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Technical Report No. 18

January 8, 1963

Contract Nonr 477(15)

Project Number NR 043 186

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**A New Type of Vector Field
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In [1] Robert Hermann introduced the concept of tangent vector fields on the space of maps of one manifold into another. A special type of these are the "k-vector fields" which were studied in [3], where this author defined their bracket and exponential. This paper explores further the analogy with classical continuous groups. Specifically, we study invariance of systems of partial differential equations under k-vector fields.

1. Introduction

Every map and manifold is C^∞ unless otherwise noted. $J^k = J^k(N, M)$ is the manifold of k-jets $j_x^k(f)$ of order k of maps $f: N \rightarrow M$ from the manifold N to the manifold M. α and β are the source and target projections, $\rho_1^{k-1}: J^{k+1} \rightarrow J^1$ the usual projection. $T(M)$ denotes the tangent bundle to M, M_y the tangent space at $y \in M$, π the tangent bundle projection. $C^\infty(Q)$ is the algebra (over the reals R) of C^∞ real-valued functions on the manifold Q.

A k-vector field is a map $\theta: C^\infty(M) \rightarrow C^\infty(J^k)$ which is linear over R and satisfies

$$\theta(FG) = (F\theta) \theta(G) + (G\theta) \theta(F) .$$

In [3] the ith prolongation $P^i \theta: C^\infty(J^1) \rightarrow C^\infty(J^{1+k})$

* Supported by ONR Contract 202736.

was defined. This satisfies $P^i \Theta(FG) = (F \circ \rho_1^{k+1}) P^i \Theta(G) + (G \circ \rho_1^{k+1}) P^i \Theta(F)$; and when $H \in C^\infty(M)$, $P^i \Theta(H \circ \rho) = \Theta(H) \circ \rho_1^{k+1}$. Using these facts one sees that if Θ and ψ are k - and i - vector fields, respectively, then $[\Theta, \psi] = P^i \Theta \circ \psi - P^k \psi \circ \Theta$ is a $k+i$ - vector field.

In local coordinates (x^i) on N , (y^λ) on M , $(x^i, y^\lambda, p_{j_1}^\lambda, \dots, p_{j_1 \dots j_k}^\lambda)$ on J^k , where $i, j_1, \dots, j_k = 1, \dots, n$; $\lambda = 1, \dots, m$, we follow Kuranishi in defining for each $F \in C^\infty(J^k)$, $\delta_j^\# F \in C^\infty(J^{k+1})$ by

$$\delta_j^\# F = \frac{\partial F}{\partial x^j} + \frac{\partial F}{\partial y^\lambda} p_{j_1}^\lambda + \dots + \frac{\partial F}{\partial p_{j_1 \dots j_k}^\lambda} p_{j_1 \dots j_k}^\lambda.$$

Then if $\Theta = a^\lambda (\partial / \partial y^\lambda)$ is a k -vector field,

$$P^i \Theta = a^\lambda \frac{\partial}{\partial y^\lambda} + \delta_j^\# a^\lambda \frac{\partial}{\partial p_j^\lambda} + \dots + \delta_{j_1}^\# \dots \delta_{j_i}^\# a^\lambda \frac{\partial}{\partial p_{j_1 \dots j_i}^\lambda}.$$

[See 3, Lemma 1.] We shall also need the following Lemma whose proof we omit.

Lemma 1. Let Θ be a k -vector field, $F_1, F_2 \in C^\infty(J^i)$, $G \in C^\infty(M)$ and $F \in C^\infty(J^{i-j})$, where $0 < j < i$. Then

- (A) $P^i \Theta(F \circ \rho_{1-j}^i) = (P^{i-j} \Theta(F)) \circ \rho_{1-j+k}^{i+k}$,
- (B) $P^i \Theta(G \circ \rho) = \Theta(G) \circ \rho_1^{i+k}$,
- (C) $P^i \Theta(F_1 F_2) = (F_1 \circ \rho_1^{i+k}) P^i \Theta(F_2) + (F_2 \circ \rho_1^{i+k}) P^i \Theta(F_1)$,
- (D) $P^i \Theta(\delta_{j_1}^\# \dots \delta_{j_r}^\# G \circ \rho_r^i) = \delta_{j_1}^\# \dots \delta_{j_r}^\# \Theta(G) \circ \rho_{r+1}^{i+k}$, $r < k$,
- (E) $P^i \Theta(\delta_{j_1}^\# \dots \delta_{j_r}^\# F \circ \rho_{1-j+r}^i) = (\delta_{j_1}^\# \dots \delta_{j_r}^\# P^{i-j} \Theta(F)) \circ \rho_{1-j+k+r}^{i+k}$,

$r < j$.

Conversely, if $\phi: C^\infty(J^i) \rightarrow C^\infty(J^{i+k})$ satisfies (A), ...,

(E) when $P^1\theta$ is replaced by θ , then $\theta = P^1\theta$.

Another important property for us is that if $F \in C^\infty(J^1)$, $f: N \rightarrow M$, then $(\partial/\partial x^i)G(j^1(f)) = (\partial_i^\#G)(j^{1+1}(f))$.

[2, Prop. 1.10]

Let $I = (-\epsilon, \epsilon)$. An integral curve of θ starting at $f_0: N \rightarrow M$ is a 1-parameter family $f: N \times I \rightarrow M$ with $f_0(x) = f(x, 0)$ and

$$\theta(j_x^k(f)) = \frac{\partial f}{\partial t}(x, t).$$

Here $(\partial f/\partial t)(x, t) \in \mathfrak{M}_{f(x, t)}$ is defined to act on any real-valued function F defined in a neighborhood of $f(x, t)$ by $dF(f(x, t))/dt$.

2. Differential Systems

A system Σ of partial differential equations (s.p.d.e.) of order h with N as independent and M as dependent variables is a finitely generated ideal in $C^\infty(J^h)$. A solution of Σ is a map $f: N \rightarrow M$ such that $F(j_x^h(f)) = 0$ for all $x \in N$, $F \in \Sigma$. $P^k\Sigma$ denotes the s.p.d.e. of order $h+k$ generated by the functions $F \circ \rho_h^{h+k}$, $\partial_j^\# F \circ \rho_{h+1}^{h+k}$, ..., $\rho_{j_1}^\# \dots \rho_{j_k}^\# F$, $1 \leq j, j_t \leq n$, $F \in \Sigma$.

Definition. A k -vector field θ leaves Σ invariant if for each $F \in \Sigma$, $P^k\theta(F) \in P^k\Sigma$.

Compare with [2] for the older theory. The intuitive meaning of invariance under a transformation group was that the transformations permute the solutions. We shall show that if f_0 is a solution of Σ which belongs to an integral curve of θ , then Σ evaluated at this integral curve has zero derivatives at f_0 of all

orders.

Lemma 2. If θ is an invariant vector field of Σ , then θ is an invariant vector field for $P^i \Sigma$, all i .

This follows from (D) and (E) in Lemma 1. Using local coordinates, a calculation proves

Lemma 3. If $F \in C^\infty(J^1)$, $f: N \times I \rightarrow M$, and $(\partial f / \partial t) = \theta(j_x^k(f))$, then

$$\frac{\partial}{\partial t} F(j_x^1(f)) = P^1 \theta(F) \Big|_{j_x^{k+1}(f)}$$

Lemma 4. If $f: N \rightarrow M$ is a solution of Σ , it is a solution of $P^i \Sigma$, all i .

Theorem 1. Suppose that

- (A) θ is an invariant k -vector field of Σ ,
- (B) $f: N \times I \rightarrow M$ satisfies $(\partial f / \partial t) = \theta(j_x^k(f))$, and
- (C) $f(\cdot, 0): N \rightarrow M$ is a solution of Σ .

Then

$$\frac{\partial^n}{\partial t^n} F(j_x^h(f)) \Big|_{t=0} = 0$$

for all $x \in N$, $F \in \Sigma$, and $n = 1, 2, \dots$

Proof: From Lemma 3,

$$\frac{\partial}{\partial t} F(j_x^h(f)) = P^h \theta(F) \Big|_{j_x^{k+h}(f)}$$

However, $P^h \theta(F) \in P^h \Sigma$, and f is a solution of $P^h \Sigma$ by Lemma 4. Hence $P^h \theta(F)(j_x^{k+h}(f)) \Big|_{t=0} = 0$, all $x \in N$.

Let $F^1 = p^h \theta(F) \in P^k \Sigma$. By Lemma 3,

$$p^{h+k} \theta(F) \Big|_{j_x^{2k+h}(f)} = \frac{d}{dt} F^1(j_x^{h+k}(f)) = \frac{d}{dt} \left[\frac{d}{dt} F(j_x^h(f)) \right].$$

Using Lemma 4 as before, $(\partial^2/\partial t^2) F(j_x^h(f)) \Big|_{t=0} = 0$.
Continuing in this way, the result follows. Q.E.D.

When the manifolds and functions are real analytic, Theorem 1 implies that integral curves of an invariant vector field which pass through one solution yield solutions for all parameter values.

3. Lie Algebra Structure

Proposition. Let θ and ψ be k - and h -vector fields, respectively. Then

$$P^i[\theta, \psi] = P^{i+h} \theta \circ P^i \psi - P^{i+k} \psi \circ P^i \theta.$$

Proof. By induction on i . A local coordinate calculation shows the result for $i = 1$. Call $\theta: C^\infty(J^1) \rightarrow C^\infty(J^{1+h+k})$ the operator on the right-hand side. We shall use Lemma 1. Let $F_1, F_2 \in C^\infty(J^1)$, $G \in C^\infty(M)$, and $F \in C^\infty(J^{1-j})$.

$$\begin{aligned} P^{i+h} \theta \circ P^i \psi (F \circ \rho_{i-j}^1) &= P^{i+h} \theta(P^{i-j} \psi(F)) \circ \rho_{i+h-j}^{1+h} \\ &= (P^{i+h-j} \theta(P^{i-j} \psi(F))) \circ \rho_{i+h+k-j}^{1+h+k} \\ &= (P^{i+h-j} \theta \circ P^{i-j} \psi)(F) \circ \rho_{i+h+k-j}^{1+h+k}, \end{aligned}$$

applying Lemma 1(A) to ψ and θ . Interchanging θ and ψ , we find

$$P^i \theta(F \circ P_{i-j}^1) = (P^{i-j} \theta(F)) \circ P_{i+h+k-j}^{i+h+k}.$$

Now, by induction, $P^{i-j} \theta(F) = P^{i-j} [\theta, \psi]$. Hence (A) holds for θ . The same technique works for (B), ..., (E). Q.E.D.

Theorem 2. If θ and ψ are k - and h -vector fields, respectively, which leave Σ invariant, then $[\theta, \psi]$ leaves Σ invariant.

Proof. If $x \in \Sigma$ and Σ is of order 1, then $P^i [\theta, \psi](F) = P^{i+h} \theta \circ P^i \psi(F) - P^{i+k} \psi \circ P^i \theta(F)$. However, $P^i \psi(F) \in P^h \Sigma$. By Lemma 2 θ is an invariant vector field of $P^h \Sigma$, so $P^{i+h} \theta \circ P^i \psi(F) \in P^{h+k} \Sigma$. Similarly $P^{i+k} \psi \circ P^i \theta(F) \in P^{h+k} \Sigma$. Q.E.D.

We conclude that the set of all k -vector fields, $k = 1, 2, \dots$, leaving Σ invariant forms a Lie algebra under the bracket.

4. An Example

Let $N = E^n$, $M = E^m$. Consider a s.p.d.e. of the type

$$\frac{\partial y^\lambda}{\partial x^n} = \theta^\lambda(x^1, \dots, x^{n-1}, y^\mu, \frac{\partial y^\mu}{\partial x^1}, \dots, \frac{\partial y^\mu}{\partial x^{n-1}}),$$

$\lambda, \mu = 1, \dots, m$. On J^1 let $F^\lambda = p_n^\lambda - \theta^\lambda(x^1, y^\mu, p_1^\mu)$,

and let Σ be generated by F^1, \dots, F^m . Then by a calculation one may check that $\Theta = \rho^\lambda (\partial / \partial y^\lambda)$ turns out to be an invariant vector field of Σ .

We can see that Θ generates solutions of the Cauchy problem associated with Σ . Since Θ is independent of x^n and p_n^λ , it can be considered a 1-vector field on E^{n-1} . Suppose $f_0: E^{n-1} \rightarrow E^n$ is the initial data at $x^n = 0$. Suppose $I = \{x^n | -\epsilon < x^n < \epsilon\}$ and $f: E^{n-1} \times I \rightarrow E^n$ is an integral curve of Θ through f_0 . But that is merely another way of saying that f is a solution of Σ .

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