DEVELOPMENT OF LEAD SELENIDE DETECTOR ARRAY

VOL II PASSIVE DETECTION STUDY

TECHNICAL DOCUMENTARY REPORT NO. SSD-TDR-63-104
APRIL 1963

Prepared Under Contract No. AF 04(695)-228

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FOREWORD

Air Force contract No. AF 04(695)-228, under Capt. S. Gavcus, Jr. as Project Officer with Dr. J. Kaspar, Aerospace Corporation as Technical Advisor, embodies three major programs: (1) a rigorous theoretical and experimental information extraction study, (2) a high bit density lead selenide array fabrication program, and (3) an experimental demonstration of data preprocessing capability based on the results of the extraction study and state-of-the-art fabrication capability.

Since the first phase of the information extraction study is of more general interest than the technique fabrication program it appears desirable to separate the two as a means of providing wider circulation to the study effort results. For this reason Vol. II of this Technical Documentary Report is devoted solely to a detailed review of the information extraction study to date. Vol. I abstracts salient features of the study program as they apply specifically to the problem at hand, as well as detailing technique development accomplished thus far.

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ABSTRACT

The reported work complements present detection theory by providing a useful mathematical statement of the ideal requirements for systems endeavoring to detect time-variant stochastic signals in the presence of Gaussian noise. The a posteriori probability approach developed by P. M. Woodward is employed to state the sufficiency conditions in general form; these are then reduced to a useful set of requirements for an ideal detection system through series representation of arbitrary signals.

The results show that a detection system can only provide estimates of relevant information appearing in a signal-plus-noise quantity; inferences drawn from these estimates subsequent to detection will say something about signal present. The results clearly state, therefore, that a detection system is fundamentally an information detector, and not a detector of signals.

Present detection methods are shown to be approximations to the ideal requirements developed. The ideal requirements are evolved in a form which permit possible logical improvements to present methods.

Subsequent signal processing of detection system output receives some attention. Maximum likelihood is considered, as well as an approximate system model based on a particular interpretation of "psuedo-constant signal" detection which leads to maximum use of detected quantity.

PREFACE

This report summarizes the first phase of a study program embodied in Air Force contract No. AF 04(695)-228, Space Systems Division, under Capt. S. Gavcus, Jr. as Project Officer and under the technical direction of Dr. J. Kaspar, Aerospace Corporation. The study aspects of this contract are aimed at establishing a firm mathematical base for signal detection from which requirements for passive infrared systems can be determined.

The basic infrared detection problem was outlined by Dr. Kaspar. It was at his request that the work of P. M. Woodward be the basis for the study approach. Credit for any valuable progress in the reported work, therefore, must largely be given to his direction. Any shortcomings in this work are the responsibility of the author.

An opinion tendered is that some of the basic building blocks needed to place a firm foundation under detection theory appear in this report. The test of time will determine any credence to this opinion.

Though the reported work sets forth a useful mathematical statement of the sufficiency conditions for a detection system, much labor must still be expended to further explore this work and reduce it to practice. Only after this expenditure of effort and after critical scrutiny by those more knowledgeable than the author in detection theory can the significance of this work be properly assessed.

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1.0 INTRODUCTION

An apparatus, a machine or a system if you wish, performs in accordance with its basic sensory capabilities. That which it seeks to obtain is information requested by the system proper; that which it does obtain varies with information climate as well as with its own detective capabilities. The better its information-collecting properties, the better is the system total. Detection is indeed a major dictate on ultimate system performance.

If the energy-sensing medium is under control, as in active sonar or radar, a unique signature attached thereto eases the selection of relevant information from the total environment. In passive detection where the source under observation has characteristics peculiar to those of its environment, a detector "tuned" to such characteristics often does well in following source behavior. Under these circumstances, there is very often reasonable certainty the received quantity is relevant. Passive detection in general, however, permits only some weighted measure of relevant information in the received signal-noise quantity.

In the limit, moreover, detection systems generally involve a weighted measure of the relevant information content because of imperfections therein. Here, most importantly, the internal noise associated with a detection system can govern uncertainty in signal detection. In infrared and stellar detection systems, the internal noise environment is so adverse at times to rule out these devices for given applications. Sometimes the problem is so severe as to delay system development until detector state-of-the-art improves.

Confronted with detection in an adverse noise environment, any one sample of the signal-noise quantity then has little meaning; that is, any one sample tells us little about the relative contributions of signal and noise to this sample. Invariably, therefore, a large population of signal-noise samples are assembled, and inferred therefrom is an average estimate of "signal" present. In the limit, this becomes the integral of the signal-noise quantity. "Signal" detection in the presence of noise, therefore, reduces to the formation of some statistical estimate of signal present by observing the signal-noise quantity for some predetermined time interval.

A brief scan of the literature reveals quickly an abundance of attention devoted to the detection of time-variant signals in the presence of noise. Analyses have proceeded in numerous ways through the use of various statistical and network theory tools, with varied degrees of success. These tools we might say comprise the area of "Statistical Theory of Signal Detection."

One approach which appears most attractive, as it clearly commences with a statement embodying all relevant information available at the output of a detector, involves the formation of the a posteriori probability of a signal event given the occurrence of a signal-noise event. This approach was applied successfully to the radar return problem some ten years back by Woodward.* Since, additional consequential work has been done thereon.**

In problems as radar return, the time shape of signal sought is presumed known, allowing the development of rather definitive statements regarding the sufficient conditions confronting any contemplated detection scheme. For the general stochastic signal case, however, such unavailable a priori knowledge of signal time shape can obstruct final definition.

Yet, the clarity of the a posteriori approach as developed by Woodward beckons us to explore this method for the general stochastic case.

Such exploration, in fact, with the aid of series representation of stochastic signals, leads to a basic revelation not heretofore treated in detection theory. A restatement of the a posteriori probability through series signal representation clearly shows that a detection system is

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* Reference 1 (Woodward).

** Reference 2 (Helstrom).

fundamentally an information detector, and not a detector of signals. In brief, the results state that the detection system can only provide estimates of relevant information appearing in the signal-noise quantity; inferences drawn from these estimates in a subsequent signal processor will say something about signal present. This latter operation, involving "guesswork" on the information measures, has nothing to do with signal detection, however.

Use of time power series is especially employed in this report to mathematically demonstrate the aforesaid statements. Of interest is the fact that the results so obtained clearly express the sufficiency conditions for an ideal detector. They fall out in a form, moreover, which are suitable for implementation. The results agree with current practices, and show them to be approximations to the ideal requirements. They also appear in a form suitable to possible systematic improvements to present practices.

The basic information detection requirements are further explored through Fourier series expansion, with the same conclusive success as with power series. These results alone, together with their product representation, permit treatment of a broad spectrum of problems.

As an aside, it is of interest to point out that the posterior statement given through Fourier-power series expansion embodies Woodward's posterior statement for radar return as a special case. Amplitude-modulated radio reception is satisfied as well. Frequency modulation can also be accommodated, though a form of Bessel expansion should provide a more useful set of requirements. This latter statement makes reference to the fact that the work in this report shows that expansions other than the Fourier and power series can be used in problem solution. The physical realization of the ideal requirements ultimately dictates the signal expansion which should be used.

The reported work centers about a Gaussian noise environment, and as such should find broad utility. Although primary attention is on the sufficiency conditions for a detection system, the results are expressed in a form which explicitly reflect what are the detection system requirements and what are the requirements for some subsequent signal processing system.

As an illustrative example, a signal processing system for the realization of maximum likelihood receives attention. The posterior statement generated through series signal representation is shown to lead to its ready implementation.

Added attention is also given to a signal processor where the signal sought is approximated by a step-like function. A particular interpretation of the constant-value steps in the signal processor provides maximum use of detected quantity.

Though the work lends itself to direct application, numerical examples are not treated herein. The primary viewpoint taken relates to passive detection; the results equally accommodate signal detection problems in active systems, however.

In summary, the reported work complements present detection theory by providing a useful mathematical statement of the sufficiency conditions for systems endeavoring to detect time-variant stochastic signals in the presence of Gaussian noise.

2.0 THE A POSTERIORI PROBABILITY

The basic approach employing the a posteriori probability as set down by Woodward for time-variant signals is first presented. A discussion regarding the difficulties, and resolution thereof, in this approach for stochastic signals then follows. With the basic passive signal detection system outlined next, the general solution for a white Gaussian noise model is subsequently presented. Discussion on the broad import of this solution and its extension then culminates this part of the work.

2.1 BASIC APPROACH*

Fundamentally, the problem of signal detection can be expressed as the extraction of a signal quantity from a signal-noise quantity. This problem can be explicitly stated in mathamatical form through the a posteriori probability for signal detection.

Development of the general a posteriori statement for such a two-variable case is derivable directly from the multiplication law of probability, by which it may be written

$$P(X, Y) = P(X)P(Y/X) = P(Y)P(X/Y) \qquad (2\text{-}1)$$

which reads as the probability of the occurrence of both X and Y is equal to both the probability of X times the conditional probability of Y given X, and the probability of Y times the conditional probability of X given Y.

If X is defined as the signal event and Y as the signal-noise event, the a posteriori probability for signal detection then is

$$P(X/Y) = \frac{P(X)P(Y/X)}{P(Y)} \qquad (2\text{-}2)$$

* Reference 1 (Woodward), Chapter 4: Statement of the basic approach is borrowed essentially directly from this text; its extension thereafter, however, is a sole claim of the author.

Since in the detection problem Y is the given quantity, equation (2-2) may be written in reduced form as

$$P(X/Y) = K\ P(X)\ P(Y/X) = K\ P(X)\ L(X) \tag{2-3}$$*

where P(Y) is absorbed into the constant K which itself is so chosen that the conditional posterior probability totals to unity for all values of X.

The a posteriori probability of equation (2-3) is, therefore, a function of both an a priori statement on the signal event X and the conditional statement P(Y/X). The latter conditional statement is known as the likelihood function, so designated in equation (2-3) by replacing P(Y/X) with L(X). This is not a probability statement, however, since it is treated as a function in X for a given Y.

Assuming noise contribution to an additive signal-noise quantity Y to be Gaussian with zero mean, the likelihood function for an arbitrary sample of Y then is

$$L(X_n) = K \exp \left\{ - \frac{(Y-X)^2}{2\sigma_N^2} \right\} \tag{2-4}$$

where σ_N^2 is the mean squared value of noise.

With this result inserted in equation (2-3), we can deduce therefrom that one such sample of Y tells little about X if noise is severe. Rather than one such spot sample, we might then ask of the requirements for observing the signal-noise quantity for some finite time, time interval (O, T) let us say, thereby endeavoring to get a better estimate of signal present. To answer this then requires formation of the likelihood function for this time interval.

To develop this likelihood function, it is convenient to consider band-limited, white Gaussian noise, that is, a white noise frequency structure

* The constant K will appear throughout the work having different values at different times, but always chosen such that the probability statement totals to unity for all possible events.

constrained within some arbitrary bandwidth W. Doing this permits noise representation over time interval (O, T) by a finite set of noise samples, 2WT such samples, in fact, taken every 1/2W seconds, as permitted by the basic sampling theorem.* Assuming statistically independent samples permits the likelihood function for interval (O, T) to be written as

$$L(X) = \prod_{n=1}^{2WT} L(X_n) = K \prod_{n=1}^{2WT} \exp\left\{-\frac{(Y_n - X_n)^2}{2\sigma_N^2}\right\}$$

$$= K \exp\left\{-\sum_{n=1}^{2WT} \frac{(Y_n - X_n)^2}{2\sigma_N^2}\right\} \tag{2-5}$$

Since the noise samples are an orthogonal set, as per the sampling theorem, the sum appearing in equation (2-5) can be written as

$$\sum_{n=1}^{2WT} \frac{(Y_n - X_n)^2}{2\sigma_N^2} = \frac{1}{N_o}\int_0^T (Y-X)^2 dt \qquad \text{(2-6)**}$$

where N_o is equal to σ_N^2/W, or the mean noise power per unit bandwidth. Thus, the likelihood function of equation (2-5) can be expressed as

$$L(X) = K \exp\left\{-\frac{1}{N_o}\int_0^T (Y-X)^2 dt\right\} \tag{2-7}$$

and the a posteriori probability for signal event X during time interval (O, T) of equation (2-3) may be written as

$$P(X/Y) = K\, P(X) \exp\left\{-\frac{1}{N_o}\int_0^T (Y-X)^2 dt\right\} \tag{2-8}$$

Expansion of the exponent term in this equation permits the reduced form

$$P(X/Y) = K\, P(X) \exp\left\{-\frac{1}{N_o}\int_0^T X^2 dt\right\} \exp\left\{\frac{2}{N_o}\int_0^T XY dt\right\} \tag{2-9}$$

* Reference 3 (Shannon) and Reference 4 (Shannon and Weaver) are recommended.

** Reference 1 (Woodward), Chapter 2.

where the Y^2 term of the integrand, with Y being given, is absorbed into the constant K.

Thus, from the final posterior statement in equation (2-9), the only operation which need be performed on the signal-noise quantity Y involves the integral of the product XY over interval (O, T). This is indeed an irreversible, noise-stripping operation on Y, in essence being the correlation of Y with all X over interval (O, T). Realization of this integral then represents a just sufficient solution to form the a posteriori probability of X for any a priori P(X) chosen for X.

Formation of the XY integral, moreover, constitutes the sufficiency conditions for a detection system. Selection of P(X) and determination of the X^2 integral in equation (2-9) are not requirements of the detection system itself, but requirements of a subsequent processing system, wherein detection system output may be interpreted as one pleases. If P(X) is known, then the posterior statement could be formed exactly (if given integral XY, that is); if some arbitrary form of P(X) is presumed, then the resulting posterior statement is only as good as the closeness between the presumed P(X) and the actual case. In neither event, however, does such data interpretation impose additional demands on the detection system itself; it still need only form the integral of XY over interval (O, T).

In the general passive detection case, time shape of X is not known; hence, the sufficiency conditions expressed through equation (2-9) are not directly realizable, that is, the integral of XY is not directly formable. Resolution of this point, in fact, is the crux of the reported work.

As an introductory exercise to subsequent work, let us at this time consider a known time shape for X; let this known shape be some constant value X_o over interval (O, T). For this simple case, equation (2-9) reduces to

$$P(X/Y) = K\,P(X)\,\exp\left\{-\frac{X_o^{\,2}T}{N_o}\right\}\exp\left\{\frac{2X_o}{N_o}\int_o^T Y\,dt\right\} \qquad (2\text{-}10)$$

The sufficiency conditions for a detection system then reduce to a simple integral of the signal-noise quantity Y.

If P(X) is assumed to be a uniform distribution, examination of equation (2-10) shows the most probable value of X equal to the average value of Y over interval (O, T). Such an assumed a priori distribution is equivalent to the application of maximum likelihood, of course, which then gives the same solution, a solution also long since intuitively derived by the electronic system designer.

In passing, we should make an observation regarding the value of observing the signal-noise quantity for interval (O, T) as opposed to a spot sample. This can be done by comparing the variances of their likelihood functions for the constant signal case. For a spot sample, equation (2-7) gives a variance of σ_N^2; for observation over interval (O, T), equation (2-10) gives a variance of $N_o/2T$, or $\sigma_N^2/2WT$. Thus, by this measure, we see an improvement in signal estimation of 2WT.

2.2 DISCUSSION OF SUFFICIENCY CONDITION REALIZABILITY

As implied heretofore, formation of the XY integral over a chosen interval (O, T) represents the sufficiency conditions for a detection system. Formation of this integral permits subsequent development of the posterior statement regarding signal present in this interval. To realize this integral in passive detection, however, poses some difficulty, as a priori knowledge of signal time shape is not generally available. Direct implementation appears out of the question, since, for a time-variant stochastic signal, the integral need be formed for an infinity of X signal time shapes.

Yet, passive detection systems are in existence today, some of which do an admirable job in picking a signal from a morass of noise. We might then conclude from such evidence that these systems must in some way approximate the sufficiency conditions set forth. If this is indeed true, it should be possible to relate present successful practices to the ideal set of requirements. Moreover, should such a relationship be realized, we might expect therefrom the direction to proceed to obtain even better approximations to the ideal set. *

To show the foregoing desired objectives to be realizable, we must delve in the subtle facets of troublesome integral XY. More specifically, we must examine ways of realizing this integral for X unknown: a noble task indeed.

It will be shortly shown that prior knowledge of X need not be available to form this integral, at least in principle, if we are judicious in the way an arbitrary X is represented. The basic argument underlying resolution of this integral appears in Appendix C of this report.

Regarding the unknown X itself, we seek a more useful way of representing this quantity, a representation which would ultimately permit solution of the XY integral. We see difficulty in forming this integral directly only because X is envisioned as an infinity of unknown time samples. In a way, the task would seem less difficult if an arbitrary X were represented by a finite set of samples through use of the sampling theorem; formation of the required integrals involve sin wt/t type functions, however, over time minus infinity to plus infinity, causing us to lose enthusiasm in this representation.

In this search for a useful form of X, we must always keep in mind the practicality of implementing the required integration resulting from any new representation of X. Realizing this, the author cannot help but reflect

* This is not idle talk, for the situation here is analogous to the problem of realizing better performance with a serial, general-purpose digital computer through better signal representation. This computer problem was resolved through reasoning similar to the above. We refer to Reference 5 and 6, even though they are only available under controlled distribution. The basic philosophy expounded therein, however, is lightly treated in Appendix A of this report.

back to the signal representation used in improving serial, general-purpose computer operation.* Therein, a power series in time was employed as the general form to represent a signal, because of its ease in implementation. Why not then assume the signal sought in the detection problem to be also representable by this form? This form can be put to good use in most cases. Its value lies in the way its coefficients are chosen, and some extremely powerful selections can be made therewith.**

If X is represented by a power series in time with arbitrary coefficients, the solution of the general XY integral is direct —— embarassingly direct to say the least. Its implementation, moreover, is straightforward, involving only successive integrals of Y. The arbitrary coefficients need not be prestated. The successive integrals of Y measure as best possible the relevant information content; these coefficients then can be determined from these measures at a later date.

We shall shortly show formally the simplicity to this general solution. Presently, we first choose to set forth a model for the general passive detection system for use in subsequent discussion.

2.3 PASSIVE DETECTION SYSTEM

A model for a general passive detection system is depicted in figure 2-1. Therein, the environment to which a detector is exposed is called the incoming energy environment. The signal energy represents that relevant information requested by the system functions; the remainder, which might pass through the detector, is termed background energy (or signal noise environment).

A filter system is inserted between the incoming energy environment and the detector to accommodate those systems employing some sort of "space

* Appendix A of this report.
** Appendix B of this report.

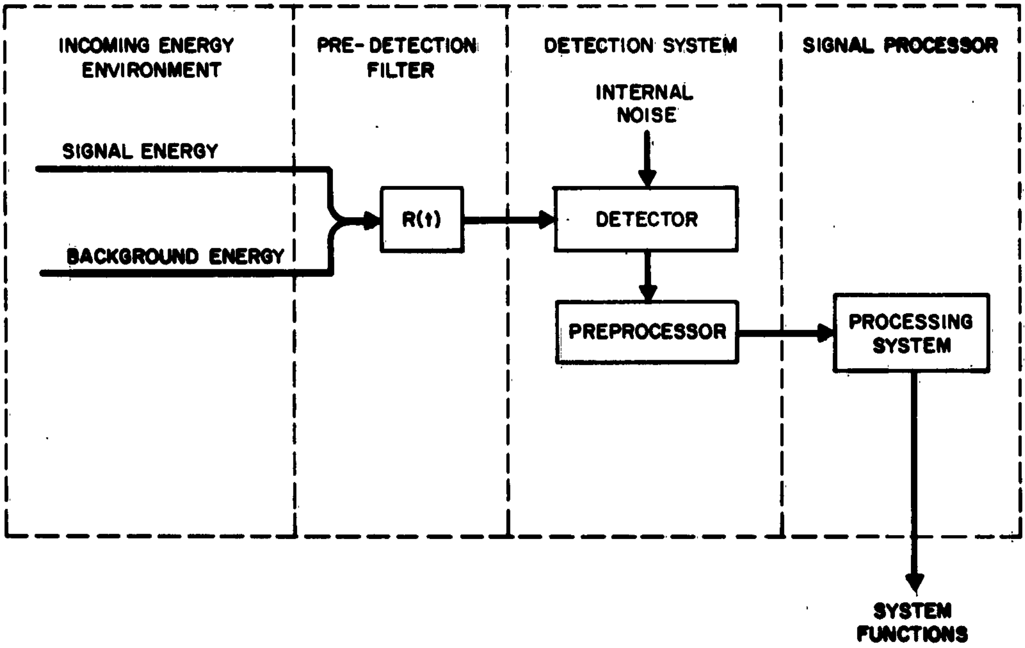

Figure 2-1. Passive Detection System.

filtering" prior to detection. This might range anywhere from simple amplitude modulation to some form of random filtering. The filter characteristic R(t) is defined such that its output is equal to the incoming energy time function multiplied by R(t). In subsequent work, R(t) is always assumed to be known explicitly.

The detection system is shown as being comprised of both a detector and preprocessor. The detection system is assumed to have consequential internal noise. A preprocessor is included as an integral part of the detection system to emphasize that provision of some unstated measure of incoming energy behavior by itself constitutes an incomplete detection system; it must also state as best possible that portion of relevant information contained in its output structure. This latter requirement falls upon the preprocessor.

In a strict sense, preprocessor is only required to provide a measure reflecting that portion of incoming energy appearing in total detector output quantity. That is, all incoming energy is treated as signal energy by the preprocessor, and noise is constrained to mean detector internal noise. When system functions request a signal with specific information structure, the preprocessor can be burdened with added special requirements. Here, background energy and detector internal noise combine to represent total noise environment.

A signal processor is shown distinct to the preprocessor of the detection system. This is done since the signal processing operation can be chosen independent of the detection system. The signal processor task is to take preprocessor measures of relevant information content in detector output and construct therefrom, or infer therefrom, the nature of the signal present. The signal processor function might be in the form of a probability statement with, or without some decision-making operation, or in the form of an inferred signal as a function of time. System functions will generally dictate the nature of signal processor operation.

2.4 GENERAL SOLUTION USING POWER SERIES (BAND-LIMITED WHITE GAUSSIAN NOISE)

To derive the general solution for a time-variant stochastic signal using power series expansion we remain with the noise environment which is band-limited, white Gaussian with zero mean. In terms of the passive detection system given in figure 2-1, all incoming energy can be assumed to constitute signal, and noise to be internal detector noise; a filter system is not employed, or, as per figure 2-1, filter transfer characteristic is unity. The object is to determine requirements for an ideal preprocessor.

Deeming spot samples of detector output to be inadequate, we then choose observation of detector output quantity for some time interval (O, T) as the basic detection mode of operation. For the noise environment chosen, previous equation (2-9) constitutes the posterior statement regarding signal detection. To aid the reader, this equation is repeated below:

$$P(X/Y) = K\, P(X) \exp \left\{ -\frac{1}{N_o} \int_o^T X^2 dt \right\} \exp \left\{ \frac{2}{N_o} \int_o^T XY dt \right\} \qquad (2\text{-}11)$$

We shall now proceed to restate this expression in more meaningful form by representing the signal sought through power series expansion.

First, X is written as a power series in time over interval (O, T):

$$X(O,T) = A_o + A_1 t + A_2 t^2 + \cdot \cdot \cdot + A_k t^k + \cdot \cdot \cdot = \sum_{n=o}^{\infty} A_n t^n \qquad (2\text{-}12)$$

where the A's are arbitrary constants left to our choosing.* Using this form of X in equation (2-11) results in

* Definition of the A's could be made a priori as per Appendix B, or made a posteriori on the basis of maximum likelihood (Section 3.0), most probable value through use of an a priori distribution, or some other method suitable to particular system application. They are, however, constants which can be treated as an arbitrary set insofar as the detection system is concerned.

$$P(X/Y) = K\, P(X) \exp\left\{ -\frac{1}{N_o} \sum_{n=o}^{\infty} \sum_{m=o}^{\infty} A_m A_n \frac{T^{m+n+1}}{(m+n+1)} \right\}$$

$$\exp\left\{ \frac{2}{N_o} \int_o^T Y \sum_{n=o}^{\infty} A_n t^n dt \right\} \tag{2-13}$$

Since P(X) and the A's are to our choosing, only the integral term needs resolution. This is the same integral we were confronted with in Appendix B of this report, wherein it was directly solved through integration by parts. Proceeding as per Appendix B, we may write

$$\int_o^T Y \sum_{n=o}^{\infty} A_n t^n dt = \sum_{n=o}^{\infty} A_n \int_o^T t^n Y dt$$

$$= A_o \left[\int_o^T Y dt \right] + A_1 \left[T \int_o^T Y dt - \int_o^T \int_o^t Y dt^2 \right]$$

$$+ A_2 \left[T^2 \int_o^T Y dt - 2T \int_o^T \int_o^t Y dt^2 + 2 \int_o^T \int_o^t \int_o^t Y dt^3 \right] + \cdots$$

$$\cdots + A_k \left[\sum_{m=1}^{k} (-)^{m+1} \frac{k!}{(k+1-m)!} T^{(k+1-m)} \right.$$

$$\left. S^{-m}[Y] \right] + \cdots \tag{2-14}$$

where $S^{-m}[Y]$ represents the mth integral of Y over interval (O, T). This solution can be written in more compact form as

$$\int_o^T Y \sum_{n=o}^{\infty} A_n t^n dt = \sum_{n=o}^{\infty} A_n \sum_{m=1}^{\infty} (-)^{m+1} \frac{n!}{(n-m+1)!} T^{(n-m+1)} S^{-m}[Y] \tag{2-15}$$

Now, P(X) can be expressed in general form in terms of the A's, that is,

$$P(X) = P(A_o, A_1, A_2, \cdots, A_k, \cdots) = P(\{A\}) \tag{2-16}$$

Utilizing equations (2-15) and (2-16) in equation (2-13) gives

$$P(X/Y) = P(\{A\}/Y)$$

$$= KP(\{A\}) \exp\left\{ -\frac{1}{N_o} \sum_{n=o}^{\infty} \sum_{m=o}^{\infty} A_m A_n \frac{T^{m+n+1}}{(m+n+1)} \right\}$$

$$\exp\left\{ \frac{2}{N_o} \sum_{n=o}^{\infty} A_n \sum_{m=1}^{n+1} (-)^{m+1} \frac{n!}{(n-m+1)!} T^{(n-m+1)} \right.$$

$$\left. S^{-m}[Y] \right\} \qquad (2\text{-}17)$$

This equation represents a useful statement of the a posteriori probability of a signal X with unknown time shape in the presence of band-limited, white Gaussian noise with zero mean. For any chosen $P(\{A\})$ and $\{A\}$, only formation of the successive integrals of the signal-noise quantity Y over interval (O, T) are needed to realize the full posterior statement for signal detection. The requirements for an ideal preprocessor then become the formation of these integrals, that is, formation of

$$\sum_{m=1}^{\infty} S^{-m}[Y] \longrightarrow \int_o^T Ydt \ ; \ \int_o^T \int_o^t Ydt^2 \ ; \ \int_o^T \int_o^t \int_o^t Ydt^3 \ ; \text{Etc.} \qquad (2\text{-}18)$$

The immediate realizability of the aforesaid integrals should be apparent, though we would hardly choose to realize an infinite set.

Note specifically what power series representation for time-variant stochastic signals has done to the detection system requirements. It has removed determination of the power series arbitrary coefficients as a requirement thereof, leaving this determination to some subsequent signal processing system. This is indeed valid since the equation (2-18) integrals measure as best possible the relevant information in detector output quantity. The arbitrary coefficients can be determined from these measures by any means we choose. Whatever method chosen, it will not be impaired by a shorted supply of information, for the infinite set of integrals reflect all that can be obtained.

Request for direct measures of a priori stated coefficients by a detection system would represent more than sufficient conditions. This would not only be a request for the best measure of relevant information appearing in detector output quantity, but also a request for some inferences to be drawn therefrom. The latter is not a task of basic information detection which is the only responsibility of a detection system, but a task involving "guesswork" on the information measures obtained which is the responsibility of some subsequent signal processing system.

We might surmise, therefore, that lack of appreciation of this basic point has obstructed treatment of time-variant stochastic signals by the theorists in the field of signal detection.

2.5 APPLICATION OF GENERAL SOLUTION OBTAINED THOUGH POWER SERIES

Using power series representation for unknown signal X, the requirements for an ideal preprocessor become the formation of an infinite set of successive integrals of detector output over observation interval (O, T), as per equation (2-18). The integrals themselves are directly realizable. In practice, moreover, signal behavior is often such that formation of the first integral alone is sufficient to satisfactorily reflect its detection in presence of noise. Here, length of observation interval T is the dominant factor, and its value is so chosen that, for all practical purposes, the signal is constant over the observation interval. Sometimes other system requirements dictate a value of T such that signal must be considered time-variant over the interval, requiring formation of the first two or first three terms of the integral set. It can be generally stated, however, that requirements for an ideal preprocessor, as given in equation (2-18), can usually be adequately approximated by but a few of the integral measures at most (that is, if the power series form is an efficient representation).

Interesting enough, realization of the required integrals by a preprocessor in no way affects subsequent signal processing. The arbitrary constants $\{A\}$ of the time power series representing signal X remain to be chosen therein as one pleases. The same is also true with regard to the a priori distribution for X.

In passing, it is of importance to relate the significance of these statements with regard to applications where signal detected is recorded (or transmitted outside the system) for use elsewhere. Its further use might take many varied, and sometimes yet to be specified forms. Retention of relevant information in this basic integral-measure form permits these subsequent operations. They in no way impair freedom in operation thereafter.

Added reference is also made to space systems, wherein signal detected is ultimately transmitted back to the earth in digital form. Here, constraints on such data transmittal are usually so severe that maximization of data compression is of primary concern. Straightforward sampling of signal quantity simply involves much too high bit rates. It then becomes a problem of finding a signal format which lends itself to the most efficient coding of signal data.

A little thought on the integral measures of equation (2-18) suggests these measures as a satisfactory signal format. Rather than a set of sampled amplitudes, we might use these measures to represent a signal as a power series for some time interval. In fact, related work in this area by the author indicates this to be highly effective. In such cases, signal is represented by an integral or a finite integral set, as per equation (2-18), of detector output quantity. Only these integrals are transmitted back to the earth. Spectral behavior of signal and other system requirements determine length of observation interval T and number of integral measures, with these so chosen that signal data bits per unit time are a minimum for transmittal. Such signal representation for transmittal, moreover, can be advantageous whether or not the noise environment is of consequence.

2.6 GENERAL SOLUTION USING FOURIER SERIES (BAND-LIMITED WHITE GAUSSIAN NOISE)

To show that signal representation through power series is not the only form of expansion which leads to a general solution, we now show the requirements for an ideal preprocessor if the Fourier series expansion is employed (though the Fourier series is embodied in the general power series). The noise model remains as band-limited, white Gaussian noise.

Suppose the arbitrary signal X is represented by the general Fourier series over the observation interval as

$$X(O,T) = \sum_{n=o}^{\infty} F_n \cos\left(\frac{2\pi n}{T}t + \theta_n\right)$$

$$= \sum_{n=o}^{\infty} \left[a_n \sin\frac{2\pi n}{T}t + b_n \cos\frac{2\pi n}{T}t\right] \tag{2-19}$$

where the F's (or the a's and b's) are arbitrary constants, whose definition can be stated at a later time (the argument used on power series applies directly). If this representation of X is inserted into the general posterior statement for band-limited, white Gaussian noise, as given in equation (2-9) and (2-11), we see the XY integral to decompose such that the requirements for an ideal preprocessor become

$$\int_o^T Y dt \; ; \; \int_o^T Y\cos\left(\frac{2\pi}{T}t+\theta_1\right)dt \; ; \; \int_o^T Y\cos\left(\frac{4\pi}{T}t+\theta_2\right)dt;$$

$$\int_o^T Y\cos\left(\frac{6\pi}{T}t+\theta_3\right)dt \; ; \; \text{Etc.} \tag{2-20}$$

or

$$\int_o^T Y dt \; ; \; \int_o^T Y\sin\frac{2\pi}{T}t dt \text{ and } \int_o^T Y\cos\frac{2\pi}{T}t dt \; ;$$

$$\int_o^T Y\sin\frac{4\pi}{T}t dt \text{ and } \int_o^T Y\cos\frac{4\pi}{T}t dt \; ; \; \text{Etc.} \tag{2-21}$$

Thus, with the signal expressed as a Fourier series, the ideal preprocessor must provide an infinite set of filters matched to the sinusoids of the harmonics representing the signal. (Note that need for synchronous detection is embodied in these ideal requirements.)

Since the Fourier series representation can often be better used to approximate a signal where similar approximation through power series is inefficient, and vice versa, the two sets of requirements for an ideal preprocessor, as given by equations (2-18) and (2-20) or (2-21), present the system designer with formidable guides by which to synthesize his system.

The comments regarding application of the power series solution in section 2.5 apply as well to the Fourier series solution given above.

2.7 SPECIAL CASE OF FOURIER SERIES SOLUTION

It is of interest to examine a special case of the Fourier series solution. Reference is to the case where the signal sought contains only one frequency and is of constant amplitude, that is, the signal can be represented by

$$X(O, T) = F \cos(\omega t + \theta) \tag{2-22}$$

The equation (2-11) posterior statement then becomes

$$P(X/Y) = K\,P(X) \exp\left\{ -\frac{F^2}{N_o} \int_o^T \cos^2(\omega t + \theta)dt \right\}$$

$$\exp\left\{ \frac{2F}{N_o} \int_o^T Y\cos(\omega t + \theta)dt \right\} \tag{2-23}$$

We see equation (2-23) is identical with Woodward's posterior statement for the radar return problem for known signal amplitude. Of course, equation (2-23) is the posterior statement for exact time behavior of signal —— phase included. Suppose we are only interested in whether signal is present. If phase is unknown, posterior statement for presence of signal requires integration of equation (2-23) over all possible θ. Assuming a uniform distribution for θ then leads to the modified Bessel form I_o as the requirement for the preprocessor.

Incidentally, if the envelope of equation (2-22) is time-variant, power series representation of envelope behavior leads to

$$\int_o^T Y \cos(\omega t + \theta)\, dt \; ; \int_o^T \int_o^t Y \cos(\omega t + \theta) dt^2;$$

$$\int_o^T \int_o^t \int_o^t Y \cos(\omega t + \theta) dt^3; \text{ Etc.} \qquad (2\text{-}24)$$

as the requirements for an ideal preprocessor. Here again, the posterior statement is for both amplitude and phase. Integration of this posterior statement over all θ then results in a solution suitable to Woodward's radar return work on signal amplitudes which are time-variant. It is, in fact, a more direct solution. This same solution is also applicable to amplitude-modulated radio reception in general.

2.8 PRE-DETECTION FILTER CONSIDERATIONS (BAND-LIMITED WHITE GAUSSIAN NOISE)

The characteristics R(t) of a pre-detection filter were defined in section 2.3, as per figure 2-1, such that its output was given by

$$X_R = X\, R(t) \qquad (2\text{-}25)$$

If X and R(t) both are represented by a power series in time, the requirements for an ideal preprocessor operating in band-limited, white Gaussian noise are those given by equation (2-18): an infinite set of successive integrals of detector output over the observation interval.

If X is represented by a power series and R(t) by a Fourier series, or vice versa, the requirements for an ideal preprocessor become

$$\sum_{n=0}^{\infty} \int_0^T Y\cos(\frac{2\pi n}{T}t + \theta_n)dt \; ; \sum_{n=0}^{\infty} \int_0^T \int_0^t Y\cos(\frac{2\pi n}{T}t + \theta_n)dt^2;$$

$$\sum_{n=0}^{\infty} \int_0^T \int_0^t \int_0^t Y\cos(\frac{2\pi n}{T}t + \theta_n)dt^3; \text{ Etc.} \qquad (2\text{-}26)$$

This solution, incidentally, also satisfies the case where a signal is represented by the product of a Fourier series with a power series.

If both X and R(t) are represented by a Fourier series, the requirements for an ideal preprocessor are

$$\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \int_0^T Y\cos(\frac{2\pi m}{T}t + \theta_m) \cos(\frac{2\pi n}{T}t + \theta_n)dt \qquad (2\text{-}27)$$

The preprocessor requirements given through equations (2-18), (2-26) and (2-27) then represent a set of design guides which should cover most cases arising in practice.

One case not covered by the above set, however, is when the pre-detection filter frequency modulates the signal. Writing the output of the pre-detection filter for this frequency modulation as

$$X_R = a(t) \cos f(t) \qquad (2\text{-}28)$$

and expressing a(t) as a power series leads to the requirements for an ideal preprocessor concerned with the general case of frequency modulation as

$$\int_0^T Y\cos f(t)dt \; ; \int_0^T \int_0^t Y\cos f(t)dt^2 \; ; \int_0^T \int_0^t \int_0^t Y\cos f(t)dt^3; \text{ etc.} \qquad (2\text{-}29)$$

In each of the above cases, explicit knowledge of R(t) is presumed known; the ideal requirements, therefore, have embodied therein a request for synchronous detection.

2.9 GENERAL GAUSSIAN NOISE CONSIDERATIONS

Though the general solutions heretofore obtained were for Gaussian noise involving independent samples, it can be readily extended to include correlated Gaussian noise with some additional labor. First, however, it is necessary to perform a transformation on the noise function in order to generate a set of independent noise samples. It can then be shown* that the posterior statement for signal detection may be written as

$$P(X/Y) = K\, P(X) \exp\left\{ -\frac{1}{2} \int_0^T XQdt \right\} \exp\left\{ \int_0^T YQdt \right\} \tag{2-30}$$

where Q is defined by the integral equation

$$XR(t) = \int_0^T \phi_N(t-s)\, Q(s)ds \tag{2-31}$$

with ϕ_N being the noise autocovariance. From equation (2-30), we see the sufficiency conditions for the preprocessor become the generation of integral YQ.

Most often ϕ_N, or its integral, is well behaved permitting its representation in Taylor's form of the time power series. With ϕ_N so represented, and Q assumed in the form of representation chosen for XR(t), equation (2-31) should generally lend itself to ready solution. This being so, the requirements for an ideal preprocessor are again dependent on the representation chosen for XR(t), being either those in equation (2-18), (2-26), or (2-27) for power and Fourier series expansions of X and R(t).

2.10 A FEW COMMENTS ON SERIES SOLUTIONS

A close examination of the series solutions presented shows their approximations to represent methods practiced today. We leave discussion of these approximations to the reader, who is probably better versed in

* Based on Reference 2 (Helstrom), Chapter 4.

current practices. A clear statement of these approximations, as per our solutions, is in itself a successful accomplishment. Note the logical structure of these series solutions, moreover; if the presently employed approximation is inadequate, they clearly express the next best approximation which can be made.

It should be emphasized that the most important analytical point developed is that the XY integral, which defines the sufficiency conditions for a preprocessor, is directly solvable if the signal X is expressed as a series expansion in time with arbitrary coefficients. Determination, or selection of these arbitrary coefficients is a task for the signal processing system, and not a task of the preprocessor.

Though only Fourier and power series are treated herein, one is not restricted to such signal representation. Many attractive expansions are available, especially those which are formed from an orthogonal set of functions. The problem application will ultimately determine the best type of expansion to be employed for solution. Use of the Bessel form, for example, appears to offer advantage for frequency modulated signals as well as certain pulse type signals.

From a practical point of view, that form of expansion which converges most rapidly for the type of signal to be accommodated and whose terms lend themselves to ready implementation should be chosen. Rapid convergence implies only a few terms of the expansion need be implemented to realize a good approximation to the ideal set of requirements.

Use of the a posteriori approach not only justifies itself, but also strongly recommends itself as the approach to be employed on signal detection problems in general. It directly identifies the sufficiency conditions for detection, which can then be readily reduced to usable form through series expansion for signal representation.

3.0 MAXIMUM LIKELIHOOD FOR POWER SERIES REPRESENTATION

Though not interested in signal processing, per se, the simplicity of realizing the maximum likelihood method through series representation for time-variant stochastic signals causes us to devote consequential attention to its implementation. It also serves as an example of numerous methods by which the arbitrary coefficients of the series expansions employed could be obtained.

Maximization of the likelihood function appearing in the posterior statement for signal detection implies that all X signal time shapes are equally likely to occur during interval (O, T), that is, any set of arbitrary constants for an expansion of X are equi-probable. The a priori X probability is then a uniform distribution representable by some arbitrary constant. This constant itself can always be absorbed by the all encompassing K which accompanies each of the posterior statements.

Thus, assuming equi-probable a priori probability permits a reduced form of the posterior statement, that is,

$$P(X/Y) = KL(X)dX \tag{3-1}$$

being only variant with the likelihood function.

3.1 BAND-LIMITED WHITE GAUSSIAN NOISE

Let us examine the band-limited, white Gaussian noise case having zero mean. As per equation (3-1) for equi-probable a priori probability, the posterior statement for signal detection becomes

$$P(X/Y) = K \exp\left\{-\frac{1}{N_o}\int_o^T (Y-X)^2 dt\right\} dX \tag{3-2}$$

Application of maximum likelihood method to equation (3-2) involves minimization of the integral term therein. Minimization of this integral term is in itself a statement of the least squares method. For the Gaussian case, maximum likelihood always encompasses least squares.* Application of least squares to a general time function X represented as an expansion in time results in a solution in terms of matrices having infinite rows and columns. Approximations to X, however, permit easy solution. This will now be shown.

Suppose the interval (O, T) is so chosen that X over this interval can be reasonably approximated as $\widetilde{X}$. Equation (3-2) can then be written as

$$P(X/Y) \cong K \exp\left\{-\frac{1}{N_o}\int_o^t (Y-\widetilde{X})^2 dt\right\} d\widetilde{X} \tag{3-3}$$

where $\widetilde{X}$ may be expressed as

$$\widetilde{X}(O,T) = A_o + A_1 t + A_2 t^2 + \cdots + A_m t^m \tag{3-4}$$

with m less than infinity.**

It can be shown that minimization of the integral term in equation (3-3) is realized by choosing the A's in accordance to the matrix equation

$$[C][A] = [B] \tag{3-5}$$

whose elements are given by

$$C_{ij} = \int_o^T t^{i+j} dt \; ; \; a_j = A_o, A_1, A_2, \cdots A_m \; ; \; b_i = \int_o^T t^i Y dt \tag{3-6}$$

or

* Reference 7 (Hald), Chapter 8.
** We could just as well choose the Fourier series expansion.

$$
[C] = \begin{bmatrix}
T & \frac{T^2}{2} & \frac{T^3}{3} & \cdots & \frac{T^{m+1}}{m+1} \\
\frac{T^2}{2} & \frac{T^3}{3} & \frac{T^4}{4} & \cdots & \frac{T^{m+2}}{m+2} \\
\frac{T^3}{3} & \frac{T^4}{4} & \frac{T^5}{5} & \cdots & \frac{T^{m+3}}{m+3} \\
\vdots & \vdots & \vdots & & \vdots \\
\frac{T^{m+1}}{m+1} & \frac{T^{m+2}}{m+2} & \frac{T^{m+3}}{m+3} & \cdots & \frac{T^{2m+2}}{2m+2}
\end{bmatrix} \tag{3-7}
$$

$$
[A] = \begin{bmatrix}
A_o \\ A_1 \\ A_2 \\ \vdots \\ A_m
\end{bmatrix} \tag{3-8}
$$

$$
[B] = \begin{bmatrix}
1 & 0 & 0 & \cdots & 0 \\
T & -1 & 0 & \cdots & 0 \\
T^2 & -2T & 2 & \cdots & 0 \\
\vdots & \vdots & \vdots & & \vdots \\
T^m & -\left\{\frac{m!}{(m-1)!}\right\}T^{m-1} & \left\{\frac{m!}{(m-2)!}\right\}T^{m-2} & \cdots & (-)^{m+1}m!
\end{bmatrix}
\begin{bmatrix}
\int_o^T Ydt \\
\int_o^T\int_o^t Ydt^2 \\
\int_o^T\int_o^t\int_o^t Ydt^3 \\
\vdots
\end{bmatrix} \tag{3-9}
$$

where the elements of the rows in the right-side matrix of equation (3-9) are generated from

$$\begin{bmatrix} \text{Sum of} \\ \text{Row Elements} \end{bmatrix} = \sum_{i=1}^{m} (-)^{i+1} \frac{m!}{(m-i+1)!} T^{(m-i+1)} \tag{3-10}$$

Thus, with m and T specified, the A's can always be determined in terms of the successive integrals of Y over interval (O, T) through equation (3-5). To illustrate this, we will now apply least squares to the three simplest cases, that is, m=o, m=1, and m=2.

3.2 SPECIFIC EXAMPLES

If m=o is chosen, then the approximation to X is

$$X(O,T) \cong \tilde{X} = A_o \tag{3-11}$$

and equation (3-5) becomes

$$\left[T\right]\left[A_o\right] = \left[\int_o^T Ydt\right] \tag{3-12}$$

or A_o is given by

$$A_o = \frac{1}{T}\int_o^T Ydt \tag{3-13}$$

If m=1 is chosen, then the approximation to X is

$$X(O,T) \cong \tilde{X} = A_o + A_1 t \tag{3-14}$$

and equation (3-5) becomes

$$\begin{bmatrix} T & \frac{T^2}{2} \\ \frac{T^2}{2} & \frac{T^2}{3} \end{bmatrix} \begin{bmatrix} A_o \\ A_1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ T & -1 \end{bmatrix} \begin{bmatrix} \int_o^T Ydt \\ \int_o^T\int_o^t Ydt^2 \end{bmatrix} \tag{3-15}$$

or A_o and A_1 are given through

$$\begin{bmatrix} A_o \\ \\ A_1 \end{bmatrix} = \begin{bmatrix} -\frac{2}{T} & \frac{6}{T^2} \\ \\ \frac{6}{T^2} & -\frac{12}{T^3} \end{bmatrix} \begin{bmatrix} \int_o^T Ydt \\ \\ \int_o^T\int_o^t Ydt^2 \end{bmatrix} \tag{3-16}$$

If m=2 is chosen, then the approximation to X is

$$X(O, T) \cong \widetilde{X} = A_o + A_1 t + A_2 t^2 \tag{3-17}$$

and equation (3-5) becomes

$$\begin{bmatrix} T & \frac{T^2}{2} & \frac{T^3}{3} \\ \\ \frac{T^2}{2} & \frac{T^3}{3} & \frac{T^4}{4} \\ \\ \frac{T^3}{3} & \frac{T^4}{4} & \frac{T^5}{5} \end{bmatrix} \begin{bmatrix} A_o \\ \\ A_1 \\ \\ A_2 \end{bmatrix} = \begin{bmatrix} 1 & o & o \\ \\ T & -1 & o \\ \\ T^2 & -2T & 2 \end{bmatrix} \begin{bmatrix} \int_o^T Ydt \\ \\ \int_o^T\int_o^t Ydt^2 \\ \\ \int_o^T\int_o^t\int_o^t Ydt^3 \end{bmatrix} \tag{3-18}$$

or A_o, A_1, and A_2 are given through

$$\begin{bmatrix} A_o \\ \\ A_1 \\ \\ A_2 \end{bmatrix} = \begin{bmatrix} \frac{3}{T} & -\frac{24}{T^2} & \frac{60}{T^3} \\ \\ -\frac{24}{T^2} & \frac{168}{T^3} & -\frac{360}{T^4} \\ \\ \frac{30}{T^3} & -\frac{180}{T^4} & \frac{360}{T^5} \end{bmatrix} \begin{bmatrix} \int_o^T Ydt \\ \\ \int_o^T\int_o^t Ydt^2 \\ \\ \int_o^T\int_o^t\int_o^t Ydt^3 \end{bmatrix} \tag{3-19}$$

3.3 COMMENTS ON IMPLEMENTATION

Note again that the preprocessor operations involve only integrals of Y. Specification of the approximation to X will determine the number of integrals to be formed by the preprocessor. Determination of the A's in the X approximation through maximum likelihood is not a preprocessor operation, however, but a part of the subsequent signal processing system. The same preprocessor integral measures could also be used to determine the A's by a method other than maximum likelihood.

It should also be noted the simplicity with which maximum likelihood can be realized in a subsequent signal processor through these preprocessor integral measures. With m and T specified, the elements in the matrix relating the A's to the integral measures, as in equations (3-13), (3-16) and (3-19), could be precalculated and stored therein. A simple set of multiplication-addition operations each observation interval would then realize maximum likelihood.

In the case of earth transmittal of signal data from a spacecraft, only the integral measures need be transmitted. Determination of the A's therefrom (or implementation of signal processor) could be done at the earth reception station.

Note also that the work given herein is applicable to least-squares curve-fitting in general. The quantity here would be some continuous function in time which can be approximated by some truncated power series in time. The constants of this truncated power series could then be realized through the same equation (3-5) matrix relation in terms of the integral measures.

4.0 APPROXIMATE MODEL FOR TIME-VARIANT STOCHASTIC SIGNAL

As heretofore implied, the time-variant stochastic signal we seek to detect in the presence of noise can often be approximated over any one observation interval (O, T). As in the previous section for maximum likelihood, that approximation consistent with information measures provided by a preprocessor would be a truncated series expansion in time.

Choosing to approximate the signal sought by some truncated time series $\widetilde{X}$, the posterior statement for signal detection can be written in approximate form as

$$P(X/Y) \cong KP(\widetilde{X})\, L(\widetilde{X}) \qquad (4\text{-}1)$$

Of primary interest is that approximation of X which assumes X to be essentially constant over any one observation interval. The simplicity in implementing the requirements for a preprocessor resulting from such an approximate signal form is its major attribute.

As will be seen herein, it is most convenient to consider the constant X signal over any one observation interval to represent the time-average of X over this interval; that is, from equation (4-1), the posterior statement for signal detection for any one observation interval becomes

$$P(X/Y) \cong KP(\widetilde{X})L(\widetilde{X}) \stackrel{\Delta}{=} KP(\overline{X}^t)L(\overline{X}^t) \qquad (4\text{-}2)$$

with $\overline{X}^t$ representing the time-average of X over an observation interval.

The latter form of the equation (4-2) posterior probability leads to a simpler statement of the general Gaussian noise case. This we shall proceed to show. In addition, it will be shown that the time-average estimate can be interpreted as an estimate of the signal at the mid-point of an observation interval. With such signal mid-point estimates, it is then possible to employ a variety of interpolation schemes in a subsequent signal processing system.

4.1 TIME-AVERAGE ESTIMATE FOR BAND-LIMITED WHITE GAUSSIAN NOISE

We shall now develop the approximate posterior statement for signal detection in the presence of band-limited, white Gaussian noise with zero mean in terms of the time-average estimates.

The density function of the time-average of noise over interval (O, T) will then be Gaussian in form, since

$$\overline{N}^t \triangleq \frac{1}{T}\int_0^T N dt \tag{4-3}$$

is but a sum of terms each of which is Gaussian. Since any one sample of noise is assumed to have zero mean, the aforesaid sum will also have zero mean. Thus, the time-average noise density distribution will be of the form

$$p(\overline{N}^t) = K \exp\left\{-\frac{(\overline{N}^t)^2}{2\sigma^2_{\overline{N}^t}}\right\} \tag{4-4}$$

where only the variance $\sigma^2_{\overline{N}^t}$ must be determined.

A direct way of obtaining the variance of equation (4-4) is through use of the basic sampling theorem. Choosing to represent the noise function N by a finite set of samples taken at 1/2W equally spaced time points across (O, T), where W is the bandwidth of the noise spectrum, the time-average of noise appearing in equation (4-3) may be written as

$$\frac{1}{T}\int_0^T N dt = \frac{1}{2WT}\sum_{\nu=1}^{2WT} N_\nu \tag{4-5}$$

where N_ν is any one noise sample of the finite set. Assuming independent samples, the variance of this sum can be determined as

$$\sigma^2_{\overline{N}^t} = \left(\frac{1}{2WT}\right)^2 \sum_{\nu=1}^{2WT} \sigma^2_{N_\nu} = \left(\frac{1}{2WT}\right)^2 2WT(N_oW) = \frac{N_o}{2T} \tag{4-6}$$

where $N_o W$ is the variance of any one noise sample and N_o the mean noise power per unit bandwidth. Thus, the density function of equation (4-4) for the time-average of band-limited, white Gaussian noise becomes

$$p(\overline{N}^t) = K \exp\left\{-\frac{(\overline{N}^t)^2}{N_o/T}\right\} \tag{4-7}$$

Since the time-average of noise is by definition

$$\overline{N}^t \triangleq \overline{Y}^t - \overline{X}^t \tag{4-8}$$

where

$$\overline{X}^t \triangleq \frac{1}{T}\int_o^T X dt \qquad \text{and} \qquad \overline{Y}^t \triangleq \frac{1}{T}\int_o^T Y dt \tag{4-9}$$

then use of equation (4-7) in equation (4-2) gives the approximate posterior statement for signal detection in presence of this noise as

$$P(X/Y) \cong K\, P(\overline{X}^t) \exp\left\{-\frac{(\overline{Y}^t - \overline{X}^t)^2}{N_o/T}\right\} \tag{4-10}$$

As in previous work, we choose to simplify this expression. We may do this by examining the exponent term more closely. By expansion, this expression becomes

$$\frac{(\overline{Y}^t - \overline{X}^t)^2}{N_o/T} = \frac{1}{N_oT}\left[\int_o^T Y dt\right]^2 - \frac{2}{N_oT}\left[\int_o^T Y dt\right]\left[\int_o^T X dt\right] + \frac{1}{N_oT}\left[\int_o^T X dt\right]^2 \tag{4-11}$$

Since the integral of Y in practice will be given, the first term on the right-hand side of equation (4-11) may be absorbed into the constant K, thereby reducing equation (4-10) to

$$P(X/Y) \cong KP(\overline{X}^t) \exp\left\{- \frac{1}{N_o T}\left[\int_o^T X dt\right]^2\right\}$$

$$\exp\left\{\frac{2}{N_o T}\left[\int_o^T X dt\right]\left[\int_o^T Y dt\right]\right\} \qquad (4\text{-}12)$$

For convenience equation (4-12) is written as

$$P(X/Y) \cong KP(\overline{X}^t) \exp\left\{- \frac{(\overline{X}^t)^2}{N_o/T}\right\} \exp\left\{\frac{2\overline{X}^t}{N_o} \int_o^T Y dt\right\} \qquad (4\text{-}13)$$

that is, we desire an estimate of the time-average of X over some observation interval (O, T). Note that our results coincide, as they should, with the results for the constant signal case as given in section 2.0, equation (2-10). Identical to the constant signal case, therefore, the sufficiency conditions for a preprocessor remain as the integral of the signal-noise quantity Y over interval (O, T). The simplicity of this implementation again is our justification for the assumed approximate form for signal X.

4.2 TIME-AVERAGE ESTIMATE FOR THE GENERAL GAUSSIAN NOISE CASE

The approximate posterior statement for the general Gaussian noise case in terms of the time-average estimates can be directly expressed as

$$P(X/Y) \cong KP(\overline{X}^t) \exp\left\{- \frac{(\overline{X}^t + \mu_N)^2}{2\sigma^2_{\overline{N}^t}}\right\} \exp\left\{\frac{(\overline{X}^t + \mu_N)}{T\sigma^2_{\overline{N}^t}} \int_o^T Y dt\right\} \qquad (4\text{-}14)$$

where μ_N is the mean value of noise (the mean of any one noise sample being equal to the time-averaged mean of noise) and $\sigma^2_{\overline{N}^t}$ is again the variance of the time-averaged noise quantity. This variance is obtainable through

$$\sigma_{\bar{N}}^{2}t = \frac{2\sigma_N^{2}}{T^{2}} \int_{o}^{T} \int_{o}^{t} \rho_N(t)dt^{2} \qquad (4\text{-}15)$$

where σ_N^2 is the variance of any one noise sample and $\rho_N(t)$ is the noise correlation coefficient. Derivation of this expression for variance is shown for an arbitrary signal in Appendix D of this report.

Again note that the requirements for a preprocessor remain as the integral of the signal-noise quantity Y over some observation interval (O, T).

As a side interest, equation (4-14) is used to re-derive the approximate posterior statement for band-limited, white Gaussian noise. This is done in Appendix E of this report.

The approximate posterior statement of equation (4-14) is also obtainable directly from the general solution for correlated noise outlined in section 2.9 by equations (2-30) and (2-31). This is not shown herein, however.

4.3 TIME-AVERAGE ESTIMATE FOR PRE-DETECTION FILTERING

The heretofore developed posterior statement for time-average quantities is now extended to encompass detection problems involving a restricted class of pre-detection filters.

As stated in section 2. 3, we constrain ourselves to filters whose output is given by

$$X_R = XR(t) \qquad (4\text{-}16)$$

Some difficulty exists if we attempt to develop the posterior statement for such filter action directly as done in the previous section for R(t) equal to unity. This difficulty exists for R(t) represented by a Fourier series, that is ,

$$R(t) = \sum_{n=0}^{\infty} F_n \cos\left(\frac{2\pi n}{T} t + \theta_n\right) \tag{4-17}$$

leading to

$$X_R = X \sum_{n=0}^{\infty} F_n \cos\left(\frac{2\pi n}{T} t + \theta_n\right) \tag{4-18}$$

Replacement of $\overline{X}^t$ in the equation (4-8) time-average noise definition by the corresponding time-average of X_R sees this latter quantity to be zero for observation interval equal to multiples of carrier period (since X itself is assumed constant over the interval of interest). Disappearance of this term causes equal disappearance of X from the eventual posterior statement. Proceeding in this fashion, therefore, causes the time-average approach to break down.

Such difficulties can be avoided by working with $(\overline{N}^t)^2$ directly rather than with $\overline{N}^t$ itself. That is, by writing

$$(\overline{N}^t)^2 = \overline{\left[Y - X \sum_{n=0}^{\infty} F_n \cos\left(\frac{2\pi n}{T} t + \theta_n\right)\right]^2}^t - \sigma_{\overline{N}^t}^2 \tag{4-19}$$

$$(\overline{N}^t)^2 = \overline{Y^2}^t - \overline{2XY \sum_{n=0}^{\infty} F_n \cos\left(\frac{2\pi n}{T} t + \theta_n\right)}^t + \overline{X^2\left[\sum_{n=0}^{\infty} F_n \cos\left(\frac{2\pi n}{T} t + \theta_n\right)\right]^2}^t - \sigma_{\overline{N}^t}^2 \tag{4-20}$$

Since we are interested in these time-averages over some interval (O, T), equation (4-20) may be written as

$$(\overline{N}^t)^2 = \frac{1}{T}\int_0^T Y^2 dt - \frac{2}{T}\int_0^T XY \sum_{n=0}^{\infty} F_n \cos\left(\frac{2\pi n}{T}t + \theta_n\right)dt$$

$$+ \frac{1}{T}\int_0^T \left[X \sum_{n=0}^{\infty} F_n \cos\left(\frac{2\pi n}{T}t + \theta_n\right)\right]^2 dt - \sigma_N^2 t \qquad (4\text{-}21)$$

Moreover, since X is assumed constant over the interval of interest, where the constant value represents the time-average of X over the interval, equation (4-21) may be written as

$$(\overline{N}^t)^2 = \frac{1}{T}\int_0^T Y^2 dt - \frac{2\overline{X}^t}{T}\int_0^T Y \sum_{n=0}^{\infty} F_n \cos\left(\frac{2\pi n}{T}t + \theta_n\right)dt$$

$$+ \frac{(\overline{X}^t)^2}{T}\int_0^T \left[\sum_{n=0}^{\infty} F_n \cos\left(\frac{2\pi n}{T}t + \theta_n\right)\right]^2 dt - \sigma_N^2 t \qquad (4\text{-}22)$$

(The ambiguous use of $\overline{X}^t$ above can be somewhat misleading, though here it is meant to represent some constant value which reflects time-average of X over the interval.)

Insertion of equation (4-22) into the approximate posterior statement for time-average estimate of the equation (4-10) type results in

$$P(X/Y) \cong KP(\overline{X}^t) \exp\left\{\frac{1}{2} - \frac{1}{2T\sigma_N^2 t}\int_0^T Y^2 dt\right\} \exp\left\{- \frac{(\overline{X}^t)^2}{2T\sigma_N^2 t}\right.$$

$$\left.\int_0^T \left[\sum_{n=0}^{\infty} F_n \cos\left(\frac{2\pi n}{T}t + \theta_n\right)\right]^2 dt\right\} \cdot$$

$$\exp\left\{\frac{\overline{X}^t}{T\sigma_N^2 t}\int_0^T Y \sum_{n=0}^{\infty} F_n \cos\left(\frac{2\pi n}{T}t + \theta_n\right)dt\right\} \qquad (4\text{-}23)$$

With Y being given, the first exponent term can be absorbed into K, resulting in

$$P(X/Y) \cong KP(\overline{X}^t) \exp\left\{ - \frac{(\overline{X}^t)^2}{2T\sigma_{\overline{N}}^2 t} \int_0^T \left[\sum_{n=0}^{\infty} F_n \cos\left(\frac{2\pi n}{T}t + \theta_n\right) \right]^2 dt \right\}$$

$$\exp\left\{ \frac{\overline{X}^t}{T\sigma_{\overline{N}}^2 t} \int_0^T Y \sum_{n=0}^{\infty} F_n \cos\left(\frac{2\pi n}{T}t + \theta_n\right) dt \right\} \qquad (4\text{-}24)$$

This is a satisfactory posterior statement. (For simplicity sake, the noise mean value was assumed zero.)

Equation (4-24) calls for preprocessor requirements to be an infinite set of filters matched to the sinusoids of the harmonics representing the R(t) Fourier expansion. (In keeping with section 2. 3, it was again assumed that time behavior of R(t) is fully known.)

4.4 MID-POINT ESTIMATE

Much as the discussion in section 3. 0 on maximum likelihood, the subsequent work also relates to signal processing which might be done with those measures provided by the detection system preprocessor. It is given as added support to the time-average interpretation utilized heretofore in sections 4. 2 and 4. 3.

Primary value in such interpretation is that signal time-average over observation interval best reflects signal value at mid-point of observation interval. With such point estimates of signal available, various interpolation schemes are possible in a signal processing operation.

To show this, we draw from Appendix F the expression describing variance of difference between signal value at any time during an observation interval (O, T) and time-average of signal over this interval. The equation for this variance is

$$\sigma^2_{X-\overline{X}^t} = \sigma^2_X \left\{ 1 - \frac{2}{T}\int_0^t \rho_X(t)dt - \frac{2}{T}\int_0^{T-t} \rho_X(t)dt + \frac{2}{T^2}\int_0^T\int_0^t \rho_X(t)dt^2 \right\} \tag{4-25}$$

where $\rho_X(t)$ is the signal correlation coefficient.

As expected, the variance of equation (4-25) varies with time over the interval, depending upon what point in time is chosen for the single signal value.

From the nature of the expression, we would expect the variance to have a minimum at some point in the interval, with intuition suggesting the mid-point as the most likely place.

Referring to the derivation in Appendix F, we see the signal correlation coefficient used therein such that it is always either zero or positive valued, that is,

$$0 \leq \rho_X(t) \leq 1 \tag{4-26}$$

Thus, the integrals appearing in equation (4-25) will always be positive, monotonically increasing functions. This being so, the variance will then be least when the sum of the first two integrals is a maximum.

Owing to the condition in equation (4-26) and that in practical situations $\rho_X(t)$ will itself be monotonically decreasing over a prescribed interval (O, T), the sum of these integrals will be maximum when their upper limits are both one half the interval length (T/2). Thus, the minimum variance of the difference between a signal value over the interval and the signal time-average over this interval occurs at the mid-point, and is

$$\left[\sigma^2_{X-\overline{X}^t}\right]_{Min.} = \sigma_X^2 \left\{ 1 - \frac{4}{T}\int_0^{T/2} \rho_X(t)\, dt + \frac{2}{T^2}\int_0^T \int_0^t \rho_X(t) dt^2 \right\} \quad (4\text{-}27)$$

This we see is in agreement with our intuition heretofore stated. As an aside, the maximum variance occurs at the end points of interval (O, T), being

$$\left[\sigma^2_{X-\overline{X}^t}\right]_{Max.} = \sigma_X^2 \left\{ 1 - \frac{2}{T}\int_0^{T} \rho_X(t) dt + \frac{2}{T^2}\int_0^T \int_0^t \rho_X(t) dt^2 \right\} \quad (4\text{-}28)$$

The most important consequence of equation (4-27) is that, if the preprocessor is constrained to only the integral of the signal-noise quantity, then utility of this preprocessor measure in a subsequent signal processor is maximized when this single-integral measure is interpreted as an estimate of signal value at mid-point of observation interval.

Assuming such an interpretation in a signal processor then permits a variety of interpolation schemes to be employed therein. If boxcar interpolation were used, the result would be the step-like form (a form which itself could have been taken as the starting point for an alternate derivation of the approximate posterior statement).

Linear interpolation between these mid-point estimates could also be used to better represent signal time form, as well as some more sophisticated scheme predicated on the more ideal sin wt/t type interpolation.

Perhaps it is important here to point out that, much like the preprocessor has the task of maximizing relevant information from signal plus noise, so must the subsequent signal processor see that preprocessor output is most effectively utilized.

As further point, let us develop the posterior statement for mid-point estimate in terms of time-average quantities. We may do this by commencing with the joint probability statement involving mid-point X_M, signal time-average $\overline{X}^t$ and signal-noise time-average $\overline{Y}^t$. First writing the identities

$$P(X_M, \overline{X}^t, \overline{Y}^t) = P(X_M, \overline{X}^t)\, P(\overline{Y}^t / X_M \overline{X}^t)$$

$$= P(X_M, \overline{X}^t) P(\overline{Y}^t / \overline{X}^t)$$

$$= P(X_M, \overline{X}^t)\, L(\overline{X}^t) \tag{4-29}$$

and

$$P(X_M, \overline{X}^t, \overline{Y}^t) = P(\overline{Y}^t) P(X_M, \overline{X}^t / \overline{Y}^t) \tag{4-30}$$

where

$$P(\overline{Y}^t / X_M \overline{X}^t) = P(\overline{Y}^t / \overline{X}^t) = L(\overline{X}^t) \tag{4-31}$$

we may then form

$$P(X_M \overline{X}^t / \overline{Y}^t) = \frac{P(X_M, \overline{X}^t) L(\overline{X}^t)}{P(\overline{Y}^t)}$$

$$= K\, P(X_M, \overline{X}^t)\, L(\overline{X}^t) \tag{4-32}$$

where $P(\overline{Y}^t)$, with $\overline{Y}^t$ given, is absorbed into K.

The desired posterior statement then is a marginal probability of equation (4-32). This marginal probability may be formed by summing over all

$\overline{X}^t$. Denoting an arbitrary value of $\overline{X}^t$ by $\overline{X}^t_i$ and summing over all i, the desired posterior statement then becomes

$$P(X_M/\overline{Y}^t) = \sum_i P(X_M, \overline{X}^t_i/\overline{Y}^t)$$

$$= K \sum_i P(X_M, \overline{X}^t_i) \, L(\overline{X}^t_i) \tag{4-33}$$

Compared with equation (4-2), we see only the a priori probability for signal time-average being replaced by a joint probability and a request made for a summation over all possible signal time-averages.

As an oft repeated reminder, we again say that any manipulations involving interpretation, interpolation and the like in a subsequent signal processor in no way affect requirements for detection system preprocessor.

5.0 A PRIORI CONSIDERATIONS FOR APPROXIMATE MODEL

To consider a detection system for any application, its effectiveness ultimately must be demonstrated. Since that employed will invariably represent an approximation to the ideal set of requirements, as implied through the series solutions of section 2.0, it must be shown to be the best compromise between performance required and allowable equipment costs. The design should reflect maximum "performance-per-equipment-cost" figures of merit.

One detection scheme can be weighed against another through the simple expedient of comparing likelihood functions, with the means and variances thereof receiving attention. Mean value will indicate sensitivity of likelihood function to signal fluctuation, and variance the deviation about the mean. For the Gaussian type likelihood functions used in our previous work, consideration of these two parameters should suffice for such comparison.

Ultimately, however, comparison of likelihood functions must be supplemented with an evaluation of total system behavior, signal processor included. This is necessary to avoid heavy emphasis on consequential changes in likelihood function which have little effect on the a posteriori probability.

To do this indeed requires knowledge of the a priori distribution, something which may not be available. If not known, a best guess must be made regarding this distribution and a hypothetical model set forth, at least for analysis purposes. The model need not necessarily be the same a priori distribution underlying the ultimate signal processor functions, however.

Subsequent work will assume the a priori distribution to be Gaussian, meaning the a posteriori probability itself will also be Gaussian. The mean and variance of the posterior statement will receive primary attention. Here the mean will be shown to reflect sensitivity of the detection system to signal fluctuation, with variance describing behavior about the mean.

The approximate posterior statement based on time-average quantities developed in section 4.0 will be used in subsequent work. Interpretation of performance obtained therethrough will be presented. For reference sake, this expression is repeated below:

$$P(X/Y) \cong KP(\bar{X}^t) \exp\left\{-\frac{(\bar{X}^t + \mu_N)^2}{2\sigma^2_{\bar{N}^t}}\right\} \exp\left\{\frac{(\bar{X}^t + \mu_N)}{T\sigma^2_{\bar{N}^t}} \int_o^T Ydt\right\} \quad (5\text{-}1)$$

where μ_N is the mean value of noise and $\sigma^2_{\bar{N}^t}$ the variance of the time-average of noise given by

$$\sigma^2_{\bar{N}^t} = \frac{2\sigma_N^2}{T^2} \int_o^T \int_o^t \rho_N(t) dt^2 \quad (5\text{-}2)$$

with σ_N^2 the variance of any one noise sample and $\rho_N(t)$ the noise correlation coefficient.

5.1 GAUSSIAN A PRIORI DISTRIBUTIONS

We now set down several forms for the a priori distribution which will receive attention in the subsequent sections.

Where behavior over any one observation interval (O, T) adequately describes the signal stochastic process, the a priori Gaussian probability can be written in terms of its density function as

$$P(\bar{X}^t) = \frac{1}{\sqrt{2\pi}\,\sigma^2_{\bar{x}^t}} \exp\left\{-\frac{(\bar{X}^t - \mu_N)^2}{2\sigma^2_{\bar{x}^t}}\right\} d\bar{X}^t \quad (5\text{-}3)$$

where μ_x is the average value of the signal and

$$\sigma_{\bar{x}}^2 t = \frac{2\sigma_x^2}{T^2} \int_o^T \int_o^t \rho_x(t) dt^2 \tag{5-4}$$

with σ_x^2 the variance of the signal and $\rho_x(t)$ the signal correlation coefficient.

Where behavior over any one observation interval is also dependent upon behavior over previous interval, the second distribution for the Gaussian a priori probability applies, which becomes

$$P(\overline{X}_n^t, \overline{X}_{n-1}^t) = \frac{1}{2\pi\sigma_{\bar{x}}^2 t \sqrt{1-\rho_{\bar{x}}^2 t}} \exp\left\{-\frac{1}{2\sigma_{\bar{x}}^2 t\,(1-\rho_{\bar{x}}^2 t)}\right.$$

$$\left.\left[(\overline{X}_n^t - \mu_x)^2 - 2\rho_{\bar{x}} t\,(\overline{X}_n^t - \mu_x)(\overline{X}_{n-1}^t - \mu_x) + (\overline{X}_{n-1}^t - \mu_h)^2\right]\right\} d\overline{X}_n^t d\overline{X}_{n-1}^t \tag{5-5}$$

where $\overline{X}_n^t$ represents time-average over present interval, $\overline{X}_{n-1}^t$ time-average over previous interval, and $\rho_{\bar{x}} t$ is defined by

$$\rho_{\bar{x}^t} = \frac{\frac{\sigma_x^2}{T^2}\int_0^T\int_t^{t+T}\rho_x(t)dt^2}{\sigma_{\bar{x}^t}^2} = \frac{\int_0^T\int_t^{t+T}\rho_x(t)dt^2}{2\int_0^T\int_0^t\rho_x(t)dt^2} \tag{5-6}$$

The derivation of $\rho_{\bar{x}^t}$ may be found in Appendix H.

If the time-average is interpreted as the mid-point value of the signal, as per section 4.4, and if behavior over any one interval adequately describes the signal process, the a priori probability is now replaced by a joint probability in the mid-point value X_M and the time-average signal value $\bar{X}^t$. This is shown in equation (4-33) of section 4.4. For a Guassian X, thus joint probability is

$$P(X_M, \bar{X}^t) = \frac{1}{2\pi\sigma_x\sigma_{\bar{x}^t}\sqrt{1-\rho^2_{x_M,\bar{x}^t}}}\exp\left\{-\frac{1}{2\sigma_x\sigma_{\bar{x}^t}(1-\rho^2_{x_M,\bar{x}^t})}\left[(X_M-\mu_x)^2 - 2\rho_{x_M,\bar{x}^t}(X_M-\mu_x)(\bar{X}^t-\mu_x) + (\bar{X}^t-\mu_x)^2\right]\right\}dX_M d\bar{X}^t \tag{5-7}$$

where $\rho_{x_M,\bar{x}^t}$ is defined by

$$\rho_{x_M,\bar{x}^t} = \frac{\frac{2\sigma_x^2}{T}\int_0^T\rho_x(t)dt}{\sigma_x\sigma_{\bar{x}^t}} = \frac{2\int_0^{T/2}\rho_x(t)dt}{\sqrt{2\int_0^T\int_0^t\rho_x(t)dt^2}} \tag{5-8}$$

The derivation of $\rho_{x_M,\bar{x}^t}$ is presented in Appendix I.

5.2 GAUSSIAN A POSTERIORI DISTRIBUTIONS

We now present the a posteriori probabilities based on the time-average likelihood function of equation (5-1) for the three a priori distributions presented in section 5.1.

For the time-average Gaussian a priori distribution for any one observation interval given in equation (5-3), the posterior probability becomes

$$
\left.
\begin{aligned}
&P(X/Y) \cong \frac{1}{\sqrt{2\pi}\,\sigma_1} \exp\left\{-\frac{(\bar{X}^t - \mu_1)^2}{2\sigma_1^2}\right\} d\bar{X}^t \\
&\text{where} \\
&\mu_1 = \frac{\left[\sigma^2_{\bar{N}^t}\,\mu_x + \sigma^2_{\bar{x}^t}\left(\frac{1}{T}\int_o^T Y dt - \mu_N\right)\right]}{\left[\sigma^2_{\bar{x}^t} + \sigma^2_{\bar{N}^t}\right]} \\
&\text{and} \\
&\sigma_1^2 = \frac{\sigma^2_{\bar{x}^t}\,\sigma^2_{\bar{N}^t}}{\left[\sigma^2_{\bar{x}^t} + \sigma^2_{\bar{N}^t}\right]}
\end{aligned}
\right\} \qquad (5\text{-}9)
$$

Where present signal behavior is conditioned on behavior during previous observation interval as described through the joint statement of equation (5-5), the posterior statement can be written as

$$
\left.
\begin{aligned}
P(\bar{X}^t_n \mid \bar{X}^t_{n-1} Y) &\cong K P(\bar{X}^t_n \mid \bar{X}^t_{n-1}) \exp\left\{-\frac{(\bar{X}^t_n + \mu_N)^2}{2\sigma^2_{\bar{N}^t}}\right\} \\
&\quad \exp\left\{\frac{(\bar{X}^t_n + \mu_N)}{T\sigma^2_{\bar{N}^t}} \int_o^T Y dt\right\} \\
&= K P(\bar{X}^t_n, \bar{X}^t_{n-1}) \exp\left\{-\frac{(\bar{X}^t_n + \mu_N)^2}{2\sigma^2_{\bar{N}^t}}\right\} \\
&\quad \exp\left\{\frac{(\bar{X}^t_n + \mu_N)}{T\sigma^2_{\bar{N}^t}} \int_o^T Y dt\right\} \\
&= \frac{1}{\sqrt{2\pi}\,\sigma_2} \exp\left\{-\frac{(\bar{X}^t_n - \mu_2)^2}{2\sigma_2^2}\right\} d\bar{X}^t_n
\end{aligned}
\right\} \qquad (5\text{-}10)
$$

where

$$\mu_2 = \frac{\sigma^2_{\bar{N}t}(1-\rho_{\bar{x}t})\mu_x + \sigma^2_{\bar{N}t}\rho_{\bar{x}t}\bar{X}^t_{n-1} + \sigma^2_{\bar{x}t}(1-\rho^2_{\bar{x}t})\left(\frac{1}{T}\int_0^T Ydt - \mu_N\right)}{\sigma^2_{\bar{N}t} + \sigma^2_{\bar{x}t}(1-\rho^2_{\bar{x}t})}$$

$$\sigma_2^{\;2} = \frac{\sigma^2_{\bar{N}t}\,\sigma^2_{\bar{x}t}(1-\rho^2_{\bar{x}t})}{\sigma^2_{\bar{N}t} + \sigma^2_{\bar{x}t}(1-\rho^2_{\bar{x}t})} \tag{5-10}$$

If the time-average of the signal is interpreted as the mid-point value, then as per equation (4-33) in section 4.4 the posterior statement becomes

$$P(X_M|\bar{Y}^t) \cong K\sum_i P(X_M, \bar{X}^t_i)\exp\left\{-\frac{(\bar{X}^t_i+\mu_N)^2}{2\sigma^2_{\bar{N}t}}\right\}$$

$$\exp\left\{\frac{(\bar{X}^t_i+\mu_N)}{T\sigma^2_{\bar{N}t}}\int_0^T Ydt\right\} \tag{5-11}$$

Upon employing equation (5-7), we then have

$$P(X_M|\bar{Y}^t) \cong \frac{1}{\sqrt{2\pi}\,\sigma_3}\exp\left\{-\frac{(X_M-\mu_3)^2}{2\sigma_3^2}\right\}dX_M$$

where

$$\mu_3 = \mu_X + \frac{\rho_{X_M,\bar{X}^t}\,\sigma_X\sigma_{\bar{X}^t}\left(\frac{1}{T}\int_0^T Ydt - \mu_N - \mu_X\right)}{(\sigma_X\sigma_{\bar{X}^t} + \sigma_{\bar{N}^t}^{\;2})}$$

and

$$\sigma_3 = \frac{\sigma_X\sigma_{\bar{X}^t}\left[\sigma_X\sigma_{\bar{X}^t}(1-\rho^2_{X_M,\bar{X}^t}) + \sigma^2_{\bar{N}^t}\right]}{(\sigma_X\sigma_{\bar{X}^t} + \sigma^2_{\bar{N}^t})} \tag{5-12}$$

5.3 PERFORMANCE FOR TIME-AVERAGE A PRIORI DISTRIBUTION

Let us now examine performance of a system employing the step-like approximation of a signal quantity. We choose first the posterior statement of equation (5-9), which is based on a Gaussian a priori distribution for any one observation interval. Examination thereof of information obtained through detection can be made by utilizing the "information change" definition developed in Appendix G. Therein information change was defined as a fractional change in variance, that is,

$$IC \triangleq \frac{[\text{Prior Variance}] - [\text{Posterior Variance}]}{[\text{Prior Variance}]} \qquad (5\text{-}13)$$

Thus, using equation (5-9) the above equation becomes

$$IC_1 = \frac{\sigma_{\overline{X}}^2 t}{\sigma_{\overline{X}}^2 t + \sigma_{\overline{N}}^2 t} \qquad (5\text{-}14)$$

From equation (5-14) we see change in information through detection is zero in the limit where noise becomes infinite. Examination of the time-average variances also sees this information change become a constant less than unity as length of observation interval becomes infinite.

In practical situations, the information change definition should be satisfactory. The formal definitions of information gain (See Appendix G) should also be applicable with no foreseeable difficulty. The simple information change should usually suffice for design purposes, though the formal definitions should be used for extensive information flow studies regarding the overall system.

Of great interest also is an understanding of how well the detection system follows signal behavior. One way of examining this is to assume the system proper will choose the most probable value of posterior statement as the signal value during the observation interval. For the Gaussian posterior statements, this will be the average value thereof.

Now, signal will fluctuate during the observation interval with a standard deviation σ_X, whereas the time-average detected quantity gradually becomes equal to the sum of signal and noise means, as length of observation goes to infinity. Thus, in the limit, the time-average detected quantity is insensitive to signal fluctuations (or noise fluctuations, for that matter).

There are many ways in which sensitivity of detected quantity could be expressed. Perhaps one of the simplest, and still satisfactory methods is to define a sensitivity factor based on ratio of change in mean of detected quantity for change in signal time-average to standard deviation of signal, that is,

$$S \triangleq \frac{1}{\sigma_X} \frac{\partial \mu}{\partial \overline{X}^t} \, d\overline{X}^t \tag{5-15}$$

or for the posterior statement of equation (5-9),

$$S_1 = \frac{1}{\sigma_X} \frac{\partial \mu_1}{\partial \overline{X}^t} d\overline{X}^t = \left[\frac{\sigma^2_{\overline{X}} t}{\sigma^2_{\overline{X}} t + \sigma^2_{\overline{N}} t} \right] \frac{d\overline{X}^t}{\sigma_X} \tag{5-16}$$

To simplify matters, equation (5-16) can be examined at one standard deviation of $\overline{X}^t$, giving

$$S_1 = \left[\frac{\sigma^2_{\overline{X}} t}{\sigma^2_{\overline{X}} t + \sigma^2_{\overline{N}} t} \right] \frac{\sigma_{\overline{X}} t}{\sigma_X} \tag{5-17}$$

This sensitivity factor then gives us a direct feeling for how well time-average detected quantity responds to signal behavior during an observation interval (Note that as T approaches infinity S_1 goes to zero). Use of both the equation (5-15) sensitivity factor and the equation (5-13) information change can act as "rule-of-thumb" guides for the system designer.

5.4 PERFORMANCE FOR CONDITIONAL TIME-AVERAGE A PRIORI DISTRIBUTION

Use of the conditional time-average a priori distribution to develop the posterior statement is now examined. The posterior statement in question is given by equation (5-10).

Using the equation (5-13) definition of information change, we have

$$IC_2 = \frac{\sigma^2_{\overline{X}t}\,(1 - \rho^2_{\overline{X}t}) + \rho^2_{\overline{X}t}\,\sigma^2_{\overline{N}t}}{\sigma^2_{\overline{N}t} + \sigma^2_{\overline{X}t}\,(1 - \rho^2_{\overline{X}t})} \tag{5-18}$$

Here, it is seen that the change in information is heavily predicated on correlation in the signal quantity, since knowledge of time-average of signal during previous interval is presumed known.

A question might arise regarding where this knowledge of previous signal behavior is obtained. Generally such knowledge will not exist. That estimate of signal time-average determined during previous observation interval could be employed. Though in error, no serious difficulty can be envisioned in its use in a properly designed system.

The sensitivity factor of equation (5-15), evaluated at one standard deviation of signal time-average, for this posterior statement becomes

$$S_2 = \left[\frac{\sigma^2_{\overline{X}t}\,(1 - \rho^2_{\overline{X}t})}{\sigma^2_{\overline{N}t} + \sigma^2_{\overline{X}t}(1 - \rho^2_{\overline{X}t})}\right] \frac{\sigma_{\overline{X}t}}{\sigma_X} \tag{5-19}$$

Notice as the correlation coefficient approaches unity the sensitivity becomes zero. This is as it should be for heavily correlated signals.

5.5 PERFORMANCE FOR MID-POINT A PRIORI DISTRIBUTION

The posterior statement for the mid-point interpretation given in equation (5-12) leads to the following expressions for information change and sensitivity:

$$IC_3 = \frac{\sigma_{\overline{N}}^2 t \left[\sigma_X - \sigma_{\overline{X}} t\right] + \sigma_X \sigma_{\overline{X}} t \left[\sigma_X - \sigma_{\overline{X}} t \, (1 - \rho_{X_M, \overline{X}}^{\,2} t)\right]}{\sigma_X \left[\sigma_{\overline{X}} t + \sigma_{\overline{N}}^2 t\right]} \tag{5-20}$$

and

$$S_3 = \left[\frac{\rho_{X_M, \overline{X}} t \, \sigma_X \, \sigma_{\overline{X}} t}{\sigma_X \, \sigma_{\overline{X}} t + \sigma_{\overline{N}}^2 t}\right] \frac{\sigma_{\overline{X}} t}{\sigma_X} \tag{5-21}$$

These expressions are not examined to determine actual gain realized through such interpretation. This can be done by comparing the foregoing quantities with equation (5-14) and (5-17).

6.0 CONCLUDING REMARKS

The a posteriori probability approach, as developed by P. M. Woodward, was shown to lead directly to sufficiency conditions for a detection system; reduction of these conditions to practical form was shown to be possible by signal representation through series expansion.

Examination thereof for cases seeking signals with unknown parameters, especially those faced by a passive detection system, revealed the "normally assumed" basic requirements for a detection system to be an improper statement of the problem. Representation of an unknown signal by an infinite series with arbitrary coefficients clearly demonstrates this point. Through this representation, a restatement of these sufficiency conditions showed detection system requirements to be the formation of an infinite set of statistical estimates reflecting all information that can be obtained, from which the arbitrary coefficients can be formed. Formation of these arbitrary coefficients, however, was shown to be a task not of the detection system, but a task of some subsequent signal processing system.

Two series expansions received attention in this report: power series and Fourier series in time. The various sets of detection system requirements obtainable through these two series expansions alone provide the system designer with rather impressive guides by which to resolve a wide variety of problems. In fact, most detection system problems, both active and passive, should lend themselves to solution therewith.

Solutions through other series expansions are also possible. In a practical sense, however, one should choose those forms of expansion which converge most rapidly for the types of signals to be accommodated and whose solutions thereby lead to ready implementation. Rapid convergence implies only a few terms of the series expansion need be considered to approximate the ideal set of requirements.

The solutions given in this report revealed present practices as approximations thereof; the exact nature of these approximations is clearly expressed. Moreover, the form of solution given allows logical improvements to present practices.

The consequences of the reported work with regard to present and future activities cannot be stated at the present moment; this work must stand the test of time. Moreover, an enormous amount of labor remains to further explore this work, and especially to reduce it to practice.

7.0 REFERENCES

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APPENDIX A

REAL-TIME COMPUTER OPERATION

As background to the work presented in the body of this report, we examine a facet of the digital computer world which finds itself confronted with the basic problem of suitable signal characterization. Specifically, we discuss how a serial, general-purpose digital computer accommodates a typical real-time problem, and roughly relate how improved performance is obtainable through better signal representation, better representation through function expansions. The similarity between this computer problem and signal detection is very striking in many respects.

First, consider the typical general-purpose digital computer. The computer has stored in its memory section a set of functions $\{F\}$ which explicitly describe some real-time problem. These functions vary in accordance to a set of variables, $\{s\}$ signal set, which are provided by the outside world. The computer, under direction of a stored program P, operates sequentially on $\{F\}$ for a given set of $\{s\}$ values to generate another set of values representing desired output quantities $\{o\}$. The operation describing provision of an $\{o\}$ value set for a given $\{s\}$ value set is called the program compute cycle. With one set of $\{o\}$ values determined, the compute cycle is repeated for another $\{s\}$ value set. In real-time applications, a time sequence of such repeated operations (or compute cycles) constitutes a "running" solution of the problem.

The time length of compute cycle for a given general-purpose computer varies with problem complexity, as well as composition of program P dictated by a particular problem. Significant here is the fact that a fixed set of $\{s\}$ values is used for each compute cycle (that is, signals from the outside world are assumed constant during each compute interval). Any signal changes over the interval thus will not reflect in the computation, and accordingly the generated output set of $\{o\}$ values will be in error.

When these errors are of a cumulative nature, extremely serious limitations can be placed on the spectrum of real-time problems suitable to such computers.

In aerospace applications especially, present state-of-the-art operational equipment places rather high cost factors on equipment weight, volume, power, reliability and the like. Many times these equipment costs are unacceptable, and the digital computer is omitted from further consideration, or worse, system development marks time until equipment state-of-the-art becomes more favorable.

The foregoing situation will always constitute an upper limit on the spectrum of real-time problems suitable to serial, general-purpose operation as described above. As the state-of-the-art improves, the limit will be raised.

This then poses a question regarding the possibility of raising the upper limit with equipment state-of-the-art remaining constant. If equipment is fixed and if we also constrain ourselves to serial, general-purpose operation, the only consequential games we can play are with the signal set $\{s\}$. Fundamentally, the games played here involve a study in signal representation, or better, whether the normally assumed constant $\{s\}$ values for a compute interval is an efficient way of representing signals. Here, we see the mapping of many signal members (representing signal behavior over the time interval) into one member of another set. This many-to-one mapping is an irreversible, information-losing proposition. So long as the relevant information lost is negligible, the mapping is acceptable. Otherwise, we must either seek a more representable member for the many-to-one mapping or consider a many-to-many mapping. Desirably, this latter many-to-many mapping should be such that its membership for any one mapping operation is controllable and its members readily formed (that is, members formable at minimum equipment costs). Since increase in mapped membership will reflect decrease in relevant information lost in mapping, such control of membership should permit minimum complexity in signal representation consistent with allowable information loss.

Choosing to represent any function expansion of each $\{s\}$ over an arbitrary compute interval as a power series in time, it is then seen that the coefficients of this power series represent a new mapped multi-membered set. Its membership for any one mapping is controlled simply by specifying the number of non-zero coefficients. These power series could represent an expansion through an orthogonal function set, an approximation thereof, or even some truncated Taylor series. The coefficients themselves might be obtained in a variety of ways. They might be determined through past differences of signal samples or some other measure of past signal behavior (as integral measures, for example). If some computation delay is permissible, signal behavior over present interval of interest could be measured to determine these coefficients.

Most important are those power series expansions whose coefficients can be determined through integral measures of signal behavior, especially that set of measures representing successive integrals of signal over some time interval. This set can be readily, and cheaply formed in a digital computer through use of a periodic, high-speed adder (in Librascope phraseology, this high-speed adder is termed the Sigmator). Moreover, this set can be readily absorbed into the problem mathematical structure. The mathematics describing expansion through integral measures is related in Appendix B.

Such expansions have been shown to be extremely effective in certain real-time computer problem formulations, especially those involving integrals of more than one variable. Orders of magnitude improvement are realizable without altering the computer structure. Their utility was first intuitively seen some 5-6 years back by R. R. Williamson, and developed since then. The present writer himself probed numerous special cases involving such expansions during essentially the same time span. Each probe resulted in success. R. R. Williamson ultimately wrote a report titled

"Real-Time Control Computers" (Reference 5) in early 1962, wherein he indicated with reasonable mathematical substance formation of these expansions. This report coupled with a number of formal and informal investigative reports by the present writer served as the foundation for a report titled "Application of GP-Sigmator Hybrid to Real-Time Problems" (Reference 6), also prepared by the writer of this report, in late 1962.

APPENDIX B

FUNCTION EXPANSION THROUGH INTEGRAL MEASURES

Consider a variable X represented by an expansion over time interval (O, T) comprised of a linear set of orthonormal functions, that is,

$$X(O,T) = F_o \varphi_o(t) + F_1 \varphi_1(t) + \cdots + F_j \varphi_j(t) + \cdots$$

$$= \sum_{n=o}^{\infty} F_n \varphi_n(t) \tag{B-1}$$

where the $\varphi(t)$'s are the orthonormal functions and the F's constants defined by

$$F_j = \int_o^T \varphi_j(t)\, X(t)dt \tag{B-2}$$

Choosing to express any orthonormal set by their corresponding power series in time, the jth orthonormal function then becomes

$$\varphi_j(t) = \sum_{k=o}^{\infty} a_{jk} t^k \tag{B-3}$$

Then inserting this form into equation (B-2) gives

$$F_j = \int_o^T \left[\sum_{k=o}^{\infty} a_{jk} t^k \right] X(t)dt \tag{B-4}$$

Assuming a uniformly convergent series in interval (O, T), we may make use of the fact that the integral of an infinite sum is equivalent to the sum of the integrals. Thus, equation (B-4) can be written as

$$F_j = \sum_{k=o}^{\infty} a_{jk} \int_o^T t^k X(t)dt \tag{B-5}$$

The a_{jk} terms are known since they describe those orthonormal functions we happen to select. Only the integral terms remain to be determined. Direct implementation of these integrals could pose a serious problem since multiplication accompanies the integration. This multiplication can be circumvented, however, by the simple expedient of expanding the integrals through integration by parts. This we illustrate below:

$$\left.\begin{aligned}
F_0^* &= \int_0^T X(t)dt = \int_0^T X(t)dt \\
F_1^* &= \int_0^T tX(t)dt = T\int_0^T X(t)dt - \int_0^T\int_0^t X(t)dt^2 \\
F_2^* &= \int_0^T t^2X(t)dt = T^2\int_0^T X(t)dt - 2T\int_0^T\int_0^t X(t)dt^2 \\
&\quad + 2\int_0^T\int_0^t\int_0^t X(t)dt^3 \\
&\vdots
\end{aligned}\right\} \qquad \text{(B-6)}$$

Etc.

or the kth term can be expressed as

$$F_k^* = \sum_{i=1}^{k+1} (-)^{i+1} \frac{k!}{(k+1-i)!} T^{(k+1-i)} s^{-i}\left[X(t)\right] \qquad \text{(B-7)}$$

where $s^{-i}[X(t)]$ represents the ith integral of X(t) over interval (O, T).

The $\{F\}$ set of equation (B-1) can then be expressed in terms of the $\{F^*\}$ set of equation (B-6) through the matrix equation

$$[F] = [a][F^*] \qquad \text{(B-8)}$$

that is,

$$
\begin{bmatrix} F_0 \\ F_1 \\ F_2 \\ \vdots \\ F_j \\ \vdots \end{bmatrix} = \begin{bmatrix} a_{00} & a_{01} & a_{02} & \cdots & a_{0k} & \cdots \\ a_{10} & a_{11} & a_{12} & \cdots & a_{1k} & \cdots \\ a_{20} & a_{21} & a_{22} & \cdots & a_{2k} & \cdots \\ \vdots & \vdots & \vdots & & \vdots & \\ a_{j0} & a_{j1} & a_{j2} & \cdots & a_{jk} & \cdots \\ \vdots & \vdots & \vdots & & \vdots & \end{bmatrix} \begin{bmatrix} F_0^* \\ F_1^* \\ F_2^* \\ \vdots \\ F_k^* \\ \vdots \end{bmatrix} \tag{B-9}
$$

Thus, we see that the variable X(t) can in general be expressed by a power series in time of the form

$$
X(O, T) = A_0 + A_1 t + A_2 t^2 + \cdots + A_k t^k + \cdots \tag{B-10}
$$

where the A's are determined through integral measures of X(t) over interval (O, T), as reflected through equations (B-1), (B-3), (B-6), and (B-9). The particular relationship between the A's and the integral measures depends upon the function expansion chosen. In general form, however, the A's may be expressed as

$$
\left.\begin{aligned}
A_0 &= F_0 a_{00} + F_1 a_{10} + F_2 a_{20} + \cdots + F_j a_{j0} + \cdots \\
A_1 &= F_0 a_{01} + F_1 a_{11} + F_2 a_{21} + \cdots + F_j a_{j1} + \cdots \\
A_2 &= F_0 a_{02} + F_1 a_{12} + F_2 a_{22} + \cdots + F_j a_{j2} + \cdots \\
&\vdots \\
A_k &= F_0 a_{0k} + F_1 a_{1k} + F_2 a_{2k} + \cdots + F_j a_{jk} + \cdots \\
&\vdots
\end{aligned}\right\} \tag{B-11}
$$

or, since the F's are given by equation (B-9), the A's become

$$
\begin{bmatrix} A_0 \\ A_1 \\ A_2 \\ \vdots \\ A_k \\ \vdots \end{bmatrix} = \begin{bmatrix} a_{00} & a_{10} & a_{20} & \cdots & a_{j0} & \cdots \\ a_{01} & a_{11} & a_{21} & \cdots & a_{j1} & \cdots \\ a_{02} & a_{12} & a_{22} & \cdots & a_{j2} & \cdots \\ \vdots & \vdots & \vdots & & \vdots & \\ a_{0k} & a_{1k} & a_{2k} & \cdots & a_{jk} & \cdots \\ \vdots & \vdots & \vdots & & \vdots & \end{bmatrix} \cdot \begin{bmatrix} a_{00} & a_{01} & a_{02} & \cdots & a_{0k} & \cdots \\ a_{10} & a_{11} & a_{12} & \cdots & a_{1k} & \cdots \\ a_{20} & a_{21} & a_{22} & \cdots & a_{2k} & \cdots \\ \vdots & \vdots & \vdots & & \vdots & \\ a_{j0} & a_{j1} & a_{j2} & \cdots & a_{jk} & \cdots \\ \vdots & \vdots & \vdots & & \vdots & \end{bmatrix} \begin{bmatrix} F_0^* \\ F_1^* \\ F_2^* \\ \vdots \\ F_k^* \\ \vdots \end{bmatrix} \qquad \text{(B-12)}
$$

where the $\{F^*\}$ are defined by equation (B-7).

As stated in Appendix A, the integral terms appearing in the $\{F^*\}$ set are easily formed in a digital computer through use of a periodic, high-speed adder (Sigmator, in Librascope language). Since such an adder is part of the normal complement of a real-time digital computer, these power series expansions can be used to excellent advantage on real-time problems to enhance computer performance at essentially zero added cost.

In a digital computer and also in our detection problem, it is the form of the expansion which is important — not the specific expansion — this we can choose to best suit the problem. Simply stated, the power series form of expansion is important because it leads to simple implementation.

In our work on signal detection, little interest is shown in specific power series expansions. We primarily deal with the general form given in equation (B-10), under the tacit assumption that particular A's can be determined if desired.

APPENDIX C

AN ARGUMENTIVE RESOLUTION OF INTEGRAL XY

Relying heavily on the reasoning developed in Appendix A, we now endeavor to resolve integral XY in much the same way. To do this, we find it instructive to commence with the constant signal case presented in section 2.1 and given by equation (2-10).

If signal is known to be constant, detection system need only form the integral of Y. This integral constitutes an irreversible operation by the detection system which endeavors to provide a reasonable estimate of signal present. Herein, the integral of the signal-noise quantity Y is seen to represent a mapping of a many-membered set of time events (occurring during the specified time interval) into a single member of another set, where a member of the latter is not definable by a unique combination of members of the initial set. This mapping of an arbitrary time set of members into a single member is a dictate of the mathematics; it may be realized directly with physical equipment because Y happens to be the quantity available and integration something readily provided by the hardware.

Now, when we move to the more general case given in section 2.1 by equation (2-9), we see the mathematics requests a similar mapping, in this case the initial set of members each representing a subset of XY time events. This mapping may not be realized directly, however, since the XY subset is unavailable. It requires explicit knowledge of X: the very thing we seek. Thus, it appears that the exact solution is not directly attainable.

For argument sake, let us set exactness aside for the moment, and say a solution which is "reasonably exact" will be permitted. Under this qualification of exactness, we may reason an approximate solution wherein the approximation is predicated on a presumed mapping operation on the unknown quantity X. We need to choose a mapping on X, however, such that the required mapping of XY is realizable, that is, the integral of XY solvable.

Suppose for the present we consider a mapping of X, as described by a subset of time events over interval (O, T), into a single member of another set where this member represents a measure of X over interval (O, T). The simplest, and most obvious such member set is that one whose members represent that measure of X defined by the time-average of X over interval (O, T). This presumed mapping of X is irreversible, with information lost in the process. It would be acceptable as long as a sub-member of the initial set does not differ significantly from a chosen member in the final set. This is to say that such an assumed mapping of X is permitted if X is reasonably behaved, with reasonable behavior determined by some fidelity criterion (as allowable variance for deviation of X during interval from the average of X over the interval).

Let us now translate this rather simple argument to a general time function for X. Such a function is shown in figure C-1. Therein, we may divide an arbitrarily long period of time into nT sub-intervals, and assume the function can be approximated by a step-like function described by a constant value for each sub-interval. Each constant value may be the time-average of X over the corresponding sub-interval. This step-like approximation of X does indeed assume an irreversible mapping on X wherein some knowledge of X is lost. As figure C-1 indicates, however, so long as the length T of each sub-interval is chosen that a prescribed fidelity criterion is satisfied, the step-like form is a reasonable representation of X itself.

This being so, the posterior statement for the constant signal case given in section 2.1, equation (2-10) is then a fair statement for this arbitrary signal X during any one of the sub-intervals. Moreover, the integral of the signal-noise quantity Y reasonably satisfies the sufficiency conditions for a detection system, that is, the general integral XY is reasonably approximated.

At a glance, there appears to be nothing profound in the foregoing argument; use of the approximate solution therefrom would not be an

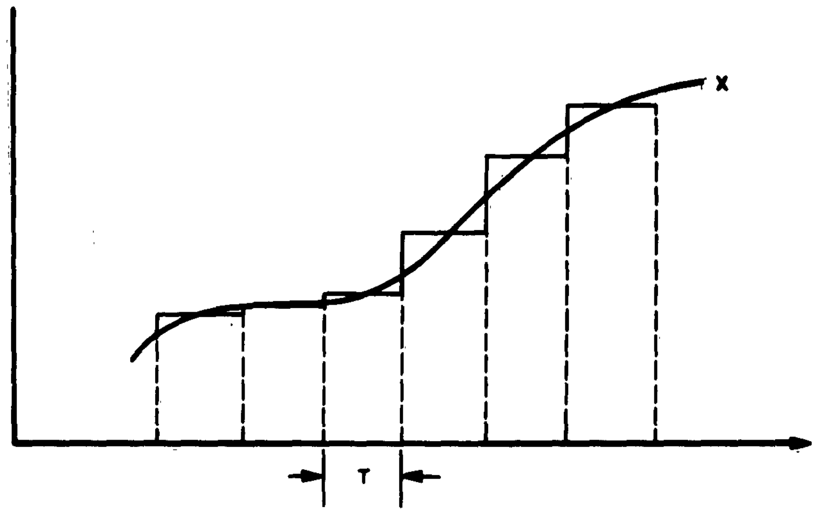

Figure C-1. Step-Like Approximation of General Time Function.

innovation — it has a long practiced history. Reflecting a moment, however, we see in our thinking the seed for general problem solution; it lies in our previous statement regarding a presumed mapping of X which is so chosen that the eventual mapping of XY is realizable, and hence the integral of XY solvable.

When we speak of a presumed mapping of X, we really are only seeking a more useful way of representing X. We see difficulty in forming the integral XY directly only because X is seen as an infinite set of unknown time samples. In a way, the task would seem less difficult if an arbitrary X were represented by a finite set of samples through use of the sampling theorem; formation of the required integrals involve sin wt/t type functions, however, not permitting direct implementation of this representation.

With an eye always on the practicality of implementing the required integration resulting from any new X representation, we might then consider the signal representation used in improving serial, general-purpose digital computer operation. Therein, a power series in time was employed as the general form to represent a signal, because of its ease in implementation.

If X is represented by a power series in time with arbitrary coefficients, the solution of the XY integral is direct. Its implementation, moreover, is straightforward, involving only successive integrals of Y. The arbitrary coefficients need not be prestated; they may be chosen at a later date.

Nor are we limited to power series alone; we may employ the Fourier series, for example. We see then the solution of the XY integral to call for an infinite set of filters matched to the sinusoid of each harmonic. The arbitrary coefficients of the Fourier series can again be chosen at a later date.

APPENDIX D

VARIANCE OF A TIME-AVERAGE QUANTITY

Suppose we request the variance of time-average of some quantity X as given by

$$\overline{X}^t = \frac{1}{T}\int_0^T X\,dt = \frac{1}{T}\sum_{\nu=0}^{\frac{T}{\Delta t}-1} X(\nu\Delta t)\Delta t \qquad \text{(D-1)}$$

wherein it is understood that the summation expression holds as Δt approaches zero.

Utilizing the infinite sum in equation (D-1) and also making use of the correlation coefficient of X, that is $\rho_x(t)$ where t represents the time separation between any two X values, the desired variance can be written in infinite sum form as

$$\begin{aligned}
\sigma^2_{\overline{x}^t} = {} & \left(\frac{\Delta t}{T}\right)^2\left[\sigma_o^{\,2} + \sigma_{\Delta t}^{\,2} + \sigma_{2\Delta t}^{\,2} + \cdots + \sigma_{(\nu-1)\Delta t}^{\,2}\right] \\
& + 2\sigma_o\left(\frac{\Delta t}{T}\right)^2\Big[\sigma_{\Delta t}\,\rho_x(\Delta t) + \sigma_{2\Delta t}\,\rho_x(2\Delta t) + \cdots \\
& \cdots + \sigma_{(\nu-1)\Delta t}\,\rho_x\big((\nu-1)\Delta t\big)\Big] \\
& + 2\sigma_{\Delta t}\left(\frac{\Delta t}{T}\right)^2\Big[\sigma_{2\Delta t}\,\rho_x(\Delta t) + \sigma_{3\Delta t}\,\rho_x(2\Delta t) + \cdots \\
& \cdots + \sigma_{(\nu-1)\Delta t}\,\rho_x\big((\nu-2)\Delta t\big)\Big] + \cdots \\
& \cdots + 2\sigma_{(\nu-3)\Delta t}\left(\frac{\Delta t}{T}\right)^2\left[\sigma_{(\nu-2)\Delta t}\,\rho_x(\Delta t) + \sigma_{(\nu-1)\Delta t}\,\rho_x(2\Delta t)\right] \\
& + 2\sigma_{(\nu-2)\Delta t}\left(\frac{\Delta t}{T}\right)^2\left[\sigma_{(\nu-1)\Delta t}\,\rho_x(\Delta t)\right] \qquad \text{(D-2)}
\end{aligned}$$

where σ_o^2, $\sigma_{\Delta t}^2$, $\sigma_{2\Delta t}^2$, $\cdots$, $\sigma_{(\upsilon-1)\Delta t}^2$ are the variances of X at times o, Δt, $2\Delta t$, $\cdots$, $(\upsilon-1)\Delta t$. The quantity υ is equal to $T/\Delta t$. Advantage was taken in the symmetry of the correlation coefficient.

Since the sum involves the same variable, then all variances appearing in equation (D-2) are equal, that is,

$$\sigma_x^2 = \sigma_o^2 = \sigma_{\Delta t}^2 = \sigma_{2\Delta t}^2 = \cdots = \sigma_{(\upsilon-1)\Delta t}^2 \tag{D-3}$$

Employing this fact reduces equation (D-2) to

$$\begin{aligned}\sigma_{\bar{x}}^2 t = &\left(\frac{\Delta t}{T}\right)\sigma_x^2 + 2\sigma_x^2\left(\frac{\Delta t}{T}\right)^2\left[\rho_x(\Delta t) + \rho_x(2\Delta t) + \cdots + \rho_x\Big((\upsilon-1)\Delta t\Big)\right] \\ &+ 2\sigma_x^2\left(\frac{\Delta t}{T}\right)^2\left[\rho_x(\Delta t) + \rho_x(2\Delta t) + \cdots + \Big(\rho_x(\upsilon-2\Delta t)\Big)\right] + \cdots \\ &\cdots + 2\sigma_x^2\left(\frac{\Delta t}{T}\right)^2\left[\rho_x(\Delta t) + \rho_x(2\Delta t)\right] \\ &+ 2\sigma_x^2\left(\frac{\Delta t}{T}\right)^2\left[\rho_x(\Delta t)\right]\end{aligned} \tag{D-4}$$

which may be written as

$$\begin{aligned}\sigma_{\bar{x}}^2 t = &\left(\frac{\Delta t}{T}\right)\sigma_x^2 + 2\sigma_x^2\left(\frac{\Delta t}{T^2}\right)\Bigg\{\sum_{n=1}^{1}\rho_x(n\Delta t)\Delta t \\ &+ \sum_{n=1}^{2}\rho_x(n\Delta t)\Delta t + \cdots + \sum_{n=1}^{\left(\frac{T}{\Delta t}-1\right)\Delta t}\rho_x(n\Delta t)\Delta t\Bigg\}\end{aligned} \tag{D-5}$$

and further as

$$\sigma_{\bar{x}}^2 t = \left(\frac{\Delta t}{T}\right)\sigma_x^2 + \frac{2\sigma_x^2}{T^2}\sum_{\upsilon=1}^{\frac{T}{\Delta t}-1}\sum_{n=1}^{\upsilon}\rho_x(n\Delta t)\Delta t^2 \tag{D-6}$$

In the limit as Δt approaches zero, the first term on the right-hand side of equation (D-6) goes to zero and the second term can be replaced by a double integral, that is, the desired variance becomes

$$\sigma^2_{\bar{x}t} = \frac{2\sigma_x^2}{T^2} \int_0^T \int_0^t \rho_x(t) dt^2 \qquad \text{(D-7)}$$

As an aside, Lee *gives an equivalent form of equation (D-7) as

$$\sigma^2_{\bar{x}t} = \sigma_x^2 \int_{-t}^{T} \frac{T-|t|}{T^2} \rho_x(t) dt \qquad \text{(D-8)}$$

wherein his notation has been altered to conform with ours. The equivalence is seen by rewriting equation (D-8) as

$$\sigma^2_{\bar{x}t} = 2\sigma_x^2 \int_0^T \frac{T-t}{T^2} \rho_x(t) dt = \frac{2\sigma_x^2}{T} \int_0^T \rho_x(t) dt - \frac{2\sigma_x^2}{T^2} \int_0^T t\rho_x(t) dt \qquad \text{(D-9)}$$

Employing integration by parts on the second term, we have

$$\frac{2\sigma_x^2}{T^2} \int_0^T t\rho_x(t) dt = \frac{2\sigma_x^2}{T} \int_0^T \rho_x(t) dt - \frac{2\sigma_x^2}{T^2} \int_0^T \int_0^t \rho_x(t) dt^2 \qquad \text{(D-10)}$$

Substituting equation (D-10) into equation (D-9) shows the equivalence

$$\sigma^2_{\bar{x}t} = \sigma_x^2 \int_{-T}^{T} \frac{T-|t|}{T^2} \rho_x(t) dt^2 = \frac{2\sigma_x^2}{T^2} \int_0^T \int_0^t \rho_x(t) dt^2 \qquad \text{(D-11)}$$

* **Reference 8 (Lee), p. 286.**

APPENDIX E

ALTERNATE DERIVATION OF TIME-AVERAGE ESTIMATE FOR BAND-LIMITED WHITE GAUSSIAN NOISE

Of some interest is an alternate derivation of the approximate posterior statement for signal detection in presence of band-limited, white Gaussian noise with zero mean in terms of the time-average estimates. Reference here is to the approximate posterior statement given in section 4.2, equation (4-14). This equation is repeated below:

$$P(X/Y) \cong KP(\overline{X}^t) \exp\left\{-\frac{(\overline{X}^t+\mu_N)^2}{2\sigma_N^2 t}\right\} \exp\left\{\frac{(\overline{X}^t+\mu_N)}{T\sigma_N^2 t}\int_o^T Y dt\right\} \qquad \text{(E-1)}$$

With noise having zero mean, μ_N equals zero. Only $\sigma_N^2 t$ remains to be determined. This variance quantity was defined in equation (4-15) of section 4.2 as

$$\sigma_N^2 t = \frac{2\sigma_N^2}{T^2}\int_o^T\int_o^t \rho_N(t)dt^2 \qquad \text{(E-2)}$$

The correlation coefficient $\rho_N(t)$ in equation (E-2) is obtainable by normalizing the autocovariance for the noise quantity, that is,

$$\rho_N(t) = \frac{\phi_N(t)}{\phi_N(0)} = \frac{\phi(t)}{\sigma_N^2} = \frac{1}{\sigma_N^2}\int_{-\infty}^{\infty}\Phi_N(\omega)\exp\left\{j\omega t\right\}d\omega \qquad \text{(E-3)}$$

where $\Phi_N(\omega)$ is the noise power density spectrum given by

$$\Phi_N(\omega) = \frac{\sigma_N^2}{4\pi W} \qquad (-2\pi W \le \omega \le 2\pi W) \qquad \text{(E-4)*}$$

Thus, equation (E-3) reduces to

$$\rho_N(t) = \frac{1}{2\pi W}\int_{-2\pi W}^{2\pi W}\exp\left\{j\omega t\right\}d\omega = \frac{1}{\pi W}\left[\frac{\sin 2\pi Wt}{t}\right] \qquad \text{(E-5)}$$

* Discussion of band-limited, white Gaussian noise may be found in Reference 9 (Bennett).

Upon substituting equation (E-5) into equation (E-2), we have

$$\sigma^2_{\overline{N}^t} = \frac{2\sigma_N^2}{T^2}\ \frac{1}{\pi W}\int_o^T\!\!\int_o^t \frac{\sin 2\pi Wt}{t}\,dt^2 = \frac{N_o}{\pi T^2}\int_o^T\left\{\int_o^t \frac{\sin 2\pi Wt}{t}dt\right\}dt \qquad \text{(E-6)}$$

where $\sigma_N^2 = N_o W$.

The results of equation (E-6) point to some inconsistency in assuming a finite bandwidth and independent noise samples. This inconsistency is considered more of academic interest, however, brought about by the noise model chosen. Normal interpretation of equation (E-6) would consider the case where bandwidth approaches infinity (analogous to zero correlation between samples), that is,

$$\lim_{W\to\infty}\int_o^t \frac{\sin 2\pi Wt}{t}\,dt = \frac{\pi}{2} \qquad \text{(E-7)}$$

or at least a very large W where equation (E-7) holds nearly always. Inserting this result into equation (E-6) provides

$$\sigma^2_{\overline{N}^t} = \frac{N_o}{2T^2}\int_o^T dt = \frac{N_o}{2T} \qquad \text{(E-8)}$$

Utilizing this quantity in equation (E-1) gives the approximate posterior statement as

$$P(X/Y) \cong KP(\overline{X}^t)\exp\left\{-\frac{(\overline{X}^t)^2}{No/T}\right\}\exp\left\{\frac{2\overline{X}^t}{N_o}\int_o^T Y dt\right\} \qquad \text{(E-9)}$$

which is in exact agreement with the previously determined result given in section 4.1 by equation (4-13).

APPENDIX F

ERROR IN STEP-LIKE REPRESENTATION OF GENERAL TIME FUNCTION

Let us determine the probable error in representing a general time function X(t) in step-like form. This representation is sketched in figure F-1. As shown, X(t) is approximated by a constant value for each T-second time interval. More specifically, these constant values are deemed to constitute the time-averages of X(t) for the corresponding time intervals.

We now inquire into the error resulting from such approximations. To satisfy our curiosity we need only concentrate on one arbitrary T-second interval. Such an interval is illustrated in figure F-2. For convenience, the time base thereon is now denoted by a starting time t equal to zero coinciding with the beginning of the chosen T interval.

The general error expression for the approximation during the arbitrary time interval of figure F-2 may be written as

$$\epsilon_{x-\bar{x}}t = X(t) - \overline{X}^t$$

$$= X(t) - \frac{1}{T}\int_0^T X(t)dt \qquad \text{(F-1)}$$

Let us assume this error to be sufficiently characterized if we ascertain its mean value and its variance. Accepting this, we first rewrite equation (F-1) to represent the integral by an infinite sum. We may do this by subdividing the interval into $T/\Delta t$ equal time increments, giving

$$\epsilon_{x-\bar{x}}t = X(t) - \frac{1}{T}\sum_{\nu=0}^{\frac{T}{\Delta t}-1} X(\nu\Delta t)\Delta t \qquad \text{(F-2)}$$

where it is understood that this equation is valid in the limit as Δt approaches zero, and that t ranges between zero and T.

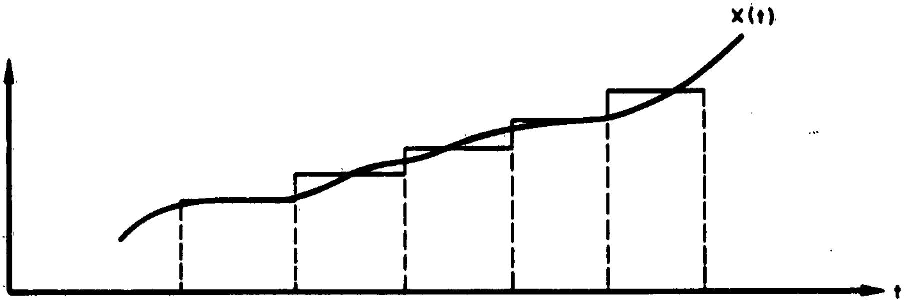

Figure F-1. Step-Like Representation of General Time Function.

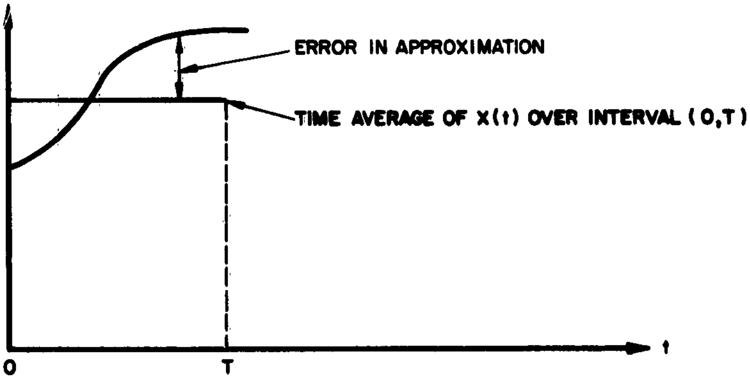

Figure F-2. Arbitrary Time Interval Describing Time Process.

The form of equation (F-2) now readily lends itself to a determination of the mean and variance of the error.

Since the mean of a sum or difference is equal to the sum or difference of the individual means, the error mean may be written as

$$\overline{\epsilon}_{x-\overline{x}^t} = \overline{X(t)} - \frac{\Delta t}{T}\left[\overline{X(0)} + \overline{X(\Delta t)} + \overline{X(2\Delta t)} + \cdot \cdot \cdot + \overline{X((\nu-1)\Delta t)}\right]$$

$$= \mu_t - \frac{\Delta t}{T}\left[\mu_o + \mu_{\Delta t} + \mu_{2\Delta t} + \cdot \cdot \cdot + \mu_{(\nu-1)\Delta t}\right] \qquad \text{(F-3)}$$

where μ_t and μ_o, $\mu_{\Delta t}$, $\mu_{2\Delta t}$, $\cdot \cdot \cdot$, $\mu_{(\nu-1)\Delta t}$ are the mean values of X(t) at times t and o, Δt, $2\Delta t$, $\cdot \cdot \cdot$, $(\nu-1)\Delta t$ respectively. Assuming a stationary stochastic process, all means appearing in equation (F-3) are the same, being the mean of X(t). Denoting this mean by μ_x allows reduction of equation (F-3) to

$$\mu_\epsilon = \overline{\epsilon}_{x-\overline{x}^t} = \mu_x - \frac{\Delta t}{T}\,\nu\mu_x = \mu_x - \mu_x = 0 \qquad \text{(F-4)}$$

Proceeding in a like manner we may write the expression for the variance of equation (F-2) as

$$\sigma_\epsilon^{\,2} = \sigma_t^{\,2} + \left(\frac{\Delta t}{T}\right)^2 \left[\sigma_o^{\,2} + \sigma_{\Delta t}^{\,2} + \sigma_{2\Delta t}^{\,2} + \cdot \cdot \cdot + \sigma_{(\nu-1)\Delta t}^{\,2}\right] \cdot$$

$$- 2\sigma_t\left(\frac{\Delta t}{T}\right)\Big[\sigma_o\,\rho_x(t) + \sigma_{\Delta t}\,\rho_x(t-\Delta t) + \sigma_{2\Delta t}\,\rho_x(t-2\Delta t) + \cdot \cdot \cdot$$

$$\cdot \cdot \cdot + \sigma_{t-\Delta t}\,\rho_x(\Delta t) + \sigma_t\,\rho_x(0)\Big]$$

$$- 2\sigma_t\left(\frac{\Delta t}{T}\right)\Big[\sigma_{t+\Delta t}\,\rho_x(\Delta t) + \sigma_{t+2\Delta t}\,\rho_x(2\Delta t) + \cdot \cdot \cdot$$

$$\cdot \cdot \cdot + \sigma_{(\nu-1)\Delta t}\,\rho_x\Big((\nu-1)\Delta t - t\Big)\Big]$$

$$+ 2\sigma_o\left(\frac{\Delta t}{T}\right)^2\Big[\sigma_{\Delta t}\,\rho_x(\Delta t) + \sigma_{2\Delta t}\,\rho_x(2\Delta t) + \cdot \cdot \cdot$$

$$\cdot \cdot \cdot + \sigma_{(\nu-1)\Delta t}\,\rho_x\Big((\nu-1)\Delta t\Big)\Big]$$

$$+ 2\sigma_{\Delta t}\left(\frac{\Delta t}{T}\right)^2 \Big[\sigma_{2\Delta t}\rho_x(\Delta t) + \sigma_{3\Delta t}\rho_x(2\Delta t) + \cdots$$

$$\cdots + \sigma_{(\nu-1)\Delta t}\rho_x((\nu-2)\Delta t)\Big] + \cdots$$

$$\cdots + 2\sigma_{(\nu-3)\Delta t}\left(\frac{\Delta t}{T}\right)^2\Big[\sigma_{(\nu-2)\Delta t}\rho_x(\Delta t) + \sigma_{(\nu-1)\Delta t}\rho_x(2\Delta t)\Big]$$

$$+ 2\sigma_{(\nu-2)\Delta t}\left(\frac{\Delta t}{T}\right)^2\Big[\sigma_{(\nu-1)\Delta t}\rho_x(\Delta t)\Big] \qquad \text{(F-5)}$$

wherein advantage is taken of the fact that the correlation coefficient $\rho_x(t)$ is an even function in t (symmetric about t equal zero). Now all σ values appearing in equation (F-5) are the same, where the common value will be denoted by σ_x. Hence, equation (F-5) can be written in more compact form as

$$\sigma_\epsilon^2 = \sigma_x^2 + \left(\frac{\Delta t}{T}\right)\sigma_x^2 - \frac{2\sigma_x^2}{T}\sum_{\nu=0}^{t/\Delta t}\rho_x(\nu\Delta t)\Delta t - \frac{2\sigma_x^2}{T}\sum_{\nu=1}^{\frac{T-t-\Delta t}{\Delta t}}\rho_x(\nu\Delta t)\Delta t \qquad \text{(F-6)}$$

$$+ 2\frac{\sigma_x^2}{T^2}\Delta t\left[\sum_{\nu=1}^{1}\rho_x(\nu\Delta t)\Delta t + \sum_{\nu=1}^{2}\rho_x(\nu\Delta t)\Delta t + \cdots + \sum_{\nu=1}^{\frac{T}{\Delta t}-1}\rho_x(\nu\Delta t)\Delta t\right]$$

which can be further compressed as

$$\sigma_\epsilon^2 = \sigma_x^2 + \left(\frac{\Delta t}{T}\right)\sigma_x^2 - \frac{2\sigma_x^2}{T}\sum_{\nu=0}^{t/\Delta t}\rho_x(\nu\Delta t)\Delta t - \frac{2\sigma_x^2}{T}\sum_{\nu=1}^{\frac{T-t-\Delta t}{\Delta t}}\rho_x(\nu\Delta t)\Delta t$$

$$+ \frac{2\sigma_x^2}{T^2}\sum_{\nu=1}^{\frac{T}{\Delta t}-1}\sum_{\eta=0}^{\nu}\rho_x(\eta\Delta t)\Delta t^2 \qquad \text{(F-7)}$$

Allowing Δt in equation (F-7) to approach zero in the limit permits the sums to be replaced by integrals, and causes the second term to go to zero, that is,

$$\sigma_\epsilon^2 = \sigma_x^2\left\{1 - \frac{2}{T}\int_0^t \rho_x(t)dt - \frac{2}{T}\int_0^{T-t}\rho_x(t)dt + \frac{2}{T^2}\int_0^T\int_0^t \rho_x(t)dt^2\right\} \qquad \text{(F-8)}$$

Since the variance of equation (F-8) varies over the interval, it is of interest to determine its minimum and maximum values. Note $\rho_x(t)$, as utilized in the derivation, is always zero or positive valued, that is,

$$0 \leq \rho_x(t) \leq 1 \tag{F-9}$$

Hence, all integrals appearing in equation (F-8) are positive, monotonically increasing functions. Moreover, from the structure of this equation we see the minimum will occur when the sum of the first two integrals is a maximum. Since in practical situations $\rho_x(t)$ itself will be a monotonically decreasing function over prescribed interval of length T, this sum has its maximum when their upper limits are both equal to T/2, or the mid-point of the interval. Thus, the minimum variance of the error becomes

$$\left[\sigma_\epsilon{}^2\right]_{Min} = \sigma_x{}^2\left\{1 - \frac{4}{T}\int_0^{T/2}\rho_x(t)dt + \frac{2}{T^2}\int_0^T\int_0^t\rho_x(t)dt^2\right\} \tag{F-10}$$

Through similar reasoning, it can be shown that the maximum variance occurs at the end-points of the interval, having a value

$$\left[\sigma_\epsilon{}^2\right]_{Max} = \sigma_x{}^2\left\{1 - \frac{2}{T}\int_0^{T}\rho_x(t)dt + \frac{2}{T^2}\int_0^T\int_0^t\rho_x(t)dt^2\right\} \tag{F-11}$$

The conclusions of equation (F-10) and (F-11) are in agreement with our intuitive notions. We would expect the error to be least at the mid-point and largest at the end-points.

As a final item, let us restate equation (F-10) for a Taylor series representation of $\rho_x(t)$. This form leads to ready interpretation, especially since the behavior of $\rho_x(t)$ over the range of interest often reduces to two or three terms of the Taylor series. The general Taylor series for $\rho_x(t)$ expanded about t equal zero is

$$\rho_x(t) = \rho_x^{(0)} + \rho_x^{(1)}\frac{t}{1!} + \rho_x^{(2)}\frac{t^2}{2!} + \cdots = \sum_{n=0}^{\infty}\rho_x^{(n)}\frac{t^n}{n!} \tag{F-12}$$

where

$$\overset{(n)}{\rho_x} = \frac{\partial^n}{\partial t^n} \rho_x(0) \qquad \text{(F-13)}$$

with

$$\overset{(0)}{\rho_x} = 1 \qquad \text{(F-14)}$$

and where it is understood the derivatives are determined by

$$\overset{(n)}{\rho_x} = \lim_{\epsilon \to o} \frac{\partial^n}{\partial t^n} \rho_x(\epsilon) \qquad \text{(F-15)}$$

with ϵ being positive. This right-hand interpretation of the derivatives circumvents arguments which might arise because derivatives at t equal zero may be undefined, since $\rho_x(t)$ is even, $\rho_x(0)$ is a maximum, and a point of discontinuity can occur at t equal zero. Actual concern with equation (F-15) is not envisioned in general application, however.

Utilizing equation (F-12) in equation (F-10) and integrating results in

$$\left[\sigma_\epsilon^2\right]_{Min} = \sigma_x^2 \left\{ 1 - \frac{4}{T} \sum_{n=o}^{\infty} \overset{(n)}{\rho_x} \left(\frac{T}{2}\right)^{n+1} \frac{1}{(n+1)!} \right.$$

$$\left. + \frac{2}{T^2} \sum_{(n+1)!}^{\infty} \overset{(n)}{\rho_x} (T)^{n+2} \frac{1}{(n+2)!} \right\}$$

$$= 2\sigma_x^2 \sum_{n=1}^{\infty} \overset{(n)}{\rho_x} T^n \frac{1}{(n+1)!} \left[\frac{1}{n+2} - \frac{1}{2^n}\right] \qquad \text{(F-16)}$$

wherein use is made of equation (F-13).

Since equation (F-13) always is true, the expression of equation (F-16) is directly applicable to a $\rho_x(t)$ comprised of a Taylor series representable function plus a delta function. The delta function would of course indicate presence of white noise. It only affects the $\rho_x^{(0)}$ term, however, which we see to be a constant equal to unity: hence the validity of equation (F-16).

APPENDIX G

INFORMATION GAIN

As we ultimately desire to evaluate effectiveness of detection operation, some measure of information gain would be most appropriate. Certain mathematical difficulties arise, however, with regard to information gain measures for continuous distributions.

In general, some feeling for information gain can be obtained by comparing some function of the prior variance with the same function of the posterior variance. This feeling can be mathematically expressed through the classical definition of entropy change, which reflects an average measure of information gain.

Since entropy for a continuous probability density distribution is defined by

$$H(X) = -\int_{\infty}^{\infty} p(X) \log p(X)\, dX \tag{G-1}$$

This definition may be used to express average information gain as the entropy differential between the a priori distribution $p(X)$ and the a posterior distribution $p(X/Y)$, that is,

$$\begin{aligned} \Delta H &= H(X) - H(X/Y) \\ &= -\int_{-\infty}^{\infty} p(X) \log p(X) dX + \int_{-\infty}^{\infty} p(X/Y) \log p(X/Y) dX \end{aligned} \tag{G-2}$$

Reza* refers to this entropy change as "transinformation".

To reflect some of the unattractive behavior of equation (G-2), let us examine the case where the distributions are Gaussian. Assuming σ_x^2 to be the a priori variance and $\sigma^2_{X/Y}$ to be the a posteriori variance, equation (G-2) becomes

* Reference 10 (Reza).

$$\Delta H = \log\left[\sqrt{2\pi e}\ \sigma_x\right] - \log\left[\sqrt{2\pi e}\ \sigma_{x/y}\right]$$

$$= \log\left[\frac{\sigma_x}{\sigma_{x/y}}\right] \qquad \text{(G-3)}$$

The normal argument associated with equation (G-3) states that one half the logarithm of the ratio of the a priori to the a posteriori variance is an average measure of information gain. A reduction, therefore, in the posterior variance from the prior variance results in a positive entropy change denoting a gain in information. Generally the equation (G-3) definition finds effective usage for continuous signal data, if care is taken to avoid some of its mathematical difficulty.

One such difficulty arises when the posterior variance becomes zero. The result is an infinite positive change in entropy, implying infinite information gain. So long as infinite entropy change is understood to mean that all information has been obtained, the definition is satisfactory. As a numerical measure where the posterior variance approaches zero, it has no quantitative meaning, however.

Actually the foregoing difficulty reflects shortcomings in extending the definition of entropy for discrete distributions to the continuous. We might say the mathematics breaks down. Entropy for continuous distributions can run the spectrum from minus infinity to plus infinity, thereby losing some of its significance in measuring information content. In a sense, equation (G-3) is only half as bad in signal detection, in that entropy change will only cover the plus side of the spectrum (that is, so long as interpretation of information does not take place). The plus infinity point usually occurs in the limit when the continuous density function becomes a discrete probability.* In equation (G-3) this occurs when the a posteriori probability becomes a delta function.

* Reference 10 (Reza), Chapter 8.

Rather than the average information gain definition of equation (G-3), information gain which is not averaged can be obtained directly from the basic definition of information, that is, by

$$IG = \log \frac{p(X)}{p(X/Y)} \tag{G-3}$$

or the prior information minus the posterior information. Again as for equation (G-3), continuous distributions can cause this expression to become infinite.

To avoid the foregoing, we need an expression for information gain whose boundaries are zero and some direct measure of actual information remaining to be gained. If we dispense with logarithmic behavior in such an expression, the straightforward relation of fractional difference between prior and posterior variance could be used, that is,

$$IC \triangleq \frac{[\text{Prior Variance}] - [\text{Posterior Variance}]}{[\text{Prior Variance}]} \tag{G-4}$$

where IC means information change (that is, a change in the information to be gained). Thus, when no information is gained, the quantity is zero; when all information is gained the quantity is unity. We should refrain from calling equation (G-4) information gain, however.

Though the equation (G-4) has its deficiencies, it is nonetheless simple and effective for comparative purposes in analysis. At least it is well behaved.

APPENDIX H

CORRELATION COEFFICIENT FOR SECOND PROBABILITY DISTRIBUTION INVOLVING TIME-AVERAGE OF SIGNAL

Let us derive the expression for the correlation coefficient appearing in a Gaussian second probability distribution for the time-average of a signal. In general form, this correlation coefficient is defined by

$$\rho_{\bar{x}^t} = \frac{\overline{\left(\bar{x}^t - \mu_{\bar{x}^t_n}\right)\left(\bar{x}^t_{n-1} - \mu_{\bar{x}^t_{n-1}}\right)}}{\sigma_{\bar{x}^t_n}\,\sigma_{\bar{x}^t_{n-1}}} = \frac{\overline{\left(\bar{X}^t_n - \mu_x\right)\left(\bar{X}^t_{n-1} - \mu_x\right)}}{\sigma^2_{\bar{x}^t}} \tag{H-1}$$

where the subscript n relates to the time-average over the interval and n-1 to the time-average over the previous interval, with advantage taken of the fact that

$$\mu_{\bar{x}^t_n} = \mu_{\bar{x}^t_{n-1}} = \mu_x \tag{H-2}$$

and

$$\sigma_{\bar{x}^t_n} = \sigma_{\bar{x}^t_{n-1}} = \sigma_{\bar{x}^t} \tag{H-3}$$

Equation (H-1) reduces to

$$\rho_{\bar{x}^t} = \frac{\overline{\left(\bar{x}^t_n\right)\left(\bar{x}^t_{n-1}\right)} - \mu_x \overline{\bar{X}^t_n} - \mu_x \overline{\bar{X}^t_{n-1}} + \mu_x^2}{\sigma^2_{\bar{x}^t}} = \frac{\overline{\left(\bar{X}^t_n\right)\left(\bar{X}^t_{n-1}\right)} - \mu_x^2}{\sigma^2_{\bar{x}^t}} \tag{H-4}$$

Only the product term remains to be resolved. Expressing this term as

$$\left(\bar{x}^t_{n-1}\right)\left(\bar{x}^t_n\right) = \left(\frac{1}{T}\int_{-T}^{o} x\,dt\right)\left(\frac{1}{T}\int_{o}^{T} x\,dt\right) \tag{H-5}$$

and subdividing both T time intervals into $\nu\Delta t$ sub-intervals, allows expression of the integral terms as sums, that is,

$$\left(\overline{x}^t_{n-1}\right)\left(\overline{x}^t_n\right) = \frac{1}{T^2}\left[\sum_{\nu=-T/\Delta t}^{-1} x(\nu\Delta t)\Delta t\right]\left[\sum_{\nu=0}^{\frac{T}{\Delta t}-1} x(\nu\Delta t)\Delta t\right]$$

$$= \left(\frac{\Delta t}{T}\right)^2\left[X(-T)+X(-T+\Delta t)+\cdots+X(-2\Delta t)+X(-\Delta t)\right]$$

$$\left[X(0)+X(\Delta t)+X(2\Delta t)+\cdots+X((\nu-1)\Delta t)\right] \tag{H-6}$$

To form the mean of the above product then involves sums of product terms of the general form expressed below for the ith sample in one term and the jth sample in the other:

$$\overline{X(i\Delta t)\ X(j\Delta t)} = \mu_x^2+\rho_x\Big((i-j)\Delta t\Big)\sigma_x^2 = \mu_x^2+\rho_x\Big((j-i)\Delta t\Big)\sigma_x^2 \tag{H-7}$$

where σ_x^2 is the variance of X and $\rho_x(t)$ its correlation coefficient.

Noticing there are ν^2 of the μ_x^2 terms, the mean of equation (H-6) may be written as

$$\overline{\left(\overline{x}^t_{n-1}\right)\left(\overline{x}^t_n\right)} = \left(\frac{\Delta t}{T}\right)^2\Big\{\nu^2\mu_x^2+\sigma_x^2\big[\rho_x(T)+\rho_x(T+\Delta t)+\rho_x(T+2\Delta t)+\cdots$$

$$\cdots+\rho_x(T+(\nu-1)\Delta t)\big]$$

$$+\sigma_x^2\big[\rho_x(T-\Delta t)+\rho_x(T)+\rho_x(T+\Delta t)+\cdots+\rho_x(T+(\nu-2)\Delta t)\big]$$

$$+\sigma_x^2\big[\rho_x(T-2\Delta t)+\rho_x(T-\Delta t)+\rho_x(T)+\cdots+\rho_x(T+(\nu-3)\Delta t)\big]$$

$$\vdots$$

$$+\sigma_x^2\big[\rho_x(2\Delta t)+\rho_x(3\Delta t)+\rho_x(4\Delta t)+\cdots+\rho_x(T+\Delta t)\big]$$

$$+\sigma_x^2\big[\rho_x(\Delta t)+\rho_x(2\Delta t)+\rho_x(3\Delta t)+\cdots+\rho_x(T)\big]\Big\} \tag{H-8}$$

which may be stated as

$$\overline{\left(\overline{x}^{t}_{n-1}\right)\left(\overline{x}^{t}_{n}\right)} = \mu_x^2 + \frac{\sigma_x^2}{T^2}(\Delta t)\left[\sum_{n=\upsilon}^{2\upsilon-1} \rho_x(n\Delta t)\Delta t + \sum_{n=\upsilon-1}^{2\upsilon-2} \rho_x(n\Delta t)\Delta t + \cdots \right.$$

$$\left. + \sum_{n=2}^{\upsilon+1} \rho_x(n\Delta t)\Delta t + \sum_{n=1}^{\upsilon} \rho_x(n\Delta t)\Delta t \right] \qquad \text{(H-9)}$$

and in integral form as

$$\overline{\left(\overline{X}^{t}_{n-1}\right)\left(\overline{X}^{t}_{n}\right)} = \mu_x^2 + \frac{\sigma_x^2}{T^2}\int_o^T\int_t^{t+T} \rho_x(t)dt^2 \qquad \text{(H-10)}$$

Upon substituting into equation (H-4), we have

$$\rho_{\overline{x}}t = \frac{\dfrac{\sigma_x^2}{T^2}\displaystyle\int_o^T\int_t^{t+T} \rho_x(t)dt^2}{\sigma_{\overline{x}}^2 t} \qquad \text{(H-11)}$$

and using the definition of $\sigma_{\overline{x}}^2 t$ from Appendix D, equation (H-11) becomes

$$\rho_{\overline{x}}t = \frac{\displaystyle\int_o^T\int_t^{t+T} \rho_x(t)dt^2}{2\displaystyle\int_o^T\int_o^t \rho_x(t)dt^2} \qquad \text{(H-12)}$$

APPENDIX I

CORRELATION COEFFICIENT FOR SECOND PROBABILITY DISTRIBUTION INVOLVING TIME-AVERAGE AND MID-POINT OF SIGNAL

The general expression for the correlation coefficient of the Gaussian second probability distribution involving time-average and mid-point of signal can be written as

$$\rho_{x_M, \bar{x}^t} = \frac{\overline{\left(X_M - \mu_{x_M}\right)\left(\bar{X}^t - \mu_{\bar{x}^t}\right)}}{\sigma_{x_M}\ \sigma_{\bar{x}^t}} = \frac{\overline{\left(X_M - \mu_x\right)\left(\bar{X}^t - \mu_x\right)}}{\sigma_x\ \sigma_{\bar{x}^t}} \tag{I-1}$$

where

$$\mu_{x_M} = \mu_{\bar{x}^t} = \mu_x \tag{I-2}$$

and

$$\sigma_{x_M} = \sigma_x \tag{I-3}$$

Equation (I-1) reduces to

$$\rho_{x_M, \bar{x}^t} = \frac{\overline{X_M \bar{X}^t} - \mu_x \overline{\bar{X}^t} - \mu_x \overline{X_M} + \mu_x^2}{\sigma_x\ \sigma_{\bar{x}^t}} = \frac{\overline{X_M \bar{X}^t} - \mu_x^2}{\sigma_x\ \sigma_{\bar{x}^t}} \tag{I-4}$$

To determine the mean of the product term, it is first rewritten as

$$X_M \bar{X}^t = X_M \left[\frac{1}{T} \int_0^T X dt\right] = \frac{1}{T} X\left(\frac{T}{2}\right) \sum_{\nu=0}^{\frac{T}{\Delta t}-1} X(\nu \Delta t) \Delta t \tag{I-5}$$

where Δt approaches zero and

$$X_M = X\left(\frac{T}{2}\right) \tag{I-6}$$

Writing out the sum, equation (I-5) becomes

$$x_M \bar{x}^t = \frac{\Delta t}{T}\, x\left(\frac{T}{2}\right)\left[x(0)+x(\Delta t)+x(2\Delta t)+\cdots + x\Big((\nu-1)\Delta t\Big)\right] \tag{I-7}$$

Recognizing that

$$\overline{x\left(\frac{T}{2}\right)x\left(\nu\Delta t\right)} = \mu_x^2 + \sigma_x^2\rho_x\left(\nu\Delta t - \frac{T}{2}\right) \tag{I-8}$$

we have

$$\overline{x_M \bar{x}^t} = \left(\frac{\Delta t}{T}\right)\left\{\nu\mu_x^2 + 2\sigma_x^2\left[\rho_x(0)+\rho_x(\Delta t)+\rho_x(2\Delta t)+\cdots + \rho_x\left(\frac{T}{2}-\Delta t\right)+\frac{1}{2}\rho_x\left(\frac{T}{2}\right)\right]\right\} \tag{I-9}$$

which may be written in integral form as

$$\overline{x_M \bar{x}^t} = \mu_x^2 + \frac{2\sigma_x^2}{T^2}\int_0^{T/2}\rho_x(t)dt \tag{I-10}$$

Upon inserting equation (I-10) into equation (I-4), we then have

$$\rho_{x_M,\bar{x}^t} = \frac{\dfrac{2\sigma_x^2}{T}\displaystyle\int_0^{T/2}\rho_x(t)dt}{\sigma_x\,\sigma_{\bar{x}}t} = \frac{2\displaystyle\int_0^{T/2}\rho_x(t)dt}{\sqrt{2\displaystyle\int_0^{T}\int_0^{t}\rho_x(t)dt^2}} \tag{I-11}$$

where the expression for $\sigma_{\bar{x}}t$ is obtained from Appendix D.