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Lie Groups Associated with Pseudo Groups

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The transformations of a pseudo group which leave a given point invariant induce a group of transformations on the space of k-jets with that point as source and target. This group is a subgroup of a Lie group. When the pseudo group contains all translations, this group, for k sufficiently large, uniquely determines the pseudo group, at least under smoothness assumptions common in the older literature.

1. Definitions. All functions, manifolds, etc., are assumed real analytic. "Transformation" means analytic homeomorphism. Let G be a pseudo group [5] of local transformations on a neighborhood Q of the origin O in E^n, Euclidean n-space. If f is in G, $U(f)$ and $V(f)$ denote the domain and range of f, respectively. Let $\underline{G}$ be the set of all elements of G containing O in their domain which leave O invariant.

$J^k(Q)$ denotes the set of all invertible k-jets $j^k{}_x(f)$ of Q into itself [1]. It has coordinates $(x^i, y^i, p^i{}_{j_1}, \ldots, p^i{}_{j_1 \ldots j_k})$, where (x^i) are those of the source x, (y^i) those of the target $f(x)$, and $p^i{}_{j_1 \ldots j_h} = (\partial^h y^i)/(\partial x^{j_1} \ldots \partial x^{j_h})(x)$. α and β denote the source and target projections, as usual. If g is in G, let $p^k(g)(j^k{}_x(f)) = j^k{}_x(g \circ f)$ for all $j^k{}_x(f)$ in $\beta^{-1}(U(g))$.

Let $\underline{B}^k$ be the set of all $j^k{}_0(f)$ where $f(0) = 0$. $\underline{B}^k$ is a Lie group under the natural product $j^k{}_0(f_1) \cdot j^k{}_0(f_2) = j^k{}_0(f_1 \circ f_2)$. Let G^k be the set of elements in $\underline{B}^k$ coming from $\underline{G}$.

Investigating relations between G^k and G is the main concern of this paper. Conditions will be imposed on G of two types. The first is special and imposes a restriction on the types of pseudo groups studied. The rest are regularity conditions and seem to have been assumed or proved for the pseudo groups studied by S. Lie and E. Cartan.

2. G^k and G.

Condition (A): G contains all translations, i.e., given any two points y_1 and y_2 in Q, there exists a translation in G defined on a neighborhood of y_1 which carries y_1 into y_2.

Proposition. *If* G *satisfies* (A) *then* G *completely determines* G.

Proof. Let t_y denote a translation in G carrying y to 0. If f is in G, y in $U(f)$, then $t^{-1}_{f(y)} \circ f \circ t_y$ is in $\underline{G}$, so the pseudo group generated by all elements $t^{-1}_{y_1} \circ \underline{G} \circ t_{y_2}$ generates G.

There is a natural analytic homeomorphism $e\colon Q \times Q \times \underline{B}^k \to J^k(Q)$ defined by $e(x_1, y_1, j^k{}_0(f)) = j^k{}_{x_1}(t^{-1}_{y_1} \circ f \circ t_{x_1})$. In coordinates, $e(x^i_1, y^i_1, 0, 0, p^i{}_{j_1}, \ldots \quad \ldots, p^i{}_{j_1 \ldots j_k}) = (x^i_1, y^i_1, p^i{}_{j_1}, \ldots, p^i{}_{j_1 \ldots j_k})$. The statement that a function F on $J^k(Q)$ is independent of $\alpha(J^k(Q))$ or $\beta(J^k(Q))$ will mean $F \circ e$ is independent of the first or second Q in $Q \times Q \times \underline{B}^k$, respectively.

Condition (B): For some k there exist independent real-valued $Y^1, \ldots, Y^{\underline{r}}$ on $J^k(Q)$ such that f belongs to G if and only if $Y^r \circ p^k(f) = Y^r$, $r = 1, \ldots, \underline{r}$. Since any function depending only on $\alpha(J^k(Q))$ is trivially invariant, it is assumed that the Y^r are independent of $\alpha(J^k(Q))$.

This regularity condition, which asserts that G is defined by "differential invariants", was proved by S. Lie for his transformation groups [2]. In some examples the Y^r may not be defined or independent except on some open subset U of $J^k(Q)$ where $\beta(U) = Q$. The arguments should be modified in this case.

Proposition. <u>If</u> G <u>satisfies</u> (A) <u>and</u> (B) <u>the functions</u> Y^r <u>are independent of</u> $\beta(J^k(Q))$, $r = 1, \ldots, \underline{r}$.

Proof. $Y^r \circ e(x_1, y_1, j^k{}_0(f)) = Y^r(j^k{}_{x_1}(t_{y_1}^{-1} \circ f \circ t_{x_1})) =$

$$Y^r \circ p^k(t_{y_1}^{-1})(j^k{}_{x_1}(f \circ t_{x_1})) = Y^r(j^k{}_{x_1}(f \circ t_{x_1})).$$

Let q be the projection of $Q \times Q \times \underline{B}^k$ onto $\underline{B}^k$. If (A) and (B) are satisfied there exist natural real-valued functions Z^r on $\underline{B}^k$ such that $Y^r \circ e = Z^r \circ q$, $r = 1, \ldots, \underline{r}$. <u>By definition</u> G^k <u>acting on</u> B^k <u>by left-multiplication leaves each</u> Z^r <u>invariant</u>.

Under condition (B), G is defined by $Y^1, \ldots, Y^{\underline{r}}$. These Y^r define a system of partial differential equations which all elements of G must satisfy. The remaining conditions are regularity conditions on these equations.

In [5] it is shown that there exist on $J^k(Q)$ 1-forms φ^i, $\varphi^i{}_{j_1 \cdots j_h}$; $i, j_1, \ldots, j_h = 1, \ldots, n$; $h = 1, \ldots, k-1$, such that a transformation $F: J^k(Q) \to J^k(Q)$ is locally a restriction of some $p^k(f)$ for $f: Q \to Q$ if and only if F^* leaves each of these 1-forms invariant and $\alpha \circ F = \alpha$. (In [5] $p^k(f)$ is defined to act on $j^k{}_x(g)$ by $j^k{}_x(g \circ f^{-1})$, so the theory there is dual to that used here.)

Let ρ_1 and ρ_2 denote the projections of $J^k(Q) \times J^k(Q)$ onto the first and second factors, respectively. Consider on $J^k(Q) \times J^k(Q)$ the exterior differential system $\Sigma(Y)$ generated by

$$Y^r \circ \rho_1 - Y^r \circ \rho_2,$$

$$\alpha \circ \rho_1 - \alpha \circ \rho_2,$$

$$\rho_1^* \varphi^i - \rho_2^* \varphi^i,$$

$$\rho_1^* \varphi^i{}_{j_1 \cdots j_h} - \rho_2^* \varphi^i{}_{j_1 \cdots j_h},$$

$r = 1, \ldots, \underline{r}$; $i, j_1, \ldots, j_h = 1, \ldots, n$; $h = 1, \ldots, k-1$, and their exterior derivatives. The second factor of $J^k(Q) \times J^k(Q)$ will be taken as independent variables. Then any solution ψ has the form $\psi(j^k{}_x(f)) = (p^k(g)(j^k{}_x(f)), j^k{}_x(f))$, where g is in G.

Condition (C): $\Sigma(Y)$ is involutive at all integral points.

Proposition. _If_ (A), (B), _and_ (C) _are all satisfied, then_ $Z^1,\dots,Z^{\underline{r}}$ _constitute a full set of invariants for_ G^k. _That is, if_ Z _is any invariant real-valued function on_ $\underline{B}^k$ _it must be locally a function of_ $Z^1,\dots,Z^{\underline{r}}$.

Proof: If $j^k{}_0(f_1)$ and $j^k{}_0(f_2)$ belong to $\underline{B}^k$ and satisfy $Z^r(j^k{}_0(f_1)) = Z^r(j^k{}_0(f_2))$, $r = 1,\dots,\underline{r}$, then $(j^k{}_0(f_1)), j^k{}_0(f_2))$ is an integral point of $\underline{\Sigma}(Y)$. Hence there exists g in G such that $p^k(g)(j^k{}_0(f_2)) = j^k{}_0(f_1)$. Since $f(0) = f(f_2(0)) = f_1(0) = 0$, g is in $\underline{G}$, and hence $j^k{}_0(g)$ is in G^k. It follows that $Z(j^k{}_0(f_1)) = Z(j^k{}_0(f_2))$ whenever $Z^r(j^k{}_0(f_1)) = Z^r(j^k{}_0(f_2))$, $r = 1,\dots,\underline{r}$.

From this one may say that G^k determines G uniquely, since its invariant functions determine the differential invariants of G. In conclusion a condition will be given under which G^k is characterized by this property, i.e., G^k contains all transformations in $\underline{B}^k$ which leave $Z^1,\dots,Z^{\underline{r}}$ invariant.

Suppose $j^k{}_0(g)$ is in $\underline{B}^k$ and leaves each $Z^1,\dots,Z^{\underline{r}}$ invariant. The manifold

$$V(j^k{}_0(g)) = \left\{(j^k{}_0(g \circ f), j^k{}_0(f)) \mid j^k{}_0(f) \text{ is in } \underline{B}^k\right\}$$

is an integral submanifold of $\underline{\Sigma}(Y)$. If one can find an integral of $\underline{\Sigma}(Y)$ of maximal dimension containing $V(j^k{}_0(g))$, it would arise from some $p^k(\underline{g})$ where $\underline{g}$ is in G. Then $j^k{}_0(g) = j^k{}_0(\underline{g})$, so $j^k{}_0(g)$ would belong to G^k.

Condition (D): **Every** $V(j^k_0(g))$, where $j^k_0(g)$ leaves $Z^1, \ldots, Z^r$ invariant, has its tangent space at each of its points contained in a regular flag of $\Sigma(Y)$. (For this terminology, see [3].)

Theorem. *If* G *satisfies* (A), (B), (C), *and* (D), *then* G^k *consists in all transformations of the Lie group* B^k *which leave invariant a finite number of real-valued functions, and* G^k *uniquely determines* G.

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