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INSTITUTE OF FLIGHT STRUCTURES

DEPARTMENT OF CIVIL ENGINEERING AND ENGINEERING MECHANICS

UPPER AND LOWER BOUNDS IN PROBLEMS OF MELTING OR SOLIDIFYING SLABS

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ABSTRACT*

The problem studied is that of a slab, heated in an arbitrary manner on one face and insulated on the other, which melts (or solidifies), the material being allowed to remain stationary after change of phase. Variable material properties are taken into account. After preliminary general considerations, it is shown that the solution to the stated problem is unique. It is then proved that higher rates of melting and higher temperatures will result from certain combinations of the magnitude of the applied heat input and of a fictitious heat source traveling with the solid-liquid interface. From this result a method is developed for the construction of upper and lower bounds to the solution of the problem; an example is also presented. It is also shown that, under the same arbitrary heat input, the rate of melting in the present problem is always lower than that in the companion problem in which the material is instantaneously removed after change of phase.

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1. Introduction

The solution of heat conduction problems of melting or solidification often presents considerable mathematical difficulties, and is therefore often approached either by numerical or approximate techniques. A different approach was described in [1], in which a method of constructing upper and lower bounds for the temperature and for the unknown location of the melting front in this type of problem was devised. The problem considered there was that of a slab, initially solid[1], insulated on one face and subjected to an arbitrary heat input on the other; after melting starts, the known heat input is applied directly to the moving boundary of the solid. In the present work the same type of approach is extended to include the problem of a melting slab, in which the melted material is not removed, but remains stationary.

Many of the theorems proved in [1] will be needed in the course of the present proofs, and are therefore restated here (with some minor extensions) for the sake of convenience, at the beginning of Section 2. The principal results required for the establishment of bounds in the present problem follow in the same section, while a statement of this particular melting problem and a proof of uniqueness of its solution are given in Section 3. Two types of upper and lower bounds are established in Section 4: the first compares melting rates and temperatures in the present problem under certain combinations of the magnitude of the applied heat input and of a fictitious variable heat source traveling with the liquid-solid interface, while the second compares the solution of the present problem with that of [1]. The construction of the first type of these bounds and their use in obtaining estimates of solution is discussed together with an illustrative example, in Section 5.

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[1] The work of [1], as well as the present one, apply equally to the problems of melting and of solidification; for convenience, however, only the former type of problem will henceforth be referred to.

2. Basic Theorems

The first two of the following theorems are listed here for ease of reference; their proofs may be found in [1].

Theorem I Let u(P) be a solution, in class $C^{(2)}$, of the equation

$$a(P)\frac{\partial^2 u}{\partial x^2} + b(P)\frac{\partial u}{\partial x} - \frac{\partial u}{\partial t} = 0 \; ; \quad a \geq 0 \tag{1}$$

at all points P(x,t) of a domain D in the xt-plane where a and b are real finite continuous functions, and let

$$\frac{\partial u}{\partial n_x} = f(P), \quad P \text{ on } B\text{-}B_{-t} \tag{2a}$$

$$u = k \quad \text{at some point } P_1 \text{ on } B\text{-}B_{-t} \tag{2b}$$

where n_x denotes the direction of the component parallel to the x-axis of the interior normal to the boundary B, k is a constant, and where B_{-t} is that part of B which includes all points P such that (a) the interior normal to the boundary exists and is directed in the negative t direction, and (b) each point P is an interior point of B_{-t}. Then:

(a) if $f(P) = 0$, $u \equiv k$ throughout D;

(b) if f(P) is prescribed throughout $B\text{-}B_{-t}$, then u is uniquely determined throughout D.

Theorem II[2] Consider a simply connected domain D in the xt-plane, whose boundary is (Fig. 1) formed by segments of the straight lines $x = x_0$, and

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[2] This is a slight extension of Theorem II of [1].

$t = t_2$ and by a line defined by a continuous single-valued function $x = F(t)$ satisfying Lipschitz conditions for $t_1 < t < t_2$ and intersecting the line $t = t_1$ at a point $P_3 : (x_1, t_1)$. Let $u(P)$ be the solution of Eq. (2) in D with

$$\frac{\partial u}{\partial x} = 0 \quad \text{on} \quad x = x_0$$
$$u = u_0(x) \quad \text{on} \quad t = t_1 \tag{3a}$$

and

$$\text{either} \quad \frac{\partial u}{\partial x} = -f(t) \quad \text{or} \quad u = f(t) \quad \text{on} \quad x = F(t) \tag{3b}$$

Then, in either of these cases, if $f(t) \geq 0$ and $u_0(x) \geq 0$, $u \geq 0$ throughout D, and if $f(t) \leq 0$, and $u_0(x) \leq 0$, then $u \leq 0$ throughout D.

<u>Theorem III</u> Let $u_1(P)$ and $u_2(P)$ be solutions of Eq. (1), respectively in the domains D_1 and D_2 of the xt-plane defined in Fig. 2, and let

$$\frac{\partial u_1(x_1,t)}{\partial x} = -f(t) , \quad t_0 < t < t_1 \tag{4a}$$

$$\frac{\partial u_1(x_1+L,t)}{\partial x} = 0 , \quad t_0 < t < t_2 \tag{4b}$$

$$\frac{\partial u_2(x_1,t)}{\partial x} = -f(t) , \quad t_1 < t < t_2 \tag{4c}$$

$$u_1(x,t_0) = 0 , \quad x_1 < x < L \tag{4d}$$

$$\left.\begin{aligned} M \frac{\partial u_1}{\partial x} &= N \frac{\partial u_2}{\partial x} \\ u_1 &= u_2 \end{aligned}\right\} \text{on } x = F(t) , \quad t_1 < t < t_2 \tag{4f}$$

where $M(P) > 0$ and $N(P) > 0$ and where $F(t)$ is a given Lipschitz continuous single-valued function of t. Then

(a) if $f(t) \equiv 0$ in $t_0 < t < t_2$, $u_1 \equiv 0$ and $u_2 \equiv 0$ throughout their domains of definition;

(b) if $f(t)$ is arbitrarily prescribed in $t_0 < t < t_2$, then u_1 and u_2 are uniquely determined throughout their respective domains;

(c) if $f(t) \leq 0$ in $t_0 < t < t_2$, then $u_1 \leq 0$ and $u_2 \leq 0$ in their respective domains, and

(d) if $f(t) \geq 0$ in $t_0 < t < t_2$, then $u_1 \geq 0$ and $u_2 \geq 0$ in their respective domains of definition.

Statement (b) is a corollary of statement (a), and it will be apparent that the proof of (c) requires only obvious modifications to apply to part (d) as well. Hence only parts (a) and (c) need be considered below.

<u>Proof of (a)</u> Clearly (Theorem I) $u_1 \equiv 0$ for $t_0 < t < t_1$. For $t_1 < t < t_2$, $\frac{\partial u_1}{\partial n_x}$ and $\frac{\partial u_2}{\partial n_x}$ have opposite signs or vanish on $x = F(t)$. If they are both zero, Theorem I again applies and $u_1 \equiv 0$, $u_2 \equiv 0$. If they were not identically zero, then on $x = F(t)$ one could find values $t'(\geq t_1)$ and $\delta > 0$ such that

$$\frac{\partial u_1}{\partial n_x} = \frac{\partial u_2}{\partial n_x} = 0 \qquad t_1 < t < t' \tag{5a}$$

and

$$\left.\begin{array}{ll} \text{either} & \dfrac{\partial u_1}{\partial n_x} > 0\,, \quad \dfrac{\partial u_2}{\partial n_x} < 0 \\[2ex] \text{or} & \dfrac{\partial u_1}{\partial n_x} < 0\,, \quad \dfrac{\partial u_2}{\partial n_x} > 0 \end{array}\right\} \quad t' < t < t' + \delta \tag{5b}$$

Then, by Theorem II, for the first of these possibilities $u_1 \leq 0$ and $u_2 \geq 0$ in $t' < t < t' + \delta$; therefore, on $x = F(t)$, $u_1 \equiv u_2 \equiv 0$ in this interval of time, and similarly for the second of the possibilities in Eqs. (5b). But then Theorem II [applied first to the domain $t' < t < t' + \delta$, $x_1 < x < F(t)$ and then to the domain $t' < t < t' + \delta$, $F(t) < x < x_1 + L$] shows that in either case $u_1 \equiv u_2 \equiv 0$ at any point within this time interval, so that neither of (5b) can exist. Hence part (a) of the theorem is proved.

Proof of (c) From Theorem II, $u_1 \leq 0$ in $t_0 < t \leq t_1$; for $t_1 < t < t_2$ two possibilities can arise, depending on whether initially $\frac{\partial u_1}{\partial x}$ is negative or positive on $x = F(t)$ in this interval. In the former case Eqs. (4f) give $\frac{\partial u_2}{\partial n_x} \geq 0$ on $x = F(t)$ and hence $u_2 \leq 0$ in domain 2, and thus $u_1 \leq 0$ on $x = F(t)$. Hence, by Theorem II, u_1 is non-positive throughout, as was to be proved. Suppose now that a later time t' exists, at which $\frac{\partial u_1}{\partial x}$ on $x = F(t)$ changes sign, or, in other words:

$$\left.\begin{array}{lll} u_1 = u_2 \leq 0 & t_1 < t \leq t' & \text{throughout} \\ \frac{\partial u_1}{\partial x} = \frac{\partial u_2}{\partial x} = 0 & t = t' & \\ \frac{\partial u_1}{\partial x} > 0 ; \quad \frac{\partial u_2}{\partial x} > 0 & t' < t < t' + \delta , \quad \delta > 0 & \end{array}\right\} \text{on } x = F(t) \tag{6}$$

Then (Theorem II) $u_1 \leq 0$ in $t' < t < t' + \delta$ for all x and so $u_2 \leq 0$ in this time interval on $x = F(t)$. Now if u_2 were non-positive on $x = x_1$, Theorem I would insure that it be non-positive throughout, as was to be proved; if, on the other hand, u_2 were positive on $x = x_1$ at some time t'' $(t' < t'' < t' + \delta)$, then some time within $t' < t < t''$ the condition $\frac{\partial u_2}{\partial t} > 0$ as well as (to avoid an interior maximum as required by Picone's theorem [1,2]) the conditions $\frac{\partial u_2}{\partial x} = 0$ and $\frac{\partial^2 u_2}{\partial x^2} \leq 0$, all on $x = x_1$, would have to prevail. The conditions are however in contradiction with Eq. (1), and therefore u_2 must be non-positive again. Now it may happen that at a later time $\frac{\partial u_1}{\partial x}$ on $x = F(t)$ once more changes sign: then the first part of this proof applies once again, and similarly for any further changes in sign. There now remains the second possibility alluded to at the beginning of this proof, namely that $\frac{\partial u_1}{\partial x}$ is initially positive on $x = F(t)$; in this case the second part of the above proof applies first, and any later changes in sign of this derivative can be taken care of as above. Then proof is thus completed.

3. Statement of the Melting Problem[3]

Consider a slab, initially (i.e. at $t = 0$) solid at zero temperature and occupying the region $0 < x < L$, and insulated at $x = L$. An arbitrarily prescribed heat input $Q(t)$ is applied at $x = 0$, so that the temperature in the slab rises and at $x = 0$ reaches the melting temperature T_m at the time $t = t_m$. Melting continues to take place thereafter, and a portion of thickness $s(t)$ is taken to have melted at any time $t \geq t_m$, while the prescribed heat input $Q(t)$ still continues to be applied at $x = 0$. Thus at any time $t > t_m$, the portion of the slab within $s(t) < x < L$ is still solid, while that within $0 < x < s(t)$ is liquid; the subscripts S and L will be used in what follows to distinguish quantities pertaining to the two phases of the material. The mathematical formulation of the problem is as follows for the temperature

[3] A statement and discussion of this and of the analogous solidification problem were given for example in [3]. An approximate solution of this problem was presented in [8].

$T_S(x,t) \leq T_m$ in the solid:

$$\frac{\partial}{\partial x}\left(k_S \frac{\partial T_S}{\partial x}\right) = \rho_S c_S \frac{\partial T_S}{\partial t}\ ; \qquad s(t) < x < L\ , \quad 0 < t < t_L \qquad (7)$$

$$T_S(x,0) = 0\ ; \qquad 0 < x < L \qquad (8)$$

$$\frac{\partial T_S}{\partial x}(L,t) = 0\ ; \qquad 0 < t < t_L \qquad (9)$$

$$-k_S \frac{\partial T_S}{\partial x}(0,t) = Q(t)\ ; \qquad 0 < t < t_m \qquad (10)$$

and for the temperature $T_L(x,t) \geq T_m$ in the liquid:

$$\frac{\partial}{\partial x}\left(k_L \frac{\partial T_L}{\partial x}\right) = \rho_L c_L \frac{\partial T_L}{\partial t}\ ; \qquad 0 < x < s(t)\ , \quad t_m < t < t_L \qquad (11)$$

$$-k_L \frac{\partial T_L}{\partial x}(0,t) = Q(t)\ ; \qquad t_m < t < t_L \qquad (12)$$

with the following interface conditions:

$$\left.\begin{aligned} &T_S[s(t),t] = T_L[s(t),t] = T_m \\ &k_{Lm} \frac{\partial T_L}{\partial x}[s(t),t] - k_{Sm} \frac{\partial T_S}{\partial x}[s(t),t] = Q^*(t) - \rho_{Sm} \ell \frac{ds}{dt}(t) \end{aligned}\right\} \quad t_m < t < t_L \qquad (13)$$

and with

$$s(t) = 0\ ; \quad 0 \leq t \leq t_m \qquad (14)$$

The times t_m and t_L are respectively defined by the equations

$$T(0,t_m) = T_m \quad \text{and} \quad s(t_L) = L \tag{15}$$

The thermal diffusivity $\kappa = k/(\rho c)$, the conductivity k, the specific heat c, and the density ρ are assumed to be functions of the temperature and therefore vary with both x and t. The subscript m, affixed to any quantity, indicates that the value at the melting temperature T_m must be used. The latent heat of melting is denoted by ℓ.

The heat input $Q^*(t)$ appearing in Eqs. (13), has the physical meaning of a variable heat source traveling with the interface, and is identically zero in the problem whose solution it is desired to find; nevertheless it will be included in all the derivations because it is a convenient quantity to deal with in the calculation of bounds for the solution of this problem. Q^* has not been defined for $t < t_m$; it will be convenient to take it as zero in this range.

It can be readily shown[4] by means of an overall heat balance that

$$\int_0^t [Q(t) + Q^*(t)]\,dt = \int_0^{s(t)} H_L dx + \int_{s(t)}^L H_S dx + (\rho_{Sm}\ell + H_m)\,s \tag{16}$$

where the heat contents H_S and H_L are defined as

$$H_S(T) = \int_0^T \rho_S(T')c_S(T')dT' \tag{17a}$$

[4] The derivation is very similar to that of [4], [5] or [6]; the extension to the present case of variable properties presents no difficulty since in terms of the heat contents H_S and H_L the right-hand sides of Eqs. (7) and (11) reduce respectively to $(\partial H_S/\partial t)$ and $(\partial H_L/\partial t)$.

and

$$H_L(T) = \int_{T_m}^{T} \rho_L(T')c_L(T')dT' \tag{17b}$$

Note that the integrands of Eqs. (17) are positive and therefore both H_S and H_L are monotonically increasing functions of $T(x,t)$. The symbol H_m stands for $H_S(T_m)$.

<u>Theorem IV</u>: <u>Uniqueness of Solution</u> It will now be proved that there exists at most one solution[5] to Eqs. (7) to (14), corresponding to prescribed functions $Q(t)$ and $Q^*(t)$. To prove this, assume that two distinct solutions exist, and denote them by the subscripts 1 and 2; then Theorem III insures uniqueness if $s_1 \equiv s_2$. Only the possibility $s_1 \not\equiv s_2$ need therefore be considered, or, without loss of generality, we may set

$$\begin{aligned} s_1 &= s_2 , \quad 0 \le t \le t' \\ s_2 &> s_1 , \quad t' < t \le t' + \delta , \quad \delta > 0 \end{aligned} \tag{18}$$

Since the solution is known to be unique before melting starts [4,7] t_m is the same for each solution; hence $t' \ge t_m$. After the start of melting, write Eq. (16) for each solution (at some time t" in the interval where $s_2 > s_1$) and subtract the results to get

$$0 = \int_0^{s_1(t'')} (H_{L2}-H_{L1})dx + \int_{s_1(t'')}^{s_2(t'')} (H_{L2}-H_{S1})dx + \int_{s_2(t'')}^{L} (H_{S2}-H_{S1})dx + \\ + (\rho_{Sm}\ell + H_m)[s_2(t'')-s_1(t'')] \tag{18a}$$

[5] That is, a twice continuously differentiable function $T(x,t)$ and a Lipschitz continuous function $s(t)$. Except in the special case of Eqs. (24), this character of the solution is assumed throughout the remainder of this paper.

On $x = s_1(t)$, $T_1 = T_m$ and $T_2 \geq T_m$, while on $x = s_2(t)$, $T_1 \leq T_m$ and $T_2 = T_m$; hence Theorem III, applied in turn to the regions $s_2 < x < L$ and $0 < x < s_1$, insures that $T_2 \geq T_1$ in these regions, so that the first and third integrals in (18a) are non-negative. Furthermore

$$\int_{s_1}^{s_2} H_{L2} dx \geq 0 \quad \text{and} \quad \int_{s_1}^{s_2} H_{S1} dx \leq H_m\,[s_2(t'')-s_1(t'')] \tag{18b}$$

Hence the right-hand side of (18a) cannot be zero, this equation cannot be satisfied, and uniqueness is assured.

4. Upper and Lower Bounds

Theorem V Consider two solutions of Eqs. (7) to (14), denoted by the subscripts 1 and 2, corresponding respectively to heat inputs $Q_1(t)$, $Q_1^*(t)$ and $Q_2(t)$, $Q_2^*(t)$ such that

$$Q_2^* \geq Q_1^* \ ; \quad Q_2(t) + Q_2^*(t) \geq Q_1(t) + Q_1^*(t) \tag{19}$$

Then

$$s_2(t) \geq s_1(t) \ ; \qquad t \geq 0 \tag{19a}$$

$$T_2(x,t) \geq T_1(x,t) \ ; \qquad 0 \leq x \leq L \tag{19b}$$

It is understood, in Eq. (19b) and elsewhere, that for the temperature T one must read the pertinent one of the functions T_S and T_L.

The case in which in both Eqs. (19) the equal signs hold throughout is covered by Theorem IV and need not be considered further. To prove the present theorem in the general case, it may first be assumed, without loss of generality, that a time t' exists such that

$$Q_1(t) + Q_1^*(t) = Q_2(t) + Q_2^*(t) \quad \text{for} \quad 0 \le t \le t'$$
$$Q_1(t) + Q_1^*(t) < Q_2(t) + Q_2^*(t) \quad \text{for} \quad t' < t \le t' + \delta \ , \ \delta > 0 \tag{20}$$

It will now be shown that a time $t''(t' < t'' \le t' + \delta)$ exists such that

$$s_2(t) > s_1(t) \quad \text{for} \quad t' < t \le t'' \tag{20a}$$

Assume in fact that this is not so; then the situation is that shown in Fig. 3a (note that it is immaterial in the proof whether $t' > t_m$ or $t' < t_m$, though only the former case is shown in the figure). Eq. (16), written for each solution at $t = t''$, gives after subtraction,

$$0 < \int_0^{t''} [(Q_2 + Q_2^*)-(Q_1 + Q_1^*)]dt = \int_0^{s_2(t'')} (H_{L2}-H_{L1})dx + \int_{s_2(t'')}^{s_1(t'')} (H_{S2}-H_{L1})dx$$
$$+ \int_{s_1(t'')}^{L} (H_{S2}-H_{S1})dx - (\rho_{Sm}\ell + H_m)[s_1(t'')-s_2(t'')] \tag{21}$$

However, on $x = s_1$, $T_1 = T_m$ and $T_2 \le T_m$, while on $x = s_2$, $T_2 = T_m$ and $T_1 \ge T_m$; hence the first and third integrals on the right-hand side of (21) are negative. Furthermore,

$$\int_{s_2}^{s_1} H_{L1}dx \ge 0 \quad \text{and} \quad \int_{s_2}^{s_1} H_{S2}dx \le H_m(s_1 - s_2) \tag{21a}$$

Hence the right-hand side of (21) cannot be positive as required, and (20a) is proved. Note that the validity of inequality (19b) for the range of times considered has also been proved.

It has been thus proved that initially s_1 cannot exceed s_2, and it must be shown next that no later time can exist at which s_1 exceeds s_2. Indeed, if there were such a time, there would have to be at least one instant at which $s_2 = s_1$; let the first of these instants be t''' ($> t'$), as shown in Fig. 3b. Note that then

$$T_1 = T_2 = T_m \;;\quad s_1 = s_2 \;;\quad \frac{ds_2}{dt} \le \frac{ds_1}{dt} \qquad \text{at } P_2 \tag{22}$$

The second of Eqs. (13), written for each solution at t''', gives, after subtraction

$$k_{Lm}\frac{\partial}{\partial x}(T_{L2}-T_{L1}) - k_{Sm}\frac{\partial}{\partial x}(T_{S2}-T_{S1}) = (Q_2^* - Q_1^*) - \rho_{Sm}\ell(\dot{s}_2-\dot{s}_1) \tag{22a}$$

where dots indicate differentiation with respect to time. Now $T_{L2}-T_{L1} \ge 0$ on $x = s_1$ and hence also to the left of P_2; in view of (22), then,

$$\frac{\partial}{\partial x}(T_{L2}-T_{L1}) \le 0 \qquad \text{at } P_2 \tag{22b}$$

Similarly, $T_{S2}-T_{S1} \ge 0$ on $x = s_2$ and hence also to the right of P_2, so that

$$\frac{\partial}{\partial x}(T_{S2}-T_{S1}) \ge 0 \qquad \text{at } P_2 \tag{22c}$$

This means that the left-hand side of (22a) is non-positive; however the right-hand side of this equation is non-negative in view of (19) and (22) and therefore this implies a contradiction if at least one of the inequalities $\dot{s}_2 < \dot{s}_1$ or $Q_2^* > Q_1^*$ hold. In the special case

$$Q_1^* = Q_2^* \; ; \quad s_1 = s_2 \; ; \quad \dot{s}_1 = \dot{s}_2 \qquad \text{at } t = t''' \tag{23a}$$

the above proof fails since then

$$k_L \frac{\partial(T_{L2}-T_{L1})}{\partial x} = k_S \frac{\partial(T_{S2}-T_{S1})}{\partial x} = 0 \qquad \text{at } P_2 \tag{23b}$$

In this case, assume that

$$\left.\begin{aligned} s(t) &= \frac{(t-t''')^2}{2!}\,\ddot{s}(t''') + \frac{(t-t''')^3}{3!}\,\dddot{s}(t''') + \cdots \\ Q^*(t) &= (t-t''')\,\dot{Q}^*(t''') + \frac{(t-t''')^2}{2!}\,\ddot{Q}^*(t''') + \cdots \end{aligned}\right\} \quad t \geq t''' \tag{24}$$

where clearly

$$\left.\begin{aligned} \ddot{Q}_2^*(t''') &\geq \ddot{Q}_1^*(t''') \\ \ddot{s}_2(t''') &\leq \ddot{s}_1(t''') \end{aligned}\right\} \tag{24a}$$

Differentiation of the second of (13) along s gives[6]

[6] In the equations which follow, all quantities must be evaluated at $x = s(t)$. The similarity of this portion of the proof with that corresponding one of Theorem IV of [1] will be readily noted.

$$k_{Lm}\left(\frac{\partial^2 T_L}{\partial x^2}\dot{s} + \frac{\partial^2 T_L}{\partial x \partial t}\right) + \left(\frac{dk_L}{dT_L}\right)_m \left(\frac{\partial T_L}{\partial x}\dot{s} + \frac{\partial T_L}{\partial t}\right)\frac{\partial T_L}{\partial x} - k_{Sm}\left(\frac{\partial^2 T_S}{\partial x^2}\dot{s} + \frac{\partial^2 T_S}{\partial x \partial t}\right)$$

$$-\left(\frac{dk_S}{dT_S}\right)_m \left(\frac{\partial T_S}{\partial x}\dot{s} + \frac{\partial T_S}{\partial t}\right) = \dot{Q}^* - \rho_{Sm}\ell\,\ddot{s} - \ell\left(\frac{d\rho_S}{dT_S}\right)_m \left(\frac{\partial T}{\partial x}\dot{s} + \frac{\partial T}{\partial t}\dot{s}\right) \qquad (25)$$

while the first of (13), written in differential form, is

$$\frac{\partial T_S}{\partial x}\dot{s} + \frac{\partial T_S}{\partial t} = \frac{\partial T_L}{\partial x}\dot{s} + \frac{\partial T_L}{\partial t} = 0 \qquad (26a)$$

With (7) and (11), Eqs. (26a) can be rewritten as

$$\frac{\partial T_S}{\partial x}\dot{s} + \frac{1}{\rho_S c_S}\left[\frac{dk_S}{dT}\left(\frac{\partial T_S}{\partial x}\right)^2 + k_S\frac{\partial^2 T_S}{\partial x^2}\right] = 0$$

$$\frac{\partial T_L}{\partial x}\dot{s} + \frac{1}{\rho_L c_L}\left[\frac{dk_L}{dT}\left(\frac{\partial T_L}{\partial x}\right)^2 + k_L\frac{\partial^2 T_L}{\partial x^2}\right] = 0 \qquad (26b)$$

Writing each of Eqs. (25b) for solutions 1 and 2 and subtracting the results in each case one obtains, with the aid of (23b),

$$\frac{\partial^2(T_{S2}-T_{S1})}{\partial x^2} = \frac{\partial^2(T_{L2}-T_{L1})}{\partial x^2} = 0 \quad \text{at } P_2 \qquad (26c)$$

The same process, applied to Eq. (25) now gives[7] at P_2, with the aid of Eqs.

[7] After use of Eq. (7) to give $\frac{\partial^2 T_S}{\partial x \partial t} = \frac{\partial}{\partial x}\left[\frac{1}{\rho_S c_S}\frac{\partial}{\partial x}\left(k_S\frac{\partial T_S}{\partial x}\right)\right] = \kappa_S\frac{\partial^3 T_S}{\partial x^3} + f\left(\frac{\partial^2 T}{\partial x^2}, \frac{\partial T}{\partial x}\right)$ and similarly for Eq. (11) and $\frac{\partial^2 T_L}{\partial x \partial t}$.

(23) and (26):

$$k_{Lm}\,\varkappa_{Lm}\,\frac{\partial^3(T_{L2}-T_{L1})}{\partial x^3}-k_{Sm}\,\varkappa_{Sm}\,\frac{\partial^3(T_{S2}-T_{S1})}{\partial x^3}=(\mathring{Q}_2^*-\mathring{Q}_1^*)-\rho_{Sm}\ell(\ddot{s}_2-\ddot{s}_1) \qquad (27)$$

In view of Eq. (23b) and (26c), the argument immediately following Eq. (22a) now requires that the left-hand side of (27) be non-positive; this again leads to a contradiction unless both equality signs hold in Eqs. (24b). The case

$$\left.\begin{array}{l} Q_1^* = Q_2^* \;;\quad \mathring{Q}_1^* = \mathring{Q}_2^* \\ \\ s_1 = s_2 \;;\quad \dot{s}_1 = \dot{s}_2 \;;\quad \ddot{s}_1 = \ddot{s}_2 \end{array}\right\} \text{ at } t = t''' \qquad (28)$$

must therefore still be investigated, as well as subsequent special cases of this type. These are treated by means of further differentiation of Eqs. (25), (26a), (7) and (11), along s; but inspection reveals that, in general, the result will be (at P_2):

$$k_{Lm}\,\varkappa_{Lm}\,\frac{\partial^{2n+1}(T_{L2}-T_{L1})}{\partial x^{2n+1}}-k_{Sm}\,\varkappa_{Sm}\,\frac{\partial^{2n+1}(T_{S2}-T_{S1})}{\partial x^{2n+1}}=\frac{d^n(Q_2^*-Q_1^*)}{dt^n}-\rho_m\ell\,\frac{d^{n+1}(s_2-s_1)}{dt^{n+1}} \qquad (29a)$$

$$\frac{\partial^m(T_{L2}-T_{L1})}{\partial x^m}=\frac{\partial^m T_{S2}-T_{S1})}{\partial x^m}=0\;;\quad m=0,\,1,\,2,\,\cdots,\,2n \qquad (29b)$$

when

$$\frac{d^m(Q_2^*-Q_1^*)}{dt^m}=0\;;\quad m=0,\,1,\,2,\,\cdots,\,n-1 \qquad (29c)$$

$$\frac{d^m(s_2-s_1)}{dt^m} = 0 \; ; \quad m = 0,\, 1,\, 2,\, \cdots,\, n \tag{29d}$$

Choose n as the smallest integer for which at least one term of the right-hand side of (29a) does not vanish, and then note that, just as before, this leads to a contradiction with the non-positive character of the left-hand side of (29a). The proof of the theorem is thus complete.

It should be noted that the converse of this theorem is false, that is, the validity of Eqs. (17) does not necessarily imply the validity of (16). As a corollary of this theorem, the statement

$$t_{L2} \leq t_{L1} \tag{30}$$

follows directly, where t_L is (cf. Eqs. 15) the time at which the entire slab has melted.

Another type of bound on the solution will now be established, namely one obtained by comparison of the rate of melting in the problem defined in Section 3 (in which the melted portion remains stationary) with that in the companion problem in which the melted portion is instantaneously removed. This is done by means of the following:

<u>Theorem VI</u> Consider two pairs of functions $T_1(x,t)$, $s_1(t)$ and $T_2(x,t)$, $s_2(t)$, such that the pair T_1, s_1 is a solution of Eqs. (7) to (14) with $Q^*(t) \equiv 0$, and the pair T_2, s_2 satisfies Eqs. (7), (8), (9), (10) and (14) in $s_2(t) < x < L$ as well as the following two equations:

$$\begin{aligned} &T_2[s_2(t),t] = T_m \\ &-k_{Sm}\frac{\partial T_2}{\partial x}[s_2(t),t] = Q(t) - \rho_{Sm}\ell\,\frac{ds_2(t)}{dt} \end{aligned} \tag{31}$$

where the same function $Q(t)$ is used in Eqs. (10), (12) and in the second of (31); then

$$s_1(t) \le s_2(t) \; , \qquad t \ge t_m \tag{32a}$$

and

$$T_1(x,t) \le T_2(x,t) \; , \qquad t \ge t_m \; , \qquad s_2(t) < x < L \tag{32b}$$

This theorem therefore states that, under the same heat input history, a more repid advance of the interface occurs when the melted material is instantaneously removed (problem 2) than when it remains stationary (problem 1). This conclusion was already reached in [3] for the special case of Q = constant; it was also noted there that for very short times after the onset of melting the solutions to the two problems are identical.

To prove this theorem, we start with the heat-balance equation (16) for problem 1 (with $Q^* = 0$) and the corresponding equation for problem 2, namely

$$\int_0^t Q(t)\,dt = \int_{s_1(t)}^{L} H_{S2}\,dx + (\rho_{Sm}\ell + H_m)\,s_2(t) \tag{33}$$

Assume now that $s_1 \ge s_2$ (Fig. 4); then subtraction of (33) from (16) gives

$$\int_0^{s_1(t)} H_{L1}\,dx + \int_{s_1(t)}^{L} (H_{S1}-H_{S2})\,dx - \int_{s_2(t)}^{s_1(t)} H_{S2}\,dx + (\rho_{Sm}\ell + H_m)[s_1(t)-s_2(t)] = 0 \tag{33a}$$

Now, on $x = s_1(t)$, $T_1 = T_m$ and $T_2 \le T_m$, so that $T_1 - T_2 \ge 0$ (and therefore $H_{S1}-H_{S2} \ge 0$) for the entire range $s_1(t) \le x \le L$, by Theorem II. Furthermore

$$\int_{s_2(t)}^{s_1(t)} H_{S2}\, dx \le H_m\, [s_1(t)-s_2(t)] \qquad (33b)$$

Therefore the left-hand side of (33a) is composed only of non-negative terms and as a consequence this equation cannot be satisfied. Hence the difference s_1-s_2 cannot be positive (and thus neither can T_1-T_2), and the theorem is established.

Problem 2 is thus seen to provide an upper bound to the solution of problem 1; a fortiori, upper bounds to the solution of problem 2 are also upper bounds to the solution of the present problem. A method for constructing bounds for problem 2 was developed in [1].

Between the two extremes of instantaneous removal and of stationary melt there may be defined intermediate problems corresponding to finite rates of ablation. It may be conjectured, as an extension of the last theorem, that a monotonic relationship exists, in the problems, between the rate of ablation and the rate of advance of the solid-liquid interface; this question is however not examined here.

5. Construction of Bounds; Example

The procedure for the use of the bounds previously derived, in estimating the solution to an actual problem, consists essentially of constructing a solution of Eqs. (7) to (14), with a heat input $\bar{Q}(t)$ at $x = 0$ which may or may not equal the prescribed input $Q(t)$, disregarding however the last of Eqs. (13). The latter equation is then used to calculate $Q^*(t)$, and Theorem V insures that either an upper or a lower bound has been found in a range $0 < t < t_1$, according to whether the relations

$$Q^*(t) \geq 0 \; ; \quad \bar{Q}(t) + Q^*(t) \geq Q(t) \tag{34}$$

are always satisfied or always violated in that range. It will be noted that, though a little more complicated in practice, this procedure is quite analogous to that described in [1] for the problem in which the melt is instantaneously removed, and therefore will be discussed only briefly here.

It is convenient to split the solution in two parts, the first pertaining to the solid portion and the second to the liquid portion. The solution for the solid is easily derived by considering a fictitious extension to a slab of the original thickness L, under an arbitrarily chosen fictitious heat input $\bar{\bar{Q}}(t)$ at $x = 0$. It is shown in [1,3] that the relation $T_S(s,t) = T_m$ then leads to the following equation for $s(t)$, for the case of constant properties in the solid:

$$\int_0^y \frac{\bar{\bar{f}}(y-y_1)}{\sqrt{y_1}} \sum_{n=0}^{\infty} \{ e^{-[2nL'+\bar{\xi}(y)]^2/y_1} + e^{-[2(n+1)L'-\bar{\xi}(y)]^2/y_1} \} \, dy_1 =$$

$$= 2\alpha - \int_0^{y+1} \frac{f(y-y_1)}{\sqrt{y_1}} \sum_{n=0}^{\infty} \{ e^{-[2nL'+\bar{\xi}(y)]^2/y_1} + e^{-[2(n+1)L'-\bar{\xi}(y)]^2/y_1} \} \, dy_1 \tag{35}$$

with the following dimensionless notation:

$$y = \frac{t}{t_m} - 1 \; ; \quad y_1 = \frac{t_1}{t_m} \; ; \quad \xi(y) = \frac{s(t)}{2\sqrt{\kappa_S t_m}} \; ; \quad \alpha = \frac{k_S T_m}{2Q_0} \sqrt{\frac{\pi}{\kappa_S t_m}} \; ;$$

$$L' = \frac{L}{2\sqrt{\kappa_S t_m}} \; ; \quad f(y) = \frac{Q(t)}{Q_0} \; ; \quad \bar{\bar{f}}(y) = \frac{\bar{\bar{Q}}(t)}{Q_0} - 1 \tag{35a}$$

where Q_0 is a constant reference heat input. In deriving this equation, the convenient choice $\bar{\bar{Q}}(t) = Q(t)$ for $t < t_m$ was made.

The solution for the liquid portion now requires the determination of the solution of Eq. (11) in the region $0 < x < \bar{s}(t)$, $t > t_m$, satisfying the conditions

$$\left.\begin{aligned} T_L\,[s(t),\,t] &= T_m \\ -k_L\,\frac{\partial T_L}{\partial x}\,(0,t) &= \bar{Q}(t) \end{aligned}\right\} \quad t \geq t_m \qquad (36)$$

where the function $s(t)$ is given by Eq. (35). Since $s(t)$ will be in general known only numerically, a numerical solution of the problem for the liquid is probably the most qppropriate; the needed solution of the heat-condition equation for a domain bounded by boundaries moving in a prescribed manner can in fact be conveniently carried out by such methods. For the purpose of illustrating the bounds, however, analytical expressions valid for short times will be derived by the integral-equation method of [3]. According to that method, $T_L(x,t)$ can be obtained as the temperature in the region[8] $0 < x < \infty$, under the heat input $\bar{Q}(t)$ at $x = 0$ and initially (i.e. at $t = t_m$) at an initial temperature distribution $T_L(x,t_m) = T_m\theta(X)$, where $\theta(1) = 1$ and $X = x/2\sqrt{\kappa_S t}$. The first of Eqs. (36) then gives the following integral equation for $\theta(X)$, for the case in which the properties of the liquid are uniform but not necessarily the same as those of the solid:

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[8] This is a special case of the a thod of [3]; more generally, one may take the region $0 < x < L_L$ $(L_L \geq L)$, under the conditions listed above and in addition under a heat unput at $x = L_L$ to be suitably specified.

$$\int_0^\infty \theta(X) \left[e^{-(\xi-X)^2/Dy} + e^{-(\xi+X)^2/Dy} \right] dX =$$

$$= \sqrt{\frac{\pi y}{D}} \{1 - \frac{K}{2\alpha\sqrt{D}} \int_0^y \bar{f}(y-y_1)\, e^{-\xi^2 D/y_1} \frac{dy_1}{\sqrt{y_1}} \} \tag{37}$$

with the following additional dimensionless notation:

$$D = \frac{\kappa_S}{\kappa_L}; \quad K = \frac{k_S}{k_L}; \quad \bar{f}(y) = \frac{\bar{Q}(t)}{Q_0} \tag{37a}$$

Once Eq. (37) is solved, the last of Eqs. (13) is used to determine $Q^*(y)$.

Consider now, for simplicity, the problem corresponding to $Q(t) = Q_0$, a constant, so that $\alpha = 1$ and so that the exact solution of [3], for short times, is:

$$\xi(y) = \frac{2m}{3\pi} y^{3/2} - \frac{m}{4\sqrt{\pi}} y + O(y^{5/2})$$

$$\bar{\bar{f}}(y) = -\frac{2}{\pi} y^{1/2} + \frac{m}{\sqrt{\pi}} y + O(y^{3/2}) \tag{38}$$

$$\theta(X) = 1 - K\sqrt{\pi}\, X + O(X^2)$$

To establish the bounds, let then

$$\bar{\bar{f}}(y) = -\frac{2}{\pi} y^{1/2} + \frac{m}{\sqrt{\pi}} ay \tag{39}$$

where a is a constant to be discussed presently. Substitution of this and of $f(y) = 1$ into Eq. (35) gives

$$\xi(y) = \frac{2m}{3\pi} ay^{1/2} + O(y) \tag{39a}$$

so that clearly for short times this is an upper or a lower bound according to whether $a > 1$ or $a < 1$, the exact result corresponding to $a = 1$. The short-time solution of (37) is, with $\bar{F} = 1$,

$$\theta(X) = 1 + O(X) \tag{39b}$$

and finally the last of Eqs. (13) gives

$$\frac{Q^*(y)}{Q_0} = \frac{2}{\pi}(a-1)\, y^{1/2} + O(y) \tag{39c}$$

Clearly, for $y \ll 1$, $Q^* = 0$ if $a = 1$, as it should be for the exact solution; furthermore, $Q^* > 0$ or $Q^* < 0$ according to whether $a > 1$ or $a < 1$. Since here $\bar{Q}(t) = Q(t)$, the former of these possibilities corresponds to an upper bound, and the latter to a lower bound, according to relations (34); this is plainly in agreement with the melting rates found in Eq. (39a).

In conclusion, it may be remarked that a simple bound to the solution of the problem defined by Eqs. (7) to (14) is easily found in the special case in which the properties of the solid and liquid are uniform and equal. If again $Q(t) = Q_0$ and for a semi-infinite solid, one may take the temperature both before and after melting as

$$T(x,t) = \frac{2Q_0\sqrt{\kappa t}}{k}\, \mathrm{ierfc}\, \frac{x}{2\sqrt{\kappa t}}, \quad 0 < x < \infty \tag{40}$$

It is then easy to show that the relation $T = T_m$ gives

$$1 = \sqrt{\pi(1+y)}\, \mathrm{ierfc}\, (\xi/\sqrt{1+y}) \tag{40a}$$

and that

$$\frac{Q^*}{Q_0} = \frac{2}{m}\frac{d\xi(y)}{dy} = \frac{e^{-\xi^2/(1+y)}}{m\sqrt{1+y}\ \mathrm{erfc}\ (\xi/\sqrt{1+y})} \tag{40b}$$

where

$$m = \frac{\sqrt{\pi}\ c\ T_m}{2\ell} \tag{41}$$

This can be readily shown to correspond to an upper bound (except in the case, here trivial, of $m = \infty$, in which it is an exact wolution with $Q^* = 0$): for example, for $y \ll 1$, Eqs. (40a) and (40b) give

$$\begin{aligned} \xi &= \frac{1}{2\sqrt{\pi}}\ y \geq \frac{2m}{3\pi}\ y^{3/2} \\ \frac{Q^*(y)}{Q_0} &= \frac{1}{m\sqrt{\pi}} > 0 \end{aligned} \tag{42}$$

in agreement with the conclusions of Theorem V.

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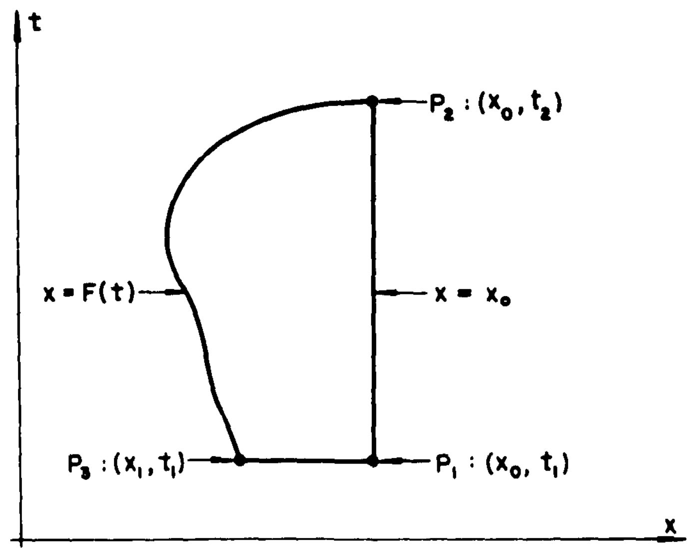
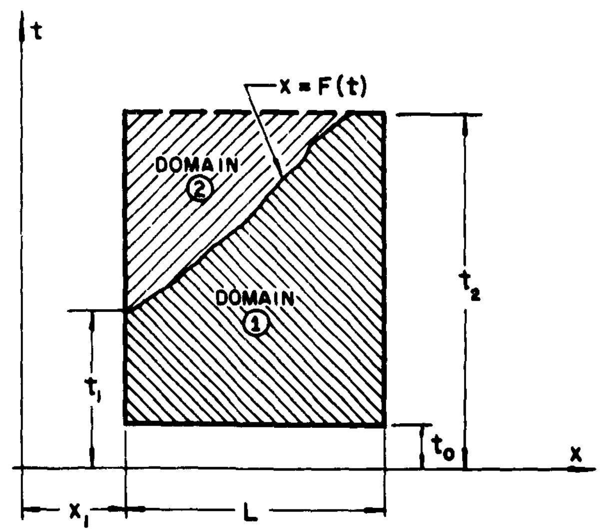

t
P₂ : (x₀, t₂)
x = F(t)
x = x₀
P₃ : (x₁, t₁)
P₁ : (x₀, t₁)
x

FIG. 1


t
x = F(t)
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t₂
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x
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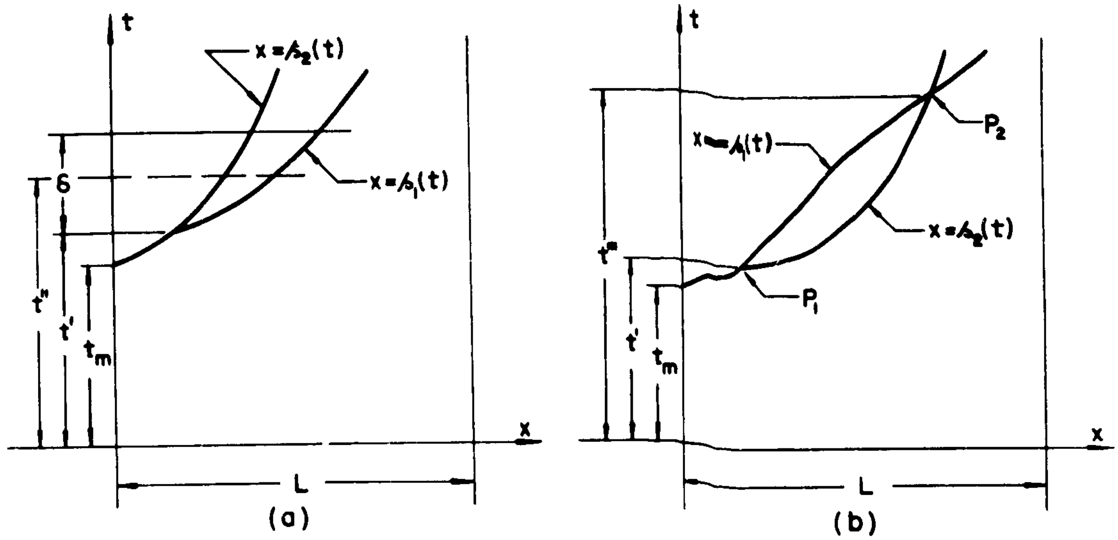
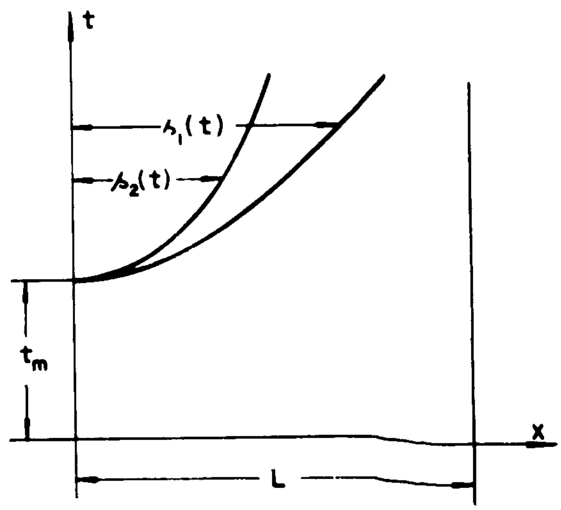
FIG. 2



F I G . 3



F I G . 4