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ERROR CONTROL IN DIGITAL COMMUNICATION

by

Arnfinn Moe Manders

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ABSTRACT

The transmission of digital information is a problem that has attracted a great deal of interest during the past decade. In the beginning, transmission rates were low and the accuracy required usually not too well specified. As digital communications grew in sophistication, the requirements for increased speed and accuracy in transmission grew also.

From the outset it was assumed that the speed and accuracy of transmission was limited by bandwidth and Gaussian noise. It was soon realized that this was not a valid assumption for many systems. In particular, in the case of digital data transmitted over a telephone channel it was observed that the accuracy was limited by noise of a great amplitude that occurred only seldom and occurred in bursts. This type of noise was named impulse noise.

It was observed that impulse noise occurs relatively seldom but when it occurs it is of an amplitude that often far exceeds the signal. These observations were put to use in the design of suitable signals to be transmitted. By coding the information in the proper ways one can always correct the bits that have been corrupted by the impulse noise. This, however, is done at the cost of adding redundant information to the transmitted signal. Since the signaling rate is limited by the bandwidth of the channel, this results in lowering the effective information carrying capacity of the system. By constructing very long codes, this lowering of the signaling rate can be made very small. Long codes on the other hand require complicated coding and decoding equipment.

This is where we presently stand. If we want to prevent burst of impulse noise of a certain length from corrupting the transmitted message, we must choose a balance between reduction in channel capacity and increased equipment complexity.

The investigation carried forward in this thesis looks for a third way out. It seeks a solution to the problem of obtaining the maximum transmission rate subject to a fixed permissible error rate and a fixed equipment complexity.

The method by which this is accomplished is to use an error correcting device (ECD) that examines the signal for signs that a burst of impulse noise is likely to have occurred. The bits that are in doubt are erased and replaced by use of a two-dimensional parity check. This method allows a few errors to pass undetected by the ECD and will therefore give a small but definite probability of errors in the presence of noise bursts. Some of these errors can be corrected if the code used has some error correction capability.

The purpose of the two-dimensional code which is used here is to break the received string of bits up into sequences that are used as rows. By making the rows sufficiently long the probability that one noise burst will cause more than one error in any column can be made very small. A figure of merit, M, is defined by which different transmission systems can be calssified as to burst length. The problem will be attacked from the following point of view.

Given a permissible probability of error and the maximum channel capacity an ECD will be constructed in such a way that will maximize the effective signaling rate.

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I. INTRODUCTION

a) Digital Data Transmission

The transmission of digital information is a problem that has attracted a great deal of interest during the last decade. In the beginning, transmission rates were low and the accuracy requirements usually were not too well specified. As an example of this we may take teletype transmission. The limits of acceptability might vary from error rates of one error every other line to one error per page between different operators.

Today with computer to computer data transmission, very much lower error rates are usually required.

In the early analysis of data transmission systems the speed and accuracy of data transmission was assumed to be limited by bandwidth and Gaussian randum noise. It was soon realized that this model was not a very accurate one. Noise of a particularly troublesome kind was observed.

This type of noise occurred in bursts with relatively long periods of quiet. It had a peak amplitude very much greater than that of the Gaussian random noise. Since the shape of these noise bursts often resembled the impulse response of the transmission channel, this type of noise was termed impulse noise.

In the case of Gaussian noise, errors occur with a random spacing. A string of consecutive errors is therefore not very likely. Error correcting codes such as the Hamming code were developed based on the Gaussian noise model. Since the most important type of noise turned out to be impulse noise where the probability of consecutive errors is very great, these codes did not perform very well.

New codes were developed that worked well in the presence of burst type noise. For these codes to be effective they had to contain a large number of digits per code block. This leads to very complicated coding and decoding equipment. For this reason these codes are not frequently used in communications systems.

If the code is to be able to correct long bursts or several bursts in a short time interval the redundary added to the information digits must be great, thereby causing the effective signaling rate to be low, or the code block must be very long resulting in very complicated equipment.

Looking for a way to avoid some of the problems associated with lowering of the signaling rate or equipment complexity this investigation seeks a different solution to the problem. The maximum signaling rate is sought subject to the constraints of a given error rate and equipment complexity.

b) Organization of the Investigation

The purpose of this investigation is to seek a more efficient solution than is presently available to the problem of reducing the effects of interference to data transmission. More efficient in the sense that it is simpler to implement or requires less redundancy to be added to the data bits caused by impulse noise. The investigation is divided into two parts. The first part deals with the nature and statistics of impulse noise and the selection of suitable mathematical tools for this task. This will be done by observing certain parameters of the noise and forming a model from which these parameters can be obtained.

The second part of the investigation is concerned with a system which we will call an Error Correcting Device (ECD). The parameters obtained in the first part of the investigation will be used to calculate the performance of an ECD system and to compare the performance of this system with a symmetric binary transmission system.

Finally there are some ideas on how the performance of this system and in general any digital data transmission system that operates over a nearly linear channel can be improved. These ideas are based on results obtained in the course of this investigation.

c) Impulse Noise Characterization

The communications channel under investigation can have either a linear or a non-linear amplitude response. In this investigation we will treat only the linear case. The non-linear case can be treated employing similar techniques but the output from the channel must be evaluated when both signal and noise are present at the same time. Many channels employing active elements are nearly linear over a substantial dynamic range and will therefore also fall into the category of linear systems.

Only two particular types of channels will be discussed here. They are the channel which has the transfer functions of an RLC circuit and a telephone circuit made up of type N carrier.

Type N carrier systems have widespread application in the Bell Telephone System.

The two most important parameters of impulse noise are the number of bit intervals it takes for a noise burst of a certain peak amplitude to decay to a certain level and the probability distribution of the peak amplitudes of the noise bursts.

To arrive at results that allow us to calculate these parameters we will use the following model for impulse noise. The impulses picked up by the channel are assumed to be perfect impulses and the resulting channel output is the impulse response of the channel between the point where the impulse entered and the observer.

A figure of merit for the system, M, is defined. M is the maximum number of bits that may be in error due to a noise burst resulting in a certain $\frac{S}{N}$ ratio. S is the peak amplitude of the signal and N is the peak amplitude of the noise. M is therefore a function of the $\frac{S}{N}$ ratio but comparison can be made between different systems once a value for $\frac{S}{N}$ ratio is selected.

M can be evaluated from the transfer function of the channel for the various data modulation methods that may be used. If a code that is sensitive to burst lengths is used, M will be useful in synthesising the code. M is a suitable figure of merit for a binary system. The ECD requires that the received signal voltages are within close limits of its expected value. Therefore a new figure of merit, M_1, is defined as the number of bit intervals the noise causes the voltage to deviate more than 10 % from its expected value. The peak amplitude distribution of the noise bursts is also of great importance but it does require further measurements to be made on the channel. In this case instruments have been built at Bell Telephone Laboratories [10] and at MIT Lincoln Labs that are triggered on each noise burst above a certain level and measures the peak amplitude and number of occurrences in a certain interval of time.

When a function that closely approximates the peak amplitude distribution has been selected and the impulse response of the channel is known the probability density function of the sample voltage at the detector output, or if an integrator is used at the integrator output, can be calculated. The probability density function of the noise output voltage will allow probable error calculations to be made. In particular, it may point the way toward improved system performance.

d) The ECD

The ECD is a system that is designed to give the maximum effective signaling rate in the presence of impulse noise and subject to limitations on error rate and equipment complexity.

The basic ECD consists of a decision device with three regions, one, zero, and erasure, used in conjunction with a relatively simple error detecting code. Ideally all zeros and ones are correct and the erasures represent suspisious bits that might have been in error. The erased bits are replaced cyclicly by a shift register, until the error detecting code is satisfied. If the number of erased bits was less than the error detecting capabilities of the code, the word is now correct since the erasures indicates the location of the suspisions bits and the error detecting code investigates if there are still errors present. A block diagram of a digital data system employing an ECD is shown in Fig. 1.

Fig. 2 shows a probability density function for the second bit interval from the start of the impulse burst when a typical telephone channel is the transmission medium. The sample voltage is taken at the integrator output when a zero was transmitted. Fig. 3 shows the probability density function at the integrator output when a one was transmitted. Actually all zeros and ones are not error free. To make the analysis more accurate these errors must be accounted for as well as the errors caused when the number of erasures exceeds the error detecting capabilities of the code. To improve the efficiency of the ECD by minimizing the effect of the errors in the zero and one decisions a code with some error correction capabilities is used.

In the example used in this investigation a code with a single error correcting single error detecting or triple error detecting capability will be used.

II. IMPULSE NOISE CHARACTERIZATION

a) Burst Length Analysis

Digital information is usually not transmitted from one place to another over large distances as a video signal. Usually the video information is used to modulate a carrier in some way. To what extent the transmitted signal is contaminated by impulse noise existing in the channel is dependent on what system of modulation is used.

At the outset it is useful to define certain quantities. The signal to noise ratio $\frac{S}{N}$, will be defined as the ratio between the peak signal and the peak impulse noise voltages. The quantity M, a figure of merit of the transmission system, is defined as the maximum number of bits that may be in error due to an impulse resulting in a certain $\frac{S}{N}$ ratio.

Two different channels will be investigated for the burst length analysis. The first will be one with the response of a simple RLC circuit. This approximation is often usable as a first approximation to systems that do not have loaded cables or much filtering. The second type channel we will consider is a type N telephone carrier channel. This type of carrier is very much used by the Bell Telephone System for medium distances.

We will assume that each noise burst is caused by one impulse and that the frequency of occurrence of impulses is low enough so that we do not get overlapping bursts. Experimental evidence obtained by Bell Telephone Laboratories[10] indicate the validity of this assumption. The bursts are assumed to be caused by perfect impulses so the received noise bursts are the impulse response of the intervening part of the channel. It will be shown later that the statistics obtained are not excessively sensitive to the actual noise burst shape so the transfer function of the intervening part of the channel can frequently be taken as the transfer function of the whole channel.

IIa. 1. Channel with the Response of an RLC Circuit

Many practical transmission systems can be approximately represented by a channel with the transfer function of an RLC circuit. Since this channel has a simple impulse response it is well suited to demonstrate the ideas behind this analysis. This type of channel is introduced here mainly to concentrate the use of the figure of merit, M. The response of such a channel to an impulse is

$$n(t) = Ae^{-\alpha(t-T)} \cos(\omega_1 t + \theta) \tag{2-1}$$

The parameters α and ω, are determined by the transfer function of the system; A is a function of the energy of the impulse. T and θ gives the timing of the occurrence of the impulse. The impulse noise is therefore treated on a mixed statistic deterministic basis. The shape of the noise burst, itself is deterministic while the amplitude, A, and time of occurrence, as given by T and θ, are random variables.

One word of caution is required here. Since the noise bursts obtained using the proposed impulse response model for the phenomenan are not exactly equal but only very similar to many of the actual noise bursts observed, any attempts to take advantage of this knowledge of the burst shape will not be very sucessful.

The figure M will now be calculated using several different modulation methods and using the RLC channel. For more complete calculations, the reader is referred to the appendix.

a. AM Envelope Defector - System I

The received signal is:

$$s(t) = \frac{S}{2}(1 + \cos \omega_m t) \cos \omega_o t \tag{2-2}$$

Bit interval $= \tau \quad \frac{2\pi}{2\omega_m}$

Time required for the noise burst to decay to a value less than $\frac{S}{2}$ as required for unique determination of the signal in an on-off carrier system.

$$t_1 = \frac{1}{\alpha} \ln \frac{2A}{S} \tag{2-3}$$

$$M = \frac{t_1}{\tau} = -\frac{1}{\pi} \ln \frac{1}{2} \frac{S}{N} \tag{2-4}$$

M is independent of whether a string of zeros or a string of ones are transmitted.

b. Vestigal Sideland. A.M. Synchronous Detector - System II

$$n(t) = Ae^{-\alpha(t-T)} \cos(\omega_2 t + \theta) \quad (2\text{-}5)$$

$$s(t) = \frac{S}{4}\left[(\cos \omega_o t + \cos(\omega_o + \omega_m) t\right] \quad (2\text{-}6)$$

$$\omega_2 = \omega_o + \alpha$$

$$M = -\frac{2}{\pi} \ln \frac{1}{4}\frac{S}{N} \quad (2\text{-}7)$$

c. AM. Synchronous Detection - System III

$$M = -\frac{1}{\pi} \ln \frac{1}{2}\frac{S}{N} \quad (2\text{-}8)$$

d. PSK. Synchronous Detection - System IV

$$s(t) = S \cos \omega_o t \quad (2\text{-}9)$$

$$M = -\frac{1}{\pi} \ln \frac{S}{N} \quad (2\text{-}10)$$

e. PSK. Differential Phase Detector - System V

$$s(t) = S \cos \omega_o t \quad (2\text{-}11)$$

$$\text{Detector output: } V(t) = S_1 \cos \omega_o t - S_2 \cos\left[\omega_o t - 2\pi\right] \quad (2\text{-}12)$$

$$M = 1 - \frac{1}{\pi} \ln \frac{S}{N} \quad (2\text{-}13)$$

When we use the simple RLC model for the channel we obtain a closed form expression for M that allows direct comparison between systems. We see for instance that the burst length in a PSK system is one bit longer when a differential phase detector is used as compared with a system employing synchronous detection.

In the following, in connection with the ECD, we shall have occasion to use a quantity M_1, M_1 is defined as the number of bit intervals it takes the detector output to settle down to within $\pm$ 10% of the correct signal value. When M is represented in closed form, M_1 is found simply by replacing $\frac{S}{N}$ by $\frac{1}{5}\frac{S}{N}$.

For the systems studied previously we obtain:

System I $$M_1 = -\frac{1}{\pi} \ln \frac{S}{N} + \frac{1}{\pi} \ln 10 \quad (2\text{-}14a)$$

System II $$M_1 = -\frac{2}{\pi} \ln \frac{S}{N} + \frac{2}{\pi} \ln 20 \quad (2\text{-}14b)$$

System III $$M_1 = -\frac{1}{\pi} \ln \frac{S}{N} + \frac{1}{\pi} \ln 10 \quad (2\text{-}14c)$$

System IV $$M_1 = -\frac{1}{\pi} \ln \frac{S}{N} + \frac{1}{\pi} \ln 5 \tag{2-14d}$$

System V $$M_1 = -\frac{1}{\pi} \ln \frac{S}{N} + \frac{1}{\pi} \ln 5 + 1 \tag{2-14e}$$

These values of M_1 together with M, for an actual telephone channel composed of type N carrier are plotted in fig. 6.

The values used for plotting the curves for the type N telephone channel with synchronous demodulation are obtained from fig. 8, 9 and 10 and from the asymptotic expression for the impulse response of a telephone channel. Since those curves are plotted for a unity peak amplitude noise burst they represent the actual noise output voltage from a symmetrical binary channel with a unity signal to noise ratio.

IIa. 2. Actual Telephone Channel.

We will now attack the problem of impulse noise bursts in an actual telephone carrier channel. If necessary the problem may be split into two parts. One transfer function can be used for the carrier channel to the central office and another from the central office to the subscriber. In this way the noise burst picked up along the carrier channel will have a form different from the one picked up in the central office. If an open wire line is used between the subscriber and the central office, this may be necessary. If on the other hand loaded cable is used for this run, use of the overall transfer function will usually suffice.

The transfer function of a typical medium distance telephone connection containing type N carrier channels is represented by the amplitude and phase response curves shown in fig. 4 and 5.[2]

The envelope delay of this channel can be approximately expressed as:

$$\Delta d = K\,(1700 - f)^2 \qquad (2\text{-}15)$$

Where $K = 10^{-9}$

The corresponding phase vs. frequency response is

$$\phi = \int \Delta d \;\; df = 10^{-9} \int (1700-f)^2 \, df$$

$$\phi = .33 \,.\, 10^{-9}\,(f - 1700)^3 + C$$

Since we are interested only in relative phase distortion we may set

$\phi\,(1700) = 0 \quad : \quad C = 0$

And obtain:

$$\phi = .33 \,.\, 10^{-9}\,(f - 1700)^3 \qquad (2\text{-}16)$$

The amplitude response of the channel is approximately 300 to 3000 cps. but this has less effect on the transient response that the phase distortion.

We will assume that the modulation system to be used is phase shift keying (PSK) with synchronous detection. This system was shown in fig. 6 to have the best figure of merit for the RLC channel and is a linear system where superpossition applies. Since the synchronous detector is a linear device, our channel will appear to have a 1300 cps band with and a phase function

$$\phi = .33 \,.\, 10^{-9}\; f^3 \qquad (2\text{-}17)$$

When viewed from the video output terminals of the demodulator, the transfer function of the channel can therefore be approximated by

$$H\,(j\omega) \doteq e^{-a^2\omega^2 - jb^3\omega^3} \qquad (2\text{-}18)$$

The Gaussian amplitude characteristic is a resonable approximation to fig. 5 and leads to a closed form answer. It is therefore chosen.

The parameters a and b are calculated as:

$$a^2 \omega_{3db}^2 = \ln \sqrt{2}$$

And $$.33 \cdot 10^{-9} = (2\pi)^3 b^3$$

Therefore

$$a = 6.7 \cdot 10^{-5}$$

$$b = 1.54 \cdot 10^{-4}$$

The impulse response of a channel with the transfer funtion [5]

$$H(j\omega) = e^{-a^2\omega^2 - j\omega^3 b^3}$$

is:

$$y(t) = \frac{\left[\frac{t}{b} - \frac{1}{3}\left(\frac{a}{b}\right)^4\right]^{\frac{1}{2}}}{3\sqrt{3}\, b} \; e^{-\frac{a^2 t}{3b^3}} \left[J_{\frac{1}{3}}(u) + J_{-\frac{1}{3}}(u)\right] \qquad 2\text{-}19$$

where:

$$U = \frac{2}{3^{3/2}} \left[\frac{t}{b} - \frac{1}{3}\left(\frac{a}{b}\right)^4\right]^{3/2} \approx 2.10^5\, t^{3/2}$$

$$y(t) \approx 10^5 t^{\frac{1}{2}} e^{-4\,10t} \left[J_{\frac{1}{3}}(2.\,10^5 t^{3/2}) + J_{-\frac{1}{3}}(2.\,10^5 t^{3/2})\right], \text{ t in sec.} \qquad 2\text{-}20$$

This expression for y(t) even though it is in closed form is still too complicated for easy evaluation. It can be shown to be closely approximated by the asymptotic expression:

$$y(t) = A\, t^{-\frac{1}{4}} e^{-.41t} \cos\left[6.3t^{3/2} + .7\right], \quad \text{t is in ms.} \qquad 2\text{-}21$$

for $t \geq 2.5$ ms.

The development of this approximation is shown in the appendix.

A plot of y(t) for $o < t \leq 2.5$ ms is shown in fig. 8. The origin has been shifted slightly to make the function causal.

IIb Peak Amplitude Distribution of Impulse Noise

When one considers the various ways by which an impulse can enter the transmission channel one concludes that impulse noise is usually picked up from the electromagnetic fields generated by the noise source. If the noise source was a piece of wire that was radiating, the potential in the vicinity would be roughly proportional to - log x where x is the distance between the pick up wire and the noise source. This is the closest form of coupling that would exist in a telephone plant.

In the majority of cases the distances between the noise sources and sinks are so great as to make them appear more nearly as point sources. This is the kind of coupling one would expect to get with an arching relay contact or with a distant thunderstorm.

In the case of point sources the resulting potential is proportional to $\frac{1}{x}$. In this investigation we will use the $\frac{1}{x}$ law for peak amplitude pickup.

Besides being an appealing physical concept it also leads to expressions which agree well with experimental evidence.

In the following we will assume that the noise sources are located with probability ε at any distance away from the point where the noise is picked up by the channel but that the noise sources are restricted to lie along the x-axis. All impulses are assumed to be radiated with the same strength. Stronger impulses can be considered as veaker impulse occuring nearer. This assumption is therefore not very restrictive. The assumptions about the geometrical arrangements are quite reasonable since the physical layout of relay contacts is usually as a row and in the case of overhead wiring it follows the surface of the earth.

We will call the peak amplitude of the observed impulse noise burst, V_o. The minimum peak amplitude that is counted as impulse noise is V_{min}. V_{min} is the voltage resulting from an impulse occurring at a distance X_o away from the channel. Based on these definitions and assumptions we obtain the following results:

$$p(x) = \varepsilon \tag{2-22}$$

$$\int_o^{x_o} p(x)\, dx = \varepsilon \; x_o = 1$$

$$v = \frac{V_o}{x} \tag{2-23}$$

$$V_{min} = \frac{V_o}{x_o} = \varepsilon \; V_o$$

$$P(v) = p(x) \left|\frac{dx}{dv}\right| \tag{2-24}$$

$$dv = -\frac{V_o}{x^2}\,dx$$

$$P(v) = \frac{\varepsilon\, x^2}{V_o}$$

$$P(v) = \frac{\varepsilon\, V_o}{v^2}$$

$$P(v) = \frac{V_{min}}{v^2} \qquad (2\text{-}25)$$

where:

$$\int_{V_{min}}^{\infty} P(v)\,dv = -V_{min} \left|\frac{1}{v}\right|_{V_{min}}^{\infty} = 1 \quad .$$

P(v) is therefore a properly normalized probability density function.

To compare this expression with available data we must find the distribution function which is usually the quantity that is measured.

The corresponding distribution function is

$$P(V_1) = Pr\left[v > V_1\right] = V_{min}\int_{V_1}^{\infty} \frac{dv}{v^2}$$

$$P(V_1) = \frac{V_{min}}{V_1} \qquad (2\text{-}26)$$

This function is of the same form as the one deduced empirically from experimental data by Mertz[1]. The function is compared with data published by Alexander[2] in fig. 7. Results obtained by J. H. Fennick[10] at Bell Telephone Laboratoies from measurements made on a large number of switched telephone circuits along the East Coast also confirms the form of P(v). From these measurements it seems like an equation of the form

$$P(v) = \frac{k}{v^2}$$

would fit the data even better. As will be shown later, in the section on Deviations from the Assumed Conditions, the original form leads to more conservative estimates of error rates and will, therefore, be used in the following calculation since it represents an upper bound on the observed phenomena. The question of the exponent of V will also be treated under "Deviations from the Assumed Conditions".

In telephone systems a 600 Ω load is normally used. In this case zero dbm corresponds to .774 volts. Using Alexander's data to determine the constant V_{min}, we obtain the expression:

$$P\left[v > V_1\right] = \frac{.0024}{V_1} \quad \text{for} \quad V_1 \geq V_{min} \tag{2-27}$$

This corresponds to a normalization such that $P\left[V > V_1\right]$ = average number of impulse counts per second that exceeds V_1. In this case a 15 min. averaging time is used.

Fig. 7 shows Alexander's results plotted together with

$$\underline{P}\left[v > V_1\right] = \frac{.0024}{V_1}$$

And

$$\underline{P}\left[v > V_1\right] = \frac{.000262}{V_1^{\ 2}} \tag{2-28}$$

IIc. Integration Output Voltage Probability Density.

This section will deal only with the telephone channel. The same exact method can be used in the case of any linear channel and modulation method. In the case of a nonlinear channel the same basic method of approach can be used but it will be necessary to handle the noise and the signal simultaneously since superposition does not apply.

It is seen from fig. 6 that the figure of merit for the system, M_1 is considerably improved if the defector output is integrated before sampling as compared to a system simply sampling the detector output. It will be shown later in the section "Effect of Integration on the Noise Output Voltage from the Demodulator" that integration after demodulation greatly reduces the expected burst lengths. This section will also go into the reason for this phenomenon.

The methods used apply to any linear system even though the spesific results are for the system shown in fig. 1.

If an impulse occurs within a bit interval, its peak amplitude and time of occurrence within the bit interval are random variables. Since we have already selected a suitable peak amplitude distribution, what remains to be evaluated is the effect of the randomness of the time of occurrence.

For this part of the investigation numerical methods were used with the help of an IBM 650 computer.

The impulse response of the channel when viewed from the detector output is

$$y(t) = A\,t^{1/2} e^{-410t} \left[J_{\frac{1}{3}} (2.\,10^5 t^{3/2}) + J_{-\frac{1}{3}} (2.\,10^5 t^{3/2}) \right], \ t \text{ in sec.} \qquad (2\text{-}20)$$

This function for $t < 2.5$ ms and unit peak amplitude is plotted in fig. 8.

For $t > 2.5$ ms, the asymtotic form

$$y(t) = -.526\, t^{-\frac{1}{4}} e^{-.41t} \cos \left[6.3t^{3/2} + .7 \right], \quad t \text{ in ms.} \qquad (2\text{-}21)$$

is used. For computational reasons and also since it represents a logical choice in view of the available bandwith, the transmission rate will be chosen equal to 1000 bits per second.

We will assume that any time of occurence of the impulse within the bit interval is equally likely. This is in agreement with data published by Mertz.

The first calculation to be done is the evaluation of the integrator output due to a unit peak amplitude noise burst occurring at different times within the bit interval. Initially a power series approximation to $y(t)$ was tried. This power series was integrated and evaluated to give the integrator output at the sampling intervals.

This method was found difficult to work with and gave results of insufficient accuracy. It was therefore abandoned and numerical integration was used instead.

The output of the integrator is designated Z. Z_n is the output at the sampling instant at the end of the n'th interval. The impulse response was sampled at 20 equally spaced times with the 1 ms bit interval and the integral evaluated as:

$$z_n = \int_{t_n}^{t_{n+1}} y\,dt = \sum_{k=1}^{20} \left[\frac{1}{20} Y_k - \frac{1}{40} (Y_k - Y_{k+1}) \right] \tag{2-29}$$

The impulse was assumed to have occurred at 20 equally spaced times within the bit interval.

The integrator output voltage z_n is evaluated for each possible time of occurrence. Fig. 9 and 10 shows z_1 and z_2 for the two first bit intervals, as a function of time of occurrence of the impulse within the first bit interval.

The 20 values of 2 for each bit interval are now arranged in increasing order. Since each of the 20 possible values are equally likely, each is assigned a probability of $\frac{1}{20}$. From these values the probability distribution function of Z for each bit interval is found. $P(z_1)$ and $P(z_2)$ are shown in fig. 11 and 12.

The functions $\underline{P}(z_n)$ are now differentiated to obtain the corresponding density functions. $p(Z_1)$ and $p(Z_2)$ are shown in fig. 13 and 14. For further calculations, it is convenient to have analytic expressions for $P_n(z)$ over the various bit intervals after the impulse has first occurred. This is conveniently done by curve fitting on a computer. In this operation considerable care must be exericsed so that the resulting polynomials are good approximations to the desired functions. Since no further differentiation of the $P_n(z)$ function is required, the requirements are not nearly as strict as if the polynomials had to be differentiated. $p_n(z)$ is now in the form

$$p_n(z) = a_k z^k + a_{k-1} z^{k-1} + \dots a_1 z + a_o \tag{2-30}$$

The output from the integrator due to a noise burst of unit peak amplitude is Z. Therefore, the integrator output due to a noise burst of peak amplitude V is given by:

$$u_n = v z_n \tag{2-31}$$

Since Z and V are independent random variables

$$P(z, v) = P(z)\ P(v)$$

$$P(u, v) = P(\tfrac{u}{v})\ P(v)\ \left| \frac{dz}{du} \right| \qquad (2\text{-}32)$$

And:

$$P(u) = \int_{-\infty}^{\infty} P(u, v)\, dv$$

$$P(u_n) = \int_{-\infty}^{\infty} P(u, v)\, dv$$

$$P(u_n) = \int_{-\infty}^{\infty} P(\frac{u_n}{v})\ P(v) \left| \frac{dv}{v} \right| \qquad (2\text{-}33)$$

On different occasions the probability density function of z in the n'th interval may be denoted.

$P(z_n)$ or $p_n(z)$ according to which notation which is most convenient.

Substituting the expression for $P_n(z)$ in the equation for $P_n(u)$ we obtain:

$$p_n(u) = V_{min} \int_R^{\infty} \left[a_k \left(\frac{u}{v}\right)^k + \ldots\ldots\ a_o \right] \frac{dv}{V^3} \qquad (2\text{-}34)$$

$$R = \text{Max of } \left[V_{min} \text{ or } \frac{u}{z}_{max} \right]$$

$$p_n(u) = V_{min} \left| \left[\frac{a_k}{k+2} \left(\frac{u}{k}\right)^k + \ldots\ \frac{a_o}{2} \right] \frac{1}{V^2} \right|_{\text{Max of } \left[V_{min} \text{ or } \frac{u}{z}_{max} \right]}^{\infty}$$

For $u \leq Z_{max} V_{min}$ we obtain a k'th order polynomial in u. Plots of $P_1(u)$ and $P_2(u)$ are shown in fig. 15 and 16.

The most interesting also most useful behaviour of $p_n(u)$ is for $u > Z_{max.} V_{min.}$ The signal levels we will be dealing with are in the .1 v range As can be seen from the graphs fig. 15 and 16 $Z_{max.} V_{min.}$ is less than 1 mv which is small compared with the expected signal leves that are in the .1 volt range. Subject to this condition we see that

$$P_n(u) = V_{min} \left[\frac{ak}{k+2} Z^k_{max} + \ldots\ \frac{a_o}{2} \right] \frac{Z^2_{max}}{U^2}$$

This equation is of particular simple form

$$P_n(u) = \frac{C_n}{u^2} \qquad (2\text{-}35)$$

We see that the form of equation 2-35 does not depend on the actual impulse waveform in the asymptotic region.

It is easy to show that this is true for any p(z) as long as the function is well behaved.

$$P_n(u) = \int_{\frac{u}{Z_{max}}}^{\infty} P_n(z)\, P(v)\, dv \tag{2-36}$$

for: $u > Z_{max} V_{min.}$

But: $u = zv$

Therefore:

$$P_n(u) = \int_{\frac{u}{Z_{max}}}^{\infty} P_n\left(\frac{u}{v}\right) P(v) \frac{dv}{|v|}$$

$$P(v) = \frac{V_{min.}}{v^2}$$

$$P_n(u) = V_{min} \int_{\frac{u}{Z_{max}}}^{\infty} P_n\left(\frac{u}{v}\right) \frac{dv}{v^3} \tag{2-37}$$

Since u is a constant in the integration

$$-u \frac{dv}{v^2} = -d\left(\frac{u}{v}\right)$$

Let: $\frac{u}{v} = L$

Equation 2-37 now becomes:

$$P_n(u) = \frac{V_{min}}{u^2} \int_{o}^{Z_{max}} P_n(L)\; L\; dL \tag{2-38}$$

The integral in equation is a constant.

Therefore:

$$P_n(u) = \frac{V_{min}}{u^2} \cdot \text{const} \tag{2-39}$$

and the statment is proved.

In the appendix an approximate method by which p_n can be quickly approximated without the use of a computer is developed. This method allows a quick estimate of the impulse noise performance of a particular system before more accurate calculations with the aid of a computer are carried out.

JOINT PROBABILITY DENSITY FUNCTIONS

$p_n(u)$ is the probability density function for the integrator output voltage that is in effect in the n'th bit interval after the start of the impulse, regardless of what happened in the previous $n-1$ bit intervals. For many types of codes it is important to have a probability function relating the various bit intervals. For this reason the joint probability density function for the noise output voltage from the detector at the sampling instants will be developed.

In developing the joint probability density function we will use numerical methods in a similar way as was done for $p_n(u)$. Let m and n denote two bit intervals.

$$m > n$$

We evaluate z_m and z_n for 20 equally spaced times of occurrence of the impulse in the first bit interval. Next we select the corresponding values of z_m and z_n and form the function

$$z_m = f(z_n) \tag{2-40}$$

If the function 2-40 is well behaved and single valued the conditional probability density of z_m given z_n is

$$p(z_m \mid z_n) = \delta \left[z_m - f(z_n) \right] \tag{2-41}$$

The probability density of z_n was calcultated in the previous section. The joint probability density of z_n and z_m is therefore:

$$p(z_n , z_m) = \delta \left[z_m - f(z_n) \right] \quad p(z_1) \tag{2-42}$$

Since z_n is independent of v we see that

$$p(z_n , z_m , v) = p(z_n , z_m)\, p(v) \tag{2.-43}$$

We are ultimately interested in the probability density of U_n and U_m and must therefore make a change of variables.

$$\left.\begin{aligned} u_m &= v\, z_m \quad , \quad z_m = \frac{u_m}{} = f_m(u) \\ u_n &= v\, z_n \quad , \quad z_n = \frac{u_n}{u_k} = f_n(u) \\ u_k &= v \quad , \quad v = u_k = f_m(u) \end{aligned}\right\} \tag{2-44}$$

Here u_k is a dummy variable introduced only for the purpose of making the transformation between equal numbers of variables. We obtain the probability densities of the u's as

$$p(u_n , u_m , u_k) = p(z_n , z_m , v) \mid J \mid \qquad (2\text{-}45)$$

where

$$J = \begin{vmatrix} \frac{\partial f_n}{\partial u_n} & \frac{\partial f_m}{\partial u_n} & \frac{\partial f_k}{\partial u_n} \\ \frac{\partial f_n}{\partial u_m} & \frac{\partial f_m}{\partial u_m} & \frac{\partial f_k}{\partial u_m} \\ \frac{\partial f_n}{\partial u_k} & \frac{\partial f_m}{\partial u_k} & \frac{\partial f_k}{\partial u_k} \end{vmatrix} = \frac{1}{u_k^2} = \frac{1}{v^2} \qquad (2\text{-}46)$$

We now are in a position to obtain our desired result

$$p(u_n , u_m) = \int_0^{\infty} p(z_n , z_m , v) \frac{1}{v^2} \, dv \qquad (2\text{-}47)$$

Because of the impulse in equation 2-41 , equation 2-47 reduces to

$$p(u_n , u_m) = p_n \left(\frac{u_n}{v_o}\right) p(v_o) \frac{1}{v_o^2} \qquad (2\text{-}49)$$

where v_o is the solution to the equation:

$$u_m - v_o \, f\left(\frac{u_n}{v_o}\right) = 0$$

From equation 2-49 we can form an expression for v_o in terms of the U's. for values of U_n and U_m subject to the conditions:

$$0 \leq u_n \quad V_{min} \, Z_{n\,max}$$

$$0 \leq u_m \quad V_{min} \, Z_{m\,max}$$

$$v_o = F_{n\,,\,m} \, (u_n , u_m) \qquad (2\text{-}49)$$

Upon substituting this solution back into equation 2-48 an introducing the probability density of v_o we obtain the final result.

$$p(u_n , u_m) = V_{min} \, (F_{n\,,\,m})^{-4} \; p_n \left(\frac{u_n}{F_{n,m}}\right) \qquad (2\text{-}50)$$

To conclude this section it is of interest to investigate $p(u_n , u_m)$

for the most critical intervals, $n = 1$, $m = 0$. As a first order approximation to equation 2-40 we obtain:

$$z_2 = .5 - z, \qquad (2\text{-}51)$$

for:

$$0 \leq z_1 \leq .5$$
$$0 \leq z_2 \leq .5$$

From equation 2-51 we obtain

$$v_o = 2(u_1 + u_2) \qquad (2\text{-}52)$$

Substituting $F_{1,2}$ back into equation 2-50 we get

$$p(u_1, u_2) = \frac{V_{min}}{16(u_1 + u_2)^4} p_n\left(\frac{u_1}{u_1 + u_2}\right) \qquad (2\text{-}53)$$

We not that equation 2-53 is of the form

$$p(u_1, u_2) = f(u_1, u_n) \frac{1}{(u_1 + u_2)^4} \qquad (2\text{-}54)$$

This expression is seen to be similar to the asymptotic expression for $p_n(U)$ which is

$$p_n(u) = \frac{C_n}{u^2}$$

Since it is a function of two bit intervals one might resonably expect equation 2-54 to be to the fourth power rather than the second.

$p(u_n, u_m)$ will not be used in the synthesis of the ECD system since the correlation between errors in adjacent bit intervals is not of importance here. It is needed for evaluate the performance of certain codes and also to evaluate the imporovement in performance that can be expected using the method described in the section "Further Improvements of the ECD".

SYNTHESIS OF THE ECD

The ECD system receiver

The ECD system receiver is shown in fig. 20. The modulation method that will be used is phase shift keying (PSK) since this system lends itself well to analysis and has the shortest burst lengths. The maximum transmitted signal level on a telephone channel is -4dbm. This will be assumed to be the transmitted power level. The average loss over a medium distance telephone circuit is 15 db resulting in a channel output level of -19dbm. Into a 600Ω load this is .123 volts peak. The output from the synchronous detector is therefore +.123 V corresponding to a zero and -.123 V corresponding to a one. Since the integretor has a gain of 1000, and the integration-time is one ms, the integrator output is +.123Vfor a one and -.123V for a zero in the absence of noise. The integrator is followed by a decision device with three output leads. The decision as to which lead should be activated is based on the integrator output voltage, u, at the sampling instant as shown in figs. 2 and 3.

Estimation

When a noise burst occurs and a digit is transmitted we do not know that the burst has occurred nor do we know what the transmitted digit was. We must rely on the detector or in this case the integrator output at the sampling instant to inform us about the likelihood of what noise voltage has been picked up and which bit was transmitted. The sample voltage may be in one of five regions corresponding to one, zero or the three erasure regions. Consequently we are faced with the problem of testing between three possible hypothesis.

This problem can be simplified by taking advantage of the symmetry involved in this situation as shown in fig.2 and 3. Since a one and a zero are mutually exclusive, only one of the two probability densitites $p_n(u/1)$ or $p_n(u/0)$, can be in effect. We will assume that zeros and ones are equally likely as is usually the case in binary communications systems. By taking advantage of the symmetry we will reduce the three possible choices to two.

We will choose as our two possible hypotheses that the output from the detector is correct, H_1, or that it is incorrect, H_o. H_1 therefore includes the cases where the received bit is the same as the transmitted bit and when an erasure is received regardless of what was transmitted. H_o only includes the case where the received bit is the opposite of the transmitted bit. This corresponds to the case where the decision device does not catch an error and is knows as an error of the first kind. We will designate this type of error by α or α_n where n refers to the bit inferval since the start of the impulse.

When a signal is received we will designate the probability of the erasure in the received bit interval or by β or β_n. We are now ready to apply the hypothesis test to the integrator output

$$\alpha_n = \int_a^b p_n\,(u/0)\,du \qquad (3\text{-}1)$$

where a and b are the decision thresholds as shown in fig. 2 and zeros and ones are equally likely. The probability of an erased bit in the n'th bit position is

$$\beta_n = \int_{-\infty}^{-b} p_n\,(u/1)\,du + \int_{-a}^{a} p_n\,(u/1)\,du + \int_{b}^{\infty} p_n\,(u/1)\,du \qquad (3\text{-}2)$$

according to fig. 3.

Since $\alpha_n << \beta_n$ we can make the approximation that

$$\beta_n = \int_{-\infty}^{a} p_n\,(u/1)\,du + \int_{b}^{\infty} p_n\,(u/1)\,du \qquad (3\text{-}3)$$

The code that will be used in connection with this investigation is a 25 x 32 bit code which breaks up the bursts as far as the coding is concerned since the error correction is done along the columns while the bits are read in along the rows. For this reason the correlation between errors or erasures is the same burst are of no importance to the average error rate since the arrangement of bursts within the code block is random. This subject is discussed by Feller and others.

As will be shown in the section on coding, errors in the output can be caused by an excess of errors of the first kind or an excess or a combination of both. It will be shown that the error rate E equals:

$$E = E_\beta + E_\alpha + E_{\alpha\beta} \qquad (3\text{-}4)$$

Where E_β are errors caused by an excess of erasures, E_α are errors caused by an excess of errors of the first kind and. The object of the decision test is to minimize E. An analytic method for this might be worked out but would be analytically complicated and successive trials has been chosen as a relatively quick way to solve the problem. The solution can be obtained the following way.

a) Select a value for α.
b) Calculate the minimum value of β for this value of α.
c) Evaluate the error rate E.
d) Repeat the calculations for a different value of α.

This method will quickly lead to an optimum choice of a and b, the decision thresholds, such that E is a minimum.

As a guide rule to the selection of a value for α that will minimize E the following statement can be made.

E will be close to its minimum value when

$$E_{\alpha} \approx E_{\beta} \approx E_{\alpha\beta} \qquad (3\text{-}5)$$

That this is so can easily be seen if one realizes that there exists an inverse relationship between E_{α} and E_{β}. Let us assume that $E_{\alpha} > E_{\beta}$. As E_{α} decreases E_{β} will increase. A point is reached when E_{β} will account for most of the errors. Since

$$E = E_{\alpha} + E_{\beta} + E_{\alpha\beta} \qquad (3\text{-}6)$$

the error rate will be almost independent of E_{α} and nothing will be gained by increasing E_{β} further.

It is therefore to be expected that the error rate is close to a minimum when the three terms are about equal.

If $E_{\alpha\beta}$ is very different from E_{α} and E_{β} the code that is used is probably not the most efficient under the given circumstances. We will now show the method by which the calculations of optimum threshold are carried out.

Since we are not concerned with corrolation between errors the probability of an error of the first kind is apprriximately:

$$\alpha = \sum_{n=1}^{n} \alpha_n \qquad (3\text{-}7)$$

As we see from fig. 19, this sum converges very rapidly. We need therefore only take into account the three first terms.

$$\alpha = \alpha_1 + \alpha_2 + \alpha_3 \qquad (3\text{-}8)$$

In the case of the telephone channel we are investigating, we have shown that the signal levels at the output of the integrator can be expected to be .123 volts at the sampling instant. The probability densities at the output of the integrator at the sampling instant due to signal plus noise when a zero was transmitted are calculated as previously explained to be:

$$\left.\begin{aligned} p_1\,(u/\,0) &= \frac{2.07.10^{-4}}{(u+.123)^2} \\ p_2\,(u/\,0) &= \frac{4.55.10^{-4}}{(u+.123)^2} \\ p_3\,(u/\,0) &= \frac{4.09.10^{-6}}{(u+.123)^2} \end{aligned}\right\} \quad \begin{matrix} u \leq -.125 \\ \text{or} \\ u \geq -.121 \end{matrix} \qquad (3\text{-}9)$$

We notice that $p_3\ (u/\,0)$ is already small campared to $p_1\ (u/\,0)$.

The probability densities of the integrator output due to signed and impulse noise for the three first bit intervals following the occurrence of the impulse are calculated to be:

$$\left.\begin{aligned} p_1\,(u/\,1) &= \frac{2.07.10^{-4}}{(u-.123)^2} \\ p_2\,(u/\,1) &= \frac{4.55.10^{-4}}{(u-.123)^2} \\ p_3\,(u/\,1) &= \frac{4.09.10^{-6}}{(u-.123)^2} \end{aligned}\right\} \quad \begin{matrix} u \geq .125 \\ \text{or} \\ u \leq .121 \end{matrix} \qquad (3\text{-}10)$$

After making several tries it was found that $\alpha = 2.10^{-7}$ results in an error rate which is close to a minimum. This value of α will therefore be used to illustrate the method by which the threshold values are selected.

The problem is, given $\alpha = 2.10^{-7}$ find threshold levels a and b such as to make β a minimum. The probability densities as given by eq. (3-9)and (3-10)are for a one sec. interval. Since the bit rate is 10^3 bits per sec. eq. (3-9) and (3-10) will be multiplied by 10^{-3} to give the respective probability densities per bit. This is in good agreement with experimental data reported both by Mertz and Fennick.

We will assume that a one was transmitted and calculate the probability of receiving a zero.

Since $p_n\ (u/\,0)$ is of the same form for all n we can sum before we integrate and obtain:

$$\alpha = \sum_{n=1}^{3} \alpha_n = c \int_a^b \frac{du}{(u + .123)^2} \qquad (3\text{-}11)$$

$$\frac{\alpha}{c} = \frac{1}{a + .123} - \frac{1}{b + .123} \qquad (3\text{-}12)$$

This equation is plotted in fig. 18 for $\alpha = 2.10^{-7}$ to show clearly the relationship between a and b for a fixed α.

Any combination of a and b along this curve will result in the same value for α. The desired value is the one that minimizes β.

$$\beta = c \int_{-\infty}^{a} \frac{du}{(u - .123)^2} + c \int_b^{\infty} \frac{du}{(u - .123)^2} \qquad (3\text{-}13)$$

where $c = c_1 + c_2 + c_3$ since : $c_n << 1$ for n = 1, 2, 3.

$$\beta = c \left[\frac{1}{.123 - a} + \frac{1}{b - .123} \right] \qquad (3\text{-}14)$$

This equation is plotted as a function of a for α = constant in fig. 18.

Fig. 18 now allows us to choose the optimum values for a and b. Fig. 18 also shows how sensitive the erasure rate is to threshold location.

From the values obtained for α and β the error rate, E, can be calculated as is shown in the section on Error correction capabilities of the code.

This process can now be repeated until a minimum value for the error rate is obtained.

For the case of the telephone carrier channel under discussion and for $\alpha = 2.10^{-7}$ the minimum erasure probability is any one bit interval is found to be $\beta = 1.5.10^{-4}$. The corresponding error rate is calculated in the appendix to be $E = 1.9.10^{-9}$. In this case the threshold values are seen from fig. 18 to be

a = .125

b = .133

with the channel centered at $\pm$ .123 volts for one and zero.

It is of interest to note that the threshold levels are not symmetrically located with respect to the signalling levels.

Coding

The code that will be used in this investigation is a two dimensional block code with one simple parity check along each row and a cyclic Hamming code along each column. This type of code was picked because of simplicity of implementation and its suitability for the task in question. The received bits are read serially in along the rows. Since the errors due to impulse noise tend to come in bursts the rows are made longer than the longest bursts that can be expected. The maximum burst length equals M for a simple bymmetric binary channel or approximately M_1 for the ECD system. From the peak amplitude distribution we can determine the maximum peak amplitude noise that will occur sufficiently seldom to be tolerable. This gives us the minimum $\frac{S}{N}$ ratio that may occur and subsequently an $M_{1\ max}$ can be found. To be on the safe side we may let the rows be from two to several times $M_{1\ max}$.

In the particular example used to illustrate this investigation it is seen from fig. 6 that a $\frac{S}{N}$ ratio of 10^{-4} is required to give rise to a burst lasting six bits. By extrapolating fig. 7 we find that the probability of this happening within a one second interval is $1.5 . 10^{-7}$ if we use curve II.

It is important to make a distinction here. $1.5 . 10^{-7}$ is not the probability of a six bit long burst but the probability that such a burst is at all possible. The major part of the error control is done along the columns. Since the probability that one burst will cause two errors or erasures in the same column is vanishingly small we can consider the errors along the columns to be independent. The code therefore allows the errors or erasures to be considered as if they were caused by random noise.

Let the code block have m rows and n columns. The redundancy along each column is:

$$R = \frac{m}{w} \tag{3-15}$$

where w is the number of information digits per column and m is the total number of digits per column. The number of check bits is k where:

$$k = m - w \tag{3-16}$$

To construct a single error correcting code we proceed as follows: Since the checking number must indicate which bit is in error and what it should have been the checking number must describe m + 1 different things.

Therefore:

$$2^k \geq w + k + 1 \tag{3-17}$$

or

$$2^w \leq \frac{2^m}{m+1} \tag{3-18}$$

By use of this inequality we can determine the least amount of redundancy required to construct a code that will be able to correct a single or detect a double error. If we add one additional check bit to the column this code will be able to detect three or correct one and detect one error.

From fig. 6 and 7 we see that a $\frac{S}{N}$ ratio of one resulting in not more than a maximum of 6 erasures will happen very infrequently. As was calculated in the beginning of this section on coding conditions for this porribility will exist only $1.5.10^{-7}$ of the second interval. A row length of 25 bits is therefore sufficient to break up the bursts.

To minimize the redundancy of the rows let us satisfy the inequality with

$$u + w = p \tag{3-19}$$

where

$$p + 1 = 2^u \tag{3-20}$$

This equation is satisfied for

$$p = 31, \quad u = 5, \quad w = 26$$

resulting in a column with 26 information bits, a 5 bit Hamming code and one parity check bit. The final code block is a 25 x 32 bit matrix. It has the capacity to detect any three or correct one and detect one bit per column.

Error Correcting Capability of the Code

The code as developed in the previous section consists of a 25 x 32 bit matrix. Each row has 24 information digits followed by a parity check bit. The code block is transmitted row by row.

Each column is constructed of 26 information bits followed by 5 bits which together forms a one bit error correcting Hamming code. The column is completed with a parity check bit. The 32 bit column vector can be viewed as a number in 32 dimensional space.

To clarify the geometric picture let us take a three digit code word abc which represents a vector in three dimensional space. The various numbers that can be represented by this word are the lattice points of the cube shown in fig. 21. We see that there are four combinations that have even and four combinations that have odd parity. To make a single error detecting three digit code we can use the last digit for parity check. Let us select odd parity. In this case all the white spheres represent valid code words. When this code is working with the ECD, a single erased bit can always be replaced cyclicly and the correct code word will be found by the parity check system. If two erasures have occurred such a replacement is no longer unique. To clarify this let us investigate a specific case.

Assume that a 100 was transmitted and that a xx0 was received. When the two first bits are replaced, both combinations 010 and 100 will be acceptable to the parity checking registers. In the first case two errors are committed, in the second case no errors are made. The average number of errors committed is therefore one per code word.

From this analysis we can generalize. In a code with an N error detection capability and Q erasures, all erasures can be replaced as long as $Q \leq N$ in the absence of errors. This is clear since the erasures indicate where a possible error is. The suspected bits are now replaced cyclically until the code indicates that no errors are present. If $Q > N$, as many as Q errors may be committed in any one code word. The average number of errors committed in a large number of code words is $Q - N$.

For the 25 x 32 bit code all cases of E ? or ??? along one column can be corrected. Here E indicates an error and ? an erasure. If four erasures occur along one column one wrong bit will result on the average. If one error and two erasures fall in the same column one error will also on the average result.

Based on this reasoning we will assume all erasures exceeding the detecting capabilities of the code to result in errors.

The probability of one erased bit per bit interval is:

$$p\ (1) = \beta_1 + \beta_2 + \beta_3 - \beta_1\beta_2 - \beta_2\beta_3 - \beta_1\beta_3 + \beta_1\beta_2\beta_3 \qquad (3\text{-}21)$$

if we neglect β_n for $u \geq 4$.

During a time less than one second p_n (u) is approximately proportional to time. For long intervals of time, $1 - p_n$ (u) has a hyperbolic distribution.[3] The probability of one erasure in any particular bit interval is therefore:

$$p\ (1) = \beta_1 + \beta_2 + \beta_3 \qquad (3\text{-}22)$$

since $\beta_n << 1$ for all n.

Since at least four erased bits are required per column to give one error on the average, the probability of an error due to this cause is:

$$p_r \left\{4\right\} = \left[p\ (1)\right]^4 \quad \frac{32!}{4!\ 28!} \qquad (3\text{-}23)$$

$p_r \left\{4\right\}$ is the probability of committing one error on the average due to four erasures. Correspondingly $p_r \left\{5\right\}$ is the probability of committing on the average two errors due to five erasures.

$$p_r \left\{5\right\} = \left[p\ (1)\right]^4 \frac{32!}{5!\ 27!} \qquad (3\text{-}24)$$

The error rate due to this erasure pattern is:

$$E_\beta = \frac{10^3}{32} \left[p_r \left\{4\right\} + 2\ p_r \left\{5\right\} + 3\ p_r \left\{6\right\} + \dots\right] \qquad (3\text{-}25)$$

$$E_\beta = \frac{10^3}{32} \left[\sum_{n=1}^{28} n\ p_r \left\{3 + n\right\}\right] \qquad (3\text{-}26)$$

Errors can also be generated if there is more than one error per column or one error and more than one erasure.

The probability of two errors in a column is:

$$p_r \left\{2E\right\} = \left[p\ (1E)\right]^2 \ \frac{32!}{2!\ 30!} \approx \left[E_\alpha\right]^2 \cdot \frac{10^3}{32} \qquad (3\text{-}27)$$

where

$$p\,(1E) = \alpha_1 + \alpha_2 + \alpha_3 - \alpha_1\alpha_2 - \alpha_2\alpha_3 - \alpha_1\alpha_3 + \alpha_1\alpha_2\alpha_3$$
$$\approx \alpha_1 + \alpha_2 + \alpha_3 \tag{3-28}$$

The probability of one erasure and two errors is:

$$p_r\left\{2,\ 1E\right\} = \left[p\,(1E)\right]\left[p\,(1)\right]^2 \frac{32!}{3!\,29!} \tag{3-29}$$

Continuing this reasoning for a greater number of erasures we obtain:

$$E_{\alpha,\beta} = \frac{10^3}{32} \sum_{n=1}^{30} n\,p\,(1E)\left[p\,(1)\right]^{1+n} \frac{32!}{(2+n)!\,(30-n)!} \tag{3-30}$$

The error rate is now

$$E = E_\alpha + E_p + E_{\alpha,\beta} \tag{3-31}$$

The ECD Integration of the Detector Output

The threshold levels that will be used here are the ones that were found to result in a minimum overall error rate

Errors of the first kind

$$\alpha_1 = \int_a^b p_1(u/0)\,du = 2.07 \left[\frac{1}{.123+.115} - \frac{1}{.123+.133}\right] .10^{-7} \tag{3-32}$$

$$\alpha_1 = .302.2.07.10^{-7} = .625.\ 10^{-7}$$

$$\alpha_2 = \int_a^b p_2\,(u/0)\,du = .302.4.55.10^{-7} \tag{3-33}$$

$$\alpha_2 = 1.374.10^{-7}$$

$$\alpha_3 = \int_a^b p_3\,(u/0)\,du = .302.4.09.10^{-9} \tag{3-34}$$

$$\alpha_3 = .0124.10^{-7}$$

$$\alpha_n \approx 0 \quad \text{for } n \geq 4$$

Erasures or errors of the second kind

$$\beta_1 = \int_{-\infty}^{a} p_1\,(u/1)\,du + \int_b^{\infty} p_1\,(u/1)\,du = 2.07.10^{-7} \left[\frac{1}{.123-.115} - \frac{1}{.123-.133}\right] \tag{3-35}$$

$$\beta_1 = 2.07.10^{-7}.225 = .\ 66.10^{-4}$$

$$\beta_2 = \int_{-\infty}^{a} p_2\,(u/1)\,du + \int_b^{\infty} p_2\,(u/1)\,du \tag{3-36}$$

$$\beta_2 = 4.55.10^{-7}.225 = 1.023.10^{-4}$$

$$\beta_3 = \int_{-\infty}^{a} p_3\,(u/1)\,du + \int_b^{\infty} p_3\,(u/1)\,du \tag{3-37}$$

$$\beta_3 = 4.09.10^{-9}.225 = .0092.10^{-4}$$

$$\beta_n << \beta_1 \quad \text{for } n \geq 4.$$

α_n and β_n represents the average number of errors or erased bits per bit interval due to the n'th bit interval following the bit when the impulse occurred. For short time intervals the probability of an error is approximately proportional to the length of the interval if only one error is counted per burst. This was shown in data published by Mertz.[3]

The probability of one incorrect bit per interval is approximately:

$$\alpha = \alpha_1 + \alpha_2 + \alpha_3$$

$$\alpha = 2.10^{-7}$$

The probability of one erased bit per bit interval is:

$$\beta = \beta_1 + \beta_2 + \beta_3 - \beta_1\beta_2 - \beta_2\beta_3 - \beta_1\beta_3 + \beta_1\beta_2\beta_3 \tag{3-38}$$

$$\beta = 1.5 \cdot 10^{-4}$$

The error rate calculated on this basis is carried out in the appendix and is found to be:

$$E = E_\alpha + E_\beta + E_{\alpha\beta}$$

$$E = (.63 + .70 + .57)\,.10^{-9} \tag{3-39}$$

$$E = 1.9\,.10^{-9}$$

E is the average number of incorrect bits in the output per second.

Comparison of Error Rates

The ECD system will be compared with two other digital data transmission systems. The first is a simple binary system, the second is the same system but with the addition of an error correcting code with the same correction capabilities as the one used with the ECD system. The row length in this code can be reduced since for a symmetrical binary channel the figure of merit is M rather than M_1. Since M_1 corresponds to a $\frac{S}{N}$ five times as great as the one needed for M we see from fig. 6 that for a telephone channel with integration of the detector output M = 5.3 bits corresponds to M_1 = 6 bits. Since the code was choosen to have $4 M_1 + 1$ bits per row for a $\frac{S}{N} = 10^{-4}$ the code used with a symmetrical binary channel could have a length of 23 rather than 25 bits for the same performance. This last system would be further simplified since no memory for the erasures would be required.

Symmetric Binary Channel

In this case all errors of the first kind are real errors and the threshold level is zero volts. Assume zeros and ones are equally likely.

$$E = \sum_{n=1}^{3} \int_{-\infty}^{0} p_n (u/1)\, du \qquad (3-40)$$

$$E = \frac{2.07.10^{-4}}{.123} + \frac{4.55.10^{-4}}{.123} + \frac{4.09.10^{-6}}{.123}$$

$$E = 5.3.10^{-3}$$

Symmetric Binary Channel with a 23 x 32 bit Code Block

$$E = \sum_{k=1}^{30} \left[\sum_{n=1}^{3} \int_{-\infty}^{0} p_n (u/1)\, du \right]^{k+1} \frac{32!\ k}{(k+1)!\ (31-k)!} \cdot \frac{10^3}{32} \qquad (3-41)$$

$$E = 4.4.10^{-7}$$

Comments

From these calculations we can see the relative performance of the three systems subject to the same amount of impulse noise. We see that the ECD system is best with about 2.10^{-9} incorrect bits per second. The symmetric binary channel using the 23 x 32 bit code is second with an error rate of 4.10^{-7}.

The symmetric binary channel with no error correction is worst with an error rate of 5.10^{-3} wrong bits per second.

The second best system has improved the error rate by a factor of 10^4 at the cost of a redundancy of 1.23 and some equipment complexity.

The ECD further improved the error rate by a factor of about 200 without additional redundancy at the cost of a modest increase in equipment complexity.

Effect of Integration on the Noise Output Voltage from the Demodulator

We will now investigate the effect on the noise voltage available at the decision device with and without first integrating the demodulator output voltage. To do this we will evaluate the probability of error for a simple symmetrical binary channel due to impulse noise for various bit positions after the impulse first occurred. We will first treat the case where we sample the demodulator output. The channel is the telephone channel we have been analyzing where

$$\text{zero} = -.123 \text{ volts}$$
$$\text{one} = +.123 \text{ volts}$$

The average error rate per second is approximately

$$a = a_1 + a_2 + a_3 + \dots = \sum_{n=1}^{N} a_n$$

since errors are mutually exclusive and a_n is a small number. Correlation between bit intervals is not important if we are interested in the average error rate.

$$\left.\begin{aligned} a_1 &= \int_0^\infty p_1 (u/0)\, du = \frac{7.2.10^{-4}}{2.123} = 2.93.10^{-3} \\ a_2 &= \int_0^\infty p_2 (u/0)\, du = \frac{3.34.10^{-4}}{2.123} = 1.36.10^{-3} \\ a_n &= \int_0^\infty p_n (u/0)\, du = \frac{5.4.10^{-4}}{.123.2^{n+1}} \qquad \text{for } n \geq 3 \end{aligned}\right\} \qquad (3\text{-}42)$$

$$a_n = \frac{4.32.10^{-3}}{2^{n+1}}$$

The probability of an error due to the impulse noise burst in the n'th bit interval is shown in fig. 19.

We will now calculate the error rate for the same channel with the difference that the detector output voltage is integrated before it is sampled. In this case:

$$\left.\begin{aligned}
a_1 &= \int_0^{\infty} p_1\,(u/\,0)\,du = \frac{4.15.10^{-4}}{2\,(u-.123)}\Big|_0^{\infty} = 1.69.10^{-3} \\
a_2 &= \int_0^{\infty} p_2\,(u/\,0)\,du = \frac{9.1.10^{-4}}{2\,(u-.123)}\Big|_0^{\infty} = 3.95.10^{-3} \\
a_3 &= \int_0^{\infty} p_3\,(u/\,0)\,du = \frac{8.19.10^{-6}}{2\,(u-.123)}\Big|_0^{\infty} = 3.33.10^{-5}
\end{aligned}\right\} \qquad (3\text{-}43)$$

For $n \geq 4$ a_n becomes neglibly small.

These values of error rate in the various bit intervals after the start of the impulse are plotted in fig. 19. The error rate for the first system is $a = 4.11.10^{-3}$ while the second system has an error rate of $5.42.10^{-3}$. The error rate is therefore not significantly different for the two system when no coding or error correction is being done. If some error control method is used that depends on short burst lengths fig. 19 shows that the situation is quite different. While there is practically no difference in the probability of making an error in the two first bit intervals after the start of the noise burst the probabilities become very much favorable for the system employing the integrator once the second bit position is passed.

The reason for this phenomenon is that once the catastrophic part of the impulse is over what remains are mainly the high frequency components of the noise burst. These components are approximately normal to the signal over the bit interval. Integration over the bit interval will therefore result in a very small detector output.

The reason why the high frequency components of the noise burst arrive late is the phase distortion of the channel.

Since integration of the detector output will tend to result in fewer wrong bits caused by a given impulse noise burst it can be considered optimum in the sense that it is better than direct sampling of the detector output voltage.

Implementation of the Code

A coding system of the type discussed in this thesis might conveniently be implemented around a magnetic core or tunnel diode memory with a capacity of 800 bits. The row vectors containing 24 information bits and one parity check bit. The generator polynomial for this code is

$$g_1 (x) = x + 1 \tag{3-44}$$

The circuit symbols that will be used are shown in fig. 23.

A circuit that will generate a code with equation 3-44 as its generating function is shown in fig. 24.

The input is connected to the transmitter coder. The row vector is being stored as it is being transmitted. Following the 24 information bits, the content of the shift register stage is transmitted to make up the parity check. The 26 row vector containing the information digits are transmitted during a 650 ms interval. After 626 ms, all the information digits of the first column vector have been transmitted and the error control bits for the first column can now be calculated. Since the column vector is constructed with 26 information bits in the first places we can form a complete column vector, F, by filling the rest of the spaces with zeros. The six parity check positions now make up an additional vector, R. By use of the Euclidian division algorithm

$$s (x) = d (x) g (x) + r (x) \tag{3-45}$$

we can form the complete column vector. If we let s (x) = F the information column vector and let r (x) = R equal the parity check vector, we can obtain the column vector:

$$f (x) - r (x) = d (x) g (x) \tag{3-46}$$

Here g (x) is the generator polynomial for this code. The parity check vector is therefore calculated as the remainder in the calculation

$$\frac{f (x)}{g (x)} \tag{3-47}$$

This calculation can be done by synthesising a circuit that will divide a polynomial by g (x). First we must determind g (x). Since there are 32 bits in the code and all bits equal to zero is not a valid code, g (x) must be a root of

$$x^{32} - 1 = 0. \tag{3-48}$$

This expression can be factored as

$$x^{32} - 1 = (x^{26} + x^{24} + x^{18} + x^{16} + x^{10} + x^{8} + x^{2} + 1)(x^{5} + x^{4} + x + 1)(x + 1) \tag{3-49}$$

bearing in mind that + and - have the same meaning modulo two. Since we have six parity check bits our division polynomial must be of sixth degree. We can therefore let

$$g(x) = (x^{5} + x^{4} + x + 1)(x + 1) = x^{6} + x^{4} + x^{2} + 1 \tag{3-50}$$

We can now proceed to synthesize a circuit that will divide a polynomial of 32 degree by g (x) and storing the remainder.

The circuit in fig. 25 operates as follows. The gate G is closed. The data bits from the column being processed are read out from the matrix. They are not needed for further use so this can be done destructively if this is most convenient. After the 26 data bits have been read in , -R has been calculated by the circuit and is stored in the shift register. Since addition and subtraction modulo two are equivalent we now have R. G is now opened and the error control bits are shifted out from the shift register and placed in the last six rows of the dolumn vector. This process continues until column 25 is reached.

To see the sequential operation of the matrix and the coding circuit more clearly let us retrace its functions.

As the data bits are being transmitted to the channel they are also being stored in the matrix. The bits are transmitted by rows. Simultaneously the circuit in Fig. 24 calculates the row check bit used to complete each row. Since this bit is not needed again they need not be stored. The transmitter memory need therefore only be a 32 x 24 matrix. As soon as the last data bit of the first column has been transmitted, the corresponding error control bits can be calculated.

This process continues for the rest of the rows as they become ready for it. While the problem of coding is fairly straight forward, the problem of decoding is much more complex. The reason for this is that while the transmission is done in a binary system, three different symbols are available at the detector output. First of all a matrix capable of storing trinary digits must be available. Such a matrix, could be made up of Esaki diodes as shown in fig. 26.

In this memory all cells would be set to zero at the beginning of the cycle. As the various bits come in, a circuit as shown in fig. 24 will calculate the row parity check. The result of this calculation is compared with the received check bit and both are stored at the end of the row. This can only be done if no E signals have occurred. If it does not check an E is stored.

This process is continued until the whole transmitted word has been received. When this has happened there are two possible groups of occurrences.

A) No erasures have been received.

B) One or more erasures have been received.

We will first consider A. The fact that no erasures are received does not necessarily mean that no errors have been committed. It is therefore necessary to check the rows. The generator polynomial for the row code was:

$$g(x) = (x + 1)(x^5 + x^4 + x + 1) \tag{3-51}$$

Since this polynomial can be factored as shown, the two factors can be calculated separately. The checking of the code can therefore be done by the circuit shown in fig. 27.

The column vector under test is fed from the matrix to the circuit shown in fig. 4. As soon as all the symbols have been read from the column, the checking calculations are finished. If $r(1) = r(a) = 0$ it is assumed that no error has occurred. If $r(1) = 1$, $r(a) \neq 0$ it is assumed that a single error has occurred. This error is corrected by reading once more the digits from the stored column vector and at the same time moving the shift register with no input. When the combination 10000 appears in A, the incorrect bit is just coming out of the matrix. This bit is inverted and the error is corrected.

If $r(1) = 1$ and $r(a) = 0$, a multiple error has occurred. This kind of error may be corrected by observing the row checks. If any of the row checks failed it will be assumed that the errors occurred at the intersection of the corresponding rows and columns. This check can not be carried out until all the columns have been checked. Now the cross check can be carried out.

If one or more erasures have occurred we are confronted with case B. In this case the column check of any columns without erasures should first be carried out and possible errors corrected. Following this the procedure explained under case A will be repeated by replacing the erased bits sequentially with zeros and ones until all the checks are found valid. This process can be simplified if in addition to storing erasures, information is stored about whether the erased bit was closest to a zero or a one.

Discussion of Other Impulse Noise Error Control Methods

As a result of this investigation into the nature and effects of impulse noise, results have been obtained that are valuable in evaluating the performance of various channels and impulse noise control methods. From the plot of the figure of merit of the system, M_1, as seen in fig. 6 we see that the system which has the shortest impulse response is the best in the sense that it will cause the least amount of consecutive wrong digits. To obtain a short impulse response, the amplitude cutoff of the channel should be gentle, and the phase distortion as small as possible. From fig. 19 we see, however, that integration of the detector output will greatly decrease the effects of phase distortion since the received impulse noise is approximately normal to the signal after the first two or three bit intervals.

Clipping of the Received Signal

Clipping is a frequently used method to combat the effects of impulse noise. If the energy from the impulse is stored in the transmission channel, such as the case is in most telephone channels, clipping will merely cause distortion of the received signal and be of little value. If, on the other hand, the receiver contains filters that can store the received energy over several bit intervals, clipping at a level slightly above the signal will probably improve the system performance considerable.

Smear Desmear Filtering[13]

The effect of smear desmear filtering is to distribute theenergy of the impulse over several bit intervals. If no error correction is done by the system, this method will cause a trade of several single bit errors for fewer longer bursts occurring more seldom since the basic principle of smear-desmear filtering is to spread the energy in the impulse over several bit intervals.

While this may be an efficient trade off if no error correction is done, the trade off is not a good one if a simple error correcting code is used.

If clipping and smear desmear filtering is employed, the performance of the system can be considerably improved. Such a system is shown in fig. 28.

In the system shown in fig. 28, data which is modulated onto an appropriate carrier is passed through a smear filter, possibly one with severe phase distortion. The output of the channel passes an equalizer which equalizes for the phase distortion of the channel. This will shape up the impulse response so that maximum advantage can be obtained from the clipper.

The clipping level should be set only slightly higher than the data level. From the clipper, the data is passed through a desmear filter, possibly one with the inverse phase response of the smear filter, and is further passed on to the demodulator.

While a system such as depicted in fig. 28 has probably great merit in any data transmission system, further investigations of it will not be carried out here since they are outside the scope of this investigation.

Deviations from the assumed conditions

It is of interest to investigate what the effects of variations from the assumed probability law for v . Let

$$p\,(v) = \frac{\epsilon x_o}{v^{2+\Delta}} = \frac{A}{v^{2+\Delta}} \qquad v \geq V_{min} \tag{3-52}$$

where $-1 < \Delta$

$$p\,(u) = 2\,a\,A \int_{V_{min}}^{\infty} p\,(a\,\frac{u}{v}) \cdot \frac{1}{v^{3+\Delta}}\,dv \tag{3-54}$$

$p\,(u/v)$ can be expanded in a power series.

$$2\,p\,(a\,\frac{u}{v}) = b_n\,(\frac{u}{v})^n + b_{n-1}\,(\frac{u}{v})^{n-1} + \ldots\, b_1(\frac{u}{v}) + b_o \tag{3-55}$$

for $\qquad Z_{min} \leq a\,\frac{u}{v} \leq Z_{max}$

$$2\,p\,(a\,\frac{u}{v}) = 0 \qquad \text{for} \qquad a\,\frac{u}{v} \geq Z_{max}, \quad a\,\frac{u}{v} \leq Z_{min}$$

where Z_{max} and Z_{min} are obtained from the given impulse response.

By use of (3-55), equation (3-54) becomes:

$$p\,(u) = a\,A \int_{V_1}^{\infty} \left[b_n\,\frac{u^n}{v^{n+3+\Delta}} + \ldots\, b_1\,\frac{u}{v^{4+\Delta}} + b_o\,\frac{1}{v^{3+\Delta}} \right] dv \tag{3-56}$$

Where $V_1 = V_{min}$ if $a\,\frac{u}{V_{min}} \leq Z_{max}$

and $\quad V_1 = a\,\frac{u}{Z_{max}}$ if $a\,\frac{u}{V_{min}} \geq Z_{max}$

Since the first condition is true only for very small values of u since $V_{min} = .0024$ and we are primarily interested in the effects on voltages of magnitudes much greater than this, we will only explore the second condition. If the second condition is applied to equation (3-56) we obtain:

$$p(u) = aA \int_{\frac{a}{Z_{max}}}^{\infty} \left[b_n \frac{u^n}{v^{n+3+\Delta}} + \dots b_1 \frac{u}{v^{4+\Delta}} + b_o \frac{1}{v^{3+\Delta}} \right] dv \qquad (3\text{-}57)$$

$$p(u) = aA \left| - \frac{b_n}{n+2+\Delta} \frac{u^n}{v^{n+2+\Delta}} \dots - \frac{b_1}{3+\Delta} \frac{u}{v^{3+\Delta}} - \frac{b_o}{2+\Delta} \frac{1}{v^{2+\Delta}} \right|_{\frac{au}{Z_{max}}}^{\infty} \qquad (3\text{-}58)$$

$$p(u) = \frac{aA}{u^{2+\Delta}} \left[\frac{b_n}{n+2+\Delta} \left(\frac{Z_{max}}{a}\right)^{n+2+\Delta} + \dots + \frac{b_1}{3+\Delta} \left(\frac{Z_{max}}{a}\right)^{3+\Delta} + \frac{b_o}{2+\Delta} \left(\frac{Z_{max}}{a}\right)^{2+\Delta} \right] \qquad (3\text{-}59)$$

If we let

$$K = aA \left[\frac{b_n}{n+2+\Delta} \left(\frac{Z_{max}}{a}\right)^{n+2+\Delta} + \dots + \frac{b_1}{3+\Delta} \left(\frac{Z_{max}}{a}\right)^{3+\Delta} + \frac{b_o}{2+\Delta} \left(\frac{Z_{max}}{a}\right)^{2+\Delta} \right] \qquad (3\text{-}60)$$

We see that K is regular if $a \neq 0$ and $\Delta > -2$. Both of these conditions agree well with our previous conditions and results. We can therefore write equation (3-49) as:

$$p(u) = \frac{K}{u^{2+\Delta}} \qquad \text{for } |u| \geq \frac{Z_{max} V_{min}}{a}$$

The probability of committing an error in any particular bit interval is:

$$a_n = \int_{a+.123}^{b+.123} p_n(u)\, du \qquad (3\text{-}61)$$

$$a_n = \frac{K_n}{1+\Delta} \left[\frac{1}{(a+.123)^{1+\Delta}} - \frac{1}{(b+.123)^{1+\Delta}} \right] \qquad (3\text{-}62)$$

As $\Delta \to -1$, the error rate as given by equation (13) becomes indeterminate. In order to determine Lim α_n ad $\Delta \to -1$ we apply L'Hospitals theorem and obtain

$$\lim_{\Delta \to -1} \alpha_n = K_n \ln \frac{b + .123}{a + .123} \tag{3-63}$$

From equation (3-62) we also see that no finite value of $\Delta > -1$ will cause any sudden changes in α.

From equation (3-63) we see that the error rate is a well behaved function of Δ that does not depend critically on any particular assumed value.

The error rate α is as can be seen from equation (3-62) proportional to K_n which again is proportional to ϵV_o. It may therefore at first glance seem like the estimate of this value is very critical. The voltage $V_{min} = \epsilon V_o$ is, however, quite well defined since it represents the amplitude threshold which is exceeded or reached on the average once a second. It should therefore not be excessively difficult to obtain a rather good estimate of this value.

It is of interest to see how much our results would have been changed if we had used $\Delta = \pm 1$.

If $\Delta = 0$:

$$\alpha_1 = K_n \left[\frac{1}{a + .123} - \frac{1}{b + .123} \right] \tag{3-64}$$

If we let $\Delta = 1$:

$$\alpha_1 = \frac{K_1}{2} \left[\frac{1}{(a + .123)^2} - \frac{1}{(b + .123)^2} \right] \tag{3-65}$$

For the limiting case $\Delta = -1$:

$$\alpha_1 = K_1 \, 1_n \frac{b + .123}{a + .123} \tag{3-66}$$

Obtaining numerical values to facilitate the evaluation of these results when

$$a = .115 , \quad b = .133$$

we obtain from equation (3-64)

$$a_1 = K_n \left[\frac{1}{.238} - \frac{1}{.256} \right] = .19\, K_1 \quad , \Delta = 0 \tag{3-67}$$

From equation (3-65)

$$a_1 = \frac{K_n}{2} \left[\frac{1}{.238^2} - \frac{1}{.256^2} \right] = 1.2\, K_1 \quad , \Delta = 1 \tag{3-68}$$

From equation 18

$$a = K_n \ln \frac{.256}{.238} = .07 K_1 \quad , \quad \Delta = -1 \tag{3-69}$$

This result is not unexpected since for all three cases, the average number of impulses per unit time and there minimum amplitude is fixed. Since a more positive Δ indicates that the picked up noise decreases more rapidly with distance, we would expect a greater number of stronger impulses.

Experimental evidence[10] indicates that $\Delta \geq 0$ is the usual case. For this reason we will consider in more detail the case where $\Delta = 1$.

$$p\,(V) = \frac{\epsilon x_o}{V^3} = \frac{A_1}{V^3} \tag{3-70}$$

$$\int_{V_{min}}^{\infty} \frac{A_1}{V^3}\, dv = - \frac{1}{2} \frac{A_1}{V^2} \Big|_{V_{min}}^{\infty} = \frac{A_1}{2\, V_{min}^{\ 2}} \tag{3-71}$$

$$V_{min} = \sqrt{\frac{A_1}{2}} \tag{3-72}$$

If the two probability distributions are fitted to the experimental data at the point where they cross:

$$\frac{A_o}{V_o} = \frac{A_1}{V_o^{\ 2}} = \frac{\epsilon x_o}{V_o} \tag{3-73}$$

$$A_1 = \epsilon x_o V_o$$

$$V_{min} = \sqrt{\frac{\epsilon x_o V_o}{2}} \tag{3-74}$$

But:

$$\left[\frac{b_n}{n+2+1}(Z_{max})^{n+2+1} + \ldots \frac{b_o}{2+1}(Z_{max})\right]^{2+1} \leq \left[\frac{b_n}{n+2}(Z_{max})^{n+2} + \ldots \frac{b_o}{2}(Z_{max})\right]^{2}$$

(3-75)

Therefore

$$K_{\Delta=1} \frac{1}{A_1} \leq \frac{1}{A_o} K_{\Delta=0} \qquad (3\text{-}76)$$

If we use the equality:

$$K_{\Delta=1} = \frac{A_1}{A_o} K_{\Delta=0}$$

$$K_{\Delta=1} = V_o K_{\Delta=0} \qquad (3\text{-}77)$$

V_o is the voltage at which the two curves intersect. In the present case $V_o = .109$ volts. Equations (3-67) and (3-68) can therefore be directly compared.

$$a_{\Delta=0} = .19 K_{\Delta=0} \qquad (3\text{-}78)$$

$$a_{\Delta=1} = 1.2 \cdot .109 \cdot K_{\Delta=0} = .131 K_{\Delta=0} \qquad (3\text{-}79)$$

$$a_{\Delta=1} \leq .69 a_{\Delta=0} \qquad (3\text{-}80)$$

We notice that for $\Delta > 0$ the error rate decreases somewhat compared to the case when $\Delta = 0$. Since not enough data is as yet available to justify using $\Delta > 0$ we will use $\Delta = 0$ and the results obtained are therefore on the conservative side.

Further Improvements of the ECD

The results obtained by use of the ECD in the previous described manner look very promising. There is, however, still room for improvements. Since in order to commit an error when a one was transmitted, u most satisfy the requirement.

$$-.123 - b = u \leq -.123 - a$$

This condition implies that the probability for the next bit to be erased is very great since noise burst of large amplitude must have occurred. We can calculate the probability that an error will be followed by an erasure by use of the joint probability function $p_{mn}(u_m, u_n)$. Instead of actually calculating this probability we can make a good estimate of it. If the one is followed by a one we see from fig. 11 and 12 that for the two first bit intervals u does not change sign. Therefore since the probability of an erasure during the second bit interval is

$$\beta_2 = 1 - \alpha_2 \tag{3-81}$$

in this particular case. An error in the first bit interval is therefore always followed by an erasure or error in the second bit interval. Since two consecutive errors are not very probable, the conclusion is that an error in the first bit interval is almost always followed by an erasure in the second bit interval. The converse is also true. An error in the second interval is almost always preceeded by an erasure in the first bit interval.

From this reasoning it is seen that the efficiency of the system could be improved by in addition to the erased bits also erasing the bits immediately preceeding and following the erased bits.

Detailed calculations of the magnitude of this improvement has not been carried out. The procedure is to calculate

$$\begin{aligned}\beta_n = &\int_{-b}^{-a}\int_{-\infty}^{-b} p_{mn}(u_m/1,\ u_n/1)\, du_n\, du_m \\ &+ \int_{-b}^{-a}\int_{-a}^{a} p_{mn}(u_m/1,\ u_n/1)\, du_n\, du_m \\ &+ \int_{-b}^{-a}\int_{b}^{\infty} p_{mn}(u_m/1,\ u_n/1)\, du_n\, du_m\end{aligned} \tag{3-82}$$

where $m = n \pm 1$

The magnitude of this improvement is uncertain and furthermore it is uncertain whether the additional equipment complexity would warrant it.

The method for finding the optimum threshold leaves $\pm$ a and $\pm$ b the same as was used for the ECD system.

Comments

Based on the calculations carried out in this investigation we can construct an ECD system that using a 25 x 32 bit block code will give an error rate of 2.10^{-9} errors per second under adverse conditions of impulse noise. The code used in this system has a redundancy of 1.23.

The symmetric binary channel operating with the same signal levels and subject to the same amount of interference would have an error rate of 5.10^{-3} errors per second. If the same 25 x 32 bit code block that is used with the ECD is used in combination with the symmetric binary channel, the error rate drops to 4.10^{-7} and the redundancy is 1.23.

The ECD system is therefore seen to give an improvement in error rate of 200 with no additional redundancy or slowing down of the signalling rate. This gain is made at the cost of a modest increase in equipment complexity.

The ECD giving this improvement in performance has threshold levels of $\pm$ .115 and $\pm$ .133 volts when the expected signal level is $\pm$ .123 volts. The average number of erasures is .15 per bit interval per second.

The example of an ECD system worked through in this thesis is a very simple one using a linear symmetric binary channel and phase shift keying with synchronous detection. This is done to show clearly the method of approach rather than to solve a specific problem. The method is quite general and can be applied to other systems such as FSK or Kineplex with equal validity. Since the results obtained using these two modulation methods will be more complex, it is felt that the chosen system is preferable in order to demonstrate the basic approach clearly.

The results obtained agree well with experimental evidence, particularly that obtained by J.H. Fennick at Bell Telephone Laboratories from measurements made on switched telephone lines at various points along the east coast. The agreement is less with data obtained from circuits where saturation effects are important. Some workers in the field have also counted errors caused by interruptions of the transmission path as errors due to impulse noise. In this case there clearly will be no agreement.

The results of this investigation show that when numerical methods are used, polynomial approximation should be used as seldom and as late in the problem as possible. This will lead to the most accurate results. Much was gained not only in accuracy but also in ease of calculation when numerical integration and differentiation rather than curve fitting was used for the first part of the calculations.

Based on the calculations carried out in this investigation and on experimental data it was found that the tail or extremal part of $p_n(u)$.The probability density at the output of the integrator at the end of the n'th bit intervals,can be represented by

$$p_n(u) = \frac{c_n}{u^k}$$

where k = 2 in this investigation. Since $p_n(u)$ is of this simple form outside of the small center region, the form of $p_n(u)$ is not influenced by the received waveshape. The integral of the received waveshape determines the magnitude of C_n and how rapidly the sequence C_n converges with increasing n values of n.

The agreement between calculations and experimental data shows definitely that the tail of $p_n(u)$ is in the form of an inverse power law rather than an exponential law. This is due to the inverse power law for p(v), the probability density of the peak amplitude voltage.

An approximate method was devised as shown in the appendix by which the coefficients C_n can be estimated directly once the integrated impulse response is known. This method is useful to obtain a first approximation to the error rate that can be expected from a channel before it is decided to carry out more detailed calculations.

The estimation test carried out to obtain the optimum threshold for the ECD is of such a nature as to keep the size of the test constant while minimizing the errors of the second kind. The test is therefore a uniformly most powerful test.

The basic idea behind the ECD is to concentrate as much as possible on the impulse noise energy in a few bit intervals as possible. Because of this, the idea of smear-desmear filters will be of no use here since their action is exactly the opposite. The use of such filters combined with clipping in the proper manner, smear-channel-equalization-clipping-desmear, will probably give considerable improvement in system performance.

To sharpen the impulse response of the channel for a given bandwidth, it is very important to keep the phase characteristics of the channel as linear as possible.

To minimize the effects on the impulse response, the band-pass characteristics of the channel from where the impulse enters the system should be as close to cosine squared as possible since this results in the smallest possible value for M.

It has been shown that integration and sampling of the detector output is a more optimum method than direct sampling of the detector output in thecase of a telephone carrier channel. This is in general true for any channel that possesses a great amount of phase distortion since in this case part of the impulse response is approximately normal to the signal over the bit interval. This phenomeon can be used to partly eliminate the effects of phase distortion in the channel.

This investigation should not be interpreted only as an investigation of errors occurring in digital date transmission over telephone circuits. It should rather be considered as a general approach to the problem where digital information is transmitted over any communications channel perturbed by impulse noise.

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Calculation of M for the RLC Channel

AM Envelope Detector, System I

Impulse noise:

$$n(t) = A\, e^{-\alpha(t-T)} \cos(\omega_1 t + \theta)$$

Assume that the detector is centered at the center frequency of the whole system.

Signals:

One : $S_1(t) = S \cos \omega_o t$

Zero: $S_o(t) = 0$

If data is transmitted at the Nyquist rate

$$s(t) = \frac{S}{2}(1 + \cos \omega_m t) \cos \omega_o t$$

$$\omega_m = \alpha$$

Bit interval : τ

$\tau = \frac{\pi}{\omega_m}$

Case I

Impulse occurs when a string of zeros are transmitted.
Detector output r (t).

$$r(t) = A\, e^{-\alpha(t-T)}$$

Time required for the noise burst to decay to less than $\frac{S}{2}$ is t_1.

$$\frac{S}{2} = A\, e^{-\alpha t_1}$$

$$M = \frac{t_1}{\tau} = \frac{1}{\alpha} \quad \frac{\omega_m}{\pi} \ln \frac{2A}{S}$$

$$M = -\frac{1}{\pi} \ln \frac{1}{2}\frac{S}{N}$$

Case II

A string of ones are transmitted.

$$p(t) = A\, e^{-\alpha(t-T)} \cos(\omega_1 t + \theta) + S \cos \omega_o t$$

Since the channel is centered at the carrier frequency

$$\omega_1 = \omega_o$$

$$p(t) = \left[A e^{-\alpha(t-T)} \cos\theta + S\right] \cos\omega_o t - A e^{-\alpha(t-T)} \sin\theta \sin\omega_o t$$

After detection and filtering:

$$r(t) = A\sqrt{e^{-2\omega_m (t-T)} + 2\,\frac{S}{N}\, e^{-\omega_m (t-T)} \cos\theta + \left(\frac{S}{N}\right)^2}$$

r (t) takes on its maximum value when $\theta = \pi\ (2n+1)$ where n is an integer. In this case:

$$r(t) = A\left[e^{-\omega_m (t-T)} + \left(\frac{S}{N}\right)\right]$$

$r(t) \geq \frac{S}{2}$ after a time t_1 since the start of the impulse.

$$\frac{S}{2A} = \frac{S}{2N} = e^{-\omega_m t_1} + \frac{S}{N}$$

$$M = \frac{t_1}{\tau} = -\frac{1}{\omega_m} \cdot \frac{\omega_m}{\pi} \ln \frac{S}{2N}$$

$$M = -\frac{1}{\pi} \ln \frac{S}{2N}$$

We see that the maximum number of bits in error after the occurrance of a noise burst is independent of whether a string of zeros or ones were transmitted when the noise burst occurred.

Vestigal Sibeband, AM, Synchronous Detection. System II

The signal is centered at ω_o.

One: $S_1(t) = \frac{S}{2} \cos\omega_o t$.

Zero: $S_o(t) = 0$.

Signalling at the Nyquist rate:

$$s(t) = \frac{S}{4}\left[\cos\omega_o t + \cos(\omega_o + \omega_m)t\right]$$

Bit interval $\tau = \frac{\pi}{2\omega_m}$

Noise:

$$n(t) = A e^{-\alpha(t-T)} \cos(\omega_1 t + \theta)$$

Where:

$$\alpha = \frac{\omega_m}{2}$$

$$\omega_1 = \omega_o + \frac{\omega_m}{2}$$

Detector output, signal only.

$$r(t) = \frac{S}{8} + \frac{S_m}{8} \cos \omega_m t$$

when a one is transmitted, m = 1

$$r(t) = \frac{S}{4}$$

Detector output, noise only:

Before filtering:

$$r'(t) = A e^{-\alpha(t-T)} \cos(\omega_1 t + \theta) \cos \omega_o t$$

$$= A e^{-\alpha(t-T)} \cos\theta \cos(\omega_o + \frac{\omega_m}{2})t \cos \omega_o t$$

$$- A e^{-\alpha(t-T)} \sin\theta \sin(\omega_o + \frac{\omega_m}{2})t \cos \omega_o t$$

After filtering:

$$r(t) = \frac{A}{2} e^{-\alpha(t-T)} \cos\theta \cos \omega_n t$$

Since the synchronous detector is a linear device, the performance of the system is not affected by whether the signal is a zero or a one.

Time required for the noise output from the demodulator to fall below half the signal value, t_1.

$$\frac{S}{8} = \frac{A}{2} e^{-\alpha t_1}$$

$$t_1 = -\frac{1}{\alpha} \ln \frac{1}{4}\frac{S}{N} \quad , \quad \tau = \frac{\pi}{2\omega_m} = \frac{\pi}{2\alpha}$$

$$M = -\frac{2}{\pi} \ln \frac{1}{4}\frac{S}{N}$$

AM Synchronous Detection. System III

Signal and impulse noise:

$$p(t) = \frac{S}{2}(1 + m \cos \omega_m t) \cos \omega_o t + A e^{-\alpha(t-T)} \cos \omega_o + \theta)$$

Demodulator output:

$$r(t) = \frac{S}{2} + \frac{A}{S} e^{-\alpha(t-T)} \cos \theta$$

Time required for $r(t) \geq \frac{S}{4}$, t_1.

$$\frac{S}{4} = \frac{S}{2} - \frac{A}{2} e^{-\alpha t_1}$$

$$M = \frac{t_1}{\tau} = -\frac{\omega_m}{\pi} \cdot \frac{1}{\omega_n} \ln \frac{1}{2} \frac{S}{N}$$

$$M = -\frac{1}{\pi} \ln \frac{1}{2} \frac{S}{N}$$

Phase Shift Keying. Synchronous Detection. System IV

Signal and impulse noise:

$$p(t) = S m \cos \omega_o t + A e^{-\alpha(t-T)} \cos(\omega_o t + \theta)$$

After demodulation before filtering.

$$r'(t)\ Sm \cos^2 \omega_o t + A e^{-\alpha(t-T)} \cos(\omega_o t + \theta) \cos \omega_o t$$

Demodulator output.

$$r(t) = \frac{Sm}{2} + \frac{A}{2} e^{-\alpha(t-T)} \cos \theta$$

The most critical condition is when: $m = -1$, $\cos \theta = 1$.
Time required for $r(t) \leq 0$ in this case, t_1.

$$\frac{S}{2} = \frac{A}{2} e^{-\alpha t_1}$$

$$t_1 = -\frac{1}{\alpha} \ln \frac{S}{N}$$

$$M = \frac{t_1}{\tau} = -\frac{\omega_m}{\pi} \frac{1}{\omega_m} \ln \frac{S}{N}$$

$$M = -\frac{1}{\pi} \ln \frac{S}{N}$$

Phase Shift Keying. Differential Phase Detector

The differential phase detector is shown in fig. 22.

$$S(t) = S \cos(\omega_o t + \beta)$$

β changes by π from one bit to the next if a one is transmitted. If a zero is transmitted there is no change.

$$x(t) = S(t) + n(t) = S \cos(\omega_o t + \beta_1) + A e^{-\alpha(t-T)} \cos(\omega_o t + \theta)$$

Bit rate: $\frac{1}{\tau} = \frac{\omega_m}{\pi}$

$$y(t) = S \cos(\omega_o t + \beta_2 - \omega_o t) + A e^{-\alpha(t-T-\tau)} \cos(\omega_o t - \omega_o t + \theta)$$

$$\begin{aligned}
Z(t) = {} & \frac{S^2}{2} \cos(-\beta_2 + \omega_o t) \\
& + \frac{S^2}{2} \cos(\omega_o t + \beta_1 + \omega_o t + \beta_2 - \omega_o t) \\
& + \frac{SA}{2} e^{-\alpha(t-T)} \cos(\theta - \beta_2 + \omega_o t) \\
& + \frac{SA}{2} e^{-\alpha(t-T)} \cos(2\omega_o t + \beta_2 - \omega_o t) \\
& + \frac{SA}{2} e^{-\alpha(t-T-\tau)} \cos(\theta - \omega_o t - \beta_1) \\
& + \frac{SA}{2} e^{-\alpha(t-T-\tau)} \cos(2\omega_o t + \theta - \omega_o t + \beta_1) \\
& + \frac{A^2}{2} e^{-\alpha(2t-2T-\tau)} \cos(2\theta + 2\omega_o t - \omega_o t) \\
& + \frac{A^2}{2} e^{-\alpha(2t-2T-\tau)} \cos \omega_o t
\end{aligned}$$

After filtering:

$$\begin{aligned}
r(t) = {} & \frac{S^2}{2} \cos(\beta_1 - \beta_2 + \omega_o t) + \frac{SA}{2} e^{-\alpha(t-T)} \cos(\theta - \beta_2 + \omega_o t) \\
& + \frac{SA}{2} e^{-\alpha(t-T-\tau)} \cos(\theta - \beta_1 - \omega_o t) + \frac{S^2}{2} e^{-\alpha(2t-2T-\tau)} \cos \omega_o t
\end{aligned}$$

since $\omega_o t = 2n\pi$

$$r(t) = \frac{S^2}{2}\cos(\beta_1 - \beta_2) + \frac{SA}{2} e^{-\alpha(t-T)} \cos(\theta - \beta_2)$$
$$+ \frac{SA}{2} e^{-\alpha(t-T-\tau)} \cos(\theta - \beta_1) + \frac{A^2}{2} e^{-\alpha(2t-2T-\tau)}$$

The worst case occurs when zeros and ones alternate. In this case the worst demodulator output is:

$$r(t) = -\frac{S^2}{2} - \frac{SA}{2} e^{-\alpha t} + \frac{SA}{2} e^{-\alpha(t-\tau)} + \frac{A^2}{2} e^{-\alpha(2t-\tau)}$$

Time required for $r(t) \leq 0$ is t_1.

$$-\frac{S^2}{2} - \frac{SA}{2} e^{-\alpha t_1} + \frac{SA}{2} e^{-\alpha(t_1-\tau)} + \frac{A^2}{2} e^{-\alpha(2t_1-\tau)} = 0$$

$$t_1 = -\frac{1}{\alpha} \ln\left[\frac{S}{N} e^{-\alpha t}\right]$$

$$t_1 = -\frac{1}{\alpha} \ln \frac{S}{N} + \tau$$

$$M = \frac{t_1}{\tau} = -\frac{\omega_m}{\pi} \frac{1}{\omega_m} \ln \frac{S}{N} + 1$$

$$M = 1 - \frac{1}{\pi} \ln \frac{S}{N}$$

Asymptotic Expansion of y (t) for $t \geq 2.5$ ms.

The asymptotic expression for $J_p(x)$ is:

$$J_p(x) \cong \sqrt{\frac{2}{\pi x}} \cos\left[x - \frac{2p+1}{4}\pi\right] + R$$

where:

$$x >> p \text{ and } |R| \approx \frac{1}{x^{3/2}}$$

Therefore:

$$J_p(x) \sqrt{\frac{2}{\pi x}} \cos(x + \theta)$$

$$J_{\frac{1}{3}}(x) \approx \sqrt{\frac{2}{\pi x}} \cos(x + \theta_1) \quad , \theta_1 = \frac{\pi}{12}$$

$$J_{-\frac{1}{3}}(x) \approx \sqrt{\frac{2}{\pi x}} \cos(x + \theta_2) \quad , \theta_2 = \frac{\pi}{3}$$

We desire to find the asymptotic expansion for:

$$y(t) = 10^5\, t^{\frac{1}{2}}\, e^{-410t} \left[J_{\frac{1}{3}}(2.10^5 t^{3/2}) + J_{-\frac{1}{3}}(2.10^5\, t^{3/2}) \right]$$

for $x >> \frac{1}{3}$ where: $x = 2.10^5\, t^{3/2}$. t is in seconds.

$$y(t) \approx 10^5\, t^{1/2}\, e^{-410t} \sqrt{\frac{1}{\pi . 10^5}}\, t^{-3/4} \left[\cos (2.10^5\, t^{3/2} + \theta) + \cos (2.10^5 t^{3/2} + \theta_2) \right]$$

$$y(t) \approx 10^5\, t^{-\frac{1}{4}}\, \frac{1}{10^5 \pi}\, e^{-410t} \cos (2.10^5\, t^{3/2} + \gamma)$$

Since the bit rate is 1000 bits per second it was decided that it is more convenient to use ms as a time unit. Upon this change of variable we obtain

$$y(t) = A\, e^{-.41t}\, t^{-\frac{1}{4}} \cos (6.3\, t^{3/2} + \gamma)$$

for $t > 2.4$ ms

upon evaluation of A and γ we obtain

$$y(t) = -.526\, t^{-\frac{1}{4}}\, e^{-.41t} \cos \left[6.3\, t^{3/2} + .7 \right]$$

Break point for $p_1(u)$

$$u_1 = Z_{1max} V_{min} = .3737 - .0024 = .888 \text{ mv}$$

$$p_1(u) = \frac{2.07.10^{-4}}{u^2} \qquad \text{for } |u| \geq .888 \text{ mv} = .888.10^{-3}$$

Break points for $p_2(u)$

$$p_2(u) = \frac{4.55.10^{-4}}{u^2} \qquad \text{for } |u| \geq 1.332.10^{-3}$$

Upper break point of $p_2(u)$:

$$u_2 = Z_{2\,max}\, V_{min} = .5532 \,.\, .0024 = 1.332 \text{ mv}$$

Lower break point of $p_2(u)$

$$u_2' = Z_{2\,min} V_{min} = .098 \,.\, .0024 = .266 \text{ mv}$$

$$p_2(u) = 0 \quad \text{for } |u| \leq u_2'$$

$$p_2(u) = \int_a^b p_2(z) \frac{1}{|v|} p(v)\, dv$$

where:

$$a = \text{Max of} \left\{ V_{min} \text{ of } \frac{u}{Z_{max}} \right\}$$

$$b = \text{Min of} \left\{ \infty \text{ or } \frac{u}{Z_{min}} \right\}$$

where $z = \frac{u}{v}$

$$p_2(z) = 2586\, z^6 - 10918\, z^5 + 13839\, z^4 - 7805\, z^3$$

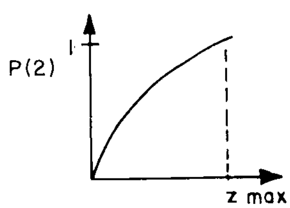
$$+ 2145\, z^2 - 2682 + 12.71$$

$$p(v) = \frac{.0024}{v^2}$$

$$p_2(u) = \frac{.0024}{v^2} \Bigg| \frac{2587}{8} \left(\frac{u}{v}\right)^6 - \frac{10918}{7} \left(\frac{u}{v}\right)^5 + \frac{13839}{6} \left(\frac{u}{v}\right)^4 - \frac{7805}{5} \left(\frac{u}{v}\right)^3 + \frac{2145}{4} \left(\frac{u}{v}\right)^2 - \frac{268}{3} \left(\frac{u}{v}\right) + \frac{12.71}{2} \Bigg|_{v = \text{Max of } .0024 \text{ or } \frac{u}{.5532}}^{v = \frac{u}{.098}}$$

$$p_2(u) = 0 \text{ for } \frac{u}{.098} \leq .0024 \quad \text{or } u \leq .236 \text{ mv}$$

Approximate evaluation of p(u)



$$p(u/v) = \frac{1}{|v|} p_1\left(\frac{u}{v}\right)$$

$$p(u) = \int_{V_{min}}^{\infty} p(u/v)\, p(v)\, dv$$

For: $\frac{u}{V_{min}} \geq Z_{max}$

$$p(u) = \int_{\frac{u}{Z_{max}}}^{\infty} p(u/v)\, p(v)\, dv = \frac{K}{u^2} = \int_{\frac{u}{Z_{max}}}^{\infty} \frac{1}{v}\, p_1\left(\frac{u}{v}\right) \frac{\varepsilon V_o}{v^2}\, dv$$

$$p(u) = \varepsilon V_o \int_{\frac{u}{Z_{max}}}^{\infty} p_1\left(\frac{u}{v}\right) \frac{dv}{v^3} = \varepsilon V_o \int_{\frac{u}{Z_{max}}}^{\infty} \left[a_n\left(\frac{u}{v}\right)^n + \ldots a_o\right] \frac{dv}{v^3}$$

$$p(u) = \varepsilon V_o \left[\frac{a_n}{n+2}(Z_{max})^n + \frac{a_{n-1}}{n+1}(Z_{max})^{n-1} + \ldots + \frac{a_1}{3} Z_{max} + \frac{a_o}{2}\right] \frac{Z_{max}^2}{u^2}$$

$$p(u) \approx \frac{\varepsilon V_o Z_{max}^2}{u^2} \quad p(Z_{max})$$

To obtain an approximate expression for p (u), proceed as follows:

calculate Z_{max}

calculate the break point: $u_B = Z_{max} V_{min}$

evaluate $p(Z_{max})$

p (u) is given by eq. 6 for $|u| \geq u_B$

Example. First Interval

$p_1(Z_{max}) = .1$

$\varepsilon V_o = .0024$

$Z_{max} = .19$

$p(u) = \frac{8.5}{u^2} \cdot 10^{-6}$ for $|u| \geq 4.56 \cdot 10^{-4}$

Estimation of required row length

To insure independence between errors in the columns we will require the row to be $4M_1 + 1$ bit long. The value of M_1 is choosen such that the possibility of M_1 corrupted bits occur sufficiently seldom.

It has been decided that the condition for M_1 corrupted bits to occur should exist only 10^{-7} of the one second intervals. From curve II fig. 7 we can extrapolate that this corresponds to an impulse of peak amplitude of 80 db. Since the transmitted amplitude.

is -4 db., this corresponds to a $\frac{S}{N}$ = -84 db. From fig. 6 we see that this corresponds with a row length of 25 bits.

Error Calculations. The ECD

$\alpha = 2.10^{-7}$

$\beta = 1.5.10^{-4}$

$p_r\left\{4\right\} \quad (1.5.10^{-4})^4 \quad \frac{32.31.30.29}{2.3.4} = .182.10^{-11}$

$E_{\beta=4} = \frac{1.82}{32} \; . \; 10^{-11}.\; 10^3 = .57.10^{-9}$

$E_{\beta=5}$ is small

$p_r\left\{2, 1E\right\} = 2.10^{-7}\,(1.5.10^{-4})^2 \quad \frac{32.21.30}{1.2.3} = 2.23.10^{-11}$

$E_{\alpha\beta} = \frac{2.23}{32} \; .10^{-8} = .7.10^{-9}$

$p_r\left\{2\,E\right\} = (2.10^{-7})^2 \quad \frac{32.31}{2} = 2.10^{-11}$

$\mathbf{E}_\alpha = \frac{20}{32} \cdot 10^{-9} = .63.10^{-9}$

$E = E_\alpha + E_\beta + E_{\alpha\beta}$

$E = 1.9.10^{-9}$

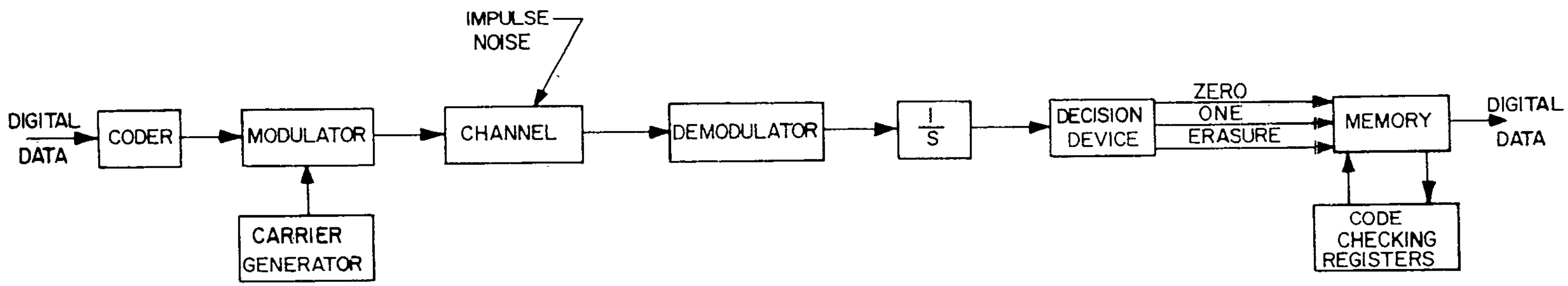


Fig. 1 The digital data communication system employing the ECD as analyzed in this investigation.

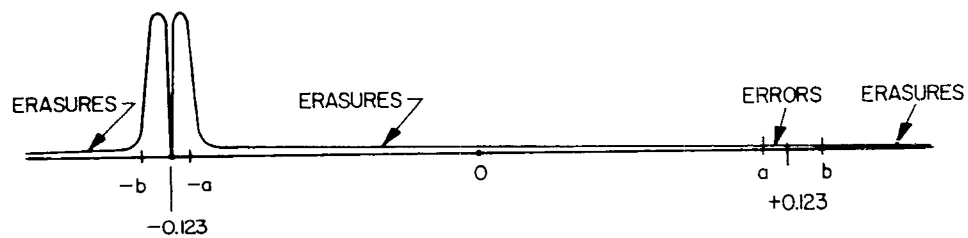


Fig. 2 Probability of receiving a one when a zero was transmitted.

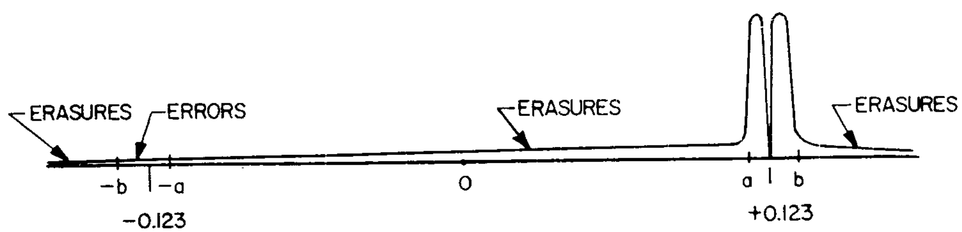


Fig. 3 The situation when a one is transmitted.

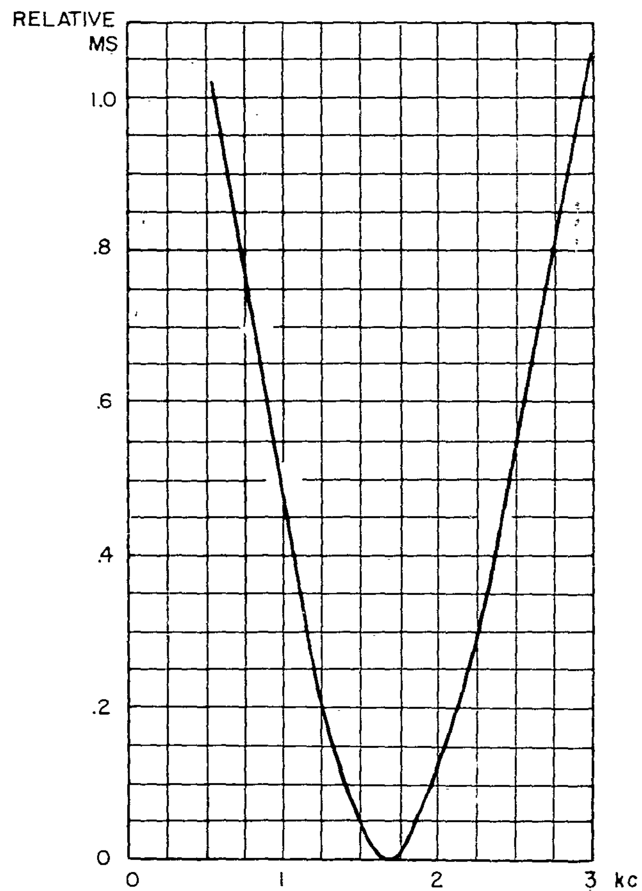


Fig. 4 Average enevelope delay for links of type N carrier circuits

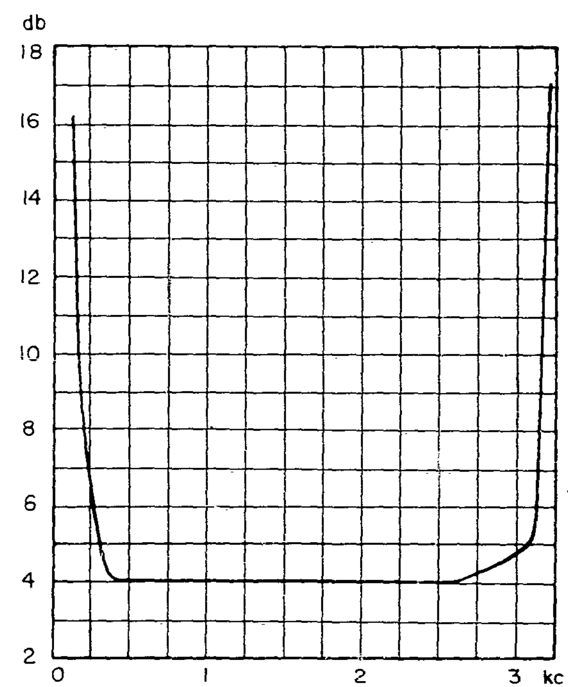
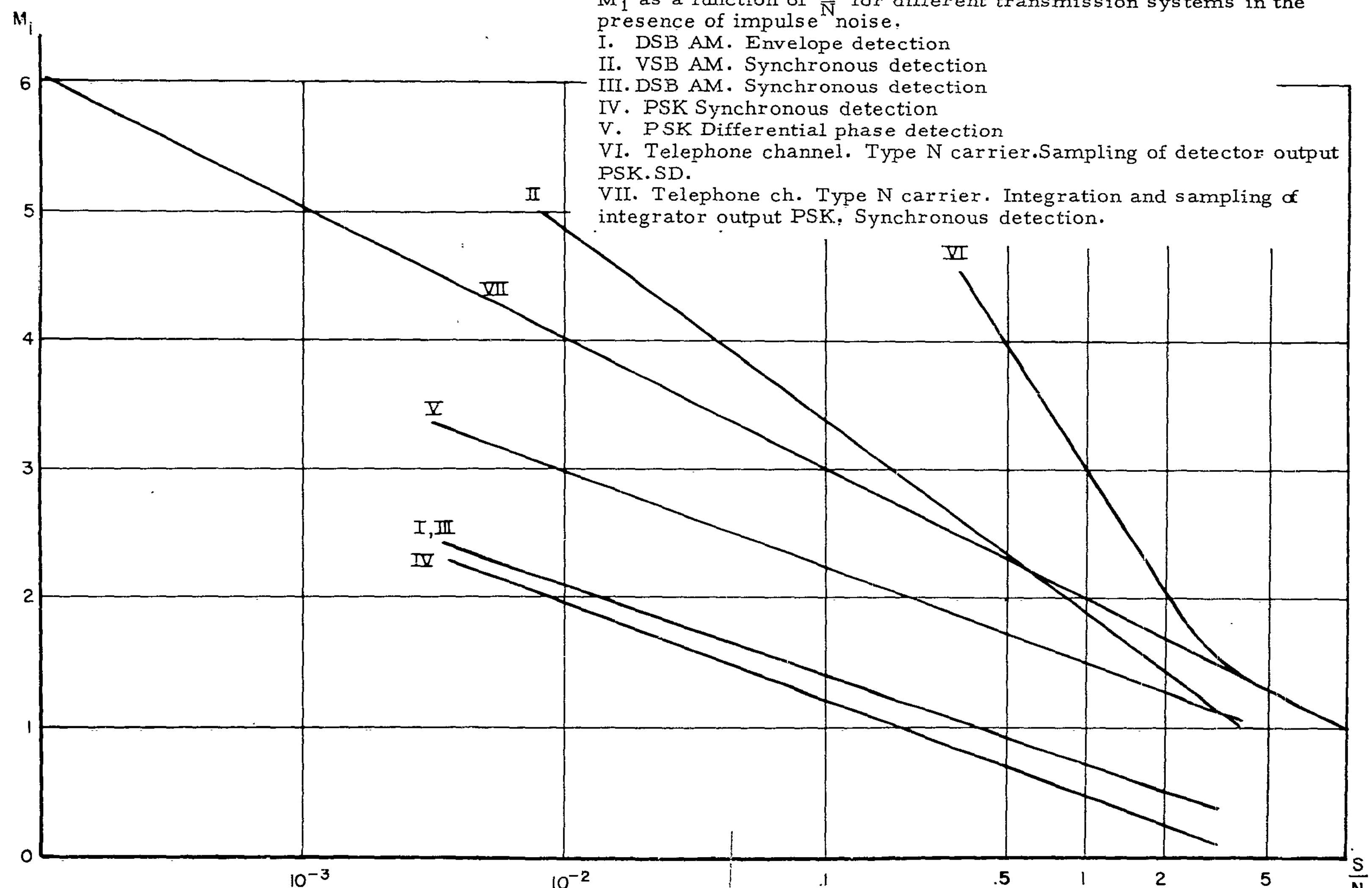


Fig. 5 Average attenuation of four links of type N carrier



M_i as a function of $\frac{S}{N}$ for *different* transmission systems in the presence of impulse noise.

I. DSB AM. Envelope detection
II. VSB AM. Synchronous detection
III. DSB AM. Synchronous detection
IV. PSK Synchronous detection
V. PSK Differential phase detection
VI. Telephone channel. Type N carrier. Sampling of detector output PSK.SD.
VII. Telephone ch. Type N carrier. Integration and sampling of integrator output PSK. Synchronous detection.

fig.6

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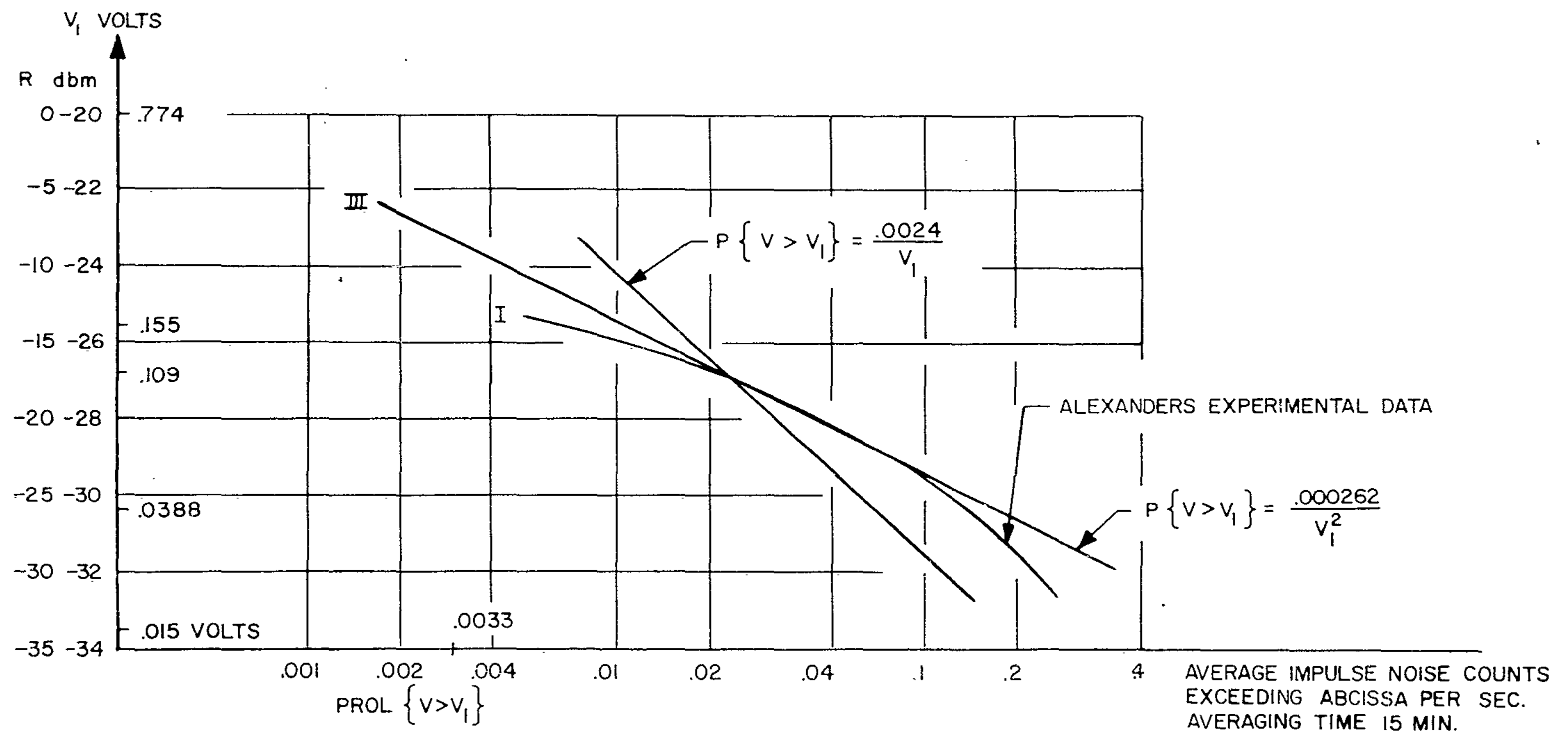
V1 VOLTS
R dbm
0 -20
-5 -22
-10 -24
-15 -26
-20 -28
-25 -30
-30 -32
-35 -34
.774
.155
.109
.0388
.015 VOLTS
.0033
.001
.002
.004
.01
.02
.04
.1
.2
4
PROL {V>V1}
III
I
P {V > V1} = .0024/V1
ALEXANDERS EXPERIMENTAL DATA
P {V > V1} = .000262/V1²
AVERAGE IMPULSE NOISE COUNTS
EXCEEDING ABCISSA PER SEC.
AVERAGING TIME 15 MIN.

fig.7

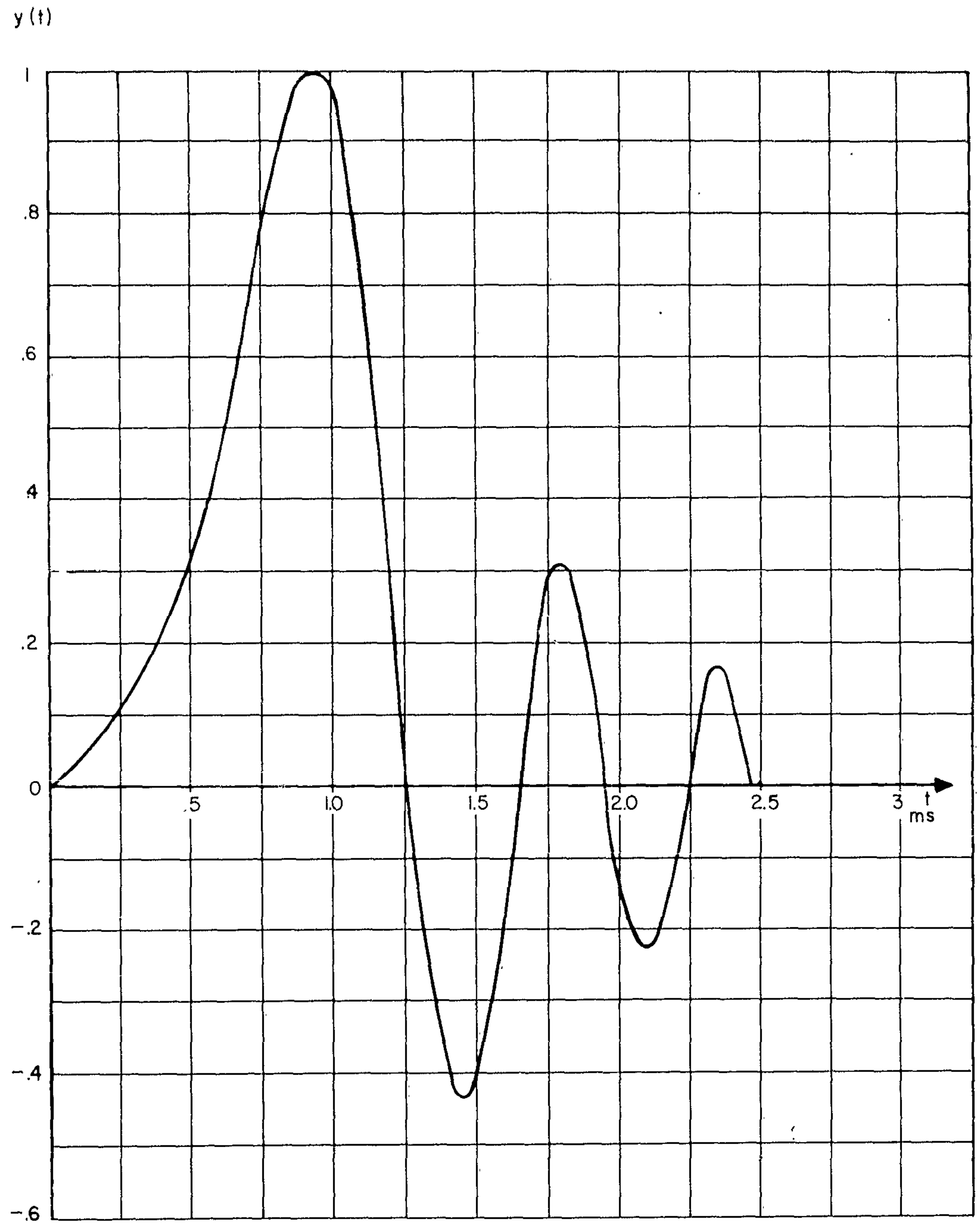

Fig.8 Normalized impulse response of a type N telephone carrier system.

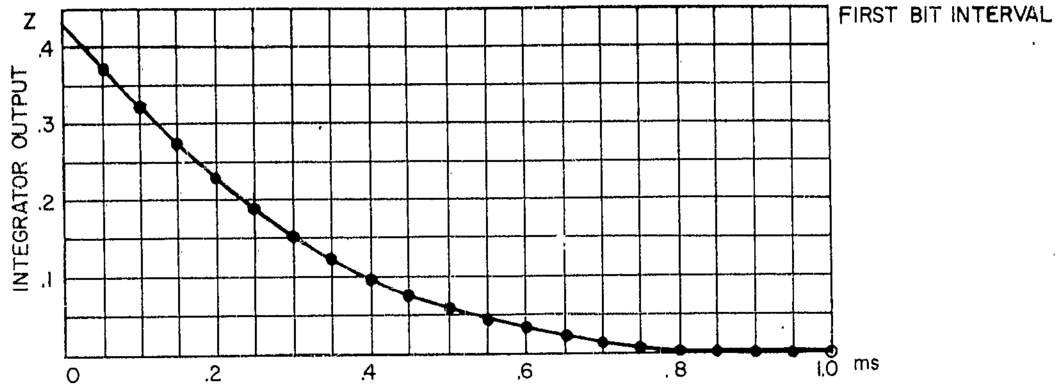

Fig. 9

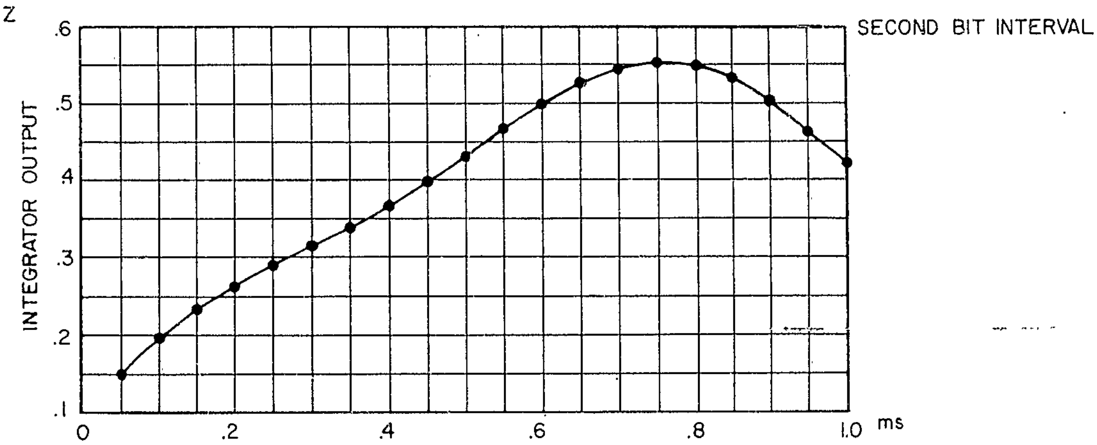

Fig. 10 Time of occurrence of the impulse referred to the first bit interval.

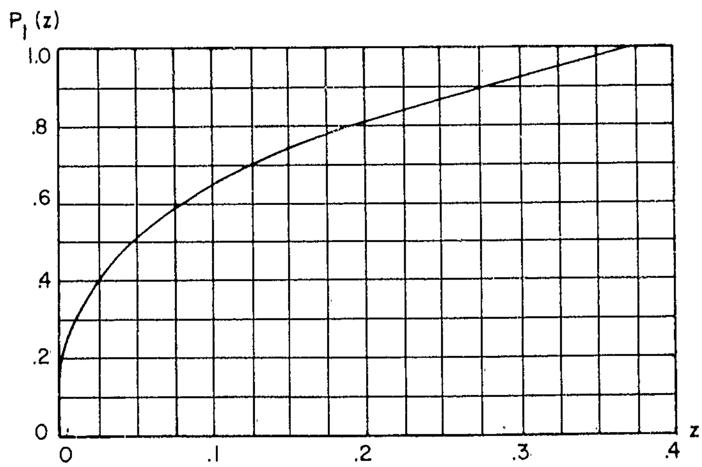

Fig. 11 P_1 (z)

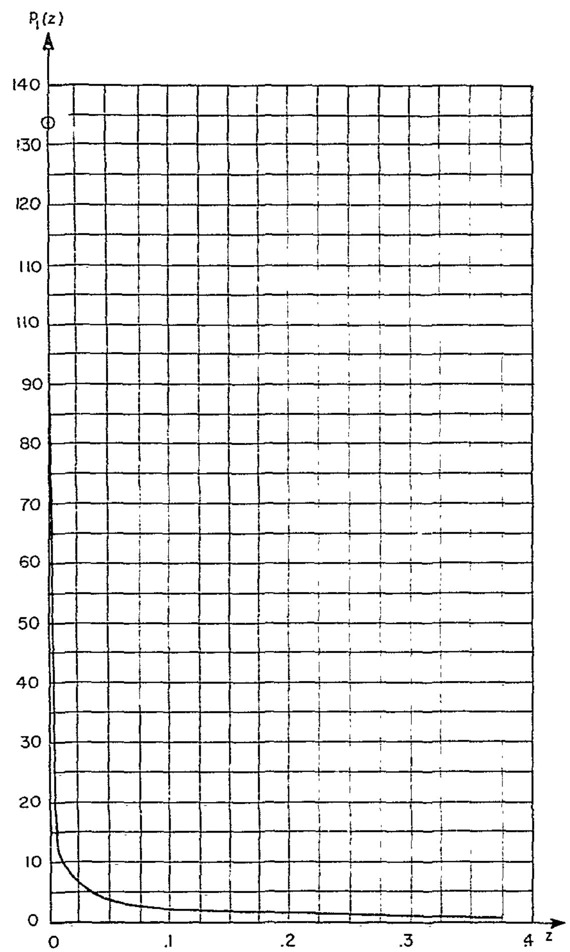

Fig. 13 $P_1(z)$ obtained by numerical methods

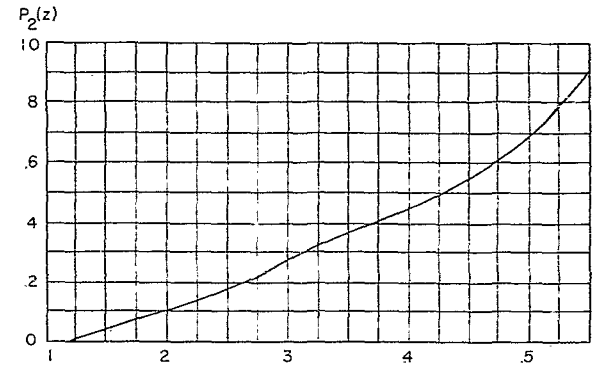

Fig. 12 $P_2(z)$

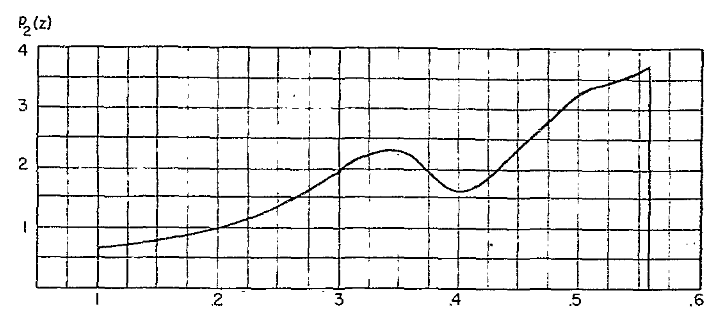

Fig. 14 $P_2(z)$ obtained by numerical methods

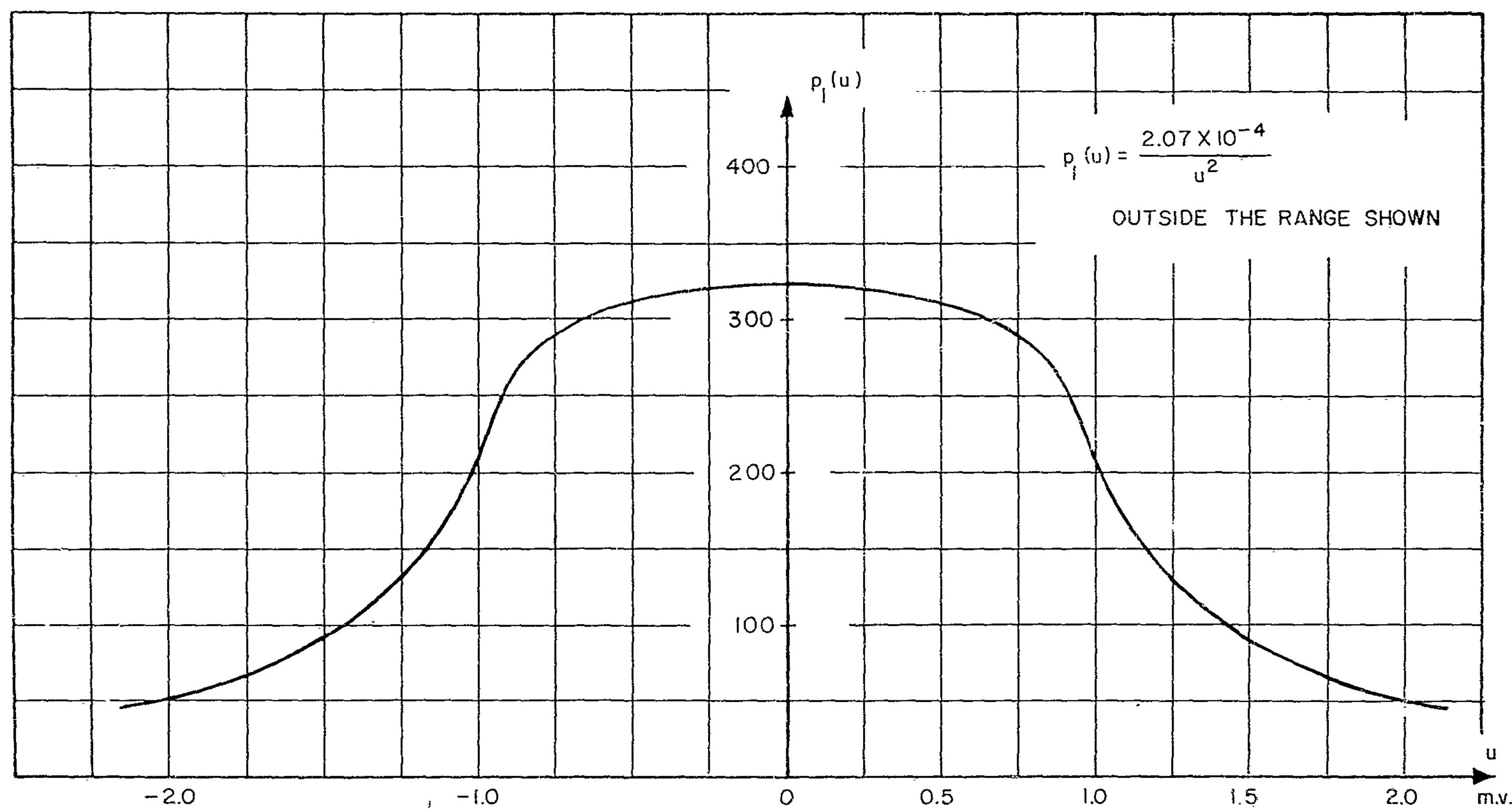

Fig. 15. Probability density at the integrator output during the first bit interval subject to the condition that an impulse with an amplitude response of at least 0.0024 has occurred in the channel.

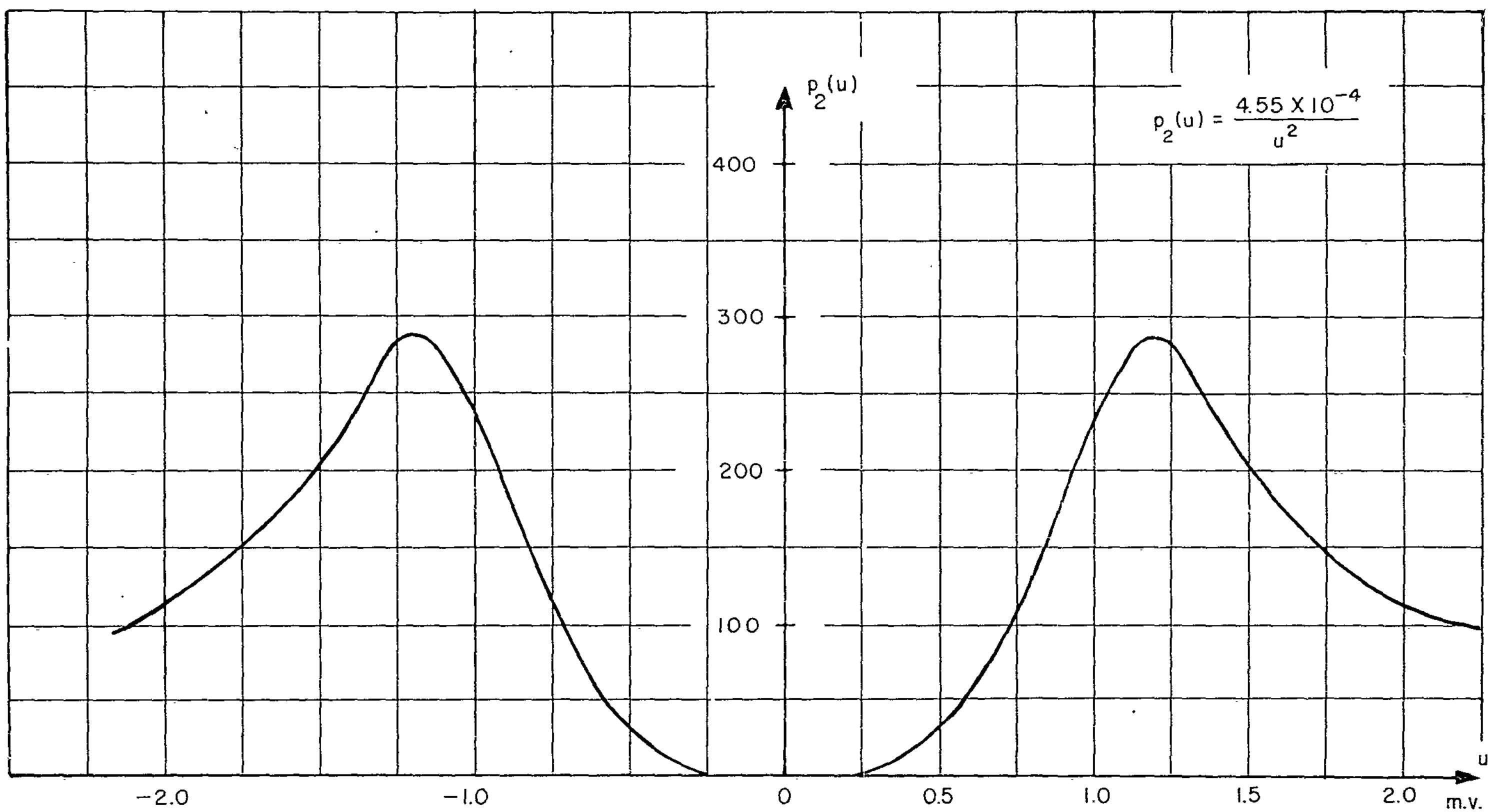

Fig. 16. Probability density at the integrator output when an impulse with a response of at least 0.0024 volt was picked up by the channel in the preceeding bit interval.

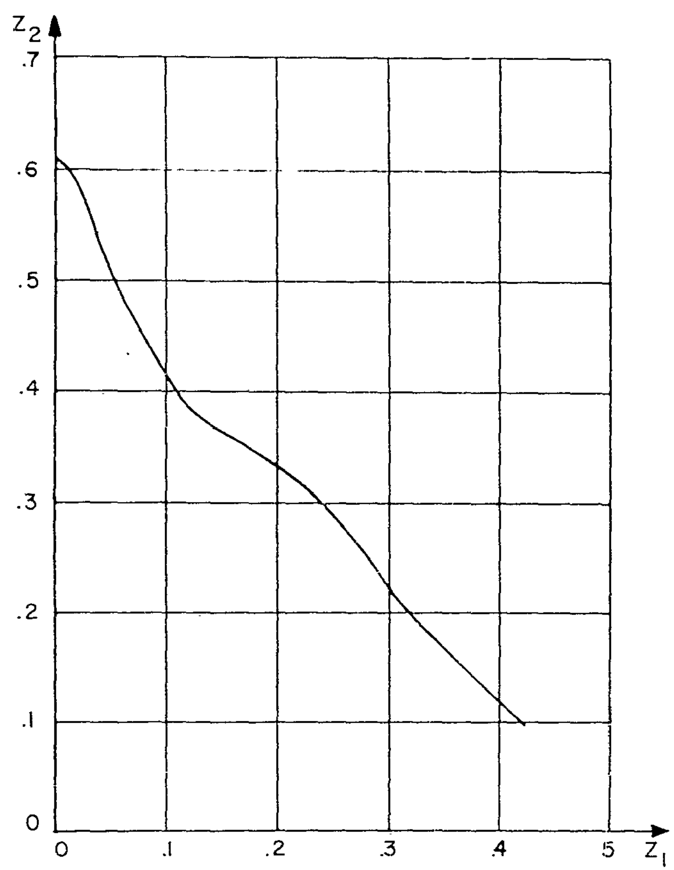

Fig. 17. Plot of $z_2 = f(z_1)$.

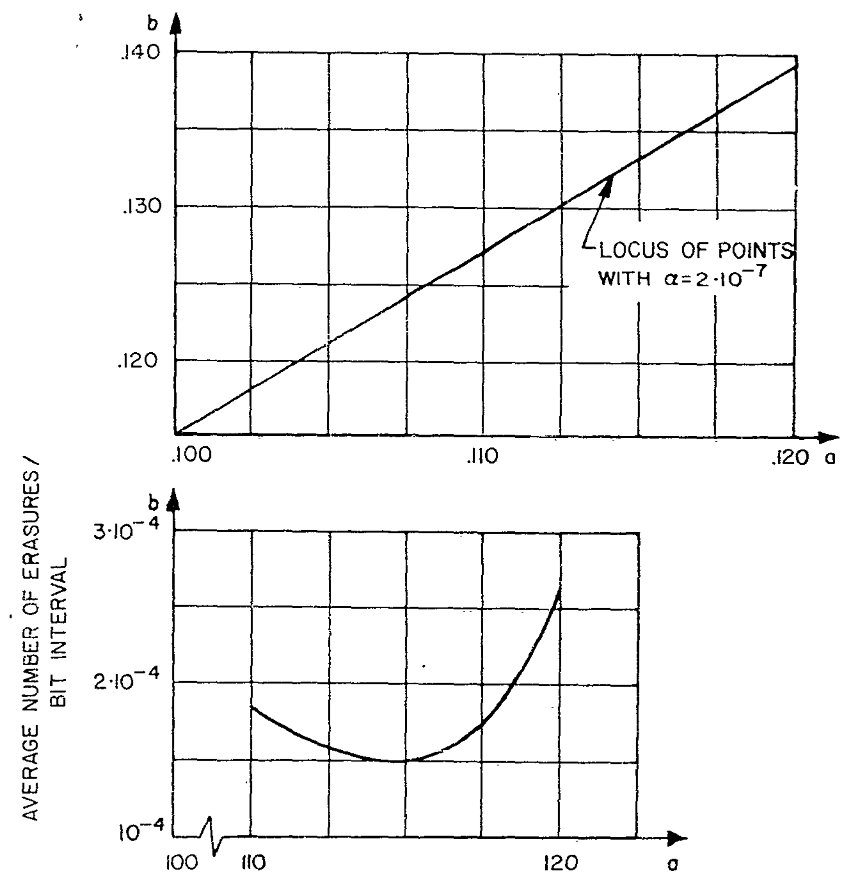

Fig. 18. Average number of erasures per bit interval as a function of threshold location for a fixed rate of errors of the first kind.

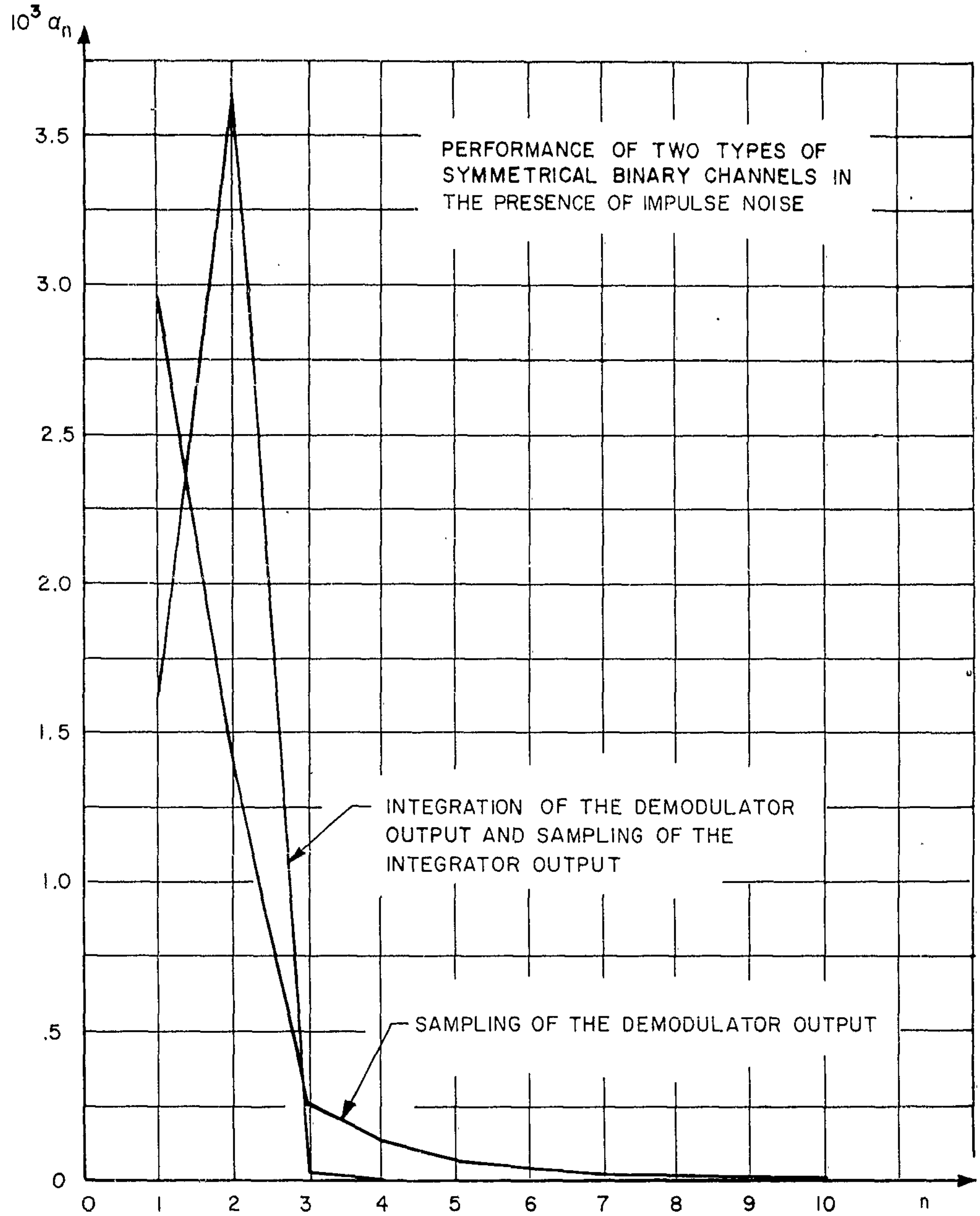

Fig. 19. Probability of a bit in error per second as a function of bit intervals from when the impulse noise burst first occurred.

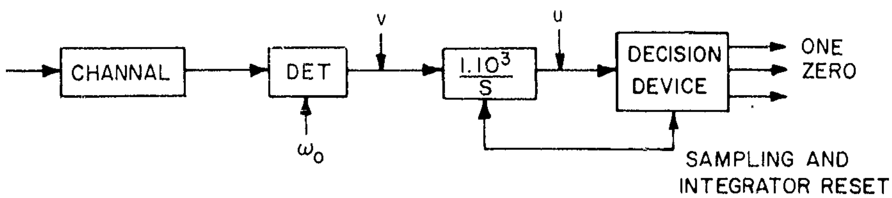

Fig.20 The ECD System Receiver

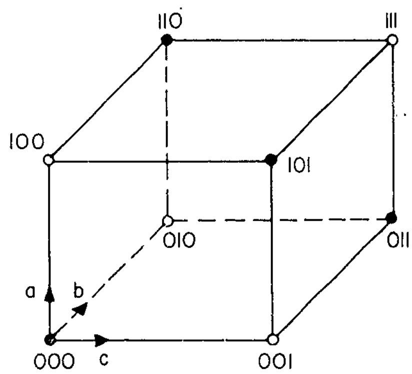

Fig. 21 The Eight Possible Numbers Represented by abc

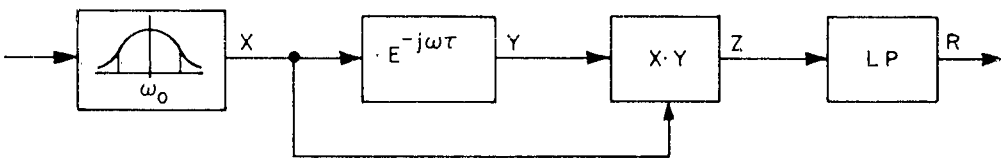

Fig. 22. Differential Phase Detector

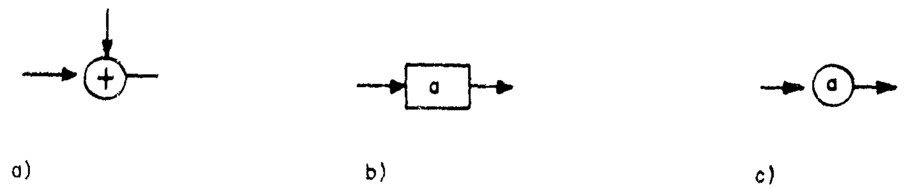

Fig. 23

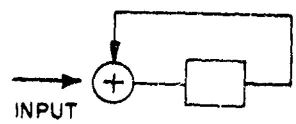

Fig. 24

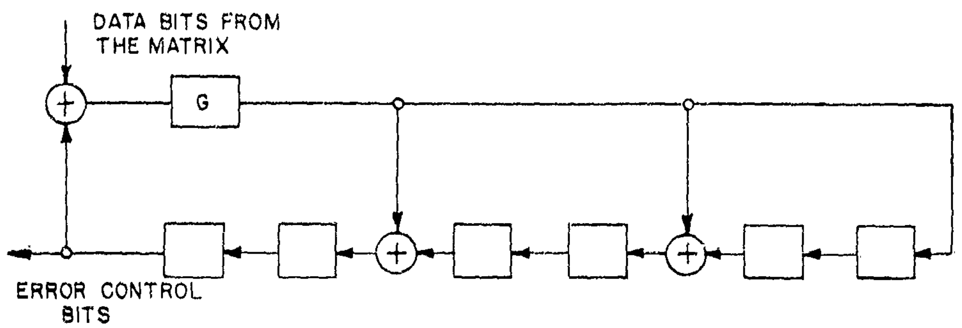

Fig. 25

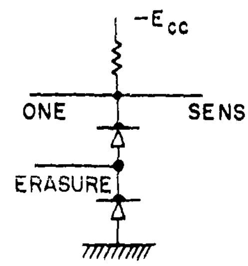

Fig. 26

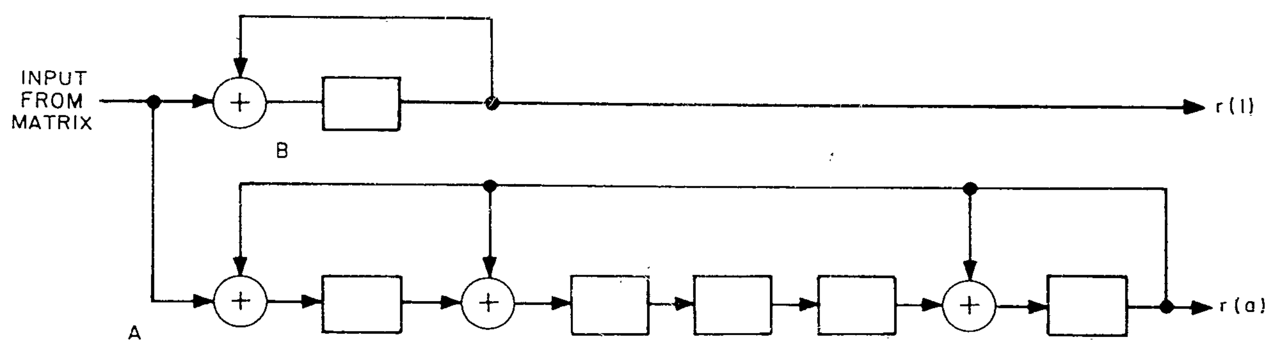
INPUT
FROM
MATRIX
B
A
r(l)
r(a)

Fig. 27


BINARY
DATA
SMEAR
FILTER
CHANNEL
CHANNEL
EQUALIZER
CLIPPER
DESMEAR
FILTER
BINARY
DATA

Fig. 28

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