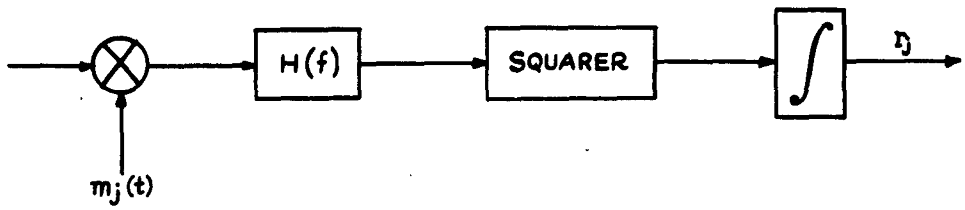
where G(t) is the complex (low pass) medium (fading) gain (see section 4). The total received waveform, including noise, is

$$\mathrm{Re}\,\{[\xi_1(t)\, G(t) + N(t)]\; e^{i\omega_o t}\}$$

The optimum receiver is an M-fold duplication of the basic weighted radiometer,** the only difference being the pre-multiplication function in each branch. A block diagram of the j-th branch of the M-ary weighted radiometer is shown below.

*[4]

**Ibid, p. 49



The output of this branch is readily demonstrated to be given by*

$$r_j = \iint [\xi_1(t_1) G(t_1) + N(t_1)] [\xi^*_1(t_2) G^*(t_2) + N^*(t_2)] \xi^*_j(t_1) \xi_j(t_2) \phi^*(t_1 - t_2) dt_1 dt_2$$

for all $1 \leq j \leq M$. (Infinite limits are allowed on the integrals since $\xi_j(t)$ is identically zero for t not in (t_a, t_b).) $\phi(\tau)$ is the complex low pass correlation of the medium.

Now we shall make two reasonable assumptions about the transmitted signals; first, the bandwidth of all the product functions $\{\xi_j(t) \xi^*_k(t)\}$, $j \neq k$, will be much wider than B, the "spread" of the medium. Since $B = 1/\tau_o$, where τ_o is the coherence interval of the medium fading, this first assumption is equivalent to many "wiggles" of the product functions in time τ_o. Secondly, the transmitted signals will have flat amplitude distribution in time; i.e.,

$$|\xi_j(t)| = \sqrt{2S_a}\ , \quad t_a < t < t_b, \text{ all } j,$$

where S_a is the average power of the received signal. Since

$$S_a = \frac{1}{2} \overline{|\xi_1(t)\, G(t)|^2} = \frac{1}{2} |\xi_1(t)|^2 \ \overline{|G(t)|^2}$$

we must have

$$\overline{|G(t)|^2} = 1$$

which is no more than a choice of normalization. More generally, we have

$$\phi(t_1 - t_2) = \overline{G(t_1)\, G^*(t_2)}$$

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*Ibid, pp. 50-54.

the correlation function of the medium fading. Using the above assumptions, we have

$$r_1 = \iint_{t_a}^{t_b} [2 S_a G(t_1) + \xi_1^*(t_1) N(t_1)] [2 S_a G^*(t_2) + \xi_1(t_2) N^*(t)] \phi^*(t_1 - t_2)\, dt_1\, dt_2$$

and

$$r_j \cong \iint_{t_a}^{t_b} N(t_1) N^*(t_2) \xi_j^*(t_1) \xi_j(t_2) \phi^*(t_1 - t_2), \quad j \geq 2.$$

Utilizing the Gaussian and white properties* of N(t), we find

$$\overline{r_1} = (2 S_a)^2 I_2 + 8 S_a TN_d$$

where

$$I_2 = \iint_{-T/2}^{T/2} |\phi(t_1 - t_2)|^2 dt_1\, dt_2 = \int_{-T}^{T} (T - |u|) |\phi(u)|^2 du$$

and N_d is the double-sided noise power density level ($N_o = 2N_d$).

Also,

$$\overline{r_j} = 8 S_a TN_d, \quad j \geq 2.$$

Continuing, it is tedious, but not difficult, to show that

$$\overline{r_1 r_j} = \overline{r_1}\, \overline{r_j}, \quad j \geq 2$$

$$\overline{r_j r_k} = \overline{r_j}\, \overline{r_k}, \quad j \neq k$$

$$\sigma^2(r_j) = (2 S_a)^2 (4 N_d)^2 I_2, \quad j \geq 2$$

$$\sigma^2(r_1) = (2 S_a)^2 (4 N_d)^2 I_2\, \alpha$$

* [3]

where

$$\alpha = 1 + \frac{S_a}{N_d} \frac{I_3}{I_2} + \left(\frac{S_a}{2N_d}\right)^2 \frac{I_4}{I_2}$$

$$I_3 = \iiint_{-T/2}^{T/2} \phi(t_1 - t_2)\, \phi(t_2 - t_3)\, \phi(t_3 - t_1)\, dt_1\, dt_2\, dt_3$$

$$I_4 = \iiiint_{-T/2}^{T/2} \phi(t_1 - t_2)\, \phi(t_2 - t_3)\, \phi(t_3 - t_4)\, \phi(t_4 - t_1)\, dt_1\, dt_2\, dt_3\, dt_4.$$

Now if $T >> \tau_o$, many independent segments of input signal will be added together (with noise) to form the variables $\{r_j\}$. Therefore $\{r_j\}$ will satisfy a joint Gaussian distribution. The probability of correct detection is then readily demonstrated to be*

$$P_c = \int \frac{\exp(-y^2/2)}{\sqrt{2\pi}}\, \Phi^{M-1}\, [\alpha y + \beta]\, dy$$

where

$$\beta = \frac{S_a}{N_o} \left[\int_{-T}^{T} (T - |u|)\, |\phi(u)|^2\, du \right]^{1/2}$$

Now since the Gaussian approximation holds only when $T >> \tau_o$, we find

$$\beta \cong \frac{S_a}{N_o} \left[\int_{-\infty}^{\infty} T\, |\phi(u)|^2\, du \right]^{1/2}$$

For a linear correlation

$$\phi(u) = 1 - \frac{|u|}{2\tau_o}, \quad |u| < 2\tau_o$$

$$\beta \cong \sqrt{\frac{4}{3}\, \frac{S_a T}{N_o}\, \frac{S_a \tau_o}{N_o}}$$

*[9]

Also $\alpha \cong 1$ if $S_a \tau_o / N_o << 1$. Therefore,

$$P_c \cong \int \frac{\exp(-y^2/2)}{\sqrt{2\pi}} \Phi^{M-1} \left[y + \sqrt{\frac{4}{3} \frac{S_a T}{N_o} \frac{S_a \tau_o}{N_o}} \right] dy.$$

This result is used extensively for plots given in the main body of the report. For M = 2, a special case results:

$$P_{c2} \cong \Phi \left(\sqrt{\frac{2}{3} \frac{S_a T}{N_o} \frac{S_a \tau_o}{N_o}} \right)$$

If the source data rate is kept fixed, and M increases without bound, we can use the result that

$$\lim_{M \to \infty} \Phi^M \left[x + \gamma \sqrt{2 \ln M} \right] \longrightarrow 1, \text{ all } x, \text{ if } \gamma > 1$$

to show that

$$P_c \longrightarrow 1 \text{ as } M \to \infty \text{ if } H < \lg e \frac{S_a}{N_o} \left(\int_0^\infty |\phi(u)|^2 \, du \frac{S_a}{N_o} \right) .$$

Thus, analogous to the non-fading case, zero error probability can be obtained in the limit, with a non-zero source rate. Thus, the capacity of a fading medium (infinite bandwidth) is bounded by

$$C_f \geq \lg e \frac{S_a}{N_o} \left(\int_0^\infty |\phi(u)|^2 \, du \frac{S_a}{N_o} \right)$$

This relation, and the others in this appendix, hold if

$$\frac{S_a}{N_o} \int_0^\infty |\phi(u)|^2 \, du << 1.$$

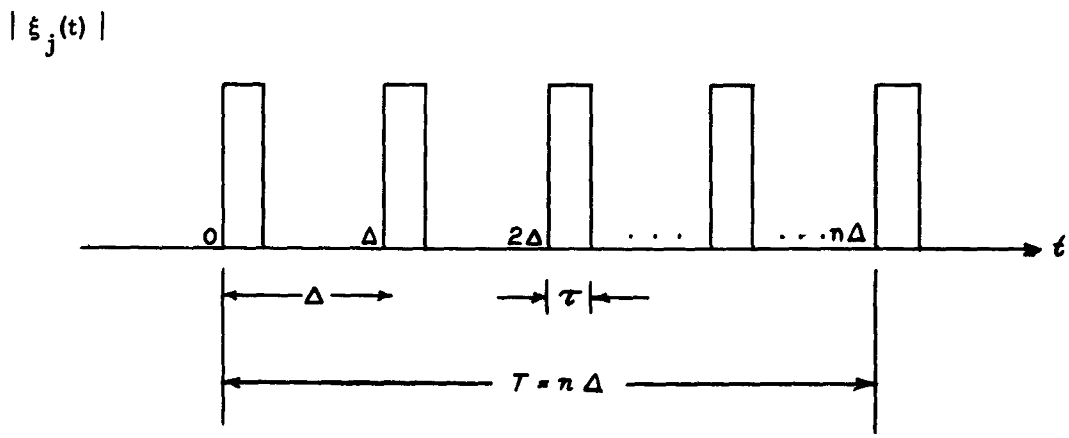
Furthermore, in this case, and an $S_p = S_a$ power limitation, this is the optimum obtainable performance.

APPENDIX B
MULTIPLE ALTERNATIVE RECEIVER

This appendix will be concerned with the derivation of the error probabilities of a multiple alternative receiver, where the same alternative (message) is repeated sequentially for the entire message interval T. The derivations are conveniently broken into two main parts: the first part deals with the case of no peak power limitation, whereas the second part considers a peak-equal-average power limitation.

B.1 No Peak Power Limitation ($S_p \geq 10\ S_a$)

Without a peak power limitation, it might appear at first that the signal energy should be packed into a time interval short compared with the coherence interval, τ_0, so as to avoid noncoherent fading throughout the message interval T. However, an immediate and obvious drawback to this proposed technique is that communication is too heavily dependent on the instantaneous fading of the medium. A much more desirable alternative would be to split the total signal energy into several pulses, each of duration less than τ_0, and then add the components after detection.* This method performs some averaging over several fading intervals. Specifically, a possible partitioning of the signal energy is depicted in the figure below.



*[10]

There are n pulses in the message interval T, of equal height A. For an average power constraint S_a, we must have

$$S_a = \frac{1}{T} \int_{t_a}^{t_b} \frac{1}{2} \left| \xi_j(t)\, G(t) \right|^2 dt = \frac{1}{2T} \int_{t_a}^{t_b} \left| \xi_j(t) \right|^2 dt$$

$$= \frac{1}{2T} A^2 n\tau$$

Therefore,

$$A^2 = 2 S_a \frac{T}{n\tau} = 2 S_a \frac{\Delta}{\tau}$$

Here we are allowed to vary Δ and τ as we please, and choose A so as to satisfy the power constraint. The received waveform is

$$\mathrm{Re} \left\{ \left[\xi_1(t)\, G(t) + N(t) \right] e^{i\omega_o t} \right\}$$

We define R_{jk} as the envelope of the output of the filter matched to the j-th message during the k-th pulse. Here, $1 \le j \le M$, $1 \le k \le n$. Then

$$R_{jk} = \left| \int_{(k-1)\Delta}^{(k-1)\Delta+\tau} \xi_j^*(t) \left[\xi_1(t)\, G(t) + N(t) \right] dt \right|, \quad \text{all } j.$$

Now for <u>constant amplitude orthogonal signals</u> over the interval of duration τ,

$$R_{1k} = \left| 2 S_a \frac{\Delta}{\tau} \int_{(k-1)\Delta}^{(k-1)\Delta+\tau} G(t)\, dt + \int_{(k-1)\Delta}^{(k-1)\Delta+\tau} \xi_1^*(t)\, N(t)\, dt \right|$$

$$= \left| 2 S_a \frac{\Delta}{\tau} \int_0^{\tau} G\left[t + (k-1)\Delta \right] dt + \int_0^{\tau} \xi_1^*(t)\, N\left[t + (k-1)\Delta \right] dt \right|$$

where we have set

$$\xi_j\,[t + (k-1)\,\Delta] = \xi_j(t), \quad 0 < t < \tau, \quad \text{all } j,$$

corresponding to the repeated signal assumption. Also

$$R_{jk} = \left| \int_0^{\tau} \xi_j^*(t)\, \xi_1(t)\, G[t + (k-1)\Delta]\, dt + \int_0^{\tau} \xi_j^*(t)\, N[t + (k-1)\Delta]\, dt \right|$$

Now we consider adding the ν-th powers of the envelopes of the j-th message, to form the decision variables:

$$y_j = \sum_{k=1}^{n} R_{jk}^{\nu}, \quad 1 \le j \le M.$$

The source information rate is given by

$$H = \frac{\lg M}{T} = \frac{\lg M}{n\Delta}$$

For fixed Δ and H, as $M \to \infty$, so also must n. Therefore, each of the y_j variables approaches a Gaussian variable, and we can compute the probability of error providing we can evaluate $\overline{y_j}$ and $\sigma^2(y_j)$. We start with

$$\overline{y_j} = \sum_{k=1}^{n} \overline{R_{jk}^{\nu}}$$

But

$$R_{jk} = |Z_{jk}| = |x_{jk} + iy_{jk}|$$

where

$$Z_{jk} = \int_0^{\tau} \xi_j^*(t)\, \{\xi_1(t)\, G\,[t + (k-1)\Delta] + N[t + (k-1)\Delta]\}\, dt, \quad \text{all } j$$

Due to the linear operation on G and N, both of which are Gaussian, x_{jk} and y_{jk} are Gaussian. Furthermore,

$$\overline{Z_{jk} Z_{\ell m}} = 0$$

and, after simplification,

$$\overline{Z_{jk} Z^*_{\ell m}} = \iint_0^\tau \xi_j^*(t_1)\, \xi_\ell(t_2)\, \xi_1(t_1)\, \xi_1^*(t_2)\, \phi[t_1 - t_2 + (k-m)\Delta]\, dt_1\, dt_2$$

$$+ 4N_d(A^2\tau)\, \delta_{km}\delta_{j\ell}$$

using the orthogonality of the signals. In order to make further substantial progress, we assume that $\Delta \geq 4\tau_o$ and $\tau \leq \frac{1}{2}\tau_o$; physically this corresponds to uncorrelated fading between pulses and negligible fading within a pulse, respectively. With these assumptions, there readily follows

$$\overline{Z_{jk} Z^*_{\ell m}} = (A^2\tau)^2\, \delta_{km}\, \delta_{j1}\, \delta_{\ell 1} + (A^2\tau)\, 4N_d\, \delta_{km}\, \delta_{j\ell}$$

This formula indicates that all the statistical variables in different pulses are independent ! Therefore we need confine our analysis to a single pulse and later scale the means and variances by n. Dropping (unnecessary) second subscript, we have, expressly,

$$\overline{Z_1 Z^*_1} = (A^2\tau)^2 + (A^2\tau)\, 4N_d$$

$$\overline{Z_j Z^*_\ell} = 0, \; j \neq \ell$$

$$\overline{Z_j Z^*_j} = (A^2\tau)\, 4N_d, \; j \neq 1.$$

Therefore

$$\overline{x_1^2} = \overline{y_1^2} = \frac{(A^2\tau)^2}{2} + (A^2\tau)\, 2N_d \equiv \sigma_1^2$$

$$\overline{x_j^2} = \overline{y_j^2} = (A^2\tau)\, 2N_d \equiv \sigma_2^2, \quad j \geq 2$$

$$\overline{x_j} = \overline{y_j} = \overline{x_j y_j} = 0$$

and

$$p(x_1, y_1) = (2\pi\sigma_1^2)^{-1} \exp\left(-\frac{x_1^2 + y_1^2}{2\sigma_1^2}\right)$$

$$p(x_j, y_j) = (2\pi\sigma_2^2)^{-1} \exp\left(-\frac{x_j^2 + y_j^2}{2\sigma_2^2}\right), \quad j \geq 2.$$

Therefore since

$$R_j^{\nu} = (x_j^2 + y_j^2)^{\nu/2},$$

$$\overline{R_j^{\nu}} = \iint_{-\infty}^{\infty} (x_j^2 + y_j^2)^{\nu/2} (2\pi\sigma_j^2)^{-1} \exp\left(-\frac{x_j^2 + y_j^2}{2\sigma_j^2}\right) dx_j\, dy_j$$

$$= 2^{\nu/2} \sigma_j^{\nu}\, \Gamma\left(\frac{\nu}{2} + 1\right)$$

Also, immediately,

$$\overline{R_j^{2\nu}} = 2^{\nu} \sigma_j^{2\nu}\, \Gamma(\nu + 1)$$

Therefore

$$\overline{y_j} = n 2^{\nu/2} \sigma_j^{\nu}\, \Gamma\left(\frac{\nu}{2} + 1\right)$$

and

$$\sigma^2(y_j) = n \left[\overline{R_j^{2\nu}} - \overline{R_j^{\nu}}^{\,2}\right]$$

$$= n 2^{\nu} \sigma_j^{2\nu} \left[\Gamma(\nu+1) - \Gamma^2 \left(\frac{\nu}{2} + 1 \right) \right]$$

Therefore the signal-to-noise ratio, upon which error probabilities critically depend, is

$$\frac{\overline{y_j}^2}{\sigma^2(y_j)} = \frac{n \Gamma^2 \left(\frac{\nu}{2} + 1 \right)}{\Gamma(\nu+1) - \Gamma^2 \left(\frac{\nu}{2} + 1 \right)}$$

and the corresponding probability of correct decision is

$$P_c = \int \frac{\exp(-x^2/2)}{\sqrt{2\pi}} \Phi^{M-1} \left[\frac{\sigma(y_1)}{\sigma(y_2)} x + \frac{\overline{y_1} - \overline{y_2}}{\sigma(y_2)} \right] dx$$

Simplifying, we have

$$\frac{\sigma(y_1)}{\sigma(y_2)} = \left(1 + \frac{A^2 \tau}{4N_d} \right)^{\nu/2} \doteq \left(1 + \frac{S_a \Delta}{N_o} \right)^{\nu/2}$$

and

$$\frac{\overline{y_1} - \overline{y_2}}{\sigma(y_2)} = \sqrt{n} \left[\left(1 + \frac{S_a \Delta}{N_o} \right)^{\nu/2} - 1 \right] \alpha_\nu \equiv \beta$$

where

$$\alpha_\nu = \frac{\Gamma \left(\frac{\nu}{2} + 1 \right)}{\sqrt{\Gamma(\nu+1) - \Gamma^2 \left(\frac{\nu}{2} + 1 \right)}}$$

In order to obtain numerical results, let us now choose $\Delta = 1/H$; that is, let the pulse repetition period equal the bit interval. Then since

$$H = \frac{\lg M}{T} = \frac{\lg e \ln M}{n\Delta}$$

we must have

$$n = \lg e \ln M$$

Then

$$\beta = \sqrt{\lg e}\left[\left(1 + \frac{S_a \frac{1}{H}}{N_o}\right)^{\nu/2} - 1\right] \alpha_\nu \sqrt{\ln M}$$

and

$$P_c = \int \frac{\exp(-x^2/2)}{\sqrt{2\pi}} \Phi^{M-1}\left[\left(1 + \frac{S_a \frac{1}{H}}{N_o}\right)^{\nu/2} x + \beta\right] dx$$

This is the general expression for the error probability as a function of M, signal-to-noise ratio, ν, and information rate. We now use the fact that

$$\lim_{M \to \infty} \Phi^M \left[y + \gamma \sqrt{2 \ln M}\right] = 1, \text{ all } y, \text{ if } \gamma > 1,$$

to show that

$$P_c \longrightarrow 1 \text{ as } M \longrightarrow \infty \text{ if}$$

$$H < \frac{S_a/N_o}{\left(\frac{\sqrt{2 \ln 2}}{\alpha_\nu} + 1\right)^{2/\nu} - 1} \equiv C_\nu$$

The right-hand side of this last equation is a lower bound on the capacity of a fading channel. The representative numbers given in the table below indicate that

ν	1	1.5	2	2.5	3	4	6	8	10	12	∞
$\frac{C_\nu}{S_a/N_o}$	.62	.75	.85	.93	1.0	1.1	1.2	1.23	1.23	1.22	1.0

the capacity of a fading channel is approximately $\lg e \; S_a/N_o = 1.443 \; S_a/N_o$, just as though no fading were present. This topic deserves much further analysis, particularly regarding the applicability of the Gaussian form as n and M simultaneously tend to infinity. However, the above results hint that proper use of a fading channel can lead to almost the same performance as in the absence of fading.

B.2 Peak-Equal-Average Power Limitation ($S_p = S_a$)

Using the same notation as before, in this case we must have $\Delta = \tau$, and $A^2 = 2S_a$; therefore,

$$\overline{Z_{jk} Z^*_{\ell m}} = \iint_0^\tau \xi_j^*(t_1)\, \xi_\ell(t_2)\, \xi_1(t_1)\, \xi_1^*(t_2)\, \phi[t_1 - t_2 + (k-m)\tau]\, dt_1\, dt_2 + 4N_d\, (2S_a\tau)\, \delta_{km}\, \delta_{j\ell}$$

Now, restricting τ to be smaller than $\frac{1}{2}\tau_o$, we get

$$\overline{Z_{jk} Z^*_{\ell m}} = (2 S_a \tau)^2\, \delta_{j1}\, \delta_{\ell 1}\, \phi((k-m)\tau) + 4N_d\, (2 S_a \tau)\, \delta_{km}\, \delta_{j\ell}$$

Now we introduce the restriction $\nu = 2$. In this case,

$$y_j = \sum_{k=1}^{n} R_{jk}^2 = \sum_{k=1}^{n} Z_{jk} Z^*_{jk}$$

and

$$\overline{y_j} = \sum_{k=1}^{n} \overline{Z_{jk} Z^*_{jk}} = n\,(2 S_a \tau)\; [(2 S_a \tau)\, \delta_{j1} + 4N_d]$$

Also,

$$\overline{y_j^2} = \sum_{k=1}^{n}\sum_{m=1}^{n} \overline{Z_{jk} Z^*_{jk} Z_{jm} Z^*_{jm}}$$

$$= \sum_{k=1}^{n}\sum_{m=1}^{n} \left(\overline{Z_{jk} Z^*_{jk}}\;\; \overline{Z_{jm} Z^*_{jm}} + \overline{Z_{jk} Z^*_{jm}}\;\; \overline{Z^*_{jk} Z_{jm}} \right)$$

since Z_{jk} is Gaussian with zero mean. Using the expression for $\overline{Z_{jk} Z^*_{\ell m}}$ above, we get $\overline{Z_{jk} Z^*_{jm}} \; \overline{Z^*_{jk} Z_{jm}} = (2 S_a \tau)^2 \; [2 S_a \tau \delta_{j1} + 4N_d]^2$

and

$$\overline{Z_{jk} Z^*_{jm}} \; \overline{Z^*_{jk} Z_{jm}} = (2 S_a \tau)^2 \, [2 S_a \tau \phi((k-m)\tau) \, \delta_{j1} + 4N_d \delta_{km}]^2$$

Thus,

$$\overline{y_j^2} = (2 S_a \tau)^2 \Big\{ n^2 \; [2 S_a \tau \delta_{j1} + 4 N_d]^2 + \sum_{k=1}^{n} \sum_{m=1}^{n} [2 S_a \tau \phi \; ((k-m)\tau) \, \delta_{j1} + 4N_d \, \delta_{km}]^2 \Big\}$$

$$= (2 S_a \tau)^2 \Big\{ n^2 \; [2 S_a \tau \delta_{j1} + 4N_d]^2 + (2 S_a \tau)^2 \sum_{k=1}^{n} \sum_{m=1}^{n} \phi^2 ((k-m)\tau) \, \delta_{j1} + 16 \, n \, S_a \tau \, N_d \, \delta_{j1} + 16 \, n \, N_d^2 \Big\}$$

Now, making use of the approximation,

$$\sum_{j_1=1}^{n} \sum_{j_2=1}^{n} f((j_1 - j_2)\tau) \cong \frac{2}{\tau^2} \int_0^{n\tau} (n\tau - x) \, f(x) \, dx,$$

we get

$$\overline{y_j^2} = (2\, S_a \tau)^2 \left\{ n^2 \left[2\, S_a \tau\, \delta_{j1} + 4N_d \right]^2 \right.$$

$$+ 2\, (2\, S_a)^2 \int_0^{n\tau} (n\tau - x)\, \phi^2(x)\, dx \cdot \delta_{j1}$$

$$\left. + 16\, n\, S_a \tau\, N_d\, \delta_{j1} + 16\, n\, N_d^2 \right\}$$

$$\therefore \sigma_j^2 = \overline{y_j^2} - \overline{y_j}^2$$

$$= n(2\, S_a \tau\, N_d)^2 \cdot 16 \left\{ \left[\left(\frac{S_a}{N_d} \right)^2 \frac{1}{2n} \int_0^{n\tau} (n\tau - x)\, \phi^2(x)\, dx + \frac{S_a \tau}{N_d} \right] \delta_{j1} + 1 \right\}$$

Thus,

$$P_c = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{x^2}{2} \right] \Phi^{M-1} \left[x \left\{ \left(\frac{S_a}{N_d} \right)^2 \frac{1}{2n} \int_0^{n\tau} (n\tau - x)\phi^2(x)\, dx + \frac{S_a \tau}{N_d} + 1 \right\} + \sqrt{n}\, \frac{S_a \tau}{N_o} \right] dx$$

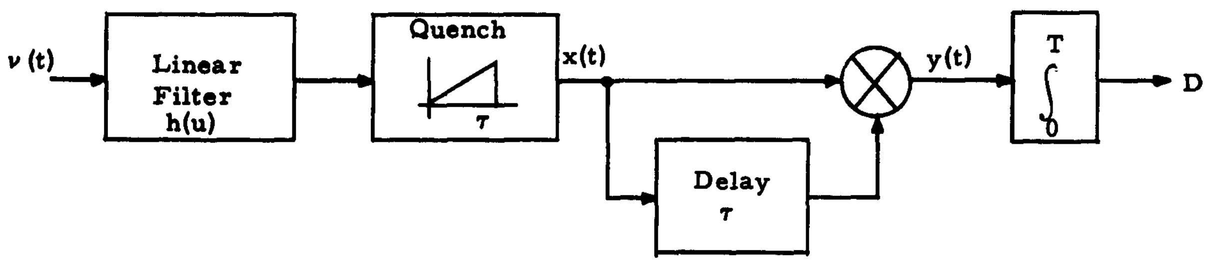
This expression is maximized by letting $\tau = \tau_o/2$ (the largest value allowed by the analysis) thus giving (for $\frac{S_a \tau}{N_o} << 1$),

$$P_c = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp \left[\frac{-x^2}{2} \right] \Phi^{M-1} \left[x + \sqrt{n}\, \frac{S_a \tau_o}{2N_o} \right] dx$$

APPENDIX C

DELAY CORRELATION SYSTEM

The block diagram of the system to be considered here is depicted below.



In this system, the quantity D is compared with a zero threshold: $D \lessgtr 0 \Rightarrow \begin{Bmatrix} \text{Mark} \\ \text{Space} \end{Bmatrix}$ transmitted.

To evaluate the performance of this system, we will derive $R = \dfrac{\overline{D}^2}{2\sigma_D^2}$,

which will provide the bit error probability for $H\tau_o << 1$. In terms of the received signal, $\nu(t)$, and the linear filter impulse response, h(t), the decision variable, D, is given by

$$D = \int_0^T y(t)\, dt$$

where $y(t) = x(t)\, x(t-\tau)$

$$\text{and} \quad x(t) = \sum_{j=1}^{n} x_j(t)$$

$$x_j(t) = F_j(t) \int_{(j-1)\tau}^{t} \nu(u)\, h(t-u)\, du$$

where it is assumed that $h(t) = 0$ for $t \leq 0$ and $t \geq \tau$.

and $F_j(t) = 1 \qquad j\tau \leq t \leq (j+1)\tau$

$\qquad = 0 \qquad$ otherwise

Substituting the explicit form of $y(t)$ in the first equation, we obtain

$$D = \sum_{j=1}^{n-1} \int_{j\tau}^{(j+1)\tau} dt \int_{j\tau}^{t} du_1 \int_{j\tau}^{t} du_2 \; \nu(u_1) \, \nu(u_2 - \tau) \, h(t-u_1) \, h(t-u_2)$$

$$= \sum_{j=1}^{n-1} \int_{j\tau}^{(j+1)\tau} dt \; A_j(t) \, B_j(t)$$

where $$A_j(t) = \frac{1}{2} \operatorname{Re} \{\alpha_j(t)\} = \int_{j\tau}^{t} du \; \nu(u) \, h(t-u)$$

$$B_j(t) = \frac{1}{2} \operatorname{Re} \{\beta(t)\} = \int_{j\tau}^{t} du \; \nu(u-\tau) \, h(t-u)$$

and $$\alpha_j(t) = e^{i2\pi f_o t} \int_{j\tau}^{t} V(u) \, H(t-u) \, du$$

$$\beta_j(t) = e^{i2\pi f_o t} \int_{j\tau}^{t} V(u-\tau) \, H(t-u) \, du,$$

and $V(t)$, $H(t)$ are the complex envelopes of the received signal $\nu(t)$, and filter impulse response, $h(t)$, respectively.

Now for the mean value of D,

$$\overline{D} = \sum_{j=1}^{n-1} \int_{j\tau}^{(j+1)\tau} dt \ \overline{A_j(t)\, B_j(t)}$$

But

$$\overline{A_j(t)\, B_j(t)} = \frac{1}{8} \ \text{Re} \ \overline{\{\alpha_j(t)\, \beta_j(t)} + \overline{\alpha_j(t)\, \beta_j{}^*(t)\}}$$

Now,

$$\overline{\alpha_{j1}(t_1)\, \beta_{j2}(t_2)} = e^{i2\pi f_o(t_1+t_2)} \int_{j_1\tau}^{t_1} du_1 \int_{j_2\tau}^{t_2} \overline{V(u_1)\, V(u_2-\tau)}\, H(t_1-u_1)\, H(t_2-u_2)$$

But, since $\overline{V(t_1)\, V(t_2)} = 0$ for all t_1, t_2 when $V(t)$ is Gaussian,*

$$\overline{\alpha_{ji}(t_1)\, \beta_{j2}(t_2)} = 0$$

Consider next $\overline{\alpha_{j1}(t_1)\, \beta_{j2}{}^*(t_2)}$ (with t_1, t_2, j_1, and j_2 general, since we will need the general result later for $\overline{D^2}$):

$$\overline{\alpha_{j1}(t_1)\, \beta_{j2}{}^*(t_2)} = e^{i2\pi f_o(t_1-t_2)} \left\{ \int_{j_1\tau}^{t_1} du_1 \int_{j_2\tau}^{t_2} du_2\, \phi(u_1-u_2+\tau)\, \xi(u_1)\, \xi^*(u_2-\tau) \right.$$

$$H(t_1-u_1)\, H^*(t_2-u_2)$$

$$\left. + (2N_o) \int_{j_1\tau}^{t_1} du_1 \int_{j_2\tau}^{t_2} du_2\, \delta(u_1-u_2+\tau)\, H(t_1-u_1)\, H^*(t_2-u_2) \right\}$$

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*See [2], p. 53.

where $\phi(u) \equiv \overline{G(t)\, G^*(t-u)}$. Since N(t) is derived from white Gaussian noise,*

$$\overline{N(t_1)\, N^*(t_2)} = 2\, N_o\, \delta(t_1 - t_2),$$

and N_o is the single sided noise power density.

Now, setting $j_1 = j_2$ and $t_1 = t_2$, we obtain

$$\overline{\alpha_j(t)\, \beta_j^*(t)} = \int_{j\tau}^{t} du_1 \int_{j\tau}^{t} du_2\, \phi(u_1 - u_2 + \tau)\, \xi(u_1)\, \xi^*(u_2 - \tau)\, H(t_1 - u_1)\, H^*(t_2 - u_2)$$

We now let (for the remainder of the appendix),

$$\begin{aligned} |\xi(t)| &= \sqrt{2S_a} \qquad 0 \le t \le T \\ &= 0 \qquad \text{otherwise} \end{aligned}$$

and

$$H(t) = F_1(t)$$

so that

$$\overline{D} = (2S_a) \sum_{j=1}^{n-1} \int_{j\tau}^{(j+1)\tau} dt\, \frac{1}{8} \int_{j\tau}^{t} du_1 \int_{j\tau}^{t} du_2\, \mathrm{Re}\, \{\phi(u_1 - u_2 + \tau)\}\, F_1(t - u_1)\, F_1(t - u_2)$$

$$= \frac{(2S_a)}{8} \sum_{j=1}^{n-1} \int_{j\tau}^{(j+1)\tau} dt \int_{j\tau}^{t} du_1 \int_{j\tau}^{t} du_2\, \mathrm{Re}\, \{\phi(u_1 - u_2 + \tau)\}$$

Now, using the identity

$$\iint_{a}^{a+b} f(x-y+c)\, g(x-y+d)\, dxdy$$

$$= \int_{-b}^{b} (b - |u|)\, f(u+c)\, g(u+d)\, du,$$

*I bid, p. 55.

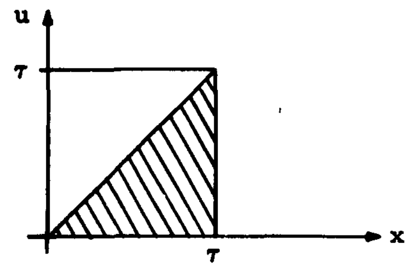
we get

$$\int_{j\tau}^{t} du_1 \int_{j\tau}^{t} du_2 \;\; \phi(u_1 - u_2 + \tau) = \int_{-(t-j\tau)}^{t-j\tau} du\, \phi\,(u+\tau)\;\; [t - j\tau - |u|] \;.$$

Substituting this integral in the expression for $\overline{D}$, and transforming the variable of integration, t, each term in the sum can be made independent of the index j, giving

$$\overline{D} = \frac{1}{4}\,(n-1)\;S_a \int_{o}^{\tau} dx \int_{o}^{x} du(x-u)\;\mathrm{Re}\,\{\,\phi\,(\tau+u)\;+\;\phi\,(\tau-u)\,\}$$

The region of integration is the u, x-plane is shown in the diagram.



Integration over this region can be performed in the form $\int_{o}^{\tau} du \int_{u}^{\tau} dx$; using this order, $\overline{D}$ becomes

$$\overline{D} = \frac{1}{4}\,(n-1)\;S_a \int_{o}^{\tau} du \int_{u}^{\tau} dx\;(x-u)\;\mathrm{Re}\{\;\phi\;(\tau+u)\;+\;\phi\;(\tau-u)\}$$

$$= \frac{1}{4}\,(n-1)\;S_a \int_{o}^{\tau} du\;\mathrm{Re}\;\{\;\;\phi\,(\tau+u)\;+\;\phi\;(\tau-u)\}\int_{o}^{\tau-u} dy\cdot\;y$$

$$= \frac{1}{8} (n-1) S_a \int_0^{\tau} du(\tau-u)^2 \operatorname{Re} \{ \phi(\tau+u) + \phi(\tau-u) \}$$

$$= \frac{1}{8} (n-1) S_a \; \psi(\tau)$$

where $$\psi(\tau) = \int_0^{\tau} du \, u^2 \operatorname{Re} \{ \phi(2\tau-u) + \phi(u) \}$$

For the second moment we have

$$\overline{D^2} = \sum_{j_1=1}^{n-1} \sum_{j_2=1}^{n-1} \int_{j_1\tau}^{(j_1+1)\tau} dt_1 \int_{j_2\tau}^{(j_2+1)\tau} dt_2 \, \overline{A_{j1}(t_1)\, B_{j1}(t_1)\, A_{j2}(t_2)\, B_{j2}(t_2)}$$

But

$$\overline{A_{j1}(t_1)\, B_{j1}(t_1)\, A_{j2}(t_2)\, B_{j2}(t_2)}$$

$$= \overline{\frac{1}{2} \operatorname{Re} \{\alpha_{j1}(t_1)\} \; \frac{1}{2} \operatorname{Re} \{\beta_{j1}(t_1)\} \; \frac{1}{2} \operatorname{Re} \{\alpha_{j2}(t_2)\} \; \frac{1}{2} \operatorname{Re} \{\beta_{j2}(t_2)\}}$$

$$= \overline{\frac{1}{2^6} \operatorname{Re} \{\alpha_1\beta_1 + \alpha_1 \beta_1{}^*\} \operatorname{Re} \{\alpha_2\beta_2 + \alpha_2\beta_2{}^*\}}$$

$$= \frac{1}{2^7} \operatorname{Re} \{\overline{\alpha_1\beta_1{}^* \alpha_2\beta_2{}^*} + \overline{\alpha_1\beta_1\, \alpha_2{}^* \beta_2{}^*} + \overline{\alpha_1\beta_1{}^* \alpha_2{}^* \beta_2}\}$$

where $\alpha_\ell \equiv \alpha_{j\ell}(t_\ell)$, and $\beta_\ell \equiv \beta_{j\ell}(t_\ell)$, $\ell = 1, 2$

(Averages of all terms not containing an equal number of conjugated and unconjugated factors are zero, since the α's and β's involve the same Gaussian process, V(t)).

Now, *

$$\overline{\alpha_1 \beta_1^* \alpha_2 \beta_2} = \overline{\alpha_1 \beta_1^*}\ \overline{\alpha_2 \beta_2^*} + \overline{\alpha_1 \beta_2^*}\ \overline{\alpha_2 \beta_1^*}$$

$$\overline{\alpha_1 \beta_1 \alpha_2^* \beta_2^*} = \overline{\alpha_1 \alpha_2^*}\ \overline{\beta_1 \beta_2^*} + \overline{\alpha_1 \beta_2^*}\ \overline{\alpha_2^* \beta_1}$$

$$\overline{\alpha_1 \beta_1^* \alpha_2^* \beta_2} = \overline{\alpha_1 \alpha_2^*}\ \overline{\beta_1^* \beta_2} + \overline{\alpha_1 \beta_1^*}\ \overline{\alpha_2^* \beta_2}$$

In the remainder of this appendix, we shall assume that the signal-to-noise ratio at the output of the linear filter is small, so that only terms involving N_o^2 are significant. With this assumption, we have, from the expression derived earlier,

$$\overline{\alpha_1 \beta_2^*} \cong e^{i2\pi f_o(t_1 - t_2)} (2 N_o) \int_{j_1\tau}^{t_1} du_1 \int_{j_2\tau}^{t_2} du_2\ \delta(u_1 - u_2 + \tau)$$

Similarly,

$$\overline{\alpha_1 \alpha_2^*} = \overline{\beta_1 \beta_2^*}$$

$$= e^{i2\pi f_o(t_1 - t_2)} \left\{ \int_{j_1\tau}^{t_1} du_1 \int_{j_2\tau}^{t_2} du_2\ \phi(u_1 - u_2) + 2 N_o \int_{j_1\tau}^{t_1} du_1 \int_{j_2\tau}^{t_2} du_2\ \delta(u_1 - u_2) \right\}$$

$$\cong (2 N_o)\ e^{i2\pi f_o(t_1 - t_2)} \int_{j_1\tau}^{t_1} du_1 \int_{j_2\tau}^{t_2} du_2\ \delta(u_1 - u_2)$$

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*[3]

Therefore,

$$\overline{\alpha_1 \beta_1^* \alpha_2 \beta_2^*} \cong (2 N_o)^2 \iiiint \delta(u_1 - w_1 + \tau)\ \delta(u_2 - w_2 + \tau)$$

$$+ (2 N_o)^2 \iiiint \delta(u_1 - u_2 + \tau)\ \delta(w_2 - w_1 + \tau)$$

where $$\iiiint \equiv \iint_{j_1\tau}^{t_1} du_1\, dw_1 \iint_{j_2\tau}^{t_2} du_2\, dw_2$$

But both of these terms are zero, since $|u_\ell - w_\ell| < \tau$ for $\ell = 1, 2$. In the first term, the δ-functions are zero unless $|u_\ell - w_\ell| > \tau$, and in the second term, at least one of the δ-functions is zero unless $|u_1 - w_1| = |u_2 - \tau - w_2 - \tau| = |u_2 - w_2 - 2\tau|$, in which case $|u_1 - w_1| \geq \tau$.

Similarly,

$$\overline{\alpha_1 \beta_1 \alpha_2^* \beta_2^*} = \overline{\alpha_1 \alpha_2^*}\ \ \overline{\beta_1 \beta_2^*} + \overline{\alpha_1 \beta_2^*}\ \ \overline{\alpha_2^* \beta_1}$$

$$\cong (\overline{\alpha_1 \alpha_2^*})^2 + 0$$

$$= (2 N_o)^2\ e^{i 2\pi f_o 2(t_1 - t_2)} \iiiint \delta(w_1 - w_2)$$

$$= (2 N_o)^2\ e^{i 2\pi f_o 2(t_1 - t_2)} (t_2 - j\tau)^2 \quad \text{if } j_1 = j_2 = j$$

$$= 0 \qquad \text{otherwise}$$

and

$$\overline{\alpha_1 \beta_1^* \alpha_2^* \beta_2} = \overline{\alpha_1 \alpha_2^*}\ \ \overline{\beta_1^* \beta_2} + \overline{\alpha_1 \beta_1^*}\ \ \overline{\alpha_2^* \beta_2}$$

$$\cong |\overline{\alpha_1 \alpha_2^*}|^2 + 0$$

$$= (2 N_o)^2 (t_2 - j\tau)^2 \qquad \text{if } j_1 = j_2 = j$$

$$= 0 \qquad \text{otherwise}$$

Therefore,

$$\overline{A_{j1}(t_1)\, B_{j1}(t_1)\, A_{j2}(t_2)\, B_{j2}(t_2)} \cong \frac{1}{2^7} \; \mathrm{Re} \left\{ (2\, N_o)^2 \, (t_2 - j\tau)^2 \left[1 + e^{i2\pi f_o \cdot 2(t_1 - t_2)} \right] \right\}$$

$$\cong 0 \qquad\qquad\qquad \text{otherwise}$$

Substituting in the expression for $\overline{D^2}$, we get

$$\overline{D^2} \cong \sum_{j=1}^{n-1} \int_{j\tau}^{(j+1)\tau} dt_1 \int_{j\tau}^{(j+1)\tau} dt_2 \; \frac{1}{2^7} \, (2\, N_o)^2 \, (t_2 - j\tau)^2 \; [\, 1 + \cos 4\pi f_o (t_1 - t_2)]$$

$u_\ell = t_\ell - j\tau, \quad \ell = 1,\ 2 \Longrightarrow$

$$= \frac{N_o^{\,2}}{2^5} \sum_{j=1}^{n-1} \int_o^{\tau} du_1 \int_o^{\tau} du_2 \, u_2^2 \;\; [\, 1 + \cos 4\pi f_o (u_1 - u_2) \,]$$

$$= \frac{N_o^{\,2}}{2^5} \, (n-1) \; \frac{\tau^4}{3} + \int_o^{\tau} \cos 4\pi f_o u_1 \, du_1 \; \int_o^{\tau} u_2^2 \cos 4\pi f_o \, u_2 du_2$$

$$+ \int_o^{\tau} \sin 4\pi f_o u_1 du_1 \; \int_o^{\tau} u_2^{\,2} \sin 4\pi f_o \, u_2 du_2$$

$$\cong \frac{N_o^{\,2} \, (n-1) \, \tau^4}{3 \cdot 2^5} \qquad\qquad \text{if } f_o \text{ is much larger than } \frac{1}{\tau}\, .$$

Now, combining the results for $\overline{D}$ and $\overline{D^2}$, using the latter as an estimate for σ_D^2 (valid for low signal-to-noise ratio), we obtain

$$R = \frac{\overline{D}^2}{\sigma_D^2} \cong \frac{3 \cdot 2^5 \left(\frac{1}{8}\right)^2 (n-1)^2 \, S_a^{\;2} \;\; \psi^2(\tau)}{N_o^{\;2} \, (n-1) \, \tau^4}$$

$$= \frac{3\,(n-1)\,S_a^{\,2}\,\psi^2(\tau)}{2\,N_o^{\,2}\,\tau^4}$$

Substituting $n = \frac{T}{\tau}$, this can be rewritten

$$R \cong \frac{3}{2}\left(1 - \frac{\tau}{T}\right)\frac{\psi^2(\tau)}{\tau_o \tau^5}\left(\frac{S_a T}{N_o}\right)\left(\frac{S_a \tau_o}{N_o}\right)$$

and if $T >> \tau$, then

$$R \cong \frac{3}{2}\,\frac{\psi^2(\tau)}{\tau_o \tau^5}\left(\frac{S_a T}{N_o}\right)\left(\frac{S_a \tau_o}{N_o}\right)$$

The triangular form of $\phi(u)$ will now be examined; i. e., let $\phi(u) = 1 - \left|\frac{u}{2\tau_o}\right|$. Then

$$\psi(\tau) = \frac{2}{3}\,\tau^3\left(1 - \frac{\tau}{2\tau_o}\right)$$

$$R \cong \frac{2}{3}\,\frac{\tau}{\tau_o}\left(1 - \frac{\tau}{2\tau_o}\right)^2\left(\frac{S_a T}{N_o}\right)\left(\frac{S_a \tau_o}{N_o}\right)$$

This expression can be maximized by setting $\tau = \frac{2\tau_o}{3}$. Making this substitution we get

$$R_{opt.} \cong \left(\frac{2}{3}\right)^4\left(\frac{S_a T}{N_o}\right)\left(\frac{S_a \tau_o}{N_o}\right)$$

Now, since this is a binary system with $S_p = S_a$, we have $T = \frac{1}{H}$, and therefore the quantity R may be written

$$R = \left(\frac{2}{3}\right)^4\left[\frac{S_a \frac{1}{H}}{N_o}\right]\left(\frac{S_a \tau_o}{N_o}\right)$$

$$= \left(\frac{2}{3}\right)^4 \rho \left(\frac{S_a \tau_o}{N_o}\right)$$

$$= \left(\frac{2}{3}\right)^4 \rho^2 (\tau_o H)$$

For $(H\tau_o) >> 1$, the decision variable, D, is Gaussian, and the bit error probability may be written

$$P_2 = \int_{-\infty}^{0} \frac{1}{\sqrt{2\pi}\,\sigma_D} \exp\left[-\frac{1}{2}\left(\frac{D-\bar{D}}{\sigma_D}\right)^2\right] dD$$

$$= \Phi\,(-\sqrt{R}).$$

For $\tau = 2/3\,\tau_o$,

$$P_2 = \Phi\left[-(2/3)^2\,\rho\sqrt{\tau_o H}\right]$$

where, as before,

$$\Phi(x) \equiv \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}}\, e^{-\frac{u^2}{2}}\, du.$$