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Research Report

Transformation of the Equations of Motion of Meteorology Into Arbitrary Orthogonal Coordinates

EUGENE M. LUKS

METEOROLOGICAL RESEARCH LABORATORY PROJECT 8628

AIR FORCE CAMBRIDGE RESEARCH LABORATORIES, OFFICE OF AEROSPACE RESEARCH, UNITED STATES AIR FORCE, L. G. HANSCOM FIELD, MASS.

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Abstract

The earth is usually represented in Cartesian Coordinates on a projection plane with the atmosphere mapped three-dimensionally. Most such maps are orthogonal and in this case the equations of motion take a particularly simple form.

Acknowledgment

This report was prepared while the author was employed under Contract No. AF19(604)7247 with Regis College. It is, in part, a generalization of remarks made by Professor N. A. Phillips in lectures at M. I. T. on Numerical Weather Predictions. It was undertaken at the suggestion and under the guidance of Dr. Louis Berkofsky of GRD.

Transformation of the Equations of Motion of Meteorology into Arbitrary Orthogonal Coordinates

1. INTRODUCTION

The flow at various levels in the atmosphere is generally represented on flat surfaces, or maps. The equations which govern this flow are derived for the spherical earth. In order to solve these equations numerically, as applied to actual maps, one must transform the flow equations into map coordinates. The coordinate system to be used depends upon the problem. For example, almost all numerical prediction work concerns itself with flow over a large part of the Northern Hemisphere represented on a stereographic projection. For some purposes, the Mercator projection, or a combination of Mercator and stereographic, have been used.

For easy reference, it seemed desirable to write the equations of motion in several orthogonal coordinate systems. In this paper the equations of motion of meteorology have been transformed from spherical coordinates into <u>any</u> orthogonal system, so that one need only substitute the transformation formulas without actually carrying out the lengthy computations to obtain the desired system.

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2. THE TRANSFORMATIONS

The equations of motion of meteorology are usually expressed in terms of distances and directions relative to the earth. Thus, we have:

$$\begin{aligned} F_e &= r \cos \phi \ddot{\lambda} + 2 (\dot{\lambda} + \Omega)(\dot{r} \cos \phi - r \sin \phi \dot{\phi}), \\ F_n &= r \ddot{\phi} + 2 \dot{r} \dot{\phi} + r \sin \phi \cos \phi \dot{\lambda}(\dot{\lambda} + 2\Omega), \\ F_u &= \ddot{r} - r \dot{\phi}^2 - r \cos^2 \phi \dot{\lambda} (\dot{\lambda} + 2\Omega). \end{aligned} \tag{1}$$

Here, F_e, F_n, F_u are respectively the eastward, northward, and upward components of the force on a particle of unit mass; λ, ϕ, r are the usual spherical coordinates—longitude, latitude, and distance to the center of the earth, respectively—and Ω is the angular velocity of the earth's rotation; the dots represent differentiation with respect to time.

However, in many situations in which the equations are to be applied, one is concerned not with the spherical earth but with a flat projection or map of the earth's surface, the atmosphere being represented three-dimensionally above the projection. For such applications it may be useful to have an equivalent set of equations which involve only quantities in the projected system. The purpose of this report is to demonstrate how, for a large class of maps, the equations of motion can be transformed into a simple equivalent set of equations relative to the map.

We consider the earth as a sphere of radius r_o. λ, ϕ, r are spherical coordinates as above. Then, if the projection plane is the x-y plane of a rectangular system x-y-z, the map is determined by expressing x and y as functions of λ and ϕ:

$$\begin{aligned} x &= x(\lambda, \phi), \\ y &= y(\lambda, \phi) \end{aligned} \tag{2}$$

and z as a function of r alone, specifically:

$$z = r - r_o, \tag{3}$$

chosen so that $z = o$, that is the x-y plane, corresponds to the earth's surface $r = r_o$. The inverse transformation is then representable in the form:

$$\begin{aligned} \phi &= \phi\,(x, y), \\ \lambda &= \lambda\,(x, y), \\ r &= z + r_o. \end{aligned} \tag{4}$$

Now let $\hat{i}, \hat{j}, \hat{k}$ be the unit vectors in the spherical system with directions:

$\hat{i}$ - eastward,

$\hat{j}$ - northward,

$\hat{k}$ - upward.

Velocity and force vectors may then be expressed in terms of these. The velocity $\vec{V}$ is seen to be:

$$\vec{V} = r \cos \phi\dot{\lambda}\hat{i} + r\dot{\phi}\hat{j} + \dot{r}\,\hat{k}, \tag{5}$$

and the force $\vec{F}$ on a unit mass, is simply:

$$\vec{F} = F_e\hat{i} + F_n\hat{j} + F_u\hat{k}, \tag{6}$$

F_e, F_n, F_u as in Eq. (1).

The position vector $\vec{R}$ (relative to the center of the earth) is:

$$\vec{R} = r\hat{k} \tag{7}$$

The partial derivatives of this expression with respect to x, y, z are vectors indicating the directions of increasing x, y, z respectively:

$$\frac{\partial \vec{R}}{\partial x} = \frac{\partial}{\partial x}(r\hat{k}) = r\frac{\partial \hat{k}}{\partial x} = r\left(\frac{\partial \hat{k}}{\partial \lambda}\frac{\partial \lambda}{\partial x} + \frac{\partial \hat{k}}{\partial \phi}\frac{\partial \phi}{\partial x} + \frac{\partial \hat{k}}{\partial r}\frac{\partial r}{\partial x}\right), \tag{8}$$

and since

$$\frac{\partial \hat{k}}{\partial \lambda} = \hat{i}\cos\phi;\ \frac{\partial \hat{k}}{\partial \phi} = \hat{j};\ \text{and}\ \frac{\partial \hat{k}}{\partial r} = 0,$$

then:

$$\frac{\partial \vec{R}}{\partial x} = r\left(\cos\phi \frac{\partial\lambda}{\partial x}\hat{i} + \frac{\partial\phi}{\partial x}\hat{j}\right) = rt_x\hat{R}_x, \tag{9}$$

where

$$t_x = \left(\cos^2\phi\left(\frac{\partial\lambda}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial x}\right)^2\right)^{1/2}$$

and $\hat{R}_x$ is now a unit vector. Similarly,

$$\frac{\partial \vec{R}}{\partial y} = r\left(\cos\phi \frac{\partial\lambda}{\partial y}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j}\right) = rt_y\ \hat{R}_y, \tag{10}$$

where

$$t_y = \left(\cos^2\phi\left(\frac{\partial\lambda}{\partial y}\right)^2 + \left(\frac{\partial\phi}{\partial y}\right)^2\right)^{1/2}$$

Finally,

$$\frac{\partial \vec{R}}{\partial z} = \frac{\partial}{\partial z}(r\hat{k}) = \left(\frac{\partial r}{\partial z}\right)\hat{k} = \hat{k} = \hat{R}_z \tag{11}$$

Obviously $\hat{R}_x \cdot \hat{R}_z = \hat{R}_y \cdot \hat{R}_z = 0$, so for $\hat{R}_x$, $\hat{R}_y$, $\hat{R}_z$ to be an orthogonal set we need only the condition $\hat{R}_x \cdot \hat{R}_y = 0$, that is

$$0 = \frac{\partial\lambda}{\partial x}\frac{\partial\lambda}{\partial y}\cos^2\phi + \frac{\partial\phi}{\partial x}\frac{\partial\phi}{\partial y}\left(= t_x t_y \hat{R}_x \cdot \hat{R}_y\right) \tag{12}$$

In the following we shall assume that this orthogonality condition holds in the given projection.

It is now possible to derive expressions for the components of the force and the velocity in the new system by dot multiplying the respective vectors with the unit vectors R_x, R_y, R_z. Thus, let F_x, F_y, F_z be the components of the force in the x, y, z directions respectively, and let u, v, w be the components of the velocity.

Then:

$$F_x = \vec{F} \cdot \hat{R}_x = \frac{F_e \cos\phi \frac{\partial\lambda}{\partial x} + F_n \frac{\partial\phi}{\partial x}}{t_x},$$

$$F_y = \vec{F} \cdot \hat{R}_y = \frac{F_e \cos\phi \frac{\partial\lambda}{\partial y} + F_n \frac{\partial\phi}{\partial y}}{t_y}, \tag{13}$$

$$F_z = \vec{F} \cdot \hat{R}_z = F_u.$$

And:

$$u = \vec{V} \cdot \hat{R}_x = \frac{r}{t_x}\left(\cos^2\phi \frac{\partial\lambda}{\partial x}\dot{\lambda} + \frac{\partial\phi}{\partial x}\dot{\phi}\right),$$

$$v = \vec{V} \cdot \hat{R}_y = \frac{r}{t_y}\left(\cos^2\phi \frac{\partial\lambda}{\partial y}\dot{\lambda} + \frac{\partial\phi}{\partial y}\dot{\phi}\right), \tag{14}$$

$$w = \vec{V} \cdot \hat{R}_z = \dot{r}.$$

Solving for $\dot{\lambda}$ and $\dot{\phi}$ in the first two equations of Eq. (14) we get:

$$\dot{\lambda} = \frac{ut_x \frac{\partial\phi}{\partial y} - vt_y \frac{\partial\phi}{\partial x}}{r\cos^2\phi\, J},$$

$$\dot{\phi} = \frac{ut_x \frac{\partial\lambda}{\partial y} - vt_y \frac{\partial\lambda}{\partial x}}{-rJ}, \tag{15}$$

where J is the Jacobian of the transformation $(\lambda, \phi) \rightarrow (x, y)$, that is,

$$J = \frac{\partial\lambda}{\partial x}\frac{\partial\phi}{\partial y} - \frac{\partial\lambda}{\partial y}\frac{\partial\phi}{\partial x}. \tag{16}$$

Substituting Eq. (15) in Eq. (13) and using Eq. (1), Eq. (14) and the condition Eq. (12) one obtains:

$$F_x = \frac{du}{dt} - v\left[2e\Omega\sin\phi - \frac{1}{rt_x t_y}\left(u\frac{\partial t_x}{\partial y} - v\frac{\partial t_y}{\partial x}\right)\right] + w\left[\frac{2\Omega\cos^2\phi}{t_x}\frac{\partial\lambda}{\partial x} + \frac{u}{r}\right],$$

$$F_y = \frac{dv}{dt} + u\left[2e\Omega\sin\phi - \frac{1}{rt_x t_y}\left(u\frac{\partial t_x}{\partial y} - v\frac{\partial t_y}{\partial x}\right)\right] + w\left[\frac{2\Omega\cos^2\phi}{t_y}\frac{\partial\lambda}{\partial y} + \frac{v}{r}\right], \quad (17)$$

$$F_z = \frac{dw}{dt} - \frac{(u^2 + v^2)}{r} - 2\Omega\cos^2\phi \quad \frac{u}{t_x}\frac{\partial\lambda}{\partial x} + \frac{v}{t_y}\frac{\partial\lambda}{\partial y} \; ,$$

where $e = \text{sign } J$

$$\left(\text{that is,} \qquad e = \begin{cases} +1 & \text{if } J > 0 \\ -1 & \text{if } J < 0 \end{cases}\right) \quad (18)$$

Equations (17) are the equations of motion in the projected system.

3. NOTE ON THE ORTHOGONALITY CONDITION, EQ. (12)

From a brief survey of the maps, or projections, in common use, it is seen that this condition is a reasonable restriction as for most such maps it is already satisfied. As a matter of fact, one of the most common restrictions map-makers wish to impose is the more general one of conformality; that is, preservation of angles or, equivalently, equality of scale in all directions at a given point (actually, then, the preservation of shapes of small regions). Analytically the conformality conditions may be expressed by Eqs. (19a) and (19b):

$$\frac{\partial\lambda}{\partial x}\frac{\partial\lambda}{\partial y}\cos^2\phi + \frac{\partial\phi}{\partial x}\frac{\partial\phi}{\partial y} = 0\, , \quad (19a)$$

$$t_x = t_y \; (t_x, \; t_y \text{ as above}). \quad (19b)$$

Eq. (19a) is of course the orthogonality condition Eq. (12), and so conformality implies orthogonality (but not conversely because of the additional condition Eq. [19b]).

One further remark on condition Eq. (12). It can be expressed equivalently by

$$\frac{\partial x}{\partial \lambda}\frac{\partial y}{\partial \lambda} + \frac{\partial x}{\partial \phi}\frac{\partial y}{\partial \phi}\cos^2\phi = 0. \tag{20}$$

Indeed, it is perhaps more useful in this form, since one is usually given x and y in terms of λ and ϕ rather than the inverse direction which would have to be computed before condition Eq. (12) could be applied.

4. THE TRANSFORMED EQUATIONS FOR SOME SPECIFIC MAPS

One of the maps most in use today is the Stereographic Projection. This consists of a projection of the earth's surface upon some secant plane from the point on the earth opposite the plane. If we take, as is usually the case, the projection plane perpendicular to the earth's axis and the point of projection as the south pole, then by geometry we can compute the equations of the map.

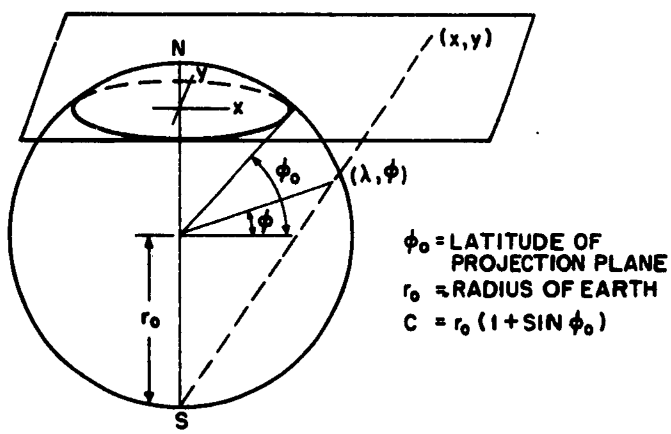

Figure 1. The Stereographic Projection

Then we have

$$x = \frac{c\cos\phi}{(1 + \sin\phi)} \cdot \cos\lambda,$$

$$y = \frac{c\cos\phi}{(1 + \sin\phi)} \cdot \sin\lambda, \tag{21}$$

$$z = r - r_o.$$

and the inverse transformation

$$\lambda = \tan^{-1}(y/x),$$

$$\phi = \sin^{-1}\left(\frac{c^2 - (x^2 + y^2)}{c^2 + (x^2 + y^2)}\right), \tag{22}$$

$$r = z + r_o .$$

The orthogonality condition is now easily verified in Eq. (12) or Eq. (20). (Actually, this is unnecessary since it is well known that the stereographic projection is conformal and therefore orthogonal.)

Applying these equations to Eq. (17) we get for the stereographic projection:

$$F_x = \frac{du}{dt} - v\left[2\Omega\left(\frac{c^2 - (x^2+y^2)}{c^2 - (x^2+y^2)}\right) + \frac{1}{c\,(z+r_o)}\,(uy - vx)\right] + w\left[-\frac{4c\,\Omega y}{c^2 + (x^2+y^2)} + \frac{u}{z+r_o}\right],$$

$$F_y = \frac{dv}{dt} + u\left[2\Omega\left(\frac{c^2 - (x^2+y^2)}{c^2 + (x^2+y^2)}\right) + \frac{1}{c\,(z+r_o)}\,(uy - vx)\right] + w\left[\frac{4c\,\Omega y}{c^2 + (x^2+y^2)} + \frac{v}{z+r_o}\right],$$

$$F_z = \frac{dw}{dt} - \frac{(u^2+v^2)}{z+r_o} + \frac{4\Omega c}{c^2 + (x^2+y^2)}\,(uy - vx). \tag{23}$$

A large number of orthogonal maps are such that the longitude and latitude lines on the earth are transformed into two sets of mutually perpendicular straight lines. In these x is a function of λ alone and y is a function of ϕ alone. Equivalently, we assume

$$\frac{\partial \lambda}{\partial y} = 0, \quad \frac{\partial \phi}{\partial x} = 0. \tag{24}$$

(Actually we need only assume the first $\frac{\partial \lambda}{\partial y} = 0$, since the orthogonality condition implies $\frac{\partial \phi}{\partial x}\frac{\partial \phi}{\partial y} = 0$, which implies $\frac{\partial \phi}{\partial x} = 0$, otherwise $\frac{\partial \phi}{\partial y} = 0$ and $\frac{\partial \lambda}{\partial y} = 0$ which means y is not a function of λ or ϕ. Conversely, if we had not assumed orthogonality, Eqs. (24) trivially imply it — see Eq. [12].) Assume also $\frac{\partial \lambda}{\partial x} > o$ (the case $\frac{\partial \lambda}{\partial x} < o$ is similar). Then Eqs. (17) become:

$$F_x = \frac{du}{dt} + (w - ve_1 \tan\phi)\,(2\,\Omega\cos\phi + \frac{u}{r}),$$

$$F_y = \frac{dv}{dt} + ue_1 \tan\phi \left(2\Omega \cos\phi + \frac{u}{r}\right) + \frac{wv}{r},$$

$$F_z = \frac{dw}{dt} - \frac{u^2 + v^2}{r} - 2\Omega \cos\phi \, u, \tag{25}$$

$$\left[= \frac{dw}{dt} - \frac{v^2}{r} - u\left(2\Omega \cos\phi + \frac{u}{r}\right)\right].$$

Here $e_1 = \text{sign}\left(\frac{\partial\phi}{\partial y}\right) = \pm 1.$

If we impose the further restriction upon the projection that it be conformal we obtain the familiar Mercator Projection. The transformation equations in this case are:

$$x = \lambda,$$
$$y = \int \sec\phi \, d\phi = \log\tan\left(\frac{\phi}{2} + \frac{\pi}{4}\right), \tag{26}$$
$$z = r - r_o,$$

or the inverse set:

$$\lambda = x,$$
$$\phi = 2\tan^{-1} e^y - \pi/2, \tag{27}$$
$$r = z + r_o.$$

Equations (25) are then easily written entirely in the new system by substituting

	tanh y	for	$\sin\phi$,
	sech y	for	$\cos\phi$,
	sinh y	for	$\tan\phi$,
and of course	$z + r_o$	for	r.

Another large class of maps is that of the so-called conic projections. These are obtained by mapping the earth onto a cone, which is secant to the earth (usually such that the cone's axis coincides with the earth's axis) subject to the condition that points on the same latitude line map into points at the same height on the cone and points on the same longitude line map into points on the same ray of the cone. The cone is then slit along a ray and flattened onto a plane.

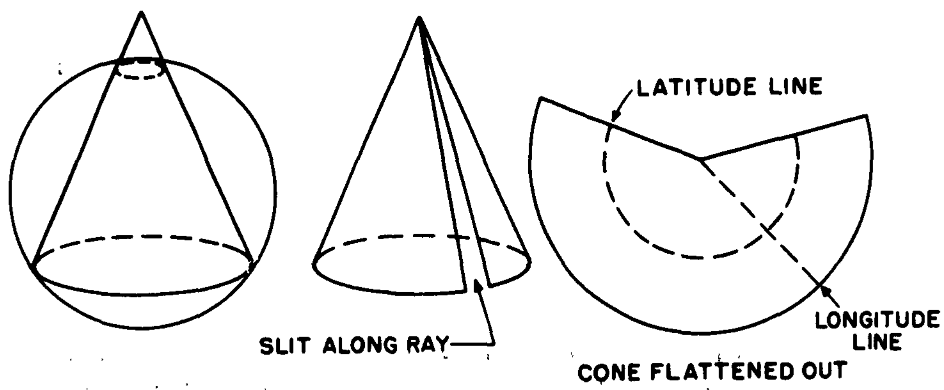

Figure 2. Construction of The Conic Projection

The transformation can be expressed:

$\rho = f(\phi)$, (f an arbitrary monotonic function),

$\alpha = \frac{1}{c}\lambda$,

$z = r - r_o$.

where ρ , α are polar coordinates on the x - y plane, that is,

$\rho = \sqrt{x^2 + y^2}$,

$\alpha = \tan^{-1}(y/x)$.

The inverse transformation is then

$$\phi = f^{-1}(\rho) = f^{-1}\left(\sqrt{x^2 + y^2}\right),$$

$$\lambda = c\alpha = c\tan^{-1}(y/x), \qquad (28)$$

$$r = z + r_o.$$

Unfortunately, such a map is not, in general, orthogonal. We can alter our point of view slightly, however, so as to apply the equations of motion. We do this by thinking in terms of the polar coordinates (ρ, α) instead of the rectangular coordinates (x, y), so that "with respect to ρ, α" the transformation is orthogonal. Indeed we are now in the situation Eq. (24) with x and y replaced by α and ρ, respectively. Thus, if we now look at radial ($\hat{e}_\rho$) and tangential ($\hat{e}_\alpha$) directions for a given point, P, and let F_α, F_ρ be the force components in

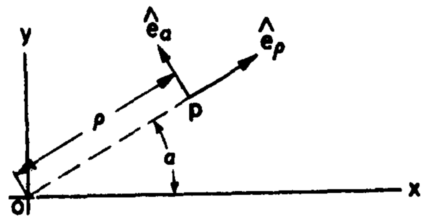

Figure 3. Radial and Tangential Directions

these directions at P and u, v, the velocity components, then Eqs. (25) apply, again with x and y replaced by α and ρ, respectively.

Getting back to rectangular coordinates, some conic projections are orthogonal in the old sense. Applying condition Eq. (12) to Eqs. (26) one finds the function f must take the form

$$\rho = f(\phi) = k \left[\tan\left(\frac{\pi}{4} + \frac{\phi}{2}\right)\right]^{-1/c} . \tag{29}$$

In this special case the transformation also turns out to be orthogonal and we have in fact the "Lambert" conformal conic. k and c are chosen usually to satisfy specific conditions, such as true scale at two standard parallels. (Note that the stereographic projection is an instance of this with $c = 1$ and $k = \underline{c}$ [that is, the "$\underline{c}$" of stereographic projection] since

$$\left[\tan\left(\frac{\pi}{4} + \frac{\phi}{2}\right)\right]^{-1} = \frac{\cos\phi}{1 + \sin\phi} .$$

Of course this was to be expected since projection on a plane is a limiting case of projection on cones.)

Thus, Eqs. (17) apply also to this oft-used class of projections, and take a form similar to that for stereographic projection.

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