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SINGULAR SOLUTIONS OF AN INTEGRO-DIFFERENTIAL EQUATION IN RADIATIVE TRANSFER

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PREFACE

Solutions to mathematical equations are often approximated by numerical solutions to approximate equations. It is helpful, in bounding numerical errors, to know general properties of solutions to the perturbed equations. In this Memorandum the author describes a class of unbounded solutions obtained by perturbing the initial data in an equation for the scattering (reflection) function in radiative transfer in a homogeneous slab.

SUMMARY

In the theory of radiative transfer in a homogeneous isotropic slab of thickness τ the scattering (reflection) function can be determined by a nonlinear integro-differential equation and initial conditions. For a numerical analysis of this equation it is often important to know the behaviour of solutions in the vicinity of the desired solution.

We extend in this Memorandum our previous treatment, RM-3548-PR, of conservative and isotropic scattering to the nonconservative case. We exhibit a set of initial conditions for which the solutions to our nonlinear integro-differential equation are infinite for finite values of the parameter τ. Some of these singular solutions first come close to the desired solution and then diverge to infinity. The nearness of approach of these singular solutions is proportional to a quantity which measures the nearness of local scattering to the conservative case. The conservative case is again found by a continuous passage from nonconservative to conservative scattering.

CONTENTS

SINGULAR SOLUTIONS OF AN INTEGRO-DIFFERENTIAL EQUATION IN RADIATIVE TRANSFER

1. INTRODUCTION

In a previous Memorandum [5] we studied an integro-differential equation satisfied by the scattering function S for a homogeneous slab of finite thickness τ with isotropic and conservative scattering. We exhibited singular solutions by imposing initial conditions different from those determining the desired solution. We showed that some of these singular solutions first come arbitrarily close to the desired solution for τ sufficiently large and then diverge to infinity for a larger, but finite, value of τ.

We now extend this analysis to isotropic and non-conservative scattering. We again exhibit singular solutions, some of which first approach the desired solution and then go to infinity. The closeness of approach depends on a measure of the nearness of local scattering to the conservative case. The conservative case is again obtained by letting the albedo of local scattering ω tend to 1.

We have been unable to think of any physical model that would correspond to the initial conditions on S that yield singular solutions. Since S goes to infinity for finite τ, we suspect that this may correspond to criticality for some type of neutron reactor in which a plane of fissionable material is backed by a reflector of scattering material.

Aside from representing any physical model, these singular solutions could be of importance in a numerical integration of the integro-differential equation with the usual initial conditions [1]. The presence of singular solutions near the desired solutions limits integration step size and permissible error accumulation since the numerical solution must be prevented from jumping to one of these singular solutions.

2. NONCONSERVATIVE SCATTERING

We consider the equation [3, p. 169]

$$\begin{aligned}(\tfrac{1}{\mu} + \tfrac{1}{\mu_o})S(\mu, \mu_o, \tau) + \frac{\partial S}{\partial \tau}(\mu, \mu_o, \tau) &= \\ \left[1 + \frac{\varpi}{2}\int_0^1 S(\mu, \sigma, \tau)\frac{d\sigma}{\sigma}\right]&\left[1 + \frac{\varpi}{2}\int_0^1 S(\sigma, \mu_o, \tau)\frac{d\sigma}{\sigma}\right],\end{aligned} \tag{2.1}$$

with $0 \leq \tau < \infty$, $0 \leq \mu, \mu_o \leq 1$. In this section we restrict ϖ to the interval $0 < \varpi < 1$. The usual initial conditions on (2.1) are

$$S(\mu, \mu_o, 0) = 0. \tag{2.2}$$

We shall determine solutions to (2.1) of the form

$$S(\mu, \mu_o, \tau) = \frac{\mu\mu_o}{\mu+\mu_o}[X(\mu,\tau)X(\mu_o, \tau) - Y(\mu, \tau)Y(\mu_o, \tau)], \tag{2.3}$$

where X and Y are solutions to Chandrasekhar's equations [3]

$$(2.4)\qquad \begin{aligned} X(\mu) &= 1 + \frac{\mu\varpi}{2}\int_0^1 \frac{X(\mu)X(\nu) - Y(\mu)Y(\nu)}{\nu + \mu}\, d\nu, \\ Y(\mu) &= e^{-\tau/\mu} + \frac{\mu\varpi}{2}\int_0^1 \frac{X(\mu)Y(\nu) - Y(\mu)X(\nu)}{\nu - \mu}\, d\nu. \end{aligned}$$

A particular solution to (2.4) is given by Busbridge [2], which we designate by (X_o, Y_o).

To exhibit all solutions to (2.4), we let k be the nonzero root of the equation

$$(2.5)\qquad 2k = \omega \ln \frac{1 + k}{1 - k}, \quad 0 < k < 1.$$

We have shown [4] that for a fixed τ all solutions to (2.4) are

$$(2.6)\qquad \begin{aligned} X(\mu) = &\left[1 + \frac{(f\alpha - g\beta)\mu + k\mu^2(f\alpha + g\beta)}{1 - (k\mu)^2}\right] X_o(\mu) \\ &+ \frac{(f\beta - g\alpha)\mu + k\mu^2(f\beta + g\alpha)}{1 - (k\mu)^2}\, Y_o(\mu), \end{aligned}$$

and

$$(2.7)\qquad \begin{aligned} Y(\mu) = &\left[1 - \frac{(f\alpha - g\beta)\mu - k\mu^2(f\alpha + g\beta)}{1 - (k\mu)^2}\right] Y_o(\mu) \\ &- \frac{(f\beta - g\alpha)\mu - k\mu^2(f\beta + g\alpha)}{1 - (k\mu)^2}\, X_o(\mu). \end{aligned}$$

The constants α and β are given by

$$\alpha = 1 - \frac{\omega}{2} \int_0^1 \frac{X_o(\nu)\,d\nu}{1 + k\nu} \,, \tag{2.8}$$

$$\beta = \frac{\omega}{2} \int_0^1 \frac{Y_o(\nu)\,d\nu}{1 + k\nu} \,.$$

The parameters f and g in (2.6) and (2.7) are constrained by

$$(f^2 - g^2)(\alpha^2 - \beta^2) = 2k(f\alpha + g\beta). \tag{2.9}$$

We shall show that Busbridge's solutions (X_o, Y_o) in (2.3) give a solution to (2.1) and (2.2). We shall show, however, that f and g can be computed as functions of τ so that (2.3) satisfies (2.1) and initial conditions different from (2.2). Some of these functions are singular.

With S given by (2.3), (2.6), and (2.7), an elementary computation shows that

$$1 + \frac{\omega}{2} \int_0^1 S(\mu, \sigma)\frac{d\sigma}{\sigma} = X(\mu) \tag{2.10}$$

$$+ \frac{\mu^2}{1 - (k\mu)^2} \left[(f^2 - g^2)(\alpha^2 - \beta^2) - 2k(f\alpha + g\beta)\right].$$

With this and (2.9) we reduce equation (2.1) to

$$\text{(2.11)} \qquad \begin{aligned} &\frac{\mu\mu_o}{\mu+\mu_o} \frac{\partial}{\partial\tau} [X(\mu, \tau)X(\mu_o, \tau) - Y(\mu, \tau)Y(\mu_o, \tau)] \\ &\quad = Y(\mu, \tau)Y(\mu_o, \tau). \end{aligned}$$

It can easily be shown [3, p. 185] that X_o and Y_o satisfy

$$\text{(2.12)} \qquad \frac{\partial X_o}{\partial\tau}(\mu, \tau) = Y_o(\mu, \tau)y_{-1}(\tau)$$

and

$$\text{(2.13)} \qquad \frac{\partial Y_o}{\partial\tau}(\mu, \tau) = -\frac{Y_o(\mu, \tau)}{\mu} + X_o(\mu, \tau)y_{-1}(\tau).$$

The quantity y_{-1} is given by

$$\text{(2.14)} \qquad y_{-1}(\tau) = \frac{\omega}{2}\int_0^1 Y_o(\nu, \tau)\frac{d\nu}{\nu}, \quad \text{for} \quad \tau > 0.$$

If we demand that f and g in (2.6) and (2.7) be functions of τ, so that X and Y satisfy (2.12) and (2.13) with y_{-1} computed for Y given by (2.13), it follows easily that (2.11) is satisfied. To use this fact in finding differential equations for f and g, we shall need properties of α and β. Using (2.12) and (2.13) in (2.8), we easily establish that

$$\text{(2.15)} \qquad \frac{d\alpha}{d\tau} = -\beta y_{-1}, \quad \frac{d\beta}{d\tau} = k\beta - \alpha y_{-1}.$$

Here and in subsequent equations, y_{-1} is computed with the Y_o function.

An elementary but detailed calculation shows that X and Y satisfy (2.12) and (2.13) if and only if f and g satisfy the equations

$$
\begin{aligned}
&\frac{df}{d\tau} - gy_{-1} = -\,(f\beta - g\alpha)g\ , \\
(2.16)\qquad & \\
&\frac{dg}{d\tau} + kg - fy_{-1} = -\,(f\beta - g\alpha)f\ .
\end{aligned}
$$

We shall see that these equations are compatible with the constraint (2.9).

We first introduce new variables so that (2.9) is in the canonical form

$$(2.17)\qquad \xi\eta = 1.$$

This is accomplished by the transformation

$$
\begin{aligned}
&\xi = \frac{\sqrt{\alpha^2 - \beta^2}}{k}\left(f + g - \frac{k}{\alpha+\beta}\right)\ , \\
(2.18)\qquad & \\
&\eta = \frac{\sqrt{\alpha^2 - \beta^2}}{k}\left(f - g - \frac{k}{\alpha-\beta}\right).
\end{aligned}
$$

By means of (2.15) we can transform the equations (2.16) to the variables ξ and η, obtaining

$$\begin{aligned}\frac{d\xi}{d\tau} - y_{-1}\xi &= -\frac{k\xi}{2\sqrt{\alpha^2-\beta^2}}\left[(\beta-\alpha)\xi + (\alpha+\beta)\eta\right] - \frac{3\alpha\beta k}{\alpha^2-\beta^2}\xi\ , \\ \frac{d\eta}{d\tau} + y_{-1}\eta &= \frac{k\eta}{2\sqrt{\alpha^2-\beta^2}}\left[(\beta-\alpha)\xi + (\alpha+\beta)\eta\right] + \frac{3\alpha\beta k}{\alpha^2-\beta^2}\eta .\end{aligned} \tag{2.19}$$

We see that <u>not only are these compatible with</u> (2.17), <u>but they uncouple along this hyperbola.</u> We then have

$$\frac{d\xi}{d\tau} - y_{-1}\xi = \frac{k}{2}\sqrt{\frac{\alpha-\beta}{\alpha+\beta}}\,\xi^2 - \frac{3\alpha\beta k}{\alpha^2-\beta^2}\xi - \frac{k}{2}\sqrt{\frac{\alpha+\beta}{\alpha-\beta}}\ , \tag{2.20}$$

$$\xi\eta = 1.$$

Since $f = g = 0$ is a particular solution of (2.9) and (2.16), a particular solution of (2.20) is given by

$$\xi_o = -\sqrt{\frac{\alpha-\beta}{\alpha+\beta}}\ , \quad \eta_o = -\sqrt{\frac{\alpha+\beta}{\alpha-\beta}}\ . \tag{2.21}$$

By the standard transformation

$$\xi = \xi_o + u^{-1}, \tag{2.22}$$

we obtain the equation

$$\frac{du}{d\tau} + pu = q, \tag{2.23}$$

with

$$p = y_{-1} - k - \frac{2k\beta^2}{\alpha^2-\beta^2} - \frac{k\alpha\beta}{\alpha^2-\beta^2}\,,$$

(2.24)

$$q = -\frac{k}{2}\sqrt{\frac{\alpha-\beta}{\alpha+\beta}}\;.$$

From (2.15) we readily obtain

$$p = -k + (\alpha^2-\beta^2)^{-1}\frac{d}{d\tau}(\alpha^2-\beta^2) + \frac{1}{2}\left(\frac{\alpha}{\beta}-1\right)^{-1}\frac{d}{d\tau}\left(\frac{\alpha}{\beta}-1\right)$$

(2.25)

$$-\frac{1}{2}\left(\frac{\alpha}{\beta}+1\right)^{-1}\frac{d}{d\tau}\left(\frac{\alpha}{\beta}+1\right).$$

With this we get from (2.23) the general solution to (2.20):

$$(2.26)\qquad \xi(\tau) = \xi_o(\tau)\left[1 - \frac{E(\alpha^2-\beta^2)e^{-k\tau}}{1 - \frac{Ek}{2}\int_0^\tau e^{-kt}[\alpha(t)-\beta(t)]^2dt}\right],$$

with E an arbitrary constant.

If in (2.26) we set $E = 0$, we obtain the solution (ξ_o, η_o) corresponding to $f = g = 0$ in (2.6) and (2.7). Thus (2.3) computed with X_o and Y_o gives the solution to (2.1) and (2.2). We now investigate other values of E corresponding to initial conditions on S other than (2.2).

First let E_o be the value of E determined by

$$(2.27)\qquad E_o\frac{k}{2}\int_0^\infty e^{-kt}[\alpha(t)-\beta(t)]^2dt = 1.$$

Then ξ is given by

$$(2.28) \qquad \xi(\tau) = -\sqrt{\frac{\alpha-\beta}{\alpha+\beta}}\left[1 - \frac{2(\alpha^2 - \beta^2)e^{-k\tau}}{k\int_\tau^\infty e^{-kt}[\alpha(t) - \beta(t)]^2 dt}\right].$$

We see that

$$(2.29) \qquad \lim_{\tau\to\infty} \xi(\tau) = \lim_{\tau\to\infty} \eta(\tau) = 1.$$

For all E other than E_o, we have

$$(2.30) \qquad \lim_{\tau\to\infty} \xi(\tau) = \lim_{\tau\to\infty} \eta(\tau) = -1.$$

With Δ defined by

$$(2.31) \qquad \Delta \equiv \alpha(0) - \beta(0) = 1 - \frac{\omega}{k}\ln(1 + k),$$

we get

$$(2.32) \qquad \xi(0) = \sqrt{\Delta}\,(E\Delta - 1).$$

<u>Therefore we have the following description of the functions</u> ξ <u>and</u> η:

(2.33)

(i) $E = E_o$ <u>implies that</u> ξ <u>and</u> η <u>bounded and positive with</u> $\lim_{\tau\to\infty} \xi(\tau) = 1 = \lim_{\tau\to\infty} \eta(\tau)$.

(ii) $\Delta^{-1} < E < E_o$ <u>implies that</u> ξ <u>tends to</u> 0, <u>and</u> η <u>tends to</u> ∞, <u>for a finite value of</u> τ.

(iii) $E_o < E$ <u>implies that</u> ξ <u>tends to</u> ∞, <u>and</u> η <u>tends to</u> 0, <u>for a finite value of</u> τ.

(iv) $E < \Delta^{-1}$ <u>implies that</u> ξ <u>and</u> η <u>bounded and</u> $\lim_{\tau\to\infty} \xi(\tau) = \lim_{\tau\to\infty} \eta(\tau) = -1$.

To establish (i) we need only to show that

$$\xi(0) > 0 \quad \text{for} \quad E = E_o; \tag{2.34}$$

that is, we need to show that

$$k \int_0^\infty e^{-kt}(\alpha - \beta)^2 dt < 2\Delta . \tag{2.35}$$

By (2.15) we have

$$(\alpha - \beta)^{-1} \frac{d}{d\tau} (\alpha - \beta) = y_{-1} - k\beta(\alpha - \beta)^{-1}, \tag{2.36}$$

and by (2.13) we have

$$\frac{dy_o}{d\tau} = - y_{-1}(1 - x_o), \tag{2.37}$$

where

$$(2.38)\qquad x_o = \frac{\omega}{2}\int_0^1 X_o(\nu)d\nu\ ,\quad y_o = \frac{\omega}{2}\int_0^1 Y_o(\nu)d\nu .$$

Two known results [2, p. 97] are

$$(2.39)\qquad (1 - x_o)^2 - y_o^2 = 1 - \omega$$

and

$$(2.40)\qquad 1 - x_o + y_o = 1/X_o(\infty\ ,\ \tau) < 1.$$

Therefore we obtain

$$(2.41)\qquad \alpha - \beta = \Delta(1 - x_o + y_o)\exp[- k\int_0^\tau \frac{\beta}{\alpha-\beta}\,dt] < \Delta\ ,$$

and

$$(2.42)\qquad 2\Delta - k\int_0^\infty e^{-kt}(\alpha - \beta)^2 dt > \Delta(2 - \Delta) > 0\ ,$$

since $0 < \Delta < 1$.

This result (2.34) shows that ξ and η are on the positive branch of $\xi\eta = 1$ and establishes (i). The validity of the other parts of (2.33) follows from the fact that the point $(\xi,\ \eta)$ can pass through ∞ at most once, and indeed must do so if $\xi(0) > 0$ and

$E \neq E_0$ since then (ξ, η) tends to $(-1, -1)$ as $\tau \to \infty$.

We now write (2.6) and (2.7) as

$$(2.43) \quad X(\mu) = [1 + \frac{c\mu + d\mu^2}{1 - (k\mu)^2}] X_0(\mu) + \frac{a\mu + b\mu^2}{1 - (k\mu)^2} Y_0(\mu),$$

and

$$(2.44) \quad Y(\mu) = [1 - \frac{c\mu - d\mu^2}{1 - (k\mu)^2}] Y_0(\mu) - \frac{a\mu - b\mu^2}{1 - (k\mu)^2} X_0(\mu).$$

The functions a, b, c, and d are given by

$$(2.45) \quad \begin{aligned} a &= \frac{k}{2} \left(-\sqrt{\frac{\alpha-\beta}{\alpha+\beta}}\, \xi + \sqrt{\frac{\alpha+\beta}{\alpha-\beta}}\, \eta\right) + \frac{2\alpha\beta k}{\alpha^2-\beta^2}, \\ b &= \frac{k^2}{2} \left(\sqrt{\frac{\alpha+\beta}{\alpha-\beta}}\, \xi - \sqrt{\frac{\alpha-\beta}{\alpha+\beta}}\, \eta\right), \\ c &= \frac{k}{2} \left(\sqrt{\frac{\alpha-\beta}{\alpha+\beta}}\, \xi + \sqrt{\frac{\alpha+\beta}{\alpha-\beta}}\, \eta\right) + \frac{\alpha^2+\beta^2}{\alpha^2-\beta^2}\, k, \\ d &= \frac{k^2}{2} \left(\sqrt{\frac{\alpha+\beta}{\alpha-\beta}}\, \xi + \sqrt{\frac{\alpha-\beta}{\alpha+\beta}}\, \eta\right) + k^2. \end{aligned}$$

Since $X_0(0) = Y_0(0) = 1$, we see that each value of the parameter E gives a solution S_E to (2.1) and the initial condition

$$(2.46) \quad S_E(\mu, \mu_0, 0) = \left[\frac{k}{\Delta} + \frac{k\eta(0)}{\sqrt{\Delta}} + \frac{k^3}{\Delta}\left(1 + \frac{\xi(0)}{\sqrt{\Delta}}\right)\mu\mu_0\right]\mu\mu_0,$$

with

(2.47) $\xi(0) = 1/\eta(0) = \sqrt{\Delta}\,(E\Delta - 1)$.

If E satisfies either (ii) or (iii) of (2.33), then S_E is infinite for a finite value of τ.

Since $\lim_{\tau \to \infty} \beta(\tau) = 0$, we can choose E satisfying $\Delta^{-1} < E < E_o$ and so near to E_o that, when (ξ, η) passes through (1, 1), β is negligible. Therefore, from (2.45) we see that there are singular solutions S_E to (2.1) and a value $\tau(E)$ of τ so that S_E is given approximately at $\tau(E)$ by

$$S_E(\mu, \mu_o, \tau(E)) \cong \frac{\mu\mu_o}{\mu+\mu_o}\left[\frac{(1 + k\mu)(1 + k\mu_o)}{(1 - k\mu)(1 - k\mu_o)} X_o(\mu,\tau(E))X_o(\mu_o,\tau(E)) - \frac{(1 - k\mu)(1 - k\mu_o)}{(1 + k\mu)(1 + k\mu_o)} Y_o(\mu, \tau(E))Y_o(\mu_o, \tau(E))\right]. \tag{2.48}$$

For a larger value of τ, this solution goes to infinity. The difference at $\tau(E)$ of this singular solution from the solution to (2.1) and (2.2) is proportional to the quantity $k/(1 - k)$. Therefore for small k, i.e., ω near 1, there are singular solutions to (2.1) that first come close to the solution to (2.1) and (2.2) and then go to infinity.

3. CONSERVATIVE SCATTERING

We wish to show that the solutions to (2.1) found in the previous sections for $0 < \omega < 1$ tend, as ω tends to 1, to the solutions given in [5] for $\omega = 1$.

We express solutions in terms of the functions a, b, c, and d. In terms of f and g, these are given by

$$
\begin{aligned}
a &= f\beta - g\alpha \,, \\
b &= k(f\beta + g\alpha) \,, \\
c &= f\alpha - g\beta \,, \\
d &= k(f\alpha + g\beta) \,.
\end{aligned}
\tag{3.1}
$$

We see then that the hyperbola (2.9) can be expressed in terms of a and b by

$$(k\zeta a)^2 + 2\delta\zeta ab - 2\zeta k^2 a + (\zeta b)^2 - 2\delta b = 0, \tag{3.2}$$

where

$$\zeta = \frac{\alpha^2-\beta^2}{2k\alpha\beta} \quad \text{and} \quad \delta = \frac{\alpha^2+\beta^2}{2\alpha\beta} \,. \tag{3.3}$$

From the fact [4] that

$$\alpha\left(1 - \frac{\omega}{2}\int_0^1 \frac{X_o(\nu)d\nu}{1-k\nu}\right) = \beta\,\frac{\omega}{2}\int_0^1 \frac{Y_o(\nu)d\nu}{1-k\nu} \quad , \tag{3.4}$$

we find

$$(3.5)\qquad \alpha^2 - \beta^2 = 2k\left(\frac{\alpha\varpi}{2}\int_0^1 \frac{\nu X_o(\nu)}{1-(k\nu)^2}\,d\nu + \frac{\beta\varpi}{2}\int_0^1 \frac{\nu Y_o(\nu)d\nu}{1-(k\nu)^2}\right).$$

Therefore we have

$$(3.6)\qquad \lim_{k\to 0} \alpha = \lim_{k\to 0} \beta = y_o\ .$$

This gives

$$(3.7)\qquad \lim_{k\to 0} \zeta \equiv \gamma = \frac{x_1 + y_1}{y_o} \quad \text{and} \quad \lim_{k\to 0} \delta = 1\ ,$$

where

$$(3.8)\qquad x_1 = \frac{1}{2}\int_0^1 \nu X_o(\nu)d\nu\ ,\quad y_1 = \frac{1}{2}\int_0^1 \nu Y_o(\nu)d\nu\ .$$

We see that the hyperbola (3.2) tends, as ϖ tends to 1, to the constraint

$$(3.9)\qquad (2\gamma a + \gamma^2 b - 2)b = 0\ .$$

This agrees with the constraint given in [5].

We now use (2.26) in (2.45) to write these functions as

$$
(3.10)\qquad
\begin{aligned}
a &= \frac{k}{2}\left[-\frac{\alpha-\beta}{\alpha+\beta}\psi + \frac{\alpha+\beta}{\alpha-\beta}\Gamma \right] , \\
b &= \frac{k^2}{2}\left[\psi - \Gamma\right] , \\
c &= \frac{k}{2}\left[\frac{\alpha-\beta}{\alpha+\beta}\psi + \frac{\alpha+\beta}{\alpha-\beta}\Gamma \right] , \\
d &= \frac{k^2}{2}\left[\psi + \Gamma\right] .
\end{aligned}
$$

Now ψ and Γ are given by

$$
(3.11)\qquad
\begin{aligned}
\psi &= \frac{(\alpha^2 - \beta^2)Ee^{-k\tau}}{1 - \frac{Ek}{2}\int_0^\tau e^{-kt}(\alpha - \beta)^2 dt} , \\
\Gamma &= \frac{\psi}{\psi - 1} .
\end{aligned}
$$

We fix τ and consider the limits in (3.10) as k decreases to 0.

We first let E depend on k as

$$
(3.12)\qquad E = \frac{2A}{A-1}\frac{1}{k} .
$$

We then find that

$$
(3.12)\qquad \lim_{k\to 0} \psi = \frac{4A}{A-1} y_o(x_1 + y_1) .
$$

This gives the solutions obtained in [5] for $k = 0$, namely,

$$a = c = \frac{4Ay_o^2}{1 - A[1 - 4y_o(x_1 + y_1)]} ,$$

(3.13)

$$b = d = 0 .$$

We now let E be given by

(3.14) $$E = \frac{4B}{k^3} ,$$

and find that

(3.15) $$\lim_{k\to 0} k^2\psi = \frac{8By_o(x_1 + y_1)}{1 - 2B\int_0^T (x_1 + y_1)^2 dt} .$$

This also gives solutions obtained in [5] for $k = 0$, namely

$$a = \gamma^{-1} - \frac{2B(x_1 + y_1)^2}{1 - 2B\int_0^T (x_1 + y_1)^2 dt} ,$$

(3.16) $$b = d = \frac{4By_o(x_1 + y_1)}{1 - 2B\int_0^T (x_1 + y_1)^2 dt} ,$$

$$c = a + \gamma b.$$

If we let E be given by

(3.17) $$E = \frac{A}{k^\alpha} , \quad 1 < \alpha < 3 ,$$

for $k = 0$ we obtain

$$\begin{aligned} a = c &= \gamma^{-1}, \\ b = d &= 0. \end{aligned} \tag{3.18}$$

This solution satisfies both of the linear constraints expressed by (3.9).

We have thus obtained all the functions discussed in our previous study of the case $\omega = 1$ as limits of solutions as ω tends to 1.

REFERENCES

1. Bellman, R., R. Kalaba, and M. Prestrud, Invariant Imbedding and Radiative Transfer in Slabs of Finite Thickness, The RAND Corporation, R-388-ARPA, February, 1962.

2. Busbridge, I. W., The Mathematics of Radiative Transfer, Cambridge Tracts, No. 50, Cambridge University Press, New York, 1960.

3. Chandrasekhar, S., Radiative Transfer, Dover, New York, 1960 (Oxford University Press, New York, 1950).

4. Mullikin, T. W., Chandrasekhar's X and Y Functions for Homogeneous Atmospheres, The RAND Corporation, RM-3376-JPL, December, 1962.

5. ________, A Type of Instability of an Integro-differential Equation for the Scattering Function in Radiative Transfer, The RAND Corporation, RM-3548-PR, March, 1963.