(SP Series)

SP-1017/001/00

Stochastic Duels with Limited Ammunition Supply

by

C. J. Ancker, Jr.

23 April 1963

SYSTEM DEVELOPMENT CORPORATION, SANTA MONICA, CALIFORNIA

Stochastic Duels with Limited Ammunition Supply

by

C. J. Ancker, Jr.

ABSTRACT

In a previous paper [1] the ground work was laid for a new theoretical model of combat called the "stochastic duel." The principal elements of the model were fixed kill probabilities on each round fired, a random time between rounds fired and unlimited ammunition supply. This paper extends the solution to those more realistic cases of limited ammunition supply. The general solution (as a quadrature) is obtained and it is applied to several specific examples. The special case of finite, fixed ammunition supply is treated in some detail.

Stochastic Duels with Limited Ammunition Supply

by

C. J. Ancker, Jr.

In a previous paper [1], the ground work was laid for a new theoretical model of combat called the "stochastic duel." The principal elements of the model were fixed kill probabilities on each round fired, a random time between rounds fired and unlimited ammunition supply. We now extend the solution to those more realistic cases in which the ammunition supply is limited. This complicates matters somewhat, but as we shall see, it is still possible to obtain the general solution (as a quadrature) and to apply it to specific examples.

ASSUMPTIONS

The duel starts with two contestants (A and B) both having unloaded weapons and a limited ammunition supply. A starts with n rounds. The probability that n is some integer k is α_k, that is

$$P\{n=k\} = \alpha_k , \qquad k=0,1,2,... \tag{1}$$

where

$$\sum_{k=0}^{\infty} \alpha_k = 1 . \tag{2}$$

Thus, the α_k form a discrete probability distribution.

Similarly, B starts with m rounds and

$$P\{m=j\} = \beta_j, \qquad j=0,1,2,\ldots \tag{3}$$

and

$$\sum_{j=0}^{\infty} \beta_j = 1 \ . \tag{4}$$

When the duel begins each contestant loads and fires and continues this process until either one or the other is killed or they both run out of ammunition. The probability of a kill on each round fired is p_A for A and p_B for B and both are fixed. The time between rounds is a random variable described by the probability density function $f_A(t)$ for A and $f_B(t)$ for B. The solutions to this duel are the probabilities that A wins or B wins or they tie (both run out of ammunition).

THE GENERAL SOLUTION

The technique for solution follows the pattern set in the first paper [1]. First, we determine the distribution of the time to a kill for each contestant. These functions are derived by considering the opponent to be a passive target. Then the probability that A wins is simply the probability that A's time to a kill is less than B's time to a kill.

If $h_A(t)$ is the density function of A's time to a kill, then

P{A hits passive target B in the interval $(t,t+dt)$} =

$$\begin{aligned} h_A(t)dt = {} & \alpha_0\delta(t-\infty)dt+\alpha_1[p_Af_A(t)+q_A\delta(t-\infty)]dt \\ & + \alpha_2[p_Af_A(t)+p_Aq_Af_A(t)*f_A(t)+q_A^2\delta(t-\infty)]dt \\ & + \alpha_3[p_Af_A(t)+p_Aq_Af_A(t)*f_A(t)+p_Aq_A^2f_A(t)*f_A(t)*f_A(t)+q_A^3\delta(t-\infty)]dt \\ & + \cdots , \end{aligned} \tag{5}$$

where * denotes the convolution and δ is the Dirac Delta Function. Thus

$$h_A(t) = p_A \sum_{k=1}^{\infty} \alpha_k \sum_{\ell=o}^{k-1} q_A^{\ell} f_A^{\ell *}(t) + \delta(t-\infty) \sum_{k=o}^{\infty} \alpha_k q_A^k$$

$$= h_{A1}(t) + \delta(t-\infty) \sum_{k=o}^{\infty} \alpha_k q_A^k \ , \tag{6}$$

where the superscript ℓ^* denotes ℓ iterated convolutions of the function with itself. The distribution and the complementary distribution functions of $h_A(t)$ are

$$H_A(t) = \int_o^t h_{A1}(\xi) d\xi + 0 = H_{A1}(t) \tag{7}$$

and

$$G_A(t) = \int_t^{\infty} h_{A1}(\xi) d\xi + \sum_{k=o}^{\infty} \alpha_k q_A^k = G_{A1}(t) + \sum_{k=o}^{\infty} \alpha_k q_A^k \ . \tag{8}$$

Similar expressions apply to B (simply replace A subscripts by B, α_k by β_j and index k by index j).

Since A cannot win if he runs out of ammunition (i.e., if his time to a kill is infinite) but he may win if B runs out (i.e., A continues to fire at B until out of ammunition), the probability that A wins is

$$P(A) = \int_o^{\infty} G_B(t) h_{A1}(t) dt$$

$$= \int_o^{\infty} G_{B1}(t) h_{A1}(t) dt + \sum_{j=o}^{\infty} \beta_j q_B^j \int_o^{\infty} h_{A1}(t) dt \ ,$$

or from (6)

$$P(A) = \int_{o}^{\infty} G_{B1}(t)h_{A1}(t)dt + \sum_{j=o}^{\infty}\beta_j q_B^j \left[1 - \sum_{k=o}^{\infty}\alpha_k q_A^k\right]. \tag{9}$$

From Parseval's formula [2], equation (9) may be written as

$$P(A) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\Phi_{A1}(-u)du\int_{o}^{\infty}e^{iut}G_{B1}(t)dt + \sum_{j=o}^{\infty}\beta_j q_B^j\left[1 - \sum_{k=o}^{\infty}\alpha_k q_A^k\right], \tag{10}$$

where $\Phi_{A1}(u)$ is the characteristic function* of $h_{A1}(t)$ and is defined as

$$\begin{aligned}\Phi_{A1}(u) &= \int_{o}^{\infty}e^{iut}h_{A1}(t)dt \\ &= p_A\int_{o}^{\infty}e^{iut}\left(\sum_{k=1}^{\infty}\alpha_k\sum_{\ell=o}^{k-1}q_A^{\ell}f_A^{\ell^*}(t)\right)dt \\ &= p_A\sum_{k=1}^{\infty}\alpha_k\sum_{\ell=o}^{k-1}q_A^{\ell}\varphi_A^{\ell+1}(u) \\ &= \frac{p_A\varphi_A(u)}{1-q_A\varphi_A(u)}\left\{1 - \sum_{k=o}^{\infty}\alpha_k[q_A\varphi_A(u)]^k\right\},\end{aligned} \tag{11}$$

and where $\varphi_A(u)$ is the characteristic function of $f_A(t)$ or

$$\varphi_A(u) = \int_{o}^{\infty}e^{iut}f_A(t)dt. \tag{12}$$

---

*$\Phi_{A1}(u)$ is not a proper characteristic function since $h_{A1}(t)$ is not a proper density function. However, the definitions and manipulations given are valid.

In the derivation of (11) the convolution property of characteristic functions has been used.

The inner integral of (10) integrates by parts to $\frac{1}{iu}\left[\Phi_{B1}(u)-1+\sum_{j=o}^{\infty}\beta_j q_B^j\right]$ where $\Phi_{B1}(u)$ is defined as in (11) and is

$$\Phi_{B1}(u) = \frac{p_B\varphi_B(u)}{1-q_B\varphi_B(u)}\left\{1-\sum_{j=o}^{\infty}\beta_j[q_B\varphi_B(u)]^j\right\} \tag{13}$$

and $\varphi_B(u)$ is the characteristic function of $f_B(t)$. Thus

$$\begin{aligned} P(A) = &\sum_{j=o}^{\infty}\beta_j q_B^j\left[1-\sum_{k=o}^{\infty}\alpha_k q_A^k\right] \\ &+\frac{1}{2\pi i}\int_{-\infty}^{+\infty}\Phi_{A1}(-u)\left[\Phi_{B1}(u)-1+\sum_{j=o}^{\infty}\beta_j q_B^j\right]\frac{du}{u}. \end{aligned} \tag{14}$$

It may be shown that functions such as $\Phi(u)$ and $\varphi(u)$ have the following properties: (1) no singularities in the upper half of the complex plane and (2) diminish like $\frac{1}{R}$ along a semicircular contour of radius R in the upper half of the complex plane. $\Phi(-u)$ and $\varphi(-u)$ have similar properties in the lower half of the complex plane. Sufficient conditions for the second property are that f(t) be a differentiable function of bounded variation and $\lim_{t\to\infty} f(t) = 0$. From (11) and (13) we also note that

$$\left.\begin{aligned} \Phi_{A1}(o) &= 1-\sum_{k=o}^{\infty}\alpha_k q_A^k \\ \Phi_{B1}(o) &= 1-\sum_{j=o}^{\infty}\beta_j q_B^j . \end{aligned}\right\} \tag{15}$$

In (14) the term $- \frac{\left(1 - \sum_{j=o}^{\infty} \beta_j q_B^j\right)}{2\pi i} \int_{-\infty}^{+\infty} \frac{\Phi_{A1}(-u)\,du}{u}$ may be integrated by noting from the foregoing that in the complex domain the integrand has no poles in the lower half plane, the integral vanishes around the contour C as $R \to \infty$ (see Figure 1), and there is only a simple pole of residue $- \frac{\left(1 - \sum_{j=o}^{\infty} \beta_j q_B^j\right)}{2\pi i} \left(1 - \sum_{k=o}^{\infty} \alpha_k q_A^k\right)$ at the origin (see equation (15)). Thus letting $\rho \to 0$ (and noting that we take $- \frac{2\pi i}{2}$

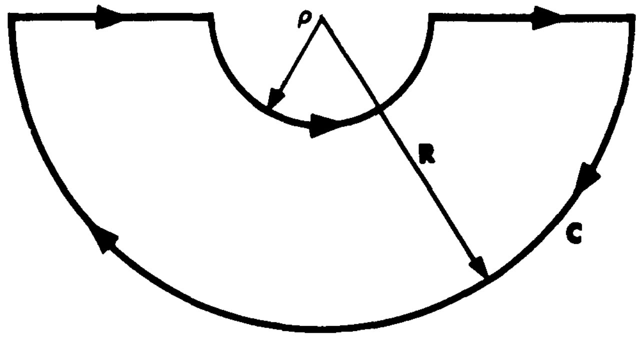

Figure 1.

times the residue because the pole is outside the small semicircle), the desired integral is $\frac{1}{2}\left(1 - \sum_{k=o}^{\infty} \alpha_k q_A^k\right) \left(1 - \sum_{j=o}^{\infty} \beta_j q_B^j\right)$ and so

$$P(A) = \frac{1}{2}\left[1 - \sum_{k=o}^{\infty} \alpha_k q_A^k\right] \left[1 + \sum_{j=o}^{\infty} \beta_j q_B^j\right] + \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \Phi_{A1}(-u) \Phi_{B1}(u) \frac{du}{u} . \tag{16}$$

The integral in (16) is the Cauchy principal value. This may be further simplified by noting that the integrand in (16) has a simple pole at u=o which contributes $-\frac{1}{2}\left(1-\sum_{k=o}^{\infty}\alpha_k q_A^k\right)\left(1-\sum_{j=o}^{\infty}\beta_j q_B^j\right)$ to the integral (path of integration as in Figure 1) so (16) may be written

$$P(A) = \sum_{j=o}^{\infty}\beta_j q_B^j\left[1-\sum_{k=o}^{\infty}\alpha_k q_A^k\right] + \frac{1}{2\pi i}\int_L \Phi_{A1}(-u)\Phi_{B1}(u)\,\frac{du}{u}\,, \tag{17}$$

where $\int_L$ means integration from $-\infty$ to $+\infty$ along the real axis excluding the simple pole at the origin. Further, it implies that the path of integration is as in Figure 1 (where $R\to\infty$), and that the integrand is analytic everywhere except at singularities and that the integrand vanishes at least like $\frac{K}{R^{1+\epsilon}}$ on C (where K is some constant), and that ρ is finite but less than the distance to the nearest singularity in the lower half-plane. The direction of integration is in the negative sense.

A still different form may be obtained by integrating around the path of Figure 2. In this case the pole at the origin contributes

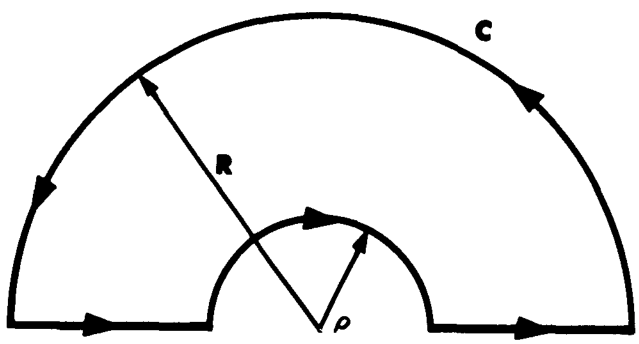

Figure 2.

$$+ \frac{1}{2}\left(1 - \sum_{k=o}^{\infty} \alpha_k q_A^k\right) \left(1 - \sum_{j=o}^{\infty} \beta_j q_B^j\right) \text{ and thus}$$

$$P(A) = 1 - \sum_{k=o}^{\infty} \alpha_k q_A^k + \frac{1}{2\pi i} \int_U \Phi_{A1}(-u)\Phi_{B1}(u) \frac{du}{u} , \tag{18}$$

where $\int_U$ means integration from $-\infty$ to $+\infty$ along the real axis excluding the simple pole at the origin. Corresponding restrictions apply to this path of integration in the upper half-plane.

In the event the integrand is not analytic or does not behave properly on C, P(A) must be evaluated from (16), rather than (17) or (18).

Similarly, the probability that B wins is,

$$P(B) = \frac{1}{2}\left[1 - \sum_{j=o}^{\infty} \beta_j q_B^j\right] \left[1 + \sum_{k=o}^{\infty} \alpha_k q_A^k\right] - \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \Phi_{A1}(-u)\Phi_{B1}(u) \frac{du}{u}$$

$$= \sum_{k=o}^{\infty} \alpha_k q_A^k \left[1 - \sum_{j=o}^{\infty} \beta_j q_B^j\right] - \frac{1}{2\pi i} \int_U \Phi_{A1}(-u)\Phi_{B1}(u) \frac{du}{u} \tag{19}$$

$$= 1 - \sum_{j=o}^{\infty} \beta_j q_B^j - \frac{1}{2\pi i} \int_L \Phi_{A1}(-u)\Phi_{B1}(u) \frac{du}{u} .$$

In addition, the probability of a draw (the probability both run out of ammunition before a kill) is

$$P(AB) = \sum_{k=o}^{\infty} \alpha_k q_A^k \sum_{j=o}^{\infty} \beta_j q_B^j , \tag{20}$$

and, of course, $P(A) + P(B) + P(AB) = 1$.

Thus, (16), (17), (18), (19), and (20) along with (11), (12) and (13) constitute the general solution to the stochastic duel with limited ammunition supply.

These results specialize to the unlimited ammunition supply case of the previous paper [1] by letting $\alpha_k = \beta_j = 0$ for all finite k, j and $\alpha_k = \beta_j = 1$ for k, j infinite.

SOME EXAMPLES

In this section the general solution is applied to certain examples. In each example there is a specific distribution for the number of rounds A starts with, the number of rounds B starts with, and specific density functions for their times between rounds. These are indicated in the heading for each example. In the following, only P(A) and P(AB) are given since $P(B) = 1 - P(A) - P(AB)$.

Example 1. Geometric-Geometric-Negative Exponential

Let

$$\alpha_k = (1-\alpha)\alpha^k, \quad \beta_j = (1-\beta)\beta^j$$

$$f_A(t) = r_A e^{-r_A t}, \quad f_B(t) = r_B e^{-r_B t}$$

where the parameters r_A and r_B are average rates of fire. Then,

$$\varphi_A(-u) = \frac{r_A}{r_A + iu}, \quad \varphi_B(u) = \frac{r_B}{r_B - iu}$$

and from (11) and (13)

$$\Phi_{A1}(-u) = \frac{\alpha p_A r_A}{(1-\alpha q_A)r_A + iu}\ , \quad \Phi_{B1}(u) = \frac{\beta p_B r_B}{(1-\beta q_B)r_B - iu}\ .$$

Thus, from (17)

$$P(A) = \frac{(1-\beta)\alpha p_A}{(1-\beta q_B)(1-\alpha q_A)} + \frac{1}{2\pi i}\int_L \left(\frac{\alpha p_A r_A}{(1-\alpha q_A)r_A + iu}\right)\left(\frac{\beta p_B r_B}{(1-\beta q_B)r_B - iu}\right)\frac{du}{u}\ . \qquad (21)$$

The integrand has one singularity in the lower half-plane which is a simple pole at $u = -i(1-\beta q_B)r_B$ with residue $-\frac{1}{2\pi i}\frac{\alpha\beta p_A p_B r_A}{(1-\beta q_B)[(1-\alpha q_A)r_A + (1-\beta q_B)r_B]}$. Finally, applying the residue theorem

$$P(A) = \frac{\alpha p_A}{1-\alpha q_A}\left[\frac{(1-\alpha q_A)r_A + (1-\beta)r_B}{(1-\alpha q_A)r_A + (1-\beta q_B)r_B}\right]. \qquad (22)$$

From (20),

$$P(AB) = \frac{(1-\alpha)(1-\beta)}{(1-\alpha q_A)(1-\beta q_B)}\ . \qquad (23)$$

In the following examples only the assumptions and the solutions are given since the procedure for derivation is the same as in Example 1.

<u>Example 2</u>. <u>Poisson-Geometric-Negative Exponential</u>

$$\alpha_k = \frac{e^{-\alpha}\alpha^k}{k!}\ , \quad \beta_j = (1-\beta)\beta^j$$

$$f_A(t) = r_A e^{-r_A t}\ , \quad f_B(t) = r_B e^{-r_B t}$$

and

$$P(A) = \frac{\beta p_A p_B r_A \left\{1-\exp-\alpha\left[\frac{p_A r_A+(1-\beta q_B)r_B}{r_A+(1-\beta q_B)r_B}\right]\right\}}{(1-\beta q_B)[p_A r_A+(1-\beta q_B)r_B]} + \left(\frac{1-\beta}{1-\beta q_B}\right)(1-e^{-\alpha p_A}) \tag{24}$$

and

$$P(AB) = \frac{(1-\beta)e^{-\alpha p_A}}{1-\beta q_B} . \tag{25}$$

The Geometric-Poisson-Negative Exponential case may of course be obtained by simply reversing the roles of A and B, that is exchanging A and B subscripts, and replacing α with β and vice versa.

<u>Example 3</u>. <u>Binomial-Geometric-Negative Exponential</u>

$$\alpha_k = \left(\frac{1}{1+\alpha}\right)^n \binom{n}{k} \alpha^k,\ k=0,1,\ldots,n;\ \beta_j = (1-\beta)\beta^j$$

$$f_A(t) = r_A e^{-r_A t},\ f_B(t) = r_B e^{-r_B t}$$

$$P(A) = \frac{\beta p_A p_B r_A}{[p_A r_A+(1-\beta q_B)r_B](1-\beta q_B)} \left\{1 - \frac{1}{(1+\alpha)^n}\left[1 + \frac{\alpha q_A r_A}{r_A+(1-\beta q_B)r_B}\right]^n\right\}$$

$$+ \left(\frac{1-\beta}{1-\beta q_B}\right)\left[1 - \left(\frac{1+\alpha q_A}{1+\alpha}\right)^n\right], \tag{26}$$

and

$$P(AB) = \left(\frac{1+\alpha q_A}{1+\alpha}\right)^n \frac{(1-\beta)}{(1-\beta q_B)} . \tag{27}$$

Again the Geometric-Binomial-Negative Exponential case may be obtained by reversing the roles of A and B, as stated in Example 2, and replacing n with m.

Example 4. Geometric-Geometric-(γ,2)

$$\alpha_k = (1-\alpha)\alpha^k, \ \beta_j = (1-\beta)\beta^j ,$$

$$f_A(t) = 4r_A^2 e^{-2r_A t}, \ f_B(t) = 4r_B^2 e^{-2r_B t}$$

$$P(A) = \frac{\alpha\beta p_A p_B r_A^2}{1-\beta q_B} \left\{ \frac{(1-\alpha q_A) r_A^2 - (1-\beta q_B) r_B^2 + 4r_B(r_A + r_B)}{[(1-\alpha q_A) r_A^2 - (1-\beta q_B) r_B^2]^2 + 4r_A r_B (r_A + r_B)[(1-\alpha q_A) r_A + (1-\beta q_B) r_B]} \right\}$$

$$+ \frac{\alpha(1-\beta) p_A}{(1-\alpha q_A)(1-\beta q_B)} , \qquad (28)$$

and

$$P(AB) = \frac{(1-\alpha)(1-\beta)}{(1-\alpha q_A)(1-\beta q_B)} . \qquad (29)$$

The important special case of a fixed limitation on ammunition supply will now be considered in more detail.

THE CASE OF FINITE, FIXED AMMUNITION SUPPLY

Suppose that A always starts with exactly n rounds and B with exactly m rounds. This might very nearly be the case where the policy is to disengage and resupply immediately after action and where there is a physical limitation on the number of rounds which can be carried such as in the interior of a tank or an interceptor.

We may go at once to the general solution for this case by letting

$$\left.\begin{aligned} \alpha_k &= 1, \quad k = n \\ &= 0 \quad \text{all other } k \\ \beta_j &= 1, \quad j = m \\ &= 0 \quad \text{all other } j. \end{aligned}\right\} \tag{30}$$

Thus, from (11), (12), (13), (16), (17), (18), (19), and (20)

$$\begin{aligned} P(A) &= \frac{1}{2}(1-q_A^n)(1+q_B^m) + \frac{1}{2\pi i}\int_{-\infty}^{+\infty} \Phi_{A1}(-u)\Phi_{B1}(u)\,\frac{du}{u} \\ &= q_B^m(1-q_A^n) + \frac{1}{2\pi i}\int_L \Phi_{A1}(-u)\Phi_{B1}(u)\,\frac{du}{u} \\ &= 1-q_A^n + \frac{1}{2\pi i}\int_U \Phi_{A1}(-u)\Phi_{B1}(u)\,\frac{du}{u}, \end{aligned} \tag{31}$$

and

$$\begin{aligned} P(B) &= \frac{1}{2}(1-q_B^m)(1+q_A^n) - \frac{1}{2\pi i}\int_{-\infty}^{+\infty} \Phi_{A1}(-u)\Phi_{B1}(u)\,\frac{du}{u} \\ &= q_A^n(1-q_B^m) - \frac{1}{2\pi i}\int_U \Phi_{A1}(-u)\Phi_{B1}(u)\,\frac{du}{u} \\ &= 1-q_B^m - \frac{1}{2\pi i}\int_L \Phi_{A1}(-u)\Phi_{B1}(u)\,\frac{du}{u} \end{aligned} \tag{32}$$

and

$$P(AB) = q_A^n q_B^m, \tag{33}$$

where

$$\Phi_{A1}(u) = \frac{p_A\varphi_A(u)}{1-q_A\varphi_A(u)}\left[1 - \left(q_A\varphi_A(u)\right)^n\right], \tag{34}$$

and

$$\Phi_{B1}(u) = \frac{p_B\varphi_B(u)}{1-q_B\varphi_B(u)}\left[1 - \left(q_B\varphi_B(u)\right)^m\right]. \tag{35}$$

$\int_L$, $\int_U$ and the φ's and Φ's all have the same meanings and restrictions as before.

The use of the general solution (equations (31) through (35)) will now be illustrated by an example. Assume that firing times are negative exponential, thus

$$f_A(t) = r_A e^{-r_A t}, \quad f_B(t) = r_B e^{-r_B t}$$

$$\varphi_A(-u) = \frac{r_A}{r_A+iu}, \quad \varphi_B(u) = \frac{r_B}{r_B-iu}$$

$$\Phi_{A1}(-u) = \left(\frac{p_A r_A}{p_A r_A+iu}\right)\left[1 - \left(\frac{q_A r_A}{r_A+iu}\right)^n\right], \quad \Phi_{B1}(u) = \left(\frac{p_B r_B}{p_B r_B-iu}\right)\left[1 - \left(\frac{q_B r_B}{r_B-iu}\right)^m\right].$$

From (31)

$$P(A) = q_B^m(1-q_A^n) + \frac{p_A p_B r_A r_B}{2\pi i}\int_L\left[\frac{(r_A+iu)^n - q_A^n r_A^n}{(p_A r_A+iu)(r_A+iu)^n}\right]\left[\frac{(r_B-iu)^m - q_B^m r_B^m}{(p_B r_B-iu)(r_B-iu)^m}\right]\frac{du}{u}. \tag{36}$$

The integrand of (36) has only one pole in the lower half-plane. This is located at $u_o = -ir_B$ and is an m'th order pole. The residue at this pole is given by $\text{Res} = \frac{1}{(m-1)!}\frac{d^{m-1}}{du^{m-1}}[g(u)]_{u=u_o}$ where $g(u) = (u-u_o)^m f(u)$ and $f(u)$ is the integrand of (36). From this

$$g(u) = \frac{p_A p_B r_A r_B}{2\pi i} \left\{ \left[\frac{(r_A+iu)^n - q_A^n r_A^n}{(p_A r_A+iu)(r_A+iu)^n} \right] \left[\frac{(r_B-iu)^m - q_B^m r_B^m}{(p_B r_B-iu)(-i)^m u} \right] \right\} . \tag{37}$$

The (m-1)st derivative of this reduces to

$$\frac{d^{m-1}}{du^{m-1}} [g(u)]_{u=u_o} = - \frac{p_A p_B r_A r_B}{2\pi i} (iq_B r_B)^m \frac{d^{m-1}}{du^{m-1}} [h(u)]_{u=u_o} \tag{38}$$

where

$$h(u) = \left[1 - \frac{q_A^n r_A^n}{(r_A+iu)^n} \right] \left[\frac{1}{(p_A r_A+iu)(p_B r_B-iu)u} \right] . \tag{39}$$

Expanding h(u) by partial fractions,

$$h(u) = \left[1 - \frac{q_A^n r_A^n}{(r_A+iu)^n} \right] \left[- \frac{i}{p_A r_A (p_A r_A+p_B r_B)(p_A r_A+iu)} + \frac{i}{p_B r_B (p_A r_A+p_B r_B)(p_B r_B-iu)} + \frac{1}{p_A p_B r_A r_B u} \right] . \tag{40}$$

The (m-1)st derivative of (40) may now be taken by the product formula. Then u_o is inserted, and the whole expression substituted into (38). Finally, going back to the residue formula, we have after much calculation

$$P(A) = \frac{p_A r_A}{p_A r_A+p_B r_B} \left[1 - \left(\frac{q_A r_A}{r_A+r_B} \right)^n \sum_{k=o}^{m-1} \binom{n+k-1}{k} \left(\frac{q_B r_B}{r_A+r_B} \right)^k \right]$$

$$+ \left(\frac{p_B r_B}{p_A r_A+p_B r_B} \right) \left(\frac{q_B r_B}{p_A r_A+r_B} \right)^m \left[1 - \left(\frac{q_A r_A}{r_A+r_B} \right)^n \sum_{k=o}^{m-1} \binom{n+k-1}{k} \left(\frac{r_B+p_A r_A}{r_A+r_B} \right)^k \right] \tag{41}$$

$$- q_A^n q_B^m \left[1 - \left(\frac{r_A}{r_A+r_B} \right)^n \sum_{k=o}^{m-1} \binom{n+k-1}{k} \left(\frac{r_B}{r_A+r_B} \right)^k \right] .$$

This expression may be put in closed form by using the known [3] relation

$$\sum_{k=0}^{m-1} \binom{n+k-1}{k} x^k = (1-x)^{-n}[1-I_x(m,n)] \ , \tag{42}$$

where $I_x(m,n)$ is the well-tabled [4] Incomplete Beta-Function Ratio defined by

$$I_x(m,n) = \frac{B_x(m,n)}{B(m,n)} = \frac{\Gamma(m+n)}{\Gamma(m)\Gamma(n)} \int_0^x \xi^{m-1}(1-\xi)^{n-1} d\xi \ .$$

The identity

$$I_x(m,n) = 1-I_{1-x}(n,m) \tag{43}$$

is also used. When (42) and (43) are applied to (41) the final result is obtained after some manipulation.

$$P(A) = \frac{p_A r_A}{p_A r_A + p_B r_B}\left[1 - \left(\frac{q_A r_A}{p_B r_B + r_A}\right)^n I_x(n,m)\right] + \left(\frac{p_B r_B}{p_A r_A + p_B r_B}\right)\left(\frac{q_B r_B}{p_A r_A + r_B}\right)^m I_y(m,n) - q_A^n q_B^m I_z(m,n) \ , \tag{44}$$

where

$$x = \frac{p_B r_B + r_A}{r_A + r_B} \ , \ y = \frac{p_A r_A + r_B}{r_A + r_B} \ , \ z = \frac{r_B}{r_A + r_B} \ . \tag{45}$$

To complete the example, from (33),

$$P(AB) = q_A^n q_B^m \ . \tag{46}$$

Two other cases are obtained rather easily from these results. If A has a fixed number of rounds, and B has an infinite number of rounds, then for negative exponential firing times, the solution is obtained by letting $m\to\infty$ in (44), (45) and (46).

$$P(A) = \frac{p_A r_A}{p_A r_A + p_B r_B}\left[1 - \left(\frac{q_A r_A}{p_B r_B + r_A}\right)^n\right], \tag{47}$$

and

$$P(AB) = 0 . \tag{48}$$

Similarly, if A has an infinite number of rounds and B is restricted, let $n\to\infty$.

$$P(A) = \frac{p_A r_A}{p_A r_A + p_B r_B} + \left(\frac{p_B r_B}{p_A r_A + p_B r_B}\right)\left(\frac{q_B r_B}{p_A r_A + r_B}\right)^m \tag{49}$$

and

$$P(AB) = 0 . \tag{50}$$

In the preceding we have used $\lim_{m\to\infty} I_x(m,n) = 0$ and $\lim_{n\to\infty} I_x(m,n) = 1$.

DISCUSSION AND CONCLUSION

The general solution to the stochastic duel with limited ammunition supply has been obtained. Several examples with specific distributions have been worked out. The general solution for the special case of fixed, finite ammunition supply has also been obtained and several examples worked out.

The results given here can be readily adapted to fit various initial conditions as described in the previous paper [1]. These include the "Classical Duel" where each contestant starts with loaded weapons and the "tactical equity" duel in which one-half the time A sights B first and fires one round which alerts B and then the duel proceeds as a fundamental duel. The other half of the time B fires first.

A limited ammunition supply may drastically affect the outcome of the duel. The precise effect depends on the exact conditions of the duel. Some insight into the nature of this effect is clearly shown from an examination of equations (47) and (49). These are duels in which firing times are exponential and only one side has a fixed, limited ammunition supply. We now compare these results with the same duel in which both sides have unlimited supplies whose outcome [1] is

$$P(A)_U = \frac{p_A r_A}{p_A r_A + p_B r_B} . \tag{51}$$

In (47), A has a limit of n rounds. The effect of this limitation is given by

$$\frac{P(A)}{P(A)_U} = 1 - \left(\frac{q_A}{\frac{p_B r_B}{r_A} + 1} \right)^n = 1 - x^n . \tag{52}$$

This expression is plotted in Figure 3. A's chances of winning are always degraded for $n < \infty$. If $p_B r_B$ is much larger than r_A, then x is nearly zero and $P(A)/P(A)_U$ is nearly 1 and the limitation on ammunition has little effect. This is explained by the fact that P(A) is very small in this case anyway. If $p_B r_B$ is of the same order as r_A or is much less than r_A then x essentially depends on q_A (A's miss probability). If q_A is small then again the effect is small but if q_A is fairly large then the ammunition limitation is important and A's chances of winning are degraded seriously by small n.

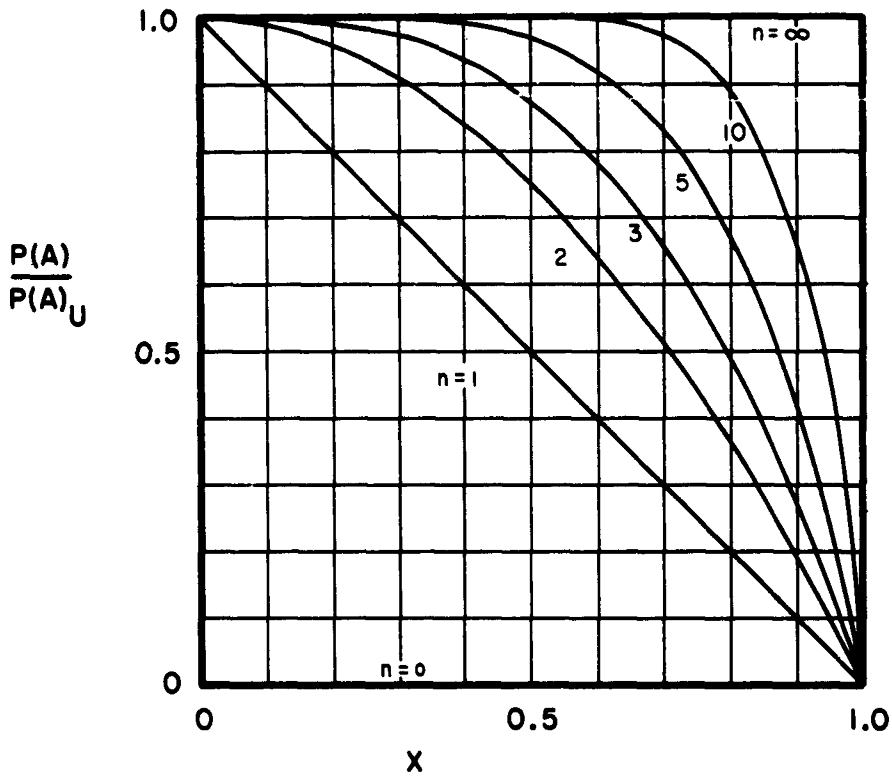

Figure 3. The Effect on A of Ammunition Limitation on A.

For the duel where B's supply is limited (equation (49)) we have

$$\frac{P(A)}{P(A)_U} = 1 + \frac{p_B r_B}{p_A r_A}\left[\frac{1-p_B}{1+p_B\left(\frac{p_A r_A}{p_B r_B}\right)}\right]^m \tag{53}$$

This expression is plotted in Figure 4. A's chances of winning are, of course, always improved for $m < \infty$. However, the major effect occurs for small p_B and this is particularly reinforced for $p_A r_A < p_B r_B$.

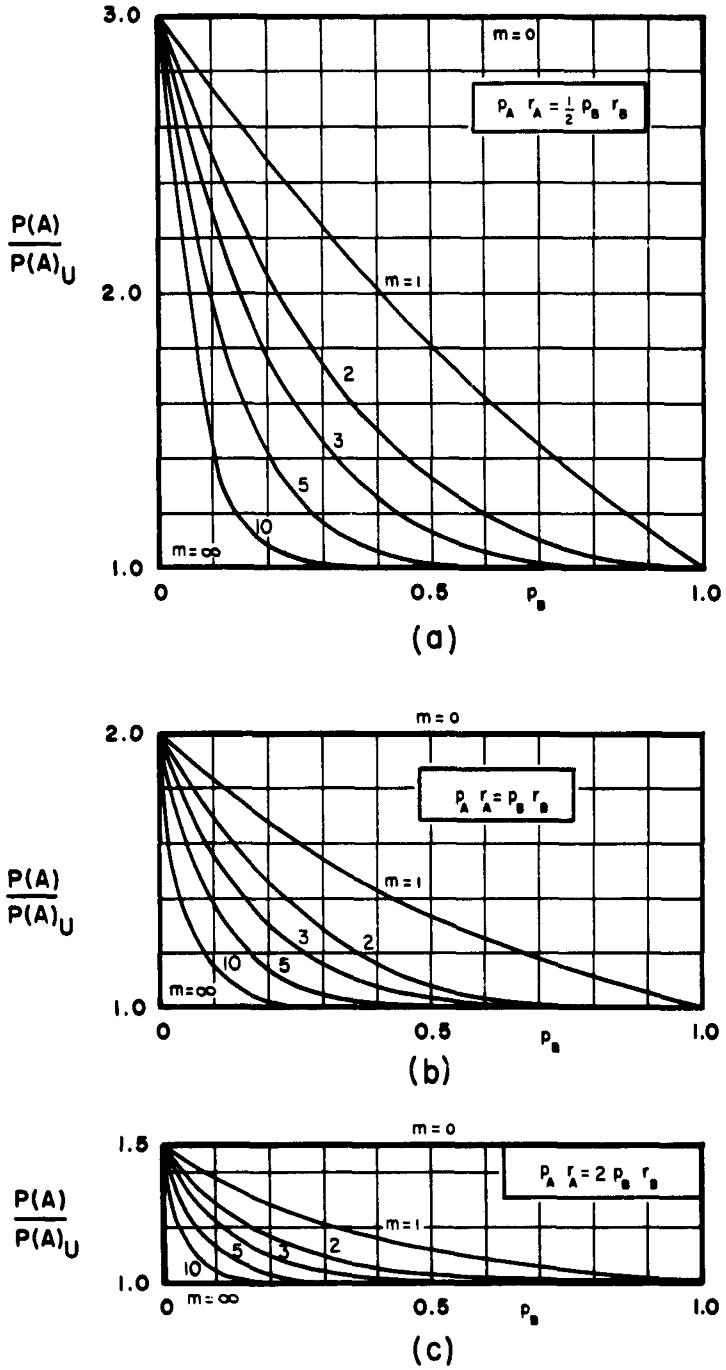

Figure 4. Effect on A of Ammunition Limitation on B.

APPENDIX

The general solution for a somewhat different duel is readily obtained from the preceding work.

In this case, the duel concludes as soon as one contestant is killed or as soon as one contestant runs out of ammunition. The last feature is new. This might be the case where the contestant who runs out takes immediate cover. The analysis proceeds as before but account is taken of the fact that no kills are possible when one contestant has no ammunition. The modified general solution is

$$\begin{aligned} P(A) &= \int_o^\infty G_{B1}(t) h_{A1}(t) dt \\ &= \frac{1}{2}\left(1 - \sum_{k=o}^{\infty} \alpha_k q_A^k\right) \left(1 - \sum_{j=o}^{\infty} \beta_j q_B^j\right) + \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \Phi_{A1}(-u)\Phi_{B1}(u) \frac{du}{u} \\ &= \frac{1}{2\pi i} \int_L \Phi_{A1}(-u)\Phi_{B1}(u) \frac{du}{u} \\ &= \left(1 - \sum_{k=o}^{\infty} \alpha_k q_A^k\right) \left(1 - \sum_{j=o}^{\infty} \beta_j q_B^j\right) + \frac{1}{2\pi i} \int_U \Phi_{A1}(-u)\Phi_{B1}(u) \frac{du}{u} , \end{aligned} \tag{54}$$

and

$$\begin{aligned} P(B) &= \frac{1}{2}\left(1 - \sum_{k=o}^{\infty} \alpha_k q_A^k\right) \left(1 - \sum_{j=o}^{\infty} \beta_j q_B^j\right) - \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \Phi_{A1}(-u)\Phi_{B1}(u) \frac{du}{u} \\ &= -\frac{1}{2\pi i} \int_U \Phi_{A1}(-u)\Phi_{B1}(u) \frac{du}{u} \\ &= \left(1 - \sum_{k=o}^{\infty} \alpha_k q_A^k\right) \left(1 - \sum_{j=o}^{\infty} \beta_j q_B^j\right) - \frac{1}{2\pi i} \int_L \Phi_{A1}(-u)\Phi_{B1}(u) \frac{du}{u} , \end{aligned} \tag{55}$$

and

$$P(AB) = \sum_{k=o}^{\infty} \alpha_k q_A^k + \sum_{j=o}^{\infty} \beta_j q_B^j - \sum_{k=o}^{\infty} \alpha_k q_A^k \sum_{j=o}^{\infty} \beta_j q_B^j . \tag{56}$$

The Φ's are the same as before. All of the examples worked out in the preceding sections are easily modified to fit this model.

REFERENCES

[1] Trevor Williams and Clinton J. Ancker, Jr., "Stochastic Duels," SP-1017/000/01, 20 March 1963. The System Development Corporation, Santa Monica, California. To be published in Operations Research.

[2] E. C. Titchmarsh, Introduction to the Theory of Fourier Integrals, 2nd Edition, Oxford University Press, London, 1948, p. 50, formula 2.1.2 (the factor of 2π arises from slightly different definitions of the transforms).

[3] J. A. Greenwood and H. O. Hartley, Guide to Tables in Mathematical Statistics, Princeton University Press, Princeton, N. J., 1962, p. 284.

[4] K. Pearson, Tables of the Incomplete Beta-Function, University Press, Cambridge, 1956.

UNCLASSIFIED

System Development Corporation,
Santa Monica, California
STOCHASTIC DUELS WITH LIMITED
AMMUNITION SUPPLY.
Scientific rept., SP-1017/001/00,
by C. J. Ancker, Jr. 23 April 1963,
24p., 4 refs.

Unclassified report

DESCRIPTORS: Sampling (Mathematics).
Statistical Analysis (Mathematics).

Reports that in a previous paper (SP-1017/000/01) the ground work was laid for a new theoretical model

UNCLASSIFIED

---

of combat called the "stochastic duel". Also reports that the principal elements of the model were fixed kill probabilities on each round fired, a random time between rounds fired and unlimited ammunition supply. Extends the solution to those more realistic cases of limited ammunition supply. Obtains the general solution (as a quadrature) and applies it to several specific examples. States that the special case of finite, fixed ammunition supply is treated in some detail.

UNCLASSIFIED

UNCLASSIFIED