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REPORT TO THE OFFICE OF NAVAL RESEARCH
ON WORK DONE UNDER CONTRACT NONR (G) 00099-62

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During the past nine months I have completed eight research papers while receiving support under contract NONR (G) 00099-62. Of these, three have been accepted for publication, three others have been submitted for publication, and the remaining two have not yet been typed. A brief general description of each of these papers follows.

1) Commutative rings containing at most two prime ideals, accepted by Mich. Math. J.

A commutative ring R containing at most two prime ideals is called a primary ring. The structure of R is known if R contains an identity. This paper gives several sets of necessary and sufficient conditions that a ring S be a primary ring. Also, a method for constructing primary rings from an arbitrary commutative ring with identity is given and it is shown that any primary domain may be realized by the method described. One result of interest proved in the paper is the following theorem. If Q is a nonzero ideal contained in the Jacobson radical M of a semi-local domain J (such a Q is a primary domain), Q is itself a noetherian ring if and only if the residue class ring J/M is finite.

2) On a classical theorem of Noether in ideal theory, accepted by Pacific Journal of Math.

This paper gives necessary and sufficient conditions that an integral domain D without unit should have the property that each ideal of D may be expressed as a product of prime ideals. These conditions are: D is the maximal ideal of a rank one discrete valuation ring D^* having a finite residue field of p elements for some prime p. It is known that such a ring D^* is isomorphic to a valuation ring contained in one of the following:

(a) Hensel's p-adic integers H_p for some prime p,

(b) an Eisenstein extension of H_p, or

(c) the ring of all formal power series in one indeterminate with coefficients in a finite prime field.

3) Integral domains with quotient overrings, written with Jack E. Ohm, accepted by Mathematische Annalen.

Let D be an integral domain with quotient field K. This paper considers domains D with identity with the following property (QR): if D' is a domain such that $D \subsetneq D' \subsetneq K$, then D' is a quotient ring of D. D has property (QR) if and only if given $x,y \in D$, there exists $z \in D$ and an integer n such that $(xy^{n-1}, y^n) \subseteq (z) \subseteq (x,y)^n$. A noetherian domain with property (QR) is a Dedekind domain having a torsion class group, and conversely. A UFD has property (QR) if and only if it is a PID. The ideal theory of domains with property (QR) is also investigated.

4) Extension of results concerning rings in which every semi-primary ideal is primary, submitted to Duke Math. J.

An ideal A of a ring R is called semi-primary if $\sqrt{A}$ is a prime ideal. This paper shows that each semi-primary ideal of a ring R is primary if and only if R is a ring with property a), b), or c):

a) every element of R is nilpotent,

b) R is a primary domain,

c) R is a u-ring (that is, if $\sqrt{A} = R$, then $A = R$) of dimension less than two and for each triple (p,P,M) where P and M are prime ideals and $p \in P \subset M \subset R$, $p \in pM$.

5) Finite rings having a cyclic multiplicative group of units, submitted to Amer. J. Math.

This paper determines all finite commutative rings R with identity such that the multiplicative group G_R of units of R is cyclic. Such a ring R is a direct sum of finite primary rings having a cyclic multiplicative group of units. The finite primary rings with this property are a) the Galois fields, b) $\{Z/(p^m)\}$ where p is an odd prime and $m > 1$, c) $Z/(4)$, d) $\Pi_p[X]/(X^2)$ where Π_p is a finite prime field of p elements and X is an indeterminate over Π_p, e) $\Pi_2[X]/(X^3)$, and f) $Z[X]/(4,2-2X,X^2)$.

6) The cancellation law for ideals in a commutative ring, submitted to Ann. of Math.

Let R be a commutative ring with identity. We say that the restricted cancellation law (RCL) holds in R if the validity of the equation $AB = AC$, where A, B, and C are ideals of R and $AB \neq (0)$, implies $B = C$. If R has proper divisors of zero, R is a primary ring with maximal ideal $M \neq (0)$ and either $M^2 = (0)$ or R is a special primary ring. The converse is also true. (RCL) holds in an integral domain D with identity if and only if D_P is a rank one discrete valuation ring for each nonzero proper prime ideal P of D. This last result is used to prove the following theorem: Let J be an integral domain with identity in which there exists a collection $\mathcal{S}$ of nonzero proper ideals such that every nonzero proper ideal of J is uniquely representable as a product of elements of $\mathcal{S}$. Then J is a Dedekind domain and $\mathcal{S}$ is the collection of nonzero proper prime ideals of J.

7) <u>Multiplication rings as rings in which ideals with prime radical are primary</u>, written with Joe Mott, being typed for submission to Trans. Amer. Math. Soc.

An ideal B of a commutative ring R will be called a multiplication ideal of R if each ideal of R contained in B is of the form BC for some ideal C of R. By definition, R is a multiplication ring if each ideal of R is a multiplication ideal. We say R is a prime multiplication ring if each prime ideal of R is a multiplication ideal.

In the first part of this paper, two new sets of necessary and sufficient conditions are given in order that a ring satisfy Condition (*) (see paper 4)). In the second section of the paper it is shown that a prime multiplication ring satisfies (*). Using results of the first section and paper 4), we then show that a prime multiplication ring is a multiplication ring. In the process of the proof, new necessary and sufficient conditions are given in order that a ring be a multiplication ring and many of the known properties of multiplication rings are derived in new ways.

8. _Some elementary results on the ideal theory of commutative rings without identity_, ready for the typist. To be submitted to _Proc. Amer. Math. Soc._

This paper generalizes to rings without identity some known results on rings with identity. The paper gives several necessary and sufficient conditions in order that a maximal ideal of a ring without identity be prime. Also, necessary and sufficient conditions are given in order that an ideal A of a ring without identity be primary when $\sqrt{A}$ is maximal.

I feel that this year has aided in a significant way my development in doing mathematical research. This help has come through research and study actually done and I feel it will continue through the many questions which have arisen in work completed and which remain unanswered. I am grateful to the Office of Naval Research for the support which it has afforded.