THIS REPORT HAS BEEN DELIMITED
AND CLEARED FOR PUBLIC RELEASE
UNDER DOD DIRECTIVE 5200.20 AND
NO RESTRICTIONS ARE IMPOSED UPON
ITS USE AND DISCLOSURE.

DISTRIBUTION STATEMENT A

APPROVED FOR PUBLIC RELEASE;
DISTRIBUTION UNLIMITED.

402107

(5) 572 900

(6) BOUNDARY VALUE PROBLEMS IN PLASMA OSCILLATIONS*

(7) (A Dissertation)

(9) [1963]
(10) 111p. incl. illus
31 refs
(11) N.A
(13) N.A.

(8) by Fred Charles Shure

N.C

(12) Nonr 1224(15)
NR 013-301

*Supported in part by the Office of Naval Research, Department of the Navy

ACKNOWLEDGEMENT

The author wishes to convey his extreme gratitude to Professor Kenneth M. Case for his indispensible support, personal as well as professional, in every aspect of this work.

TABLE OF CONTENTS


| | | Page |
|---|---|---|
| ACKNOWLEDGEMENTS | | ii |
| CHAPTER | | |
| I. | INTRODUCTION | 1 |
| | 1. Outline | 1 |
| | 2. The Plasma as Dielectric | 2 |
| | 3. The Vlasov Equation | 5 |
| | 4. The Vlasov Equation (continued) | 6 |
| | 5. Linearization | 13 |
| | 6. Separation into Longitudinal and Transverse Modes | 15 |
| II. | THE NORMAL MODE EXPANSION FOR FIXED FREQUENCY TRANSVERSE WAVES | 18 |
| | 1. Introduction | 18 |
| | 2. Matrix Notation, the Adjoint Equation and Orthogonality | 18 |
| | 3. The Form of the Normal Modes | 22 |
| | 4. Solutions of the Adjoint Equation | 26 |
| | 5. The Normalization Coefficients $N(\nu)$ | 28 |
| | 6. The Normal Mode Expansion and the Full Range Completeness Theorem | 33 |
| | 7. Specialization to the Boltzmann Distribution | 41 |
| III. | BOUNDARY VALUE PROBLEMS FOR TRANSVERSE WAVES | 48 |
| | 1. Introduction | 48 |
| | 2. Boundary Conditions | 49 |
| | 3. The Electromagnetic Fields at the Boundary | 50 |
| | 4. Reflection From a Plasma Half Space | 52 |
| | 5. Reflection From a Slab | 58 |
| | 6. Reflection From a Half-Space: The Diffuse Absorber | 61 |
| | 7. Digression on the Outgoing-Wave Boundary Condition | 67 |
| | 8. Propagation Along a Constant Magnetic Field | 76 |
| IV. | OTHER APPLICATIONS | 85 |
| | 1. Initial Value Problems | 85 |
| | 2. The Initial Value for Transverse Modes | 85 |
| | 3. Stability and Landau Damping | 89 |

TABLE OF CONTENTS (CONT'D)

| | Page |
|---|---|
| CHAPTER | |
| 4. Landau Damping of Transverse Modes . . . . . . | 93 |
| 5. Longitudinal Oscillations: Fixed Frequency Modes. | 94 |
| 6. Boundary Value Problems (Longitudinal Oscillations) | 99 |
| 7. Conclusion - General Considerations . . . . . . . | 105 |
| BIBLIOGRAPHY. . . . . . . . . . . . . . . . . . . . . . | 110 |

## Chapter I. INTRODUCTION

### 1. Outline

We shall present ~~a~~ unified treatment of boundary value problems involving a system of ionized particles (i.e. a plasma) whose singlet phase-space density is presumed to satisfy the linearized form of the Landau-Vlasov equation. ~~Specifically, we shall make use of a~~ normal mode expansion in the singular eigenfunctions of the coupled Maxwell and Landau-Vlasov equations.

This first chapter, by way of introduction, contains a description of the nonstatistical approach to plasma oscillations, followed by an outline of the ideas of various writers who have derived the Vlasov equation using a variety of approaches. In the last section of this chapter we explicitly present the basic equations to be considered in the remainder of the text.

In chapter II we exhibit normal mode solutions of the coupled equations for the case of transverse waves of fixed frequency. The spectrum and orthogonality properties of these modes are discussed and a fundamental completeness theorem is proved.

In chapter III we treat the problem of reflection of electromagnetic radiation from a plasma half space and slab, and discuss the modifications necessary when the plasma is located in a steady-state magnetic field.

Chapter IV includes an assortment of topics. The initial value problem for transverse modes is considered from the standpoint of both the singular eigenfunction expansion and the more conventional Laplace transform treatment. Since this is the most familiar problem is plasma oscillations, only the results are presented and these only in outline form. The long-time "Landau" damping that results is discussed along with the corresponding damping in the transverse mode initial value problem. The normal modes for fixed frequency longitudinal oscillations are derived and the corresponding-completeness theorem is proved. The eigenfunction expansion is then used to solve longitudinal mode boundary value problems, resulting in expressions for the penetration of an oscillatory longitudinal electric field into a plasma half-space and the impedance of a plasma-filled parallel plate capacitor. Finally, we mention a rather indirect application of the singular eigenfunction method to the problem of electron migration in a discharge tube.

## 2. The Plasma as Dielectric

The most well known theory of plasma oscillations is Langmuir's[1] characterization of a plasma as a collection of mobile electrons at zero temperature, and essentially immobile ions.* According to this very

---

*In fact, Langmuir[2] was the first to use the term "plasma" to denote a region in an ionized gas containing equal numbers of ions and electrons.

Chapter IV includes an assortment of topics. The initial value problem for transverse modes is considered from the standpoint of both the singular eigenfunction expansion and the more conventional Laplace transform treatment. Since this is the most familiar problem is plasma oscillations, only the results are presented and these only in outline form. The long-time "Landau" damping that results is discussed along with the corresponding damping in the transverse mode initial value problem. The normal modes for fixed frequency longitudinal oscillations are derived and the corresponding completeness theorem is proved. The eigenfunction expansion is then used to solve longitudinal mode boundary value problems, resulting in expressions for the penetration of an oscillatory longitudinal electric field into a plasma half-space and the impedance of a plasma-filled parallel plate capacitor. Finally, we mention a rather indirect application of the singular eigenfunction method to the problem of electron migration in a discharge tube.

## 2. The Plasma as Dielectric

The most well known theory of plasma oscillations is Langmuir's[1] characterization of a plasma as a collection of mobile electrons at zero temperature, and essentially immobile ions.* According to this very

---

*In fact, Langmuir[2] was the first to use the term "plasma" to denote a region in an ionized gas containing equal numbers of ions and electrons.

rudimentary model, an electron, normally at rest, moves under the influence of the electric field which arises from the displacement of all other electrons from their equilibrium positions. From this interdependence of electric field and electron positions originates the characteristic collective behavior of the plasma. In particular, in this simple model the plasma behaves like a homogeneous dielectric medium (with respect to sinusoidal disturbances of frequency $\omega$ ) with permeability

$$\varepsilon = 1 - \frac{4\pi n e^2}{m\omega^2} \tag{1.1}$$

e and m are the electron's charge and mass respectively, and n is the electron density.

It follows that the plasma can support longitudinal oscillations--"plasma oscillations"--of the frequency

$$\omega = \omega_p \equiv \left(\frac{4\pi n e^2}{m}\right)^{1/2} \tag{1.2}$$

for which the permeability vanishes. That is, Maxwell's equations

$$\vec{\nabla}\times\vec{E} = -\frac{1}{c}\frac{\partial\vec{B}}{\partial t} \qquad \vec{\nabla}\times\vec{B} = \varepsilon\frac{1}{c}\frac{\partial\vec{E}}{\partial t}$$

$$\varepsilon\,\vec{\nabla}\cdot\vec{E} = \vec{\nabla}\cdot\vec{B} = 0 \tag{1.3}$$

are satisfied for disturbances of the form

$$\vec{E}(\vec{r},t) = E_0 \hat{k}\, e^{i(\vec{k}\cdot\vec{r} \pm \omega_p t)}$$
$$\vec{B}(\vec{r},t) = 0 \tag{1.4}$$

for any wave vector $\vec{k}$.

These oscillations can never be utilized to propagate a signal (the group velocity $\partial\omega_p/\partial k$ vanishes identically), but a plasma can be caused to oscillate in such a manner that surfaces of constant phase will move forward - "a situation like the familiar barber poles which appear to move steadily without rising"[1].

The characterization of a plasma as a dielectric medium whose dielectric properties are determined by relating electron positions to the electromagnetic field and vice-versa, places plasma problems on an equal footing with problems of the electromagnetic behavior of more familiar materials. In this approach, the plasma is characterized entirely by its permeability, and its unique behavior stems from the peculiar frequency dependence of the permeability.

For example, consider the case of an electromagnetic wave normally incident on a plasma half space. The familiar formulas of electromagnetic theory yield

$$\eta(\omega) = \sqrt{\varepsilon(\omega)} \qquad \text{(the index of refraction)} \tag{1.5}$$

and

$$T(\omega) = \frac{4\,\mathrm{Re}\, n(\omega)}{|1+n(\omega)|^2} \text{ (the transmission coefficient)} \tag{1.6}$$

$$= \begin{cases} \dfrac{4\omega\sqrt{\omega^2-\omega_p^2}}{\left(\omega+\sqrt{\omega^2-\omega_p^2}\right)^2} & \omega \geqslant \omega_p \\ 0 & \omega \leqslant \omega_p \end{cases}$$

A complete exposition of the applications of this approach to problems of wave propagation in ionized atmospheres may be found in the review article by Mimno[3]. Also, Ford[4] exhibits the various modes of oscillation of a zero temperature plasma under the influence of a constant magnetic field, and utilizes them in the solution of boundary value problems in the transmission and reflection of electromagnetic radiation from a plasma half space.

## 3. The Vlasov Equation

A more nearly rigorous description of the plasma as a collection of charged particles with random motions demands a statistical treatment. This is given through the Vlasov, or "collisionless-Boltzmann" equation.

Let $\mathcal{F}(\vec{r}, \vec{u}, t)$ denote the phase space single particle density. It obeys the Boltzmann equation[5]

$$\frac{\partial \mathcal{F}}{\partial t} + \vec{u}\cdot\vec{\nabla}\mathcal{F} + \vec{a}\cdot\vec{\nabla}_u\mathcal{F} = \left(\frac{\partial \mathcal{F}}{\partial t}\right)_{collision} \tag{1.7}$$

where $\vec{a}$ is the local acceleration produced by external forces (if any) and the right-hand side may be evaluated explicitly by examining the binary collision mechanisms in detail. Vlasov[6] noted that if the plasma is sufficiently ionized and not too dense (as is the case, for example, in the ionosphere), the individual electron collisions via both short range and Coulomb interactions occur with frequencies far lower than the plasma frequency. On the other hand, the long-range effects of the electromagnetic forces due to every other charged particle ("many-particle" collisions) may be taken into account through the $\vec{a} \cdot \vec{\nabla}_u$ term. Thus Vlasov asserted:

$$\left(\frac{\partial \mathcal{F}}{\partial t}\right)_{collision} \cong 0 \tag{1.8}$$

$$\vec{a} = \frac{e}{m}\left[\vec{E} + \frac{1}{c}\vec{u} \times \vec{B}\right] \tag{1.9}$$

where the plasma particles themselves serve as the source of the electromagnetic fields $\vec{E}$ and $\vec{B}$. Hence the electromagnetic fields are made self-consistent through the coupling of the Vlasov equation

$$\frac{\partial \mathcal{F}}{\partial t} + \vec{u} \cdot \vec{\nabla} \mathcal{F} + \frac{e}{m}\left[\vec{E} + \frac{1}{c}\vec{u} \times \vec{B}\right] \cdot \vec{\nabla}_u \mathcal{F} = 0 \tag{1.10}$$

and Maxwell's equations.

## 4. The Vlasov Equation (continued)

Vlasov's heuristic "derivation", despite its aesthetic appeal, is actually quite unrealistic. The Boltzmann equation rests on the hypothesis that the particles spend a negligible fraction of their time colliding, whereas the electrons are constantly interacting through the Coulomb force. On the other hand, the separation of Coulomb effects into binary collisions and an "external" interaction via the sel - consistent field possesses no rigorous meaning. In actual fact, the "smearing" of an electric field produced by a collection of point charges into a smoothly varying self-consistent field must be viewed as some sort of an approximation.

The nature of this approximation appears directly from consideration of the Liouville equation for an N particle system with binary interactions:

$$\frac{\partial}{\partial t}\mathcal{F}_N(\vec{r}_1,\vec{u}_1,\cdots\vec{r}_N,\vec{u}_N,t)+\sum_{i=1}^{N}\vec{u}_i\cdot\vec{\nabla}_i\mathcal{F}_N-\frac{1}{m}\sum_{i\neq j=1}^{N}\vec{\nabla}_i\phi(|\vec{r}_i-\vec{r}_j|)\cdot\vec{\nabla}_{u_i}\mathcal{F}_N=0 \qquad (1.11)$$

where $\phi$ is the binary interaction potential.

Integration of this master equation over position and momentum coordinates of all but s particles (s = 1, 2, ... N) leads to a hierarchy of equations, the familiar B-B-G-K-Y hierarchy. If we define reduced distribution functions

$$\mathcal{F}_s(\vec{r}_1,\vec{u}_1\cdots\vec{r}_s,\vec{u}_s,t)=V^s\int\mathcal{F}_N\prod_{i=s+1}^{N}d^3\vec{r}_i\,d^3\vec{u}_i \qquad (1.12)$$

and set

$$\phi(|\vec{r}_i-\vec{r}_j|)=\frac{e^2}{|\vec{r}_i-\vec{r}_j|} \qquad (1.13)$$

we obtain the equation for $\mathcal{F}_1$ :

$$\frac{\partial}{\partial t}\mathcal{F}_1(\vec{r},\vec{u},t) + \vec{u}\cdot\vec{\nabla}\mathcal{F}_1 = \frac{ne}{m}\int d^3\vec{r}\,' d^3\vec{u}\,' \vec{\nabla}\frac{e}{|\vec{r}-\vec{r}\,'|}\cdot\vec{\nabla}_u\mathcal{F}_2(\vec{r}_1,\vec{u}_1,\vec{r}_2,\vec{u}_2,t) \quad (1.14)$$

and similar equations expressing $\mathcal{F}_s$ in terms of $\mathcal{F}_{s+1}$ :

$$\frac{\partial\mathcal{F}_s}{\partial t} + \sum_{i=1}^{s}\vec{u}_i\cdot\vec{\nabla}_i\mathcal{F}_s - \frac{e^2}{m}\sum_{i\neq j=1}^{s}\vec{\nabla}_i\frac{1}{|\vec{r}_i-\vec{r}_j|}\cdot\vec{\nabla}_{u_i}\mathcal{F}_s =$$

$$\frac{ne}{m}\sum_{i=1}^{s}\int d^3\vec{r}_{s+1}d^3\vec{u}_{s+1}\vec{\nabla}_{u_i}\frac{e}{|\vec{r}_i-\vec{r}_{s+1}|}\cdot\vec{\nabla}_{u_i}\mathcal{F}_{s+1} \quad (1.15)$$

In order to close the hierarchy, some statistical assumption is necessary. In particular, if we assume that there are no correlations between particle motions, then the two-particle probability distribution is the product of the respective single-particle probability distributions

$$\mathcal{F}_2(\vec{r},\vec{u},\vec{r}\,',\vec{u}\,',t) = \mathcal{F}_1(\vec{r},\vec{u},t)\mathcal{F}_1(\vec{r}\,',\vec{u}\,',t) \quad (1.16)$$

and the $\mathcal{F}_1$ equation decouples from the rest of the hierarchy:

$$\frac{\partial\mathcal{F}_1}{\partial t} + \vec{u}\cdot\vec{\nabla}\mathcal{F}_1 = -\frac{e}{m}\left[-\vec{\nabla}\int\frac{ne\,\mathcal{F}_1(\vec{r}\,',\vec{u}\,',t)d^3\vec{r}\,'d^3\vec{u}\,'}{|\vec{r}-\vec{r}\,'|}\right]\cdot\vec{\nabla}_u\mathcal{F}_1 \quad (1.17)$$

This is just Vlasov's equation with the self-consistent field obtained through the solution of Poisson's equation. Harris [7], starting from Liouville's equation with the correct (retarded) electromagnetic interaction, instead of (1.13), shows that the resulting Vlasov equation contains the Lorentz force term as well, as might be expected.

Intuitively, the assumption of no correlations breaks down is two particles spend much time "close" to each other. Hence, the factorization (1.16) rests on the assumption that the density is low enough so that individual particle effects may be neglected with respect to long-range effects. The criterion for this may be taken to be

$$n\lambda_D^3 \gg 1 \tag{1.18}$$

where

$$\lambda_D = \left(\frac{\Theta}{4\pi n e^2}\right)^{1/2} \quad \text{the Debye shielding length.} \tag{1.19}$$

Equation (1.18) implies that the long-range effects predominate, since

a) it insures that many particles lie within the range of the shielded potential of a single electron, and

b) by virtue of the relation

$$\frac{1}{4\pi}\left(\frac{1}{n\lambda_D^3}\right)^{2/3} = \frac{n^{\frac{1}{3}} e^2}{\Theta} \tag{1.20}$$

it has the effect that the average potential energy of a binary interaction is much less than the average kinetic energy of a particle.

Laboratory plasmas and plasmas of astrophysical interest (solar corona, gaseous nebulae, etc.) satisfy the criterion (1.18) with several orders of magnitude to spare.

Rosenbluth and Rostocker[8] have developed a consistent expansion of the hierarchy (1.15) in powers of $g = 1/n\,\lambda_D^3$ similar to the Mayer cluster expansion. Write:

$$\mathcal{F}_s(\vec{r}_1,\vec{u}_1,\ldots\vec{r}_s,\vec{u}_s,t) = \prod_{i=1}^{s} \mathcal{F}_1(\vec{r}_i,\vec{u}_i,t) + \sum_{\substack{\text{pairs}\\(j,k)}} P(\vec{r}_j,\vec{u}_j,\vec{r}_k,\vec{u}_k,t) \prod_{\substack{i\neq j,k\\=1}}^{s} \mathcal{F}_1(\vec{r}_i,\vec{u}_i,t)$$

$$+\sum_{\substack{\text{triads}\\(j,k,\ell)}} T(j,k,\ell) \prod_{\substack{i\neq j,k,\ell\\=1}}^{s} \mathcal{F}_1(i) + \sum_{\substack{\text{pairs}\\\text{of pairs}}} P(j,k)P(\ell,m) \prod_{i\neq j,k,\ell,m} \mathcal{F}_1(i) + \cdots \tag{1.21}$$

where the pair, triplet ..... correlations are defined recursively. For example:

$$P(\vec{r},\vec{u},\vec{r}',\vec{u}',t) = \mathcal{F}_2(\vec{r},\vec{u},\vec{r}',u',t) - \mathcal{F}_1(\vec{r},\vec{u},t)\mathcal{F}_1(\vec{r}',\vec{u}',t) \tag{1.22}$$

Then if dimensionless units are chosen in terms of the basic length $\lambda_D$ and the basic time $1/\omega_p$, (1.15) becomes:

$$\frac{\partial \mathcal{F}_s}{\partial t} + \sum_{i=1}^{s} \vec{u}_i \cdot \vec{\nabla}_i \mathcal{F}_s - \frac{g}{4\pi} \sum_{i\neq j=1}^{s} \vec{\nabla}_i \frac{1}{|\vec{r}_i - \vec{r}_j|} \cdot \vec{\nabla}_{u_i} \mathcal{F}_s$$

$$= \frac{1}{4\pi} \sum_{i=1}^{s} \int d^3\vec{r}_{s+1}\, d^3\vec{u}_{s+1}\; \vec{\nabla}_i \frac{1}{|\vec{r}_i - \vec{r}_{s+1}|} \cdot \vec{\nabla}_{u_i} \mathcal{F}_{s+1} \tag{1.23}$$

and one may assume an expansion of the form

$$\mathcal{F}_s = \mathcal{F}_s^{(0)} + \mathcal{F}_s^{(1)} + \mathcal{F}_s^{(2)} + \cdots \tag{1.24}$$

where $\mathcal{F}_s^{(n)}$ is of order $g^n$. To order $g^o$, the third term in (1.23) may be neglected and a solution is seen to be:

$$\mathcal{F}_s^{(o)} = \prod_{i=1}^{s} \mathcal{F}_i^{(o)} \tag{1.25}$$

where $\mathcal{F}_1^{(o)}$ satisfies the Vlasov equation.

Although this is enough for our purposes, it should be noted that the expansion may be carried to higher orders in $g$ quite straightforwardly. In particular, Rosenbluth and Rostocker obtain, but do not solve, the integro-differential equation for $P$ in terms of $\mathcal{F}_1$, correct to order $g^1$.

Bogoliubov[9] treats the deeper question of whether or not a kinetic description is even possible. That is, under what circumstances does the single particle distribution obey an equation of the form

$$\frac{\partial}{\partial t}\mathcal{F}_1 = A(\vec{r},\vec{u}\,|\,\mathcal{F}_1) \tag{1.26}$$

In this notation, $A(\vec{r}, \vec{u}\,|\,\mathcal{F}_1)$ is an expression which depends upon time functionally through $\mathcal{F}_1$, i.e. $\partial \mathcal{F}_1/\partial t$ is completely determined by the distribution $\mathcal{F}_1$ at time $t$.

Bogoliubov notes that after a time $t_s$ after the system is set into motion the correlations become synchronized with the single particle distributions to the extent that the s-particle distributions depend upon time only through $\mathcal{F}_1$

$$\mathcal{F}_s(\vec{r}_1 \cdots \vec{u}_s, t) = \mathcal{F}_s(\vec{r}_1 \cdots\cdots \vec{u}_s | \mathcal{F}_1) \tag{1.27}$$

Of course, for long times, the particle motions become entirely uncorrelated.

$$\mathcal{F}_s \xrightarrow[t \to \infty]{} \prod_{i=1}^{\infty} \mathcal{F}_1(\vec{r}_i, \vec{u}_i, t) \tag{1.28}$$

an expression which is indeed of the form (1.27). However, Bogoliubov argues that this process of synchronization takes place very rapidly, in the time it takes an electron to travel a Debye length

$$t_s \sim \frac{(\Theta/m)^{1/2}}{\lambda_D} = \frac{1}{\omega_p} \tag{1.29}$$

and it is from this time onward that a kinetic description exists.

Granting this assumption, one then expands

$$\mathcal{F}_s(\vec{r}_1 \cdots \vec{u}_s, t) = \mathcal{F}_s^{(0)}(\vec{r}_1 \cdots\cdots \vec{u}_s | \mathcal{F}_1) + g\,\mathcal{F}_s^{(1)}(\vec{r}_1 \cdots \vec{u}_s | \mathcal{F}_1) + \cdots \tag{1.30}$$

$$\frac{\partial \mathcal{F}_1}{\partial t} = A_0(\vec{r}, \vec{u} | \mathcal{F}_1) + g\,A_1(\vec{r}, \vec{u} | \mathcal{F}_1) + g^2 A_2 + \cdots \tag{1.31}$$

and requires that (1.23) be satisfied to all orders in $g$. This results in a set of differential equations for the functionals $\mathcal{F}_s^{(n)}$ subject to the boundary condition (1.28). The zeroth order result is again

$$\mathcal{F}_s^{(0)} = \prod_{i=1}^{s} \mathcal{F}_1 \tag{1.32}$$

along with the Vlasov equation.

Guernsey[10] extends Bogoliubov's treatment to a detailed study of the approximation to order $g^1$. He obtains an integral equation for $\mathcal{F}_2^{(1)}$ in terms of $\mathcal{F}_1$, and solves it by means of the technique described in Chapter II, section 5 of this work. The result when substituted into (1.14) yields a correction to the Vlasov equation in the form of an additional collision-like term. In fact, he shows that the presence of this collision term insures that an H-theorem is satisfied and the system is driven to equilibrium. I.e.

$$\frac{d}{dt}\int d^3\vec{r}, d^3\vec{u}\ \mathcal{F}_1(\vec{r},\vec{u},t)\ \ln \mathcal{F}_1(\vec{r},\vec{u},t) \geqslant 0 \tag{1.33}$$

with equality only when $\mathcal{F}_1$ is Maxwellian.

Balescu and Prigogine[11, 12] obtain the same results as Guernsey by means of a direct expansion of the full Liouville equation using diagram techniques to evaluate the complicated sums that arise.

## 5. Linearization

Our starting point then is the Vlasov equation which represents the effect of the Lorentz force on the electron distribution function,

$$\frac{\partial \mathcal{F}}{\partial t} + \vec{u}\cdot\vec{\nabla}\mathcal{F} = -\frac{e}{m}\left[\vec{E}(\vec{r},t) + \frac{1}{c}\vec{u}\times\vec{B}(\vec{r},t)\right]\cdot\vec{\nabla}_u\mathcal{F} \tag{1.34}$$

and Maxwell's equations, with the charges and currents being given explicitly in terms of the velocity moments of the electron distribution.

$$\vec{\nabla}\times\vec{E} = -\frac{1}{c}\frac{\partial\vec{B}}{\partial t} \qquad \vec{\nabla}\cdot\vec{E} = 4\pi e\left[\int \mathcal{F}\, d^3\vec{u} - N_i\right]$$

$$\vec{\nabla}\times\vec{B} = \frac{1}{c}\frac{\partial\vec{E}}{\partial t} + \frac{4\pi e}{c}\int \vec{u}\,\mathcal{F}\, d^3\vec{u} \qquad \vec{\nabla}\cdot\vec{B} = 0 \tag{1.35}$$

We have assumed that the ions, by virtue of their greater mass, are relatively immobile and simply comprise a uniform background of density $N_i$. This restriction is relaxed in chapter IV. These equations may be linearized under the assumption that the electron distribution does not deviate substantially from its spatially uniform steady-state value. I.e.

$$\mathcal{F}(\vec{r},\vec{u},t) = n\mathcal{F}_0(\vec{u}) + f(\vec{r},\vec{u},t) \tag{1.36}$$

( $n$ is the electron density)

$f(\vec{r}, \vec{u}, t)$ is to be regarded as a "small" quantity.

Since $n = N_i$ for a neutral plasma, and since we will be dealing with situations in which the steady-state electron distribution does not contain currents, the charge-current distribution will be due entirely to $f(\vec{r}, \vec{u}, t)$.

$$\rho(\vec{r},t) = e\int f(\vec{r},\vec{u},t)\, d^3\vec{u} \tag{1.37}$$

$$\vec{j}(\vec{r},t) = e\int \vec{u}\, f(\vec{r},u,t)\, d^3\vec{u} \tag{1.38}$$

Thus the electric and magnetic fields are also "small" quantities and to first order in small quantities we obtain.

$$\frac{\partial}{\partial t} f(\vec{r},\vec{u},t) + \vec{u}\cdot\vec{\nabla} f = -\frac{ne}{m}\left[E(\vec{r},t) + \frac{1}{c}\vec{u}\times B(\vec{r},t)\right]\cdot\vec{\nabla}_u \mathcal{F}_o(\vec{u}) \tag{1.39a}$$

$$\vec{\nabla}\times\vec{E} = -\frac{1}{c}\frac{\partial\vec{B}}{\partial t} \qquad \vec{\nabla}\times\vec{B} = \frac{1}{c}\frac{\partial\vec{E}}{\partial t} + \frac{4\pi e}{c}\int \vec{u} f\, d^3\vec{u} \tag{1.39b}$$

$$\vec{\nabla}\cdot\vec{E} = 4\pi e\int f\, d^3\vec{u} \qquad \vec{\nabla}\cdot\vec{B} = 0 \tag{1.39c}$$

## 6. Separation into Longitudinal and Transverse Modes

It is easy to see that for isotropic $\mathcal{F}_o(\vec{u})$, i.e.

$$\mathcal{F}_o(\vec{u}) = \mathcal{F}_o(|\vec{u}|) \tag{1.40}$$

(the Maxwell-Boltzmann distribution is such a function), the equations (1.39) may be separated into three independent sets of equations. First note that in view of (1.40)

$$(\vec{u}\times\vec{B})\cdot\vec{\nabla}_u\mathcal{F}_o = -\vec{B}\cdot\left[\vec{u}\times\vec{\nabla}_u\,\mathcal{F}_o(|\vec{u}|)\right] = 0 \tag{1.41}$$

We shall seek plane wave solutions of the coupled equations (1.39). Set $\partial/\partial x = \partial/\partial y = 0$, write $\vec{u} = (u_x, u_y, u)$ and define

$$g_i(z,u,t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} du_x\, du_y\; u_i\, f(z,\vec{u},t) \tag{1.42}$$

Now for isotropic $\mathcal{F}_o$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u_i \vec{E}\cdot\vec{\nabla}_u \mathcal{F}_o \, du_x du_y = E_i(z,t) F_i(u) \tag{1.43}$$

(no sum convention implied)

where

$$F_x(u) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} du_x du_y \, u_x \frac{\partial \mathcal{F}_o}{\partial u_x} = -\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathcal{F}_o du_x du_y \equiv -F(u) \tag{1.44}$$

$$F_y(u) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} du_x du_y \, u_y \frac{\partial \mathcal{F}_o}{\partial u_y} = -F(u) \tag{1.45}$$

$$F_z(u) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} du_x du_y \, u \frac{\partial \mathcal{F}_o}{\partial u} = u \frac{dF(u)}{du} \tag{1.46}$$

We obtain three equations satisfied by the $g_i(z,u,t)$ by multiplying (1.39a) by $u_i$ and integrating over $du_x \, du_y$. These three equations, together with Eqs. (1.39b) written componentwise, comprise the three independent sets of equations to be considered in the next chapter.

$$E_x \text{ mode} \quad \begin{cases} \dfrac{\partial g_x(z,u,t)}{\partial t} + u\dfrac{\partial g_x}{\partial z} = n\dfrac{e}{m}E_x(z,t)F(u) \\[2ex] \dfrac{\partial E_x}{\partial z} = -\dfrac{1}{c}\dfrac{\partial B_y(z,t)}{\partial t} \\[2ex] \dfrac{\partial B_y}{\partial z} = -\dfrac{1}{c}\dfrac{\partial E_x}{\partial t} - \dfrac{4\pi e}{c}\displaystyle\int_{-\infty}^{\infty} g_x \, du \end{cases} \tag{1.47x}$$

$E_y$ mode

$$\left\{\begin{aligned} \frac{\partial}{\partial t} g_y(z,u,t) + u\frac{\partial g_y}{\partial z} &= \frac{ne}{m} E_y(z,t)\,F(u) \\ \frac{\partial E_y}{\partial z} &= \frac{1}{c}\frac{\partial B_x}{\partial t}(z,t) \\ \frac{\partial B_x}{\partial z} &= \frac{1}{c}\frac{\partial E_y}{\partial t} + \frac{4\pi e}{c}\int_{-\infty}^{\infty} g_y\,du \end{aligned}\right. \tag{1.47y}$$

longitudinal or "plasma" mode

$$\left\{\begin{aligned} \frac{\partial g_z}{\partial t}(z,u,t) + u\frac{\partial g_z}{\partial z} &= -\frac{ne}{m} u E_z \frac{dF(u)}{du} \\ \frac{\partial E_z}{\partial t} &= 4\pi e\int_{-\infty}^{\infty} g_z\,du \\ B_z &= 0 \end{aligned}\right. \tag{1.47z}$$

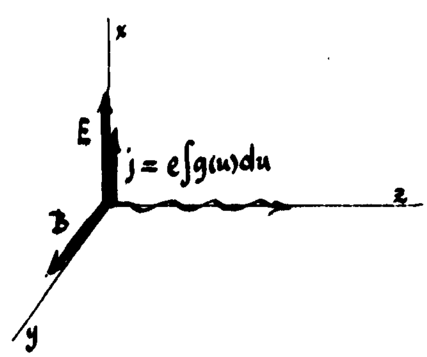
## Chapter II. THE NORMAL MODE EXPANSION FOR FIXED FREQUENCY TRANSVERSE WAVES

### 1. Introduction

In this section we present the complete normal mode decomposition for transverse disturbances of fixed frequency. This specialization is made in order to illustrate the main features of the method which is actually quite general. It will be seen that a normal mode decomposition is feasible provided we do not restrict ourselves to "functions" in the ordinary sense of the word, but allow generalized functions[13] or "distributions" as well. Except for this, the normal modes of the Vlasov equation are similar in character to the normal modes of any nonsingular problem. They are orthogonal to the solutions of the corresponding adjoint equation, and they form a complete set in that any well-behaved function may be expanded in terms of them with the expansion coefficients determined through the orthogonality relations.

### 2. Matrix Notation, the Adjoint Equation and Orthogonality

We choose the electric field to be plane polarized in the x-direction,



and take the Fourier time transform of Eqs. (1.47x). I.e. we seek solutions whose time dependence is of the form $\sim e^{-i\omega t}$ throughout. Dropping subscripts we have

$$
\begin{aligned}
-i\omega g(z,u) + u\frac{\partial}{\partial z} g(z,u) &= \frac{ne}{m} E(z) F(u) \\
\frac{dE(z)}{dz} &= \frac{i\omega}{c} B(z) \\
\frac{dB(z)}{dz} &= \frac{i\omega}{c} E(z) - \frac{4\pi e}{c}\int_{-\infty}^{\infty} g(z,u)\,du
\end{aligned}
\tag{2.1}
$$

It is convenient to combine the coupled Vlasov and Maxwell equations into a symbolic matrix equation for the "state function" $\Psi$, where $\Psi$ is written in the form of an array containing the three field quantities.

$$
\Psi = \begin{pmatrix} g(u) \\ E \\ B \end{pmatrix}
\tag{2.2}
$$

(the z-dependence is suppressed)

Equations (2.1), when combined,

$$
\begin{pmatrix} u & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \frac{\partial}{\partial z} \begin{pmatrix} g(u) \\ E \\ B \end{pmatrix} = \begin{pmatrix} i\omega & \frac{ne}{m}F(u) & 0 \\ 0 & 0 & \frac{i\omega}{c} \\ -\frac{4\pi e}{c}\int & \frac{i\omega}{c} & 0 \end{pmatrix} \begin{pmatrix} g(u) \\ E \\ B \end{pmatrix}
\tag{2.3}
$$

become, in this notation

$$\rho \frac{\partial \Psi}{\partial z} = H \tag{2.4}$$

(the operator $\int$ is defined by

$$\int g = \int_{-\infty}^{\infty} g(u)du \quad ) \tag{2.5}$$

The scalar product is defined in a natural manner. I.e. if

$$\chi^{\dagger} = \underbrace{G^{\dagger}(u) \quad e^{\dagger} \quad b^{\dagger}} \tag{2.6}$$

then

$$(\chi^{\dagger}, \Psi) = \int_{-\infty}^{\infty} G^{\dagger}(u) g(u) du + e^{\dagger}E + b^{\dagger}B \tag{2.7}$$

In this notation, the adjoint equation for the row matrix $\Psi^{\dagger}$ corresponding to $\Psi$ becomes

$$\frac{\partial \Psi^{\dagger}}{\partial z} \rho = \Psi^{\dagger} H^{\dagger} \tag{2.8}$$

with the adjoint operator $H^{\dagger}$ defined by the requirement

$$(\Psi^{\dagger}, H\Psi) = (\Psi^{\dagger} H^{\dagger}, \Psi) \tag{2.9}$$

By direct computation we find that $H^{\dagger}$ is given by

$$H^{\dagger} = \begin{pmatrix} i\omega & \frac{ne}{m}\int_F & 0 \\ 0 & 0 & \frac{i\omega}{c} \\ -\frac{4\pi e}{c} & \frac{i\omega}{c} & 0 \end{pmatrix} \tag{2.10}$$

where the operator $\int_F$ is defined by

$$\int_F g^\dagger = \int_{-\infty}^{\infty} F(u)\, g(u)\, du \tag{2.11}$$

The reason that we write the adjoint equation is that solutions of Eq. (2.4) and the adjoint equation (2.8) are complementary in the following sense: If we seek solutions $\psi_\nu$, $\psi_\nu^\dagger$ to Eqs. (2.4) and (2.8) with spatial dependence $\sim e^{i\omega z/\nu}$, then $\partial/\partial z \equiv i\,\omega/\nu$ and

$$H \Psi_\nu = \frac{i\omega}{\nu} \rho \Psi_\nu \tag{2.12}$$

$$\Psi_{\nu'}^\dagger H^\dagger = \frac{i\omega}{\nu} \Psi_{\nu'}^\dagger \rho \tag{2.13}$$

Using (2.9) and the fact that $\rho$ is self-adjoint, we obtain

$$\frac{i\omega}{\nu} \left(\Psi_{\nu'}^\dagger, \rho \Psi_\nu\right) = \frac{i\omega}{\nu'} \left(\Psi_{\nu'}^\dagger, \rho \Psi_\nu\right) \tag{2.14}$$

or

$$\left(\Psi_{\nu'}, \rho \Psi_\nu\right) = 0 \qquad \nu \neq \nu' \tag{2.15}$$

This is the basic orthogonality relation.

The orthogonality relation will turn out to be of practical value in computing expansion coefficients. That is, the composite quantities $\psi_\nu$ are to be thought of as the normal modes of the plasma, and will serve as the basis for normal mode expansions. The orthogonality relation implies that

$$\left(\Psi^{\dagger}_{\nu'}, \rho\, \Psi_{\nu}\right) = N(\nu)\,\delta(\nu-\nu') \tag{2.16}$$

(it will be necessary to make precise the meaning of the $\delta$- symbol). Then if

$$\Psi = \sum_{\nu} A(\nu)\Psi_{\nu} \tag{2.17}$$

(the sum may run over continuous as well as a discrete range of values of the variable $\nu$), we immediately obtain

$$A(\nu) = \frac{1}{N(\nu)}\left(\Psi^{\dagger}_{\nu}, \rho\, \Psi\right) \tag{2.18}$$

In the following sections we shall discuss the spectrum of allowed values of the wave velocity $\nu$, and obtain explicit expressions for the normal modes $\Psi_{\nu}$ and the normalization function N ($\nu$). Finally, we shall prove that the $\Psi_{\nu}$ form a complete set--that an expansion of the kind considered in Eq. (2.17) is, in general, possible.

## 2. The Form of the Normal Modes

Let us choose:

$$B_{\nu} = \frac{4\pi i e \nu c}{\omega} \tag{2.19}$$

This choice is quite arbitrary inasmuch as Eq. (2.12) is linear and homogeneous in the three field quantities, and is made for convenience.* The three rows of the matrix equation (2.12) then become, after rearranging:

---

*It is simple to see that there can be no non-trivial solutions with $B_{\nu} = 0$ (cf. eqs. (2.20)).

$$E_\nu = \frac{4\pi i e \nu^2}{\omega} \tag{2.20a}$$

$$(u-\nu) g_\nu(u) = \frac{\omega_p^2}{\omega^2} \nu^3 F(u) \tag{2.20b}$$

$$\int_{-\infty}^{\infty} g_\nu(u) du = c^2 - \nu^2 \tag{2.20c}$$

($\omega_p = (4\pi n e^2/m)^{1/2}$ is the plasma frequency.) For a given value of $\nu$ Eq. (2.20b) determines $g_\nu(u)$. Eq. (2.20c) then serves as a subsidiary condition restricting the possible allowed values of $\nu$. The solutions are similar in form to those exhibited by Case[14] for the longitudinal electric field problem. It is convenient to classify them in four groups:

Class 1.

These are solutions for complex $\nu$. We have

$$g_{\nu_j}(u) = \frac{\omega_p^2}{\omega^2} \nu_j^3 \frac{F(u)}{u - \nu_j} \tag{2.21}$$

where, as required by Eq. (2.20c), $\nu_j$ is one of the roots of the characteristic equation

$$\Lambda(\nu) \equiv c^2 - \nu^2 - \frac{\omega_p^2}{\omega^2} \nu^3 \int_{-\infty}^{\infty} \frac{F(u) du}{u - \nu} = 0 \tag{2.22}$$

It is not difficult to show that there are only a finite number of such roots. Note that the characteristic function $\Lambda(\nu)$ is analytic in the complex $\nu$-plane cut along these parts of the real axis where $F(\nu) \neq 0$.

Class 2 a.

For these solutions $\nu$ is real and $F(\nu) \neq 0$. Then the solution represented by (2.21) is indefinite until we give a prescription for treating the singularity in the denominator. The general solution of (2.20b) for real $\nu$ is

$$g_\nu(u) = \frac{\omega_p^2}{\omega^2} \nu^3 P \frac{F(u)}{u-\nu} + \lambda(\nu)\delta(u-\nu) \tag{2.23}$$

The P indicates that in integrals involving $g_\nu(u)$, the Cauchy principal value is implied. $\lambda(\nu)$ is arbitrary and is chosen so that (2.20c) is satisfied. This condition becomes:

$$\lambda(\nu) = c^2 - \nu^2 - \frac{\omega_p^2}{\omega^2}\nu^3 P\int_{-\infty}^{\infty} \frac{F(u)du}{u-\nu} = \frac{1}{2}\left[\Lambda^+(\nu) + \Lambda^-(\nu)\right] \tag{2.24}$$

$\Lambda^+(\nu)$ and $\Lambda^-(\nu)$ are the boundary values of $\Lambda(\nu)$ as $\nu$ approaches the cut from above and below respectively.

Class 2 b.

Here we have $\nu$ real and $F(\nu) = 0$. The results are essentially the same as above: $g_\nu(u)$ is given by (2.23) and $\lambda(\nu)$ by (2.24). In this case $\Lambda^+(\nu)$ and $\Lambda^-(\nu)$ take on the same value, since the integral contained in the definition of $\Lambda(\nu)$ is continuous across the real axis when $F(\nu) = 0$. Also, the principal value sign is not now necessary, but it is carried along as a reminder that when integrating with respect to $\nu$ we are to omit points of class 2 c.

Class 2 c.

For these solutions $\nu$ is real, $F(\nu) = 0$ and $\lambda(\nu) = 0$ as well. Just as in class 1, there are only a finite number of $\nu_i$ satisfying these conditions. The solution for $g_{\nu_i}(u)$ is

$$g_{\nu_i}(u) = \frac{\omega_p^2}{\omega^2} \nu_i^3 \frac{F(u)}{u-\nu_i} \tag{2.25}$$

If $F(u)$ vanishes at least linearly as $u \to \nu_i$, then $g_{\nu_i}(u)$ is a perfectly well-behaved function. We shall assume this to be the case. At any rate, the (possibly improper) integral $\int g_{\nu_i} \, du$ is well defined.

To recapitulate: the fundamental equation (2.12) possesses solutions

$$\Psi_\nu = \begin{pmatrix} g_\nu(u) \\ E_\nu \\ B_\nu \end{pmatrix} \tag{2.26}$$

of two basically different types. There is a discrete set of values $\nu_j$, $\nu_i$ with either a) $\nu_j$ complex, $\Lambda(\nu_j) = 0$

or b) $\nu_i$ real, $\Lambda(\nu_i) = F(\nu_i) = 0$

The respective $g_{\nu_j}(u)$ and $g_{\nu_i}(u)$ are well-behaved functions. The remaining (continuum) real values of $\nu$ are associated with solutions for which the $g_\nu(u)$ are singular and are interpreted as distributions in the Schwarzian sense.

## 3. Solutions of the Adjoint Equation

For every allowed eigenvalue of the fundamental equation (2.12), there is also a solution $\psi_\nu$ of the adjoint equation (2.13). The decomposition into classes is essentially the same.

Class 1 : $\nu$ complex

We take, for convenience

$$E^\dagger_{\nu_j} = \frac{ne\nu_j c^2}{mi\omega} \tag{2.27}$$

Then the three columns of the matrix equation (2.13) read

$$B^\dagger_{\nu_j} = \frac{ne\nu_j^2 c}{mi\omega} \tag{2.28a}$$

$$(u-\nu_j)\, g_{\nu_j}(u) = \frac{\omega_p^2}{\omega^2}\nu_j^3 \tag{2.28b}$$

$$\int_{-\infty}^{\infty} F(u)\, g_{\nu_j}(u)\, du = c^2 - \nu_j^2 \tag{2.28c}$$

The solution of (2.28 b) is

$$g_{\nu_j}(u) = \frac{\omega_p^2}{\omega^2}\nu_j^3 \frac{1}{u-\nu_j} \tag{2.29}$$

which when substituted into (2.28 c) yields the condition

$$\Lambda(\nu_j) = 0 \tag{2.30}$$

as before.

Class 2 a. $\nu$ real and $F(\nu) \neq 0$

Again we choose

$$E_\nu^\dagger = \frac{ne\nu c^2}{mi\omega} \tag{2.27}$$

which implies

$$B_\nu^\dagger = \frac{ne\nu^2 c}{mi\omega} \tag{2.28 a}$$

Equations (2.28 b, c) are also still valid, but the solution for this case is

$$g_\nu^\dagger(u) = \frac{\omega_p^2}{\omega^2}\nu^3 P\frac{1}{u-\nu} + \tilde{\lambda}(\nu)\,\delta(u-\nu) \tag{2.31}$$

with

$$\tilde{\lambda}(\nu) = \frac{\lambda(\nu)}{F(\nu)} \tag{2.32}$$

Class 2 b. $\nu$ real, $F(\nu) = 0$, but $\lambda(\nu) \neq 0$

Here we must take

$$E_\nu^\dagger = 0 \tag{2.33}$$

A nonzero choice for $E_\nu$ must be avoided. It would lead to solutions of the same form as those of class 2 a. But this is impossible, since Eq. (2.32) cannot be satisfied. The matrix equation (2.13) becomes:

$$B_\nu^\dagger = 0 \tag{2.34 a}$$

$$(u-\nu)\,g_\nu^\dagger(u) = 0 \tag{2.34 b}$$

$$\int_{-\infty}^{\infty} F(u)\,g_\nu^\dagger(u)\,du = 0 \tag{2.34 c}$$

The solution of Eq. (2.34 b) is

$$g_\nu(u) = \delta(u-\nu) \tag{2.35}$$

which when substituted in Eq. (2.34 c) yields an identity.

Class 2 c. $\nu_i$ real, $F(\nu_i) = 0$ and $\lambda(\nu_i) = 0$

As before, we take

$$E^\dagger_{\nu_i} = \frac{ne\nu_i c^2}{mi\omega} \tag{2.27}$$

which implies

$$B^\dagger_{\nu_i} = \frac{ne\nu_i^2 c}{mi\omega} \tag{2.28 a}$$

$$(u-\nu_i)\, g_{\nu_i}(u) = \frac{\omega_p^2}{\omega^2}\nu_i^3 \tag{2.28 b}$$

$$\int_{-\infty}^{\infty} F(u)\, g_{\nu_i}(u)\, du = c^2 - \nu_i^2 \tag{2.28 c}$$

The solution for $g_{\nu_i}(u)$ is

$$g_{\nu_i}(u) = \frac{\omega_p^2}{\omega^2}\nu_i^3\, P\frac{1}{u-\nu_i} \tag{2.36}$$

## 4. The Normalization Coefficients $N(\nu)$

The general orthogonality relation (2.15) may be verified by direct calculation, using the solutions obtained in the preceding sections. It remains to define precisely and determine the normalization coefficients $N(\nu)$ referred to in Eq. (2.16). It is simplest to consider the discrete spectrum first.

Suppose we have a $\nu_j$ . Then the orthogonality relation implies

$$(\Psi_\nu^\dagger, \rho \Psi_{\nu_j}) = 0 \qquad \text{all } \nu \neq \nu_j \tag{2.15'}$$

and since

$$(\Psi_{\nu_j}^\dagger, \rho \Psi_{\nu_j}) = \int_{-\infty}^{\infty} \left[\frac{\omega_p^2}{\omega^2} \nu_j^3 \frac{1}{u-\nu_j}\right] u \left[\frac{\omega_p^2}{\omega^2} \nu_j^3 \frac{F(u)}{u-\nu_j}\right] du \; +$$

$$+ \frac{ne\nu_j c^2}{mi\omega} \frac{4\pi i e \nu_j^2}{\omega} + \frac{ne\nu_j^2 c}{mi\omega} \frac{4\pi i e \nu_j c}{\omega} \tag{2.37}$$

$$= \frac{\omega_p^2}{\omega^2} \nu_j^3 \left[\frac{\omega_p^2}{\omega^2} \nu_j^3 \int_{-\infty}^{\infty} \frac{u F(u) du}{(u-\nu_j)^2} + 2c^2\right]$$

which becomes after a little algebra,

$$(\Psi_{\nu_j}, \rho \Psi_{\nu_j}) = -\frac{\omega_p^2}{\omega^2} \nu_j^4 \frac{d}{d\nu} \Lambda(\nu)\Big|_{\nu=\nu_j} \tag{2.38}$$

Thus we say

$$(\Psi_\nu^\dagger, \rho \Psi_{\nu_j}) = N(\nu_j)\, \delta_{\nu,\nu_j} \tag{2.39}$$

with

$$N(\nu_j) = -\frac{\omega_p^2}{\omega^2} \nu_j^4 \frac{d}{d\nu} \Lambda(\nu)\Big|_{\nu=\nu_j} \tag{2.40}$$

$\delta_{\nu,\nu_j}$ is the Kronecker $\delta$-symbol. Since $\nu_j$ is, by assumption, a simple zero of the characteristic equation, $N(\nu_j)$ will not vanish (cf. (2.18)).

The result is essentially the same when we consider a $\nu_i$.

$$\left(\Psi^{\dagger}_{\nu}, \rho\, \Psi_{\nu_i}\right) = N(\nu_i)\, \delta_{\nu,\nu_i} \tag{2.39'}$$

$$N(\nu_i) = \frac{\omega_p^2}{\omega^2} \nu_i^3 \left[ \frac{\omega_p^2}{\omega^2} P\int_{-\infty}^{\infty} \frac{u\, F(u)\, du}{(u-\nu)^2} + 2c^2 \right]$$

$$= -\frac{\omega_p^2}{\omega^2} \nu_i^4 \, \frac{1}{2} \left\{ \frac{d}{d\nu}\Lambda(\nu)\Big|_{\nu=\nu_i+i\epsilon} + \frac{d}{d\nu}\Lambda(\nu)\Big|_{\nu=\nu_i-i\epsilon} \right\}_{\epsilon\to 0^+} \tag{2.41}$$

When $\nu$ belongs to the continuum, the situation is somewhat different. The corresponding normalization integrals are undefined. We write, however,

$$\left(\Psi^{\dagger}_{\nu'}, \rho\, \Psi_{\nu}\right) = N(\nu)\, \delta(\nu-\nu') \tag{2.42}$$

where the (Dirac) $\delta$-symbol is nothing more or less than an abbreviation for the following statement: if

$$\Psi = \int_a^b A(\nu')\Psi_{\nu'}\, d\nu' \qquad -\infty \leq a < b \leq \infty \tag{2.43}$$

then

$$\left(\Psi^{\dagger}_{\nu}, \rho\, \Psi\right) = \begin{cases} A(\nu)N(\nu) & a < \nu < b \\ 0 & \text{otherwise} \end{cases} \tag{2.44}$$

It is a simple (and familiar) matter to calculate $N(\nu)$ when $\nu$ belongs to class 2b. Then $\Psi$ is of the form

$$\Psi = \begin{pmatrix} g(u) \\ E \\ B \end{pmatrix} = \begin{pmatrix} \int_a^b g_{\nu'}(u) A(\nu')\,d\nu' \\ \int_a^b E_{\nu'} A(\nu')\,d\nu' \\ \int_a^b B_{\nu'} A(\nu')\,d\nu' \end{pmatrix} \tag{2.45}$$

But since $E_\nu = B_\nu = 0$ we have

$$(\Psi_\nu^\dagger, \rho\Psi) = \int_{-\infty}^{\infty} g_\nu^\dagger(u)\, u\, g(u)\,du = \int_{-\infty}^{\infty} g_\nu^\dagger(u)\, u\,du \int_a^b g_{\nu'}(u) A(\nu')\,d\nu'$$

(2.46)

$$= \int_{-\infty}^{\infty} u\,\delta(u-\nu)\,du \left\{ \frac{\omega_p^2}{\omega^2} F(u)\, P\!\int_a^b \frac{A(\nu')\,d\nu'}{u-\nu'} + \begin{bmatrix} \lambda(u)A(u) & a<u<b \\ 0 & \text{otherwise} \end{bmatrix} \right\}$$

The first term in the braces does not contribute to the final result since $F(\nu) = 0$. Thus

$$(\Psi_\nu^\dagger, \rho\Psi) = \begin{bmatrix} \nu\lambda(\nu)A(\nu) & a<\nu<b \\ 0 & \text{otherwise} \end{bmatrix} \tag{2.47}$$

which implies

$$N(\nu) = \nu\lambda(\nu) \qquad \nu \text{ in class 2b.} \tag{2.48}$$

The preceding result is little more than a partly rigorous justification of the often used symbolic relation

$$\int_{-\infty}^{\infty} f(u)\,\delta(u-\nu)\,\delta(u-\nu')\,du = f(\nu)\,\delta(\nu-\nu') \tag{2.49}$$

but it is important to make precise the meaning of the normalization for the continuum functions. This requires that we consider not the functions themselves but "wave packets"--superpositions of the $\psi_\nu$ where $\nu$ varies over a continuous range. Equations (2.43) and (2.44) follow immediately. It would be tempting to conclude (incorrectly) from (2.42) that

$$N(\nu) = \int_a^b (\Psi_{\nu'}^\dagger, \rho \Psi_\nu) d\nu' \qquad a<\nu<b \tag{2.50}$$

but in fact this is just Eq. (2.44) with the orders of integration interchanged. This point is emphasized because when $\nu$ belongs to class 2a, we come upon one of the rare instances that care must be taken to perform the integrations in the correct order.

In calculating $N(\nu)$ for $\nu$ in class 2a, we note that there is no objection to letting $a = -\infty$, $b = \infty$ in (2.43), just so long as we make sure to choose $A(\nu')$ so that all the integrals exist. Assuming this to be done, we have,

$$A(\nu)N(\nu) = \left(\Psi_\nu^\dagger, \rho \int_{-\infty}^{\infty} A(\nu')\Psi_{\nu'} d\nu'\right)$$

$$= \int_{-\infty}^{\infty}\left[\frac{\omega_p^2}{\omega^2}\nu^3 P\frac{1}{u-\nu} + \tilde{\lambda}(\nu)\delta(u-\nu)\right] u\,du \int_{-\infty}^{\infty}\left[\frac{\omega_p^2}{\omega^2}\nu'^3 P\frac{F(u)}{u-\nu'} + \lambda(\nu')\delta(u-\nu')\right] A(\nu')d\nu'$$

$$+ \frac{ne\nu c^2}{mi\omega}\int_{-\infty}^{\infty}\frac{4\pi ie\nu'^2 A(\nu')d\nu'}{\omega} + \frac{ne\nu^2 c}{mi\omega}\int_{-\infty}^{\infty}\frac{4\pi ie\nu' c\, A(\nu')d\nu'}{\omega}$$

$$= -\left(\frac{\omega_p^2}{\omega^2}\right)^2 \nu^3\, P\int_{-\infty}^{\infty}\frac{u\,F(u)du}{u-\nu}\, P\int_{-\infty}^{\infty}\frac{\nu'^3 A(\nu')d\nu'}{\nu'-u} \quad +$$

$$+\frac{\omega_p^2}{\omega^2}\nu^3 P\int_{-\infty}^{\infty}\frac{u\lambda(u)A(u)du}{u-\nu} - \frac{\omega_p^2}{\omega^2}\lambda(\nu)\nu P\int_{-\infty}^{\infty}\frac{\nu'^3 A(\nu')d\nu'}{\nu'-\nu}$$

$$+\nu\frac{\lambda^2(\nu)}{F(\nu)}A(\nu) + \frac{\omega_p^2}{\omega^2}C^2\int_{-\infty}^{\infty}(\nu\nu'^2+\nu'\nu^2)A(\nu')d\nu' \qquad (2.51)$$

The first term, containing the iterated principal value integrations, is the only one that presents any difficulty. Here we use the Poincare-Bertrand formula[15] to obtain

$$-\left(\frac{\omega_p^2}{\omega^2}\right)^2\nu^3 P\int_{-\infty}^{\infty}\frac{uF(u)du}{u-\nu}P\int_{-\infty}^{\infty}\frac{\nu'^3A(\nu')d\nu'}{\nu'-u}$$

$$(2.52)$$

$$= \left(\frac{\omega_p^2}{\omega^2}\nu^3\right)^2\pi^2\nu F(\nu)A(\nu) + \left(\frac{\omega_p^2}{\omega^2}\right)^2\nu^3 P\int_{-\infty}^{\infty}\nu'^3A(\nu')d\nu' \, P\int_{-\infty}^{\infty}\frac{uF(u)\,du}{(u-\nu)(u-\nu')}$$

The integration over u may now be performed using partial fractions and the definition of $\lambda(\nu)$. The result exactly cancels all the remaining integrals in Eq. (2.51) leaving

$$N(\nu) = \nu\,\frac{\lambda^2(\nu)+\pi^2\left(\frac{\omega_p^2}{\omega^2}\nu^3\right)^2F^2(\nu)}{F(\nu)} \qquad (2.53)$$

$\nu$ in class 2a.

## 5. The Normal Mode Expansion and the Full Range Completeness Theorem

Here we prove that the normal modes $\psi_\nu$ form a complete set, in the sense that an arbitrary state function $\Psi$ may be expanded in terms of them.

$$\Psi = \sum_i a_i \Psi_{\nu_i} + \sum_j a_j \Psi_{\nu_j} + \int_{-\infty}^{\infty} A(\nu) \Psi_\nu \, d\nu \tag{2.54}$$

Once we know that the expansion is possible, then the expansion coefficients are determined using the orthogonality relation (2.15) and the known normalization functions. viz.

$$a_i = \frac{1}{N(\nu_i)} (\Psi^\dagger_{\nu_i}, \rho\Psi) = \frac{1}{N(\nu_i)} \left[ \frac{\omega_p^2}{\omega^2} \nu_i^3 P\int_{-\infty}^{\infty} \frac{g(u) u\,du}{u-\nu_i} + \frac{ne\nu_i c}{mi\omega}\left[cE + \nu_i B\right] \right] \tag{2.55}$$

with $N(\nu_i)$ given by (2.41)

$$a_j = \frac{1}{N(\nu_j)} (\Psi^\dagger_{\nu_j}, \rho\Psi) = \frac{1}{N(\nu_j)} \left[ \frac{\omega_p^2}{\omega^2} \nu_j^3 P\int_{-\infty}^{\infty} \frac{u g(u) du}{u-\nu_j} + \frac{ne\nu_j c}{mi\omega}\left[cE + \nu_j B\right] \right] \tag{2.56}$$

with $N(\nu_j)$ given by (2.40). When $\nu$ is in class 2 a,

$$A(\nu) = \frac{1}{N(\nu)} (\Psi^\dagger_{\nu}, \rho\Psi) = \frac{\lambda(\nu) g(\nu) + F(\nu)\left[\frac{\omega_p^2}{\omega^2} P\int_{-\infty}^{\infty} \frac{\nu^2 u g(u) du}{u-\nu} + \frac{nec}{mi\omega}\left[cE + \nu B\right]\right]}{\pi^2 \left(\frac{\omega_p^2}{\omega^2}\nu^3\right)^2 F^2(\nu) + \lambda^2(\nu)} \tag{2.57}$$

and this expression is also correct for $\nu$ in class 2 b, since in this case

$$A(\nu) = \frac{1}{N(\nu)} \int g^\dagger_\nu(u) u g(u) du = \frac{g(\nu)}{\lambda(\nu)} \tag{2.58}$$

which is just Eq. (2.57) with $F(\nu) = 0$ .

The full range completeness theorem can be simply stated as follows:

THEOREM I.

Given a state function $\Psi$ with $g(u)$ everywhere well-behaved, $\int g(u)\,du < \infty$ but otherwise arbitrary, then an expansion of the form presented by Eq. (2.54) exists, and the expansion coefficients $a_i$, $a_j$ and $A(\nu)$ are unique.

In the proof "well-behaved" will be taken to mean "satisfying a Hölder condition". That is there exists a pair of numbers $M$ and $\gamma$ $(\gamma > 0)$ such that for any $u_o$

$$\left| g(u) - g(u_0) \right| < M \left| u - u_0 \right|^{\gamma} \tag{2.59}$$

Actually, the theorem holds for a much wider class of functions $g(u)$. For example, $g(u)$ may itself be a Schwarzian distribution. In the same spirit we shall assume that $F(u)$ also satisfies a Holder condition.

The proof of the theorem involves an actual construction of the expansion coefficients. That is, we show that (2.54) possesses a solution by solving it. Write

$$\Psi' = \Psi - \sum_i a_i \Psi_{\nu_i} - \sum_j a_j \Psi_{\nu_j} \qquad \text{with } a_i, a_j \text{ given by (2.55,56)} \tag{2.60}$$

We note that the expansion of $\Psi'$ in terms of the continuum functions alone

$$\Psi' = \int_{-\infty}^{\infty} A(\nu)\, \Psi_\nu \, d\nu \tag{2.61}$$

is equivalent to the expansion (2.54) of $\Psi$ in terms of all the $\psi_\nu$.

$\Psi'$ must satisfy

$$(\Psi_{\nu_i}^{\dagger}, \rho\Psi') = (\Psi_{\nu_j}^{\dagger}, \rho\Psi') = 0 \tag{2.62}$$

as well as the conditions imposed on $\Psi$ in the statement of the theorem, but is otherwise arbitrary. The three components of Eq. (2.61) are

$$g'(u) = \frac{\omega_p^2}{\omega^2} F(u)\, P\int_{-\infty}^{\infty} \frac{\nu^3 A(\nu)d\nu}{u-\nu} + \lambda(u) A(u) \tag{2.63a}$$

$$E' = \frac{4\pi i e}{\omega} \int_{-\infty}^{\infty} \nu^2 A(\nu) d\nu \tag{2.63b}$$

$$B' = \frac{4\pi i e c}{\omega} \int_{-\infty}^{\infty} \nu A(\nu) d\nu \tag{2.63c}$$

Equation (2.63a) is a singular integral equation of a type treated extensively by several Russian mathematicians. The solution closely follows Muskhelishvili.[15] We shall make use of the following properties of singular integrals of the Cauchy type:

Suppose $M(x)$ satisfies a Hölder condition and $\int_a^b M(x')\,dx' < \infty$ then

a) $$\mathcal{M}(z) \overset{\text{def}}{=} \frac{1}{2\pi i}\int_a^b \frac{M(x')dx'}{x'-z}$$

is an analytic function in the complex $z$-plane cut along the real axis form $a$ to $b$, vanishing as $z \to \infty$ at least as fast as $1/z$.

b) $\mathcal{M}^+(x)$ and $\mathcal{M}^-(x)$, the boundary values of $\mathcal{M}(z)$ as $z$ approaches the cut from above and below respectively, each satisfy a Hölder condition.

c) $$\mathcal{M}^+(x) - \mathcal{M}^-(x) = M(x) \tag{2.64a}$$

$$\pi i\left(\mathcal{M}^+(x) + \mathcal{M}^-(x)\right) = P\int_a^b \frac{M(x')dx'}{x'-x} \tag{2.64b}$$

(the Plemelj formulae)

d) $\bar{\mathcal{M}}(z) \stackrel{\text{def}}{=} \frac{1}{2\pi i}\int_a^b \frac{x' M(x')dx'}{x'-z}$ vanishes as $z \to \infty$

The converse also holds: if $\mathcal{M}(z)$ satisfies a) and b), then it may be represented as a Cauchy integral of the form

$$\mathcal{M}(z) = \frac{1}{2\pi i}\int_a^b \frac{M(x')dx'}{x'-z} \tag{2.65}$$

$M(x)$ will, of course, be given by the first Plemelj formula.

We combine (2.63a-c) into a single equation for $A(\nu)$.

$$u g'(u) = -\frac{\omega_p^2}{\omega^2} u F(u)\left[\frac{\omega}{4\pi i e}\left(E' + \frac{u}{c}B'\right) + u^2 P\int_{-\infty}^{\infty} \frac{\nu A(\nu)d\nu}{\nu - u}\right] + \lambda(u)\, u\, A(u) \tag{2.66}$$

Now define

$$\mathcal{G}(z) \stackrel{\text{def}}{=} \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{u\, g'(u)du}{u - z} \tag{2.67}$$

$$\mathcal{F}(z) \stackrel{\text{def}}{=} \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{F(u)du}{u - z} \tag{2.68}$$

$$\mathcal{A}(z) \stackrel{\text{def}}{=} \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{\nu A(\nu)d\nu}{\nu - z} \tag{2.69}$$

$\mathcal{G}(z)$ and $\mathcal{F}(z)$ are analytic in the cut plane and vanish at infinity. Their boundary values $\mathcal{A}^{\pm}(u)$ and $\mathcal{G}^{\pm}(u)$ satisfy a Hőlder condition. If a solution $A(\nu)$ to (2.66) exists, then $\mathcal{A}(z)$ will have those same properties. We shall show that such an $A(\nu)$ exists by constructing the appropriate $\mathcal{A}(z)$ and showing that it does in fact possess the properties we require it to have. We would then find $A(\nu)$ by using the Plemelj formula

$$A(\nu) = \frac{1}{\nu}\left[\mathcal{A}^{+}(\nu) - \mathcal{A}^{-}(\nu)\right] \tag{2.70}$$

if we didn't already have the orthogonality relations. In terms of the boundary values of the functions $\mathcal{G}$, $\mathcal{F}$ and $\mathcal{A}$, Eq. (2.66) becomes, after some rearranging and dropping primes

$$\begin{aligned} &\mathcal{G}^{+}(u) + \frac{\omega_p^2}{\omega^2}\left[\frac{\omega}{4\pi i e}\left(E+\frac{u}{c}B\right) + 2\pi i u^2 \mathcal{A}^{+}(u)\right]\mathcal{F}^{+}(u) - (c^2-u^2)\mathcal{A}^{+}(u) \\ &= \mathcal{G}^{-}(u) + \frac{\omega_p^2}{\omega^2}\left[\frac{\omega}{4\pi i e}\left(E+\frac{u}{c}B\right) + 2\pi i\, \mathcal{A}^{-}(u)\right]\mathcal{F}^{-}(u) - (c^2-u^2)\mathcal{A}^{-}(u) \end{aligned} \tag{2.71}$$

Now consider the function

$$J(z) \overset{\text{def}}{=} \mathcal{G}(z) + \frac{\omega_p^2}{\omega^2}\left[\frac{\omega}{4\pi i e}\left(E+\frac{z}{c}B\right) + 2\pi i\, \mathcal{A}(z)\right]\mathcal{F}(z) - (c^2-z^2)\mathcal{A}(z) \tag{2.72}$$

$J(z)$ is analytic in the cut plane, since $\mathcal{F}$ and $\mathcal{G}$ are, and $\mathcal{A}$ is assumed to be. But according to (2.71), $J(z)$ is continuous across the cut. Therefore, $J(z)$ is an entire function, and may be found simply by observing its behavior for large values of $|z|$. We note that

$$\mathcal{G}(z) \xrightarrow[|z|\to\infty]{} 0 \tag{2.73}$$

$$z\,\mathcal{F}(z) = \frac{1}{2\pi i}\, z \int_{-\infty}^{\infty} \frac{F(u)\,du}{u - z} = -\frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{F(u)\,du}{1 - u/z} \xrightarrow[|z|\to\infty]{} -\frac{1}{2\pi i} \tag{2.74}$$

$$z^2 \mathcal{A}(z) = \frac{1}{2\pi i}\, z^2 \int_{-\infty}^{\infty} \frac{\nu A(\nu)\,d\nu}{\nu - z} = -\frac{1}{2\pi i}\left[\int_{-\infty}^{\infty} \nu^2 A(\nu)\,d\nu + z\int_{-\infty}^{\infty} \nu A(\nu)\,d\nu\right] +$$

$$+ \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{\nu^3 A(\nu)\,d\nu}{\nu - z} = -\frac{1}{2\pi i}\,\frac{\omega}{4\pi i e}\left(E + \frac{z}{c}B\right) + \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{\nu^3 A(\nu)\,d\nu}{\nu - z} \xrightarrow[|z|\to\infty]{} -\frac{1}{2\pi i}\,\frac{\omega}{4\pi i e}\left(E + \frac{z}{c}B\right) \tag{2.75}$$

(the fact that $\int \nu^2 A(\nu)\,d\nu < \infty$ will be verified later). Thus

$$J(z) \xrightarrow[|z|\to\infty]{} -\frac{1}{2\pi i}\,\frac{\omega}{4\pi i e}\left(E + \frac{z}{c}B\right) \tag{2.76}$$

We conclude, from Liouville's theorem

$$J(z) = -\frac{1}{2\pi i}\,\frac{\omega}{4\pi i e}\left(E + \frac{z}{c}B\right) \tag{2.77}$$

which gives the solution

$$\mathcal{A}(z) = \frac{\mathcal{G}(z) + \frac{\omega}{4\pi i e}\left(E + \frac{z}{c}B\right)\left[\frac{1}{2\pi i} + \frac{\omega_p^2}{\omega^2}\, z\,\mathcal{F}(z)\right]}{c^2 - z^2 - 2\pi i\,\frac{\omega_p^2}{\omega^2}\, z^3\,\mathcal{F}(z)} \tag{2.78}$$

Now we must examine the function $\mathcal{A}(z)$ and see if it has the properties demanded of it. We note that $\mathcal{A}(z)$ vanishes as $|z| \to \infty$ and is analytic in the cut plane except possible at the zeroes of the denominator, where $\mathcal{A}(z)$ could have simple poles. But these points are just the $\nu_j$, and when $\nu = \nu_j$, the numerator may be shown to vanish. viz.

$$[\text{numerator}] \times \nu_j^3 2\pi i \frac{\omega_p^2}{\omega^2} = \nu_j^3 \int_{-\infty}^{\infty} \frac{u g(u) du}{u - \nu_j} + \frac{\omega}{4\pi i e}\left[E\nu_j + B\frac{\nu_j^2}{c}\right]\underbrace{\left[\nu_j^2 + \frac{\omega_p^2}{\omega^2}\nu_j^3 \int_{-\infty}^{\infty}\frac{F(u)du}{u-\nu_j}\right]}_{c^2}$$

$$= (\psi^{\dagger}_{\nu_j}, \rho \Psi') = 0 \qquad (2.79)$$

Thus the singularity is removable, and $\mathcal{A}(z)$ is analytic in the cut plane. We need look only at its boundary values. As $z$ approaches the cut from $\left\{\begin{matrix}\text{above}\\ \text{below}\end{matrix}\right\}$ the denominator becomes

$$[\text{denominator}] = c^2 - \nu^2 - \frac{\omega_p^2}{\omega^2}\nu^3 2\pi i \mathcal{F}^{\pm}(\nu) = \lambda(\nu) \mp \frac{\omega_p^2}{\omega^2}\pi i \nu^3 F(\nu) \qquad (2.80)$$

The boundary values are well-defined except possibly where the denominator vanishes, so the points in question are just the $\nu_i$. And, at these points, the numerator also vanishes, since by direct calculation

$$[\text{numerator}] \times \nu_i^3 2\pi i \frac{\omega_p^2}{\omega^2} = (\psi^{\dagger}_{\nu_i}, \rho \Psi') = 0 \qquad (2.81)$$

Finally, we must show that the various integrals involving $A(\nu)$ actually exist. By direct calculation, we find

$$\mathcal{A}(z) \xrightarrow[|z| \to \infty]{} -\frac{\omega B}{4\pi i e c}\frac{1}{2\pi i}\frac{1}{z} \qquad (2.82)$$

This implies (cf. condition a), page 36) that $\int \nu A(\nu)\, d\nu < \infty$. (It is, in fact, given by $\omega B/4\pi i e c$.) Moreover, by condition d), the Cauchy integral

$$\bar{\mathcal{A}}(z) = \int_{-\infty}^{\infty} \frac{\nu^2 A(\nu) d\nu}{\nu - z} \qquad (2.83)$$

converges, and another straightforward calculation yields

$$\bar{a}(z) = z\,a(z) + \frac{1}{2\pi i}\int_{-\infty}^{\infty} \nu A(\nu)\,d\nu \xrightarrow[|z|\to\infty]{} -\frac{\omega E}{4\pi i e c}\,\frac{1}{2\pi i}\,\frac{1}{z} \tag{2.84}$$

which implies that $\int \nu^2 A(\nu)\,d\nu$ converges and its value is $\omega E/4\pi i e$. Finally, we note that this means that the Cauchy integral $\int \nu^3 A(\nu)/(\nu - z)\,d\nu$ converges. Hence $a(z)$, the solution of (2.71) does indeed possess the properties we assumed it had. This completes the proof.

For future reference we note that by virtue of the Plemelj formulas $N(\nu)$ may be expressed in terms of the boundary values of the analytic function $\Lambda(z)$

$$\frac{1}{N(\nu)} = \frac{F(\nu)}{\nu\left[\lambda(\nu) + \pi i \nu^3 \frac{\omega_p^2}{\omega^2} F(\nu)\right]\left[\lambda(\nu) - \pi i \nu^3 \frac{\omega_p^2}{\omega^2} F(\nu)\right]}$$

$$= \frac{F(\nu)}{\nu\,\Lambda^+(\nu)\,\Lambda^-(\nu)} = \frac{\Lambda^-(\nu) - \Lambda^+(\nu)}{2\pi i\,\frac{\omega_p^2}{\omega^2}\,\nu^4 \Lambda^+(\nu)\Lambda^-(\nu)} \tag{2.85}$$

$$= \mathcal{T}^+(\nu) - \mathcal{T}^-(\nu)$$

where

$$\mathcal{T}(z) = \frac{1}{2\pi i\,\frac{\omega_p^2}{\omega^2}\,z^4 \Lambda(z)} \tag{2.86}$$

## 6. Specialization to the Boltzmann Distribution

We consider now the particular case where $\mathcal{F}_0(\hat{u})$ is the Boltzmann distribution for electrons at temperature $T = \Theta/k$.

$$\mathcal{F}_0(\vec{u}) = \left(\frac{m}{2\pi\Theta}\right)^{3/2} e^{-\frac{m\vec{u}\cdot\vec{u}}{2\Theta}} \tag{2.87}$$

Then

$$F(u) = \left(\frac{m}{2\pi\Theta}\right)^{1/2} e^{-\frac{mu^2}{2\Theta}} \tag{2.88}$$

Some of the results of this section will depend only on the general shape of $F(u)$ and not on its exact functional form.

It is possible to ascertain the number of elements in the discrete spectrum; i.e. the number of roots of

$$\Lambda(z) = c^2 - z^2 - \frac{\omega_p^2}{\omega^2} z^3 \int_{-\infty}^{\infty} \frac{F(u)\,du}{u-z} = 0 \tag{2.89}$$

First note that since $\Lambda(z)$ is a real function of $z$, the roots occur in complex conjugate pairs. But since $F(u)$ is an even function, $\Lambda(z)$ is also even and $-z_o$ is a root along with $z_o$. Thus in general, the roots occur in groups of four: $z_o$ and $-z_o^*$ in the upper half plane, symmetric about the imaginary axis, and $-z_o$ and $z_o^*$ in the lower half plane. We shall show that there is at most one root in the upper half plane. This root, when it occurs, must then lie on the imaginary axis.

We seek to find the number of zeros of $\Lambda(z)$ in the upper half plane, using the argument principle. Consider the contour C in the upper half z-plane. (See Fig. 1.) Let us follow the behavior of $\Lambda(z)$ as z traverses the contour C. The change of argument of $\Lambda(z)$ will be equal to $2\pi \times$ (number of zeroes of $\Lambda(z)$ inside C). The semicircle

1-2-3 is large enough so that $\Lambda(z)$ assumes its asymptotic behavior along it.

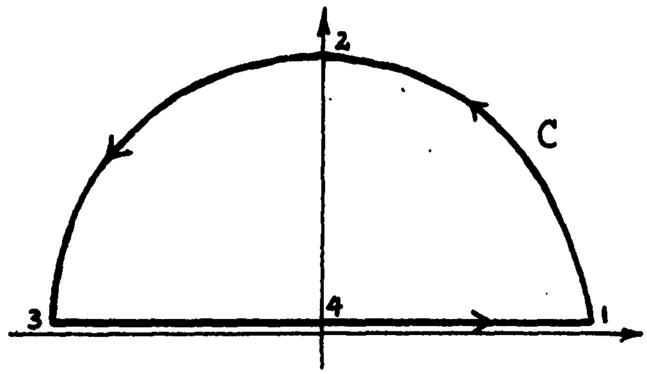



Fig. 1 The contour C

For $z$ on the large semi-circle

$$\Lambda(z) \cong -\left(1-\frac{\omega_p^2}{\omega^2}\right)z^2 \tag{2.90}$$

and when $z = x + i\epsilon$ is just above the real axis, we have

$$\Lambda(z) = \Lambda^+(x) = \lambda(x) - \pi i \frac{\omega_p^2}{\omega^2} x^3 F(x) \tag{2.91}$$

$$\Lambda(0) = c^2 \tag{2.92}$$

Since $F(x)$ is positive and nonvanishing, when $z$ is on the upper lip of the real axis, $\operatorname{Im}\Lambda(z)$ is $\left\{\begin{matrix}\text{positive}\\ \text{negative}\end{matrix}\right\}$ for $\left\{\begin{matrix}\text{negative}\\ \text{positive}\end{matrix}\right\}$ $x$ and vanishes only when $z = 0$. Thus, as $z$ traverses the path 3-4-1, $\Lambda(z)$ crosses the real axis only once. There are, then, only two possibilities.

1) $\omega > \omega_p$ The image of the contour C is shown in Fig. 2. $\Delta \arg \Lambda(z) = 0$ and there are no zeroes of $\Lambda(z)$ inside the contour C; hence no zeroes in any finite part of the upper half plane.

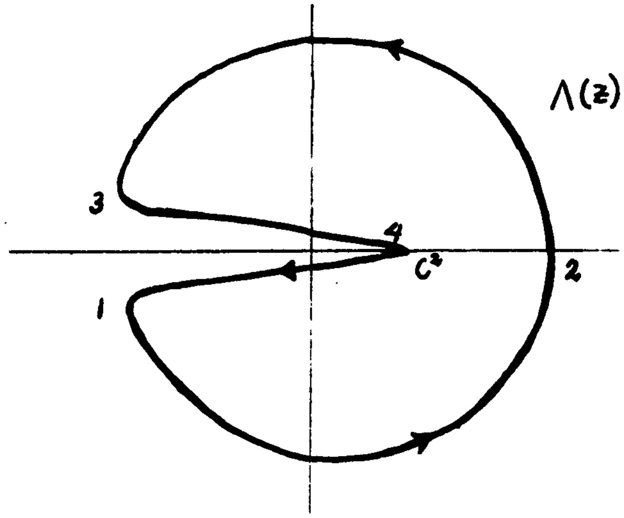



Fig. 2

2) $\omega < \omega_p$ The image of the contour C is shown in Fig. 3. $\Delta \arg \Lambda(z) = 2\pi$ and there is one zero in the upper half plane.

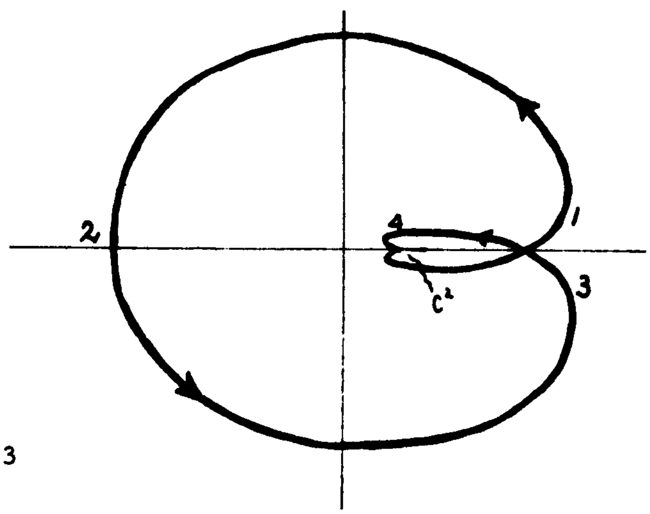



Fig. 3

Thus, for Boltzmann F(u), the set of allowed values of $\nu$ includes the entire real axis (class 2 a) and sometimes the two discrete roots $\pm\ i\nu_0$. (class 2 b)

We may also obtain an estimate of the value of the discrete root $i\nu_o$. We have, from Eq. (2.89) using the explicit form of F (u)

$$\left(\sqrt{\frac{m}{2\Theta}}\,c\right)^2 - \left(\sqrt{\frac{m}{2\Theta}}\,z\right)^2 - \frac{\omega_p^2}{\omega^2}\left(\sqrt{\frac{m}{2\Theta}}\,z\right)^3 \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}\frac{e^{-x^2}dx}{x-\sqrt{\frac{m}{2\Theta}}\,z} = 0 \tag{2.93}$$

Assume, for the moment, that $|z|$ is large enough so that the asymptotic expression

$$\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}\frac{e^{-x^2}dx}{x-\sqrt{\frac{m}{2\Theta}}\,z} \sim \frac{-1}{\sqrt{\frac{m}{2\Theta}}\,z} \tag{2.94}$$

is valid. Then

$$\left(\sqrt{\frac{m}{2\Theta}}\,c\right)^2 - \left(1-\frac{\omega_p^2}{\omega^2}\right)\left(\sqrt{\frac{m}{2\Theta}}\,z\right)^2 = 0 \tag{2.95}$$

or

$$z^2 = \frac{c^2}{1-\omega_p^2/\omega^2}\,, \qquad \nu_o = \frac{\omega}{\sqrt{\omega_p^2-\omega^2}}\,c \tag{2.96}$$

This result is correct so long as (2.94) holds, which amounts to

$$\left|\sqrt{\frac{m}{2\Theta}}\,z\right| \gtrsim 3 \tag{2.97}$$

or

$$\frac{mc^2/2\Theta}{\frac{\omega_p^2}{\omega^2}-1} \gtrsim 9 \tag{2.98}$$

which, for $T \cong 6000^\circ K$, becomes

$$\omega > 4 \times 10^{-3}\,\omega_p \tag{2.99}$$

Below this frequency, we may try to approximate the result by assuming that $|(m/2\Theta)^{1/2} z|$ turns out to be small. Then we may use the power series expansion

$$\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}\frac{e^{-x^2}dx}{x-\sqrt{\frac{m}{2\Theta}}\,z} \cong i\sqrt{\pi}\, e^{-\frac{mz^2}{2\Theta}} - 2\sqrt{\frac{m}{2\Theta}}\,z\{1+\cdots\} \tag{2.100}$$

To lowest order, this yields

$$\nu_0 = \left[\frac{\omega^2}{\pi\omega_p^2}\sqrt{\frac{2\pi\Theta}{mc^2}}\right]^{1/3} c \tag{2.101}$$

valid (at $T = 6000^\circ$K) when $\omega \lesssim 4 \times 10^{-5}\,\omega_p$, and an iteration gives

$$\nu_0 = \left\{\frac{\frac{\omega^2}{\pi\omega_p^2}\sqrt{\frac{2\pi\Theta}{mc^2}}}{1-\left(\frac{\omega^2}{\omega_p^2}\frac{mc^2}{2\pi\Theta}\right)^{1/3}}\right\}^{1/3} \tag{2.102}$$

We note that except at extremely high temperatures or very low frequencies, the location of the two modes with imaginary wave number is correctly given by the zero temperature treatment.

Finally, we may obtain an approximate expression for the normalization coefficient $N(\pm i\nu_0)$ defined by

$$N(\pm i\nu_0) = -\frac{\omega_p^2}{\omega^2}(\pm i\nu_0)^4 \frac{d}{d\nu}\Lambda(\nu)\bigg|_{\nu=\pm i\nu_0} \tag{2.103}$$

By direct calculation, for Boltzmann $F(u)$, this becomes

$$N(\pm i\nu_0) = -\frac{\omega_p^2}{\omega^2}(\pm i\nu_0)^3\left\{\frac{m}{\Theta}\left[\left(\frac{\omega_p^2}{\omega^2}-1\right)(\pm i\nu_0)^2+c^2\right](\pm i\nu_0)^2+\left[(\pm i\nu_0)^2-3c^2\right]\right\} \tag{2.104}$$

Except at very low frequencies, to first order in $2\,\Theta/mc^2$ we get

$$N(\pm i\nu_0) = \mp 2i\,\frac{\omega_p^2}{\omega^2}\left[1-\frac{\omega_p^2}{4\omega^2}\,\frac{2\Theta}{mc^2}\right]\frac{\left(1+\frac{\omega_p^2}{2\omega^2}\,\frac{2\Theta}{mc^2}\right)^{3/2}}{\left(\frac{\omega_p^2}{\omega^2}-1\right)^{3/2}}\,c^5 \tag{2.105}$$

# Chapter III. BOUNDARY VALUE PROBLEMS FOR TRANSVERSE WAVES

## 1. Introduction

In this section we shall utilize the normal modes of the Vlasov equation in the solution of some typical boundary value problems. The method is straightforward: we write the state function of the medium under consideration as a superposition of normal modes, and apply the boundary conditions to determine the expansion coefficients. In certain cases (the degenerate problem) the expansion to be performed will utilize the full set of normal modes, and the orthogonality relations may be used to determine the coefficients. In any event, we will arrive at an integral equation for the expansion coefficients, part of which (the so-called "dominant" part), by virtue of the singular nature of the $g_{\nu}(u)$, will contain singular integrals.

We shall consider the reflection-transmission problem. An electromagnetic wave of frequency $\omega$ is incident on a plasma. What is the strength of the reflected wave? It will be convenient to ignore the fact that the actual plasma oscillations are "forced" by the impinging wave. This avoids the problem presented by the presence of an external force field whose effect has not been taken into account in the linearization of the basic equations. The electromagnetic fields throughout the plasma, in particular at the boundary, are to be viewed as "small" in the sense of the original linearization (Chapter I). We simply construct a solution representing a plasma configuration consistent with the presence of an incident and reflected wave, and only afterwards do we interpret the plasma motions as the effect and the impinging wave as the cause.

## 2. Boundary Conditions

The boundary conditions for the electromagnetic fields are simple. Since all disturbances are small, there can be no surface changes or currents and the fields must be continuous.

$$E\,(z=0+) = E\,(z=0-)$$
$$B\,(z=0+) = B\,(z=0-) \qquad (3.1)$$

We must specify, in addition, the behavior of the electron distribution function at the boundary. It is simplest to assume, with Landau,[16] that electrons experience specular reflection back into the plasma at the boundary. This automatically implies that at the boundary the distribution function is symmetric in the normal component of the velocity

$$f(u_x, u_y, u, z=0) = f(u_x, u_y, -u, z=0) \qquad (3.2)$$

or

$$g\,(0, u) = g\,(0, -u) \qquad (3.3)$$

This is the degenerate case, and the solution may be written in terms of the normalization integrals using the orthogonality relations.

The opposite extreme, one might term the "diffuse absorber." In this model, the electrons are returned to the plasma at the boundary in a manner essentially independent of the velocity distribution of the electrons which leave. This implies:

$$g(u, z=0)\Big|_{u>0} = G(u) \quad \text{(a given function)} \tag{3.4}$$

This boundary condition leads to a singular integral equation which is equivalent to the inhomogeneous Hilbert problem.

The intermediate case, too complicated to be considered here, allows the distribution function of the incoming electrons to depend on the outgoing electron distribution in an arbitrary (but linear) manner. This leads to a singular integral equation for the expansion coefficients which contains a Fredholm term as well.

## 3. The Electromagnetic Fields at the Boundary

We consider a plane polarized $E_x$ electromagnetic wave impinging from the left on the surface of a plasma (interface at $z = 0$). In Chapter I it was pointed out that the transmission and reflection coefficients for an ordinary dielectric medium may be written in terms of the index of refraction of the medium. (cf. Eq. (1.6).) Of course in this general treatment, the plasma is not simply a dielectric medium. For a given frequency, waves of all wave numbers can propagate. Nevertheless, it is an easy matter to define a quantity which plays the same role as the index of refraction.

The incident wave is of the form

$$\begin{aligned} E_{ix} &= E_i \, e^{i\omega\left(\frac{z}{c}-t\right)} \\ B_{iy} &= B_i \, e^{i\omega\left(\frac{z}{c}-t\right)} = E_{ix} \end{aligned} \tag{3.5}$$

The reflected wave is of the form

$$\begin{aligned} E_{rx} &= E_r \, e^{-i\omega\left(\frac{z}{c}+t\right)} \\ B_{ry} &= B_r \, e^{-i\omega\left(\frac{z}{c}+t\right)} = -E_{rx} \end{aligned} \tag{3.6}$$

Hence, by (3.1) the electric and magnetic fields at the boundary are given by

$$\begin{aligned} E_x(z=0) &= E_o e^{-i\omega t} = (E_i + E_r) e^{-i\omega t} \\ B_y(z=0) &= B_o e^{-i\omega t} = (E_i - E_r) e^{-i\omega t} \end{aligned} \tag{3.7}$$

Thus

$$\frac{E_r}{E_i} = \frac{1-n}{1+n} \tag{3.8}$$

where n is the ratio of the magnetic and electric fields at the boundary

$$n = \frac{B_o}{E_o} \tag{3.9}$$

Note that this is a reasonable analogy to the index of refraction of an ordinary dielectric medium, since if we assume a disturbance in such a medium to be of the form

$$E_x\,,\,B_y \sim e^{i\omega\left(\frac{nz}{c}-t\right)} \tag{3.10}$$

then it follows from Maxwell's equations that n represents the ratio of magnetic and electric fields throughout the medium.

4. Reflection From a Plasma Half Space

We consider the state function of the plasma to be expanded in terms of the normal modes

$$\Psi(z,t) = e^{-i\omega t} \mathop{S}_{\nu} A(\nu)\,\Psi_\nu\, e^{i\frac{\omega}{\nu}z} \tag{3.11}$$

The summation incorporates an integral over the continuum modes and a discrete sum. The boundary condition

$$g(0,u) = g(0,-u) \tag{3.12}$$

implies that

$$\mathop{S}_{\nu} A(\nu)\left[g_\nu(u) - g_\nu(-u)\right] = 0 \tag{3.13}$$

We will deal here only with the case treated in Chap. II, sec. 6, where $F_o(u)$ is assumed to have such a form that there are either no discrete eigenvalues (when $\omega > \omega_p$) or two imaginary eigenvalues $\nu_{\pm} = \pm i \nu_o$ (when $\omega < \omega_p$). In this case

$$g_\nu(-u) = g_{-\nu}(u) \tag{3.14}$$

for both the discrete and continuum modes, and hence from (3.13)

$$\mathop{S}_{\nu} [A(\nu) - A(-\nu)] g_\nu(u) = 0 \tag{3.15}$$

Moreover,

$$\mathop{S}_{\nu} [A(\nu) - A(-\nu)] E_\nu = 0 \tag{3.16}$$

simply because $E_\nu$ is even in $\nu$. Also, since $B_\nu$ is odd in $\nu$

$$\mathop{S}_{\nu} [A(\nu) - A(-\nu)] B_\nu = 2B_o \tag{3.17}$$

Therefore, if we define

$$C(\nu) = A(\nu) - A(-\nu) \tag{3.18}$$

we obtain the concise relation

$$\underset{\nu}{S}\, C(\nu)\,\Psi_\nu = \begin{pmatrix} 0 \\ 0 \\ 2B_0 \end{pmatrix} \tag{3.19}$$

which implies that $C(\nu)$ is the coefficient in the expansion of

$$2B_0\,\mathcal{X} = 2B_0 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \tag{3.20}$$

in terms of the $\psi_\nu$ . Hence, by the orthogonality relation (2.18)

$$C(\nu) = 2B_0 \frac{(\Psi_\nu^\dagger, \rho\,\mathcal{X})}{N(\nu)} \tag{3.21}$$

Having obtained $C(\nu)$, in order to obtain the desired coefficient $A(\nu)$ we make use of the fact that the system must be well behaved as $z \rightarrow \infty$.

$$a)\quad A(+i\nu_0) = 0 \tag{3.22a}$$

$$b)\quad A(\nu) = 0 \qquad \nu < 0 \tag{3.22b}$$

Condition a) insures that we exclude the exponentially growing modes, and b) may be thought of as a radiation condition ( no incoming waves) or a regularity condition which results if we assign to $\omega$ a small imaginary part. This last point will be discussed in section 7 of this chapter.

Combining (3.18), (3.21) and (3.22), we obtain the expansion coefficients:

$$A(\nu) = \begin{cases} C(\nu) = \dfrac{2B_0 B_\nu^\dagger}{N(\nu)} & \nu > 0 \\ 0 & \nu < 0 \end{cases} \tag{3.23}$$

$$A(-i\nu_0) = C(-i\nu_0) = 2B_0 B_{-i\nu_0}^\dagger$$

$$A(i\nu_0) = 0$$

from which g (z, u), and E (z), and B (z) may be computed. In particular, since

$$E_0 = \underset{\nu}{S}\, A(\nu) E_\nu \tag{3.24}$$

we obtain for the index of refraction n

$$\frac{1}{n} \equiv \frac{E_0}{B_0} = 2\left[\int_0^\infty \frac{B_\nu^\dagger E_\nu \, d\nu}{N(\nu)} + \frac{B_{-i\nu_0}^\dagger E_{-i\nu_0}}{N(-i\nu_0)}\right] \tag{3.25}$$

where the discrete term is to be included only when appropriate (i.e. when $\omega < \omega_p$ ).

Using (2.85-6), the integral term in (3.25) may be written as a contour integral.

$$2\int_0^\infty \frac{B_\nu^\dagger E_\nu \, d\nu}{N(\nu)} = 2c\int_0^\infty \frac{\omega_p^2}{\omega^2}\nu^4 \frac{d\nu}{N(\nu)}$$

$$= \frac{2c}{2\pi i}\int_0^\infty \left(\frac{1}{\Lambda^+(\nu)} - \frac{1}{\Lambda^-(\nu)}\right)d\nu = \frac{2c}{2\pi i}\int \frac{dz}{\Lambda(z)} \tag{3.26}$$

Moreover, in view of (2.40), the discrete term becomes simply the residue of $[-2c/\Lambda(z)]$ at $z = -i\nu_0$. This leads to the compact expression

$$\frac{1}{n} = \frac{2c}{2\pi i}\int \frac{dz}{\Lambda(z)} \tag{3.27}$$

In the zero temperature limit

$$F(u) = \delta(u) \tag{3.28}$$

This implies

$$\Lambda(z) = c^2 - \left(1 - \frac{\omega_p^2}{\omega^2}\right)z^2 \tag{3.29}$$

and the contour integration may be performed easily, to yield the familiar result

$$n = \begin{cases} \sqrt{1-\dfrac{\omega_p^2}{\omega^2}} & \omega > \omega_p \\ i\sqrt{\dfrac{\omega_p^2}{\omega^2}-1} & \omega < \omega_p \end{cases} \tag{3.30}$$

The zero temperature result, which predicts a purely imaginary index of refraction (and corresponding complete reflection) for frequencies below the plasma frequency, has well-known experimental consequences. It provides the explanation for the "plasma blackout" which inhibits communication with a reentering space vehicle. It is evidenced also in the relative ease with which the low frequency AM signals may be transmitted over large distances by reflection from the ionosphere, whereas transmission of the higher frequency FM signals is limited by the curvature of the earth. Nevertheless, at finite temperatures, since the integrand in (3.26) is explicitly positive definite, there will be a small contribution to the real part of the index of refraction at frequencies below the plasma frequency, the real expression (3.26) serves as a slight modification to the zero-temperature result $n = (1 - \omega_p^2/\omega^2)^{1/2}$.

For purposes of computation, it is convenient to write (3.27) in the form

$$\frac{1}{n} = \frac{2\nu}{2\pi i}\int \frac{dz}{\Lambda^{\nu}(z)} \tag{3.31}$$

where $U$ is the velocity of light in units of $(2\,\Theta/m)^{1/2}$, and

$$\Lambda^{U}(z) = U^2 - z^2 - \frac{\omega_p^2}{\omega^2} z^3 \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{e^{-x^2}\,dx}{x - z} = U^2 - z^2 - \frac{\omega_p^2}{\omega^2} z^3 Z(z) \tag{3.32}$$

$Z(z)$, the "plasma dispersion function," is tabulated.[17]

## 5. Reflection From a Slab

We consider the reflection of an electromagnetic wave of frequency $\omega$ incident on a slab of thickness $a$. As in the case of a half space, we write the state function as a superposition of normal modes and apply the boundary conditions. We again obtain:

$$C(\nu) \equiv A(\nu) - A(-\nu) = 2B_0 \frac{B_\nu^\dagger}{N(\nu)} \tag{3.33}$$

We must also apply the condition of specular reflection to the far side of the slab.

$$g(a, u) = g(a, -u) \tag{3.34}$$

This gives

$$D(\nu) \equiv A(\nu) e^{i\frac{\omega}{\nu}a} - A(-\nu) e^{-i\frac{\omega}{\nu}a} = 2B_a \frac{B_\nu^\dagger}{N(\nu)} \tag{3.35}$$

where $B_a$ is the magnetic field at the surface $z = a$, Combining (3.33) and (3.35) we obtain the expansion coefficient

$$A(\nu) = (1 - e^{2i\omega a/\nu})^{-1} 2\left(B_0 - B_a e^{i\omega a/\nu}\right) \frac{B_\nu^\dagger}{N(\nu)} \tag{3.36}$$

which we use to obtain the electric fields at the boundaries in terms of the magnetic fields

$$E_0 = \underset{\nu}{S} A(\nu) E_\nu = B_0\, i \underset{\nu}{S} \frac{B_\nu^\dagger E_\nu}{N(\nu) \tan \frac{\omega a}{\nu}} + B_a (-i) \underset{\nu}{S} \frac{B_\nu^\dagger E_\nu}{N(\nu) \sin \frac{\omega a}{\nu}}$$

$$= \alpha B_0 + \beta B_a \tag{3.37}$$

$$E_a = \underset{\nu}{S} A(\nu) e^{i\frac{\omega}{\nu}a} E_\nu = B_0\, i \underset{\nu}{S} \frac{B_\nu^\dagger E_\nu}{N(\nu) \sin \frac{\omega a}{\nu}} - B_a\, i \underset{\nu}{S} \frac{B_\nu^\dagger E_\nu}{N(\nu) \tan \frac{\omega a}{\nu}}$$

$$= -\beta B_0 - \alpha B_a \tag{3.38}$$

The above expressions have been simplified by the use of the fact that $B_\nu$ and $N(\nu)$ are even and odd functions of $\nu$ respectively. The last boundary condition to be applied follows from the requirement that the electromagnetic field in the space behind the plasma slab represents an outgoing monchromatic wave

$$E_a = B_a \tag{3.39}$$

Combining the last three equations, we obtain the result

$$E_0 = \left[\alpha - \frac{\beta^2}{1-\alpha}\right] B_0 \tag{3.40}$$

The singularities in the integrals defining $\alpha$ and $\beta$ are to be resolved by ascribing to $\omega$ a small positive imaginary part, as will be discussed later.

Therefore, with respect to a normally incident plane wave, the slab behaves like a half space with index of refraction given by

$$\frac{1}{n} = \left[\alpha - \frac{\beta^2}{1-\alpha}\right] \tag{3.41}$$

The coefficients $\alpha$, $\beta$ may be simplified via contour integration

$$\alpha = -\frac{i\omega a}{c\pi^2}\sum_{n=-\infty}^{\infty}\frac{1}{n^2\Lambda^+\left(\frac{\omega a}{n\pi}\right)} \tag{3.42}$$

$$\beta = \frac{i\omega a}{c\pi^2}\sum_{n=-\infty}^{\infty}\frac{(-1)^n}{n^2\Lambda^+\left(\frac{\omega a}{n\pi}\right)} \tag{3.43}$$

For the zero temperature model $[\ \Lambda(z) = c^2 - \epsilon z^2\ ]$, we obtain

$$\alpha = \frac{i}{\sqrt{\epsilon}\tan\left(\sqrt{\epsilon}\frac{\omega a}{c}\right)} \qquad \beta = \frac{-i}{\sqrt{\epsilon}\sin\left(\sqrt{\epsilon}\frac{\omega a}{c}\right)} \tag{3.44}$$

which is in agreement with the result obtained by elementary methods.

## 6. Reflection From a Half-Space: The Diffuse Absorber

The problem of the diffuse absorber is of interest because it illustrates a situation where the eigenfunction expansion must be performed without the aid of an orthoganality relation.

We assume that

$$f(0,\vec{u}) = \left(\frac{m}{2\pi\Theta}\right)^{3/2} e^{-\frac{m\vec{u}^2}{2\Theta}} \tag{3.45}$$

which implies

$$g(o,u) = 0 \qquad u > o \tag{3.46}$$

Then with $\omega > \omega_p$ so that there are no discrete $\nu_j$, the eigenfunction expansion becomes

$$\begin{pmatrix} 0 \\ E_o \\ B_o \end{pmatrix} = \int_0^\infty A(\nu)\Psi_\nu d\nu \qquad u > o \tag{3.47}$$

Thus, we are presented with a "half-range" expansion--the expansion of $\Psi$ $(o < u < \infty)$ in terms of the $\psi_\nu$ $(o < \nu < \infty)$. The resulting integral equation is

$$0 = -\frac{\omega_p^2}{\omega^2} F(u) P\int_0^\infty \frac{\nu^3 A(\nu) d\nu}{\nu - u} + \lambda(u) A(u) \qquad u > o \tag{3.48}$$

subject to

$$\int_0^\infty \nu A(\nu) d\nu = \frac{\omega B_0}{4\pi i e c} \tag{3.49}$$

$$\int_0^\infty \nu^2 A(\nu) d\nu = \frac{\omega E_0}{4\pi i e} \tag{3.50}$$

Now define

$$\mathcal{a}(z) = \frac{1}{2\pi i} \int_0^\infty \frac{\nu^3 A(\nu) d\nu}{\nu - z} \tag{3.51}$$

$\mathcal{a}(z)$ should be analytic in the complex plane cut from 0 to $\infty$ (the "cut plane"), vanishing as $|z| \longrightarrow \infty$. (3.48) becomes

$$\mathcal{a}^+(u)\Lambda^+(u) - \mathcal{a}^-(u)\Lambda^-(u) = 0 \qquad u > 0 \tag{3.52}$$

To solve this, we construct a function X (z) which is analytic, and, along with its boundary values $X^{\pm}(u)$, non-vanishing in the cut plane, such that

$$\frac{X^+(u)}{X^-(u)} = \frac{\Lambda^+(u)}{\Lambda^-(u)} \qquad u \geqslant 0 \tag{3.53}$$

Consider the function

$$\Gamma(z) \equiv \frac{1}{2\pi i} \int_0^\infty \frac{\ln\left[\Lambda^+(u)/\Lambda^-(u)\right] du}{u - z} \tag{3.54}$$

$\ln(\Lambda^+/\Lambda^-)$ is that branch which vanishes as $u \longrightarrow \infty$. Then $e^{\Gamma(z)}$ satisfies (3.53), since

$$\frac{(e^\Gamma)^+}{(e^\Gamma)^-} = e^{\Gamma^+ - \Gamma^-} = e^{\ln[\Lambda^+/\Lambda^-]} = \frac{\Lambda^+}{\Lambda^-} \tag{3.55}$$

However, since $\ln[\Lambda^+(0)/\Lambda^-(0)] = 2\pi i$, we find

$$e^{\Gamma(z)} \sim \frac{1}{z} \tag{3.56}$$

in the neighborhood of $z = 0$. Thus an appropriate $X(z)$ is given by

$$X(z) = z\, e^{\Gamma(z)} \tag{3.57}$$

Combining eqs. (3.52) and (3.53), we find

$$X^+(u)\, a^+(u) = X^-(u)\, a^-(u) \tag{3.58}$$

which shows that $X(z)\ a(z)$ must be an entire function. But since

$$e^{\Gamma(\infty)} = 1\ , \quad a(\infty) = 0 \tag{3.59}$$

$X(z)\,\mathcal{A}(z)$ must be a bounded function. Thus

$$\mathcal{A}(z) = \frac{\sigma}{X(z)} \qquad \text{(Liouville's theorem)} \tag{3.60}$$

We may obtain the index of refraction from this general result simply by noting that

$$\frac{\dfrac{1}{2\pi i}\displaystyle\int_0^\infty \frac{\nu^3 A(\nu)\,d\nu}{(\nu - z)^2}}{\dfrac{1}{2\pi i}\displaystyle\int_0^\infty \frac{\nu^3 A(\nu)\,d\nu}{\nu - z}} = \frac{\dfrac{d}{dz}\left(\dfrac{1}{X(z)}\right)}{\left(\dfrac{1}{X(z)}\right)} = -\frac{X'(z)}{X(z)} \tag{3.61}$$

Thus

$$n = \frac{B_0}{E_0} = \frac{c\displaystyle\int_0^\infty \nu A(\nu)\,d\nu}{\displaystyle\int_0^\infty \nu^2 A(\nu)\,d\nu} = \frac{-c\,X'(0)}{X(0)} \tag{3.62}$$

This may be simplified by means of the identity[18]

$$X(z)X(-z) = \frac{1}{\epsilon}\Lambda(z) \tag{3.63}$$

which is most easily verified by noting that in view of (3.53), $R(z)$ defined by

$$R(z) = \frac{\Lambda(z)}{\epsilon\, X(z)X(-z)} \tag{3.64}$$

is an entire function. But

$$\lim_{|z|\to\infty} R(z) = \lim_{|z|\to\infty} \frac{\Lambda(z)}{-\epsilon z^2}\cdot e^{-\Gamma(z)}\cdot e^{-\Gamma(-z)} = 1\cdot 1\cdot 1 = 1 \tag{3.65}$$

Hence,

$$R(z) = 1 \qquad \text{(Liouville's theorem)} \tag{3.66}$$

It follows that

$$X(0) = \frac{-c}{\sqrt{\epsilon}} \tag{3.67}$$

(the phase is easy to verify). Thus

$$\eta = X'(0)\sqrt{\epsilon} \tag{3.68}$$

which is just the zero temperature result with the correction factor $X'(0)$. $X'(0)$ may be approximated through the use of the identity[18]

$$X(z) = X(0) + z + \frac{z}{2\pi i\epsilon}\int_0^\infty \frac{\Lambda^+(u)-\Lambda^-(u)}{u(u-z)X(-u)}\,du \tag{3.69}$$

To prove this second identity we make use of the fact that $[X(z) - z - X(0)]\, z^{-1}$ is analytic in the cut plane, vanishing as $|z| \to \infty$. Hence, by Cauchy's theorem:

$$\frac{X(z)-z-X(0)}{z} = \frac{1}{2\pi i}\oint \frac{X(z')-z'-X(0)}{z'(z'-z)}\,dz' = \frac{1}{2\pi i}\int_0^\infty \frac{X^+(u)-X^-(u)}{u(u-z)}\,du \tag{3.70}$$

But

$$X^+(u) - X^-(u) = \left(\frac{X^+(u)}{X^-(u)} - 1\right) X^-(u) = \left(\frac{\Lambda^+(u)}{\Lambda^-(u)} - 1\right) X^-(u)$$

$$= \left[\Lambda^+(u) - \Lambda^-(u)\right] \frac{X^-(u)}{\Lambda^-(u)} = \left[\Lambda^+(u) - \Lambda^-(u)\right] \frac{1}{\epsilon X(-u)} \tag{3.71}$$

from which the identity follows.

Using (3.69), we obtain the result

$$X'(0) = 1 + \frac{1}{2\pi i \epsilon} \int_0^\infty \frac{\Lambda^+(u) - \Lambda^-(u)}{u^2 X(-u)} du = 1 - \frac{1}{\epsilon} \frac{\omega_p^2}{\omega^2} \int_0^\infty \frac{u F(u) du}{X(-u)} \tag{3.72}$$

Since $F(u) \sim e^{-mu^2/2\Theta}$, it is reasonable to approximate $X(-u) = X(0)$ in the integrand. Thus

$$X'(0) \cong 1 - \frac{1}{2\epsilon X(0)} \frac{\omega_p^2}{\omega^2} \sqrt{\frac{2\Theta}{m}} = 1 + \frac{1}{2\sqrt{\pi}\epsilon} \frac{\omega_p^2}{\omega^2} \sqrt{\frac{2\Theta}{mc^2}} \tag{3.73}$$

to first order in $(2\Theta/mc^2)^{1/2}$. Higher order corrections may be obtained by iteration.

## 7. Digression on the Outgoing-Wave Boundary Condition

In treatment of half-space problems in this chapter, we have somewhat arbitrarily required that there shall be only "outgoing waves" (section 4) or that the frequency should be considered to contain a small positive imaginary part (section 5). The second prescription actually incorporates the first, and both are the results of attempts to correct the mistake made in posing the problem of a wave which has been in existence for an infinite length of time, rather than treating the actual build-up of the wave as an initial value problem.

By way of illustration we solve the initial value problem for the zero temperature plasma, a case where the results are well-known. The plasma fills the half-space $z > 0$ and for $t < 0$ the electromagnetic fields are confined to a finite region on the vacuum side. In general, for x-polarized disturbances

$$\begin{aligned} \frac{\partial}{\partial z} E(z,t) &= -\frac{1}{c}\frac{\partial}{\partial t} B(z,t) \\ \frac{\partial}{\partial z} B(z,t) &= -\frac{1}{c}\frac{\partial}{\partial t} D(z,t) \\ D(z,t) &= E(z,t) \qquad z < 0 \end{aligned} \tag{3.74}$$

Taking the Laplace transform with respect to time, we obtain

$$\begin{aligned} \frac{\partial E_p(z)}{\partial z} &= -\frac{1}{c}\left[pB_p(z) - B(z,0)\right] \\ \frac{\partial B_p(z)}{\partial z} &= -\frac{1}{c}\left[pD_p(z) - E(z,0)\right] \end{aligned} \tag{3.75}$$

We need, of course, the constitutive relation between D and E. As a rule one writes

$$D_\omega = \left(1 + \frac{\omega_p^2}{\omega^2}\right) E_\omega \tag{3.76}$$

The rationale for this is well known. If we consider a disturbance of time dependence $e^{-i\omega t}$, then the equation of motion for each electron,

$$\ddot{\vec{r}} = -\omega^2 \vec{r} = \frac{e\vec{E}}{m} \tag{3.77}$$

automatically yields as a solution for the polarization

$$\vec{P} \equiv ne\vec{r} = \frac{-ne^2}{m\omega^2}\vec{E} \tag{3.78}$$

and hence

$$\vec{D} = \vec{E} + 4\pi\vec{P} = \left(1 - \frac{\omega_p^2}{\omega^2}\right)\vec{E} \tag{3.79}$$

This, of course, assumes that the oscillation was initiated far enough in the past for the polarization to build up to its final value.

In the initial value approach we write

$$\vec{r}(t) = \frac{e}{m}\int_0^t d\tau \int_0^\tau \vec{E}(t')dt'$$

$$= \frac{e}{m}\int_0^t (t-t')\vec{E}(t')\,dt' \qquad \text{(Euler's identity)} \tag{3.80}$$

$$\vec{D}(t) = \vec{E}(t) + 4\pi n e \vec{r}(t)$$

$$= \vec{E}(t) + \omega_p^2 \int_0^t (t-t') E(t') dt' \tag{3.81}$$

and

$$D_p = \left(1 + \frac{\omega_p^2}{p^2}\right) E_p \tag{3.82}$$

where we have used the convolution formula to write

$$\int_0^\infty e^{-pt} dt \int_0^t (t-t') E(t') dt' = \frac{1}{p^2} E_p \tag{3.83}$$

Thus the expression (3.76) is valid if it is interpreted as the complex Fourier time transform of Eq. (3.81) where we have the usual correspondence $\omega = ip$. As an indication of what we can expect, we note that for $p$ on a typical Laplace inversion contour, (parallel to the imaginary axis in the right hand plane) $\omega$ will have a positive imaginary part.

In order to avoid loss of generally, we shall write instead of (3.82)

$$D_p = \varepsilon_p E_p \tag{3.84}$$

We now take the Fourier transform of (3.75) with respect to the spatial variable $z$.

$$-ik\tilde{E}_p(k) = -\frac{1}{c}\left\{p\tilde{B}_p(k) - \tilde{B}(k,0)\right\}$$

$$-ik\tilde{B}_p(k) = -\frac{1}{c}\left\{p\tilde{D}_p(k) - \tilde{E}(k,0)\right\} \tag{3.85}$$

where the Fourier transform is defined by

$$\tilde{E}_p(k) = \int_{-\infty}^{\infty} E_p(z)\, e^{ikz}\, dz \qquad \text{etc.} \tag{3.86}$$

Eliminating $B_p(k)$, we obtain

$$k^2\tilde{E}_p(k) + \frac{p^2}{c^2}\tilde{D}_p(k) = \frac{p}{c^2}\tilde{E}(k,0) + \frac{ik}{c}\tilde{B}(k,0) = \frac{i}{c}\left(k - \frac{ip}{c}\right)\tilde{E}(k,0) \tag{3.87}$$

where we have used

$$\tilde{B}(k,0) = \tilde{E}(k,0) \tag{3.88}$$

which represents the fact that the initial disturbance consists of waves traveling towards the interface with velocity $c$. Now make the separation

$$\tilde{E}_p(k) = \tilde{E}_p^+(k) + \tilde{E}_p^-(k) = \int_0^{\infty} E_p(z)e^{ikz}dz + \int_{-\infty}^{0} E_p(z)e^{ikz}dz$$

$$\tilde{D}_p(k) = \tilde{D}_p^+(k) + \tilde{D}_p^-(k) = \varepsilon_p\tilde{E}_p^+(k) + \tilde{E}_p^-(k) \tag{3.89}$$

This yields

$$\left(k^2 + \frac{p^2}{c^2}\right)\tilde{E}_p^-(k) + \left(k^2 + \frac{\varepsilon_p p^2}{c^2}\right)\tilde{E}_p^+(k) = \frac{i}{c}\left(k - \frac{ip}{c}\right)\tilde{E}^-(k,0) \tag{3.90}$$

Besides enabling us to express D conveniently in terms of E, the separation (3. 89) places our equation in a form suitable for treatment by standard complex variable techniques. The function $\tilde{E}_p^+(k)$ may be analytically continued into the upper half k-plane, and is, in fact, analytic throughout the upper half plane and vanishes as $k \rightarrow i\infty$. Similarly, $\tilde{E}_p^-(k)$ is the boundary value, as k approaches the real axis from below, of a function analytic throughout the lower half k-plane, vanishing as $k \rightarrow -i\infty$. Since the initial disturbance is confined to the region $z < 0$, $\tilde{E}(k, t=0)$ is a boundary value in the same sense as $\tilde{E}_p^-(k)$, and the (-) superscript on the right hand side of Eq. (3. 90) serves to remind us of that fact.

Equation (3. 90) is of the form

$$\Lambda^+(k)\tilde{E}_p^+(k) + \Lambda^-(k)\tilde{E}_p^-(k) = G(k) \tag{3.91}$$

which may be solved by the same technique used in section 8. We must factor the function

$$\frac{\Lambda^+(k)}{\Lambda^-(k)} = \frac{k^2 + \frac{\epsilon_p p^2}{c^2}}{k^2 + \frac{p^2}{c^2}} = \frac{X^+(k)}{-X^-(k)} \tag{3.92}$$

where $X^{\pm}(k)$ is the boundary value of a function analytic and nonvanishing in the $\left[\begin{smallmatrix}\text{upper}\\ \text{lower}\end{smallmatrix}\right]$ half plane. In this case, the result is obtained by inspection:

$$\frac{k^2 + \frac{\epsilon_p p^2}{c^2}}{k^2 + \frac{p^2}{c^2}} = \frac{\left(k + \frac{ip\sqrt{\epsilon_p}}{c}\right) / \left(k + \frac{ip}{c}\right)}{\left(k - \frac{ip}{c}\right) / \left(k - \frac{ip\sqrt{\epsilon_p}}{c}\right)} = \frac{X^+(k)}{-X^-(k)} \tag{3.93}$$

(It is assumed that we may define the square root in such a manner that $(p\,\epsilon_p)^{1/2}$ lies in the right half plane whenever p lies in some appropriately chosen half plane Re $p > p_o$.) Then

$$X^+(k)\tilde{E}_p^+(k) - X^-(k)\tilde{E}_p^-(k) = \frac{-X^-(k)}{\Lambda^-(k)}G(k) = \frac{i}{c}\frac{(k-\frac{ip}{c})\,\tilde{E}^-(k,0)}{(k+\frac{ip}{c})(k-\frac{ip\sqrt{\epsilon_p}}{c})} \tag{3.94}$$

from which

$$X^\pm(k)\tilde{E}_p^\pm(k) = \lim_{\alpha\to 0+}\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{i}{c}\frac{(k'-\frac{ip}{c})\;\tilde{E}^-(k',0)\,dk'}{(k'+\frac{ip}{c})(k'-\frac{ip\sqrt{\epsilon_p}}{c})(k'-k\mp i\alpha)} \tag{3.95}$$

or

$$\tilde{E}_p^+(k) = \frac{i}{c}\,\frac{2}{1+\sqrt{\epsilon_p}}\;\frac{1}{k+\frac{ip\sqrt{\epsilon_p}}{c}}\;\tilde{E}^-(-\tfrac{ip}{c},0)$$

$$\tilde{E}_p^-(k) = -\frac{i}{c}\frac{2}{1+\sqrt{\epsilon_p}}\frac{k-\frac{ip\sqrt{\epsilon_p}}{c}}{(k+\frac{ip}{c})(k-\frac{ip}{c})}\tilde{E}^-(-\tfrac{ip}{c},0) + \frac{i}{c}\frac{1}{k+\frac{ip}{c}}\tilde{E}^-(k,0) \tag{3.96}$$

Inversion of the Fourier-Laplace transform is straight-forward:

$$E(z,t) = \begin{cases} \dfrac{1}{2\pi}\dfrac{1}{c}\displaystyle\int_{-\infty}^{\infty}\frac{2}{1+\sqrt{\epsilon(\omega+)}}\tilde{E}^-(-\tfrac{\omega}{c},0)\,e^{i\omega_+\sqrt{\epsilon(\omega_+)}\frac{z}{c}}e^{-i\omega_+ t}d\omega & (z\geq 0) \\[2ex] \dfrac{1}{2\pi}\dfrac{1}{c}\displaystyle\int_{-\infty}^{\infty}\tilde{E}^-(-\tfrac{\omega}{c},0)\,e^{i\frac{\omega}{c}z}e^{-i\omega t}d\omega \\ \quad + \dfrac{1}{2\pi}\dfrac{1}{c}\displaystyle\int_{-\infty}^{\infty}\frac{1-\sqrt{\epsilon(\omega_+)}}{1+\sqrt{\epsilon(\omega_+)}}\tilde{E}^-(-\tfrac{\omega}{c},0)\,e^{-i\omega\frac{z}{c}}e^{-i\omega t}d\omega & (z\leq 0) \end{cases} \tag{3.97}$$

where

$$\epsilon(\omega) = \epsilon_{-i\omega} \quad \text{(the dielectric coefficient)} \tag{3.98}$$

and where we have written $\omega_+$ throughout to indicate that whenever $\omega$ appears in the integrand, it should be thought of as having a small imaginary part which is allowed to tend to zero after the integration is performed.

We interpret the different terms in $E(z,t)$ as superpositions of transmitted, incident and reflected waves respectively. Having done this we see that the transmitted and reflected electric field amplitudes for waves of a given frequency are given in terms of the amplitude of the incident field, viz:

$$
\begin{aligned}
E_t &= \frac{2}{1+\sqrt{\epsilon(\omega_+)}} E_i = \frac{2}{1+n(\omega)} E_i \\
E_r &= \frac{1-\sqrt{\epsilon(\omega_+)}}{1+\sqrt{\epsilon(\omega_+)}} E_i = \frac{1-n(\omega)}{1+n(\omega)} E_i
\end{aligned}
\tag{3.99}
$$

($n(\omega)$ is the index of refraction)

Note that the solution to the initial value problem automatically satisfies the condition that the electric field inside the plasma does not contain incoming waves. This condition is usually introduced ad hoc.

We have shown that the initial value problem has a unique solution, without the imposition of arbitrary subsidiary conditions. In fact, we have constructed the solution. This, therefore, is the "correct" way to solve the problem, or rather, it is the correct way to pose the problem. What then is meant by a solution to the artificial problem, which specifies steady-state time dependence $e^{-i\omega t}$? Nothing more than the interpretation of the above result as a superposition of steady-state solutions.

Working backwards, we will agree that we have a correct solution for the reflected and transmitted amplitudes $E_r(\omega)$, $E_t(\omega)$, if a superposition of such solutions yields the correct solution to the well-posed initial value problem. This, of course, is a general criterion which should be applicable to the case where the medium fills a slab rather than a half-space, or even if the medium does not satisfy phenomenological equations of the form (3.74).

The above result (3.97) suggests the assertion that in considering more general problems from the steady-state (time dependence $e^{-i\omega t}$) point of view, all ambiguities will be correctly resolved by assuming that $\omega$ has a small positive imaginary part and requiring that the fields vanish as $z \to \infty$. Of course, in any particular practical case, one would use this prescription to solve for $E_r(\omega)$ and $E_t(\omega)$, and then justify it a forteriori by examining the solutions so obtained in the light of the corresponding initial value problem.

The preceeding remarks may be seen in a more general light by understanding that causality lies at the root of all these considerations. Suppose we restrict our attention to the vacuum side of the interface, without any restriction on what lies beyond, except that it be a linear system. Then, quite generally

$$E(z,t) = \int_{-\infty}^{\infty} E_i(\omega)\, e^{i\frac{\omega}{c}z} e^{-i\omega t} d\omega + \int_{-\infty}^{\infty} E_r(\omega) e^{-i\frac{\omega}{c}z} e^{-i\omega t} d\omega \qquad z \leq 0 \tag{3.100}$$

$$= F_1(z-ct) + F_2(z+ct)$$

Mathematically, the$\left[\begin{smallmatrix}\text{first}\\ \text{second}\end{smallmatrix}\right]$ term represents a disturbance which is traveling to the $\left[\begin{smallmatrix}\text{right}\\ \text{left}\end{smallmatrix}\right]$ without change in spatial form. The condition that the initial disturbance consists entirely of waves propagation towards the interface from the left and is localized to the left of the interface requires that at $t = 0$ the function $F_1 (z - c\ t)$ is non-vanishing only for $z < 0$, and the function $F_2 (z + c\ t)$ is non-vanishing only for $z > 0$. (Of course, the expression (3.10) is valid only for $z < 0$.) Hence, $E_i(\omega)$ and $E_r(\omega)$ are boundary values of functions analytic in the upper half plane.

As before, we shall interpret the functions $F_1$ and $F_2$ as incident and reflected fields, the reflected fields being caused by the response of the medium behind the interface to the incident field. Causality then, is simply the requirement that there can be no reflected field until the incident field distrubance has reached the interface. Moreover, in this interpretation, since the medium is linear, the reflected field $F_2$ must depen d linearly upon the incident field $F_2$ which causes it. Thus

$$E_r(\omega) = \mathcal{R}(\omega) E_i(\omega) \tag{3.101}$$

where $\mathcal{R}(\omega)$ the "response function" depends only upon the properties of the medium, and is the boundary value of a function analytic in the upper half plane.

All this serves as a justification for including a small imaginary part in $\omega$ when solving the steady-state problem--time dependence $e^{-i\omega t}$.

Since all field amplitudes are boundary values of analytic functions, we may seek steady-state solutions with $\omega$ in the upper half plane, with the assurance that the resulting $\mathcal{R}(\omega)$ will be the analytic continuation of the desired response function. In other words, we insure that the field amplitudes will possess the required analytically properties by working in a region of the $\omega$ plane in which they are known to be analytic.

## 8. Propagation Along a Constant Magnetic Field

If the plasma is located in a steady magnetic field (not assumed "small"), then the Vlasov equation takes on an added degree of complexity--in particular, it becomes a differential equation in the velocity variable.

The general treatment may be approached through the "Fessel" transformation,[19] which transforms away the magnetic field term at the start. However, in the special case of wave propagation along the steady-state magnetic field, the problem becomes almost identical to the case of zero field. We sketch the results here.

In the linearization of (1.34) we require

$$\begin{aligned} \mathcal{F} &= n\mathcal{F}_0(\vec{u}) + f(\vec{r},\vec{u},t) \quad \text{"small"} \\ \vec{E} &= \vec{E}(\vec{r},t) \quad \text{"small"} \end{aligned} \tag{3.102}$$

as before. On the other hand, now:

$$\vec{B} = B_0\hat{k} + B(\vec{r},t) \quad \text{"small"} \tag{3.103}$$

where $\hat{k}$ is a unit vector in the z-direction. Note that if $F_o(\vec{u})$ is a function of energy alone, it will be independent of $B_o$ since there is no magnetic potential energy. Hence, if $F_o$ is isotropic in the zero field case, it will also be isotropic in this case. The linearized equations are then

$$\frac{\partial f(\vec{r},\vec{u},t)}{\partial t} + \vec{u}\cdot\vec{\nabla} f + \frac{eB_o}{mc}(\hat{u}\times\hat{k})\cdot\vec{\nabla}_u f = -\frac{ne}{m}\vec{E}\cdot\vec{\nabla}_u \mathcal{F}_o$$

$$\vec{\nabla}\times\vec{E}(\vec{r},t) = -\frac{1}{c}\frac{\partial\vec{B}(\vec{r},t)}{\partial t} \qquad \vec{\nabla}\cdot\vec{E} = 4\pi e\int f\,d^3\vec{u} \tag{3.104}$$

$$\vec{\nabla}\times\vec{B} = \frac{1}{c}\frac{\partial\vec{E}}{\partial t} + \frac{4\pi e}{c}\int\vec{u}\,f\,d^3\vec{u} \qquad \vec{\nabla}\cdot\vec{B} = 0$$

We have used Eq. (1.41) to simplify. Otherwise, except for the $B_o$ term, these equations are identical with Eqs. (1.39). Because of the presence of the $B_o$ term, the Vlasov equation does not separate into plane polarized modes. The equations corresponding to (1.47x) and (1.47y) are

$$\frac{\partial g_x(z,u,t)}{\partial t} + u\frac{\partial g_x}{\partial z} + \Omega\, g_y(z,u,t) = \frac{ne}{m}E_x(z,t)F(u) \tag{3.105x}$$

$$\frac{\partial g_y}{\partial t} + u\frac{\partial g_y}{\partial z} - \Omega\, g_x = \frac{ne}{m}E_y(z,t)F(u) \tag{3.105y}$$

where we have introduced $\Omega = -eB_o/mc$ (the cyclotron frequency). However, these do separate into right and left circularly polarized modes: writing

$$E_{\pm} = E_x \pm i E_y$$

$$B_{\pm} = B_x \pm i E_y \tag{3.106}$$

$$g_{\pm} = g_x \pm i g_y$$

we get

right circularly polarized modes

$$\begin{cases} \frac{\partial}{\partial t} g_+ + u \frac{\partial}{\partial z} g_+ - i\Omega g_+ = \frac{ne}{m} E_+ F(u) & (3.107+) \\ \frac{\partial}{\partial z} E_+ = \frac{i}{c} \frac{\partial}{\partial t} B_+ \\ i \frac{\partial}{\partial z} B_+ = \frac{1}{c} \frac{\partial}{\partial t} E_+ + \frac{4\pi e}{c} \int g_+ du \end{cases}$$

left circularly polarized modes

$$\begin{cases} \frac{\partial}{\partial t} g_- + u \frac{\partial}{\partial z} g_- + i\Omega g_- = \frac{ne}{m} E_- F(u) & (3.107-) \\ \frac{\partial}{\partial z} E_- = -\frac{i}{c} \frac{\partial B_-}{\partial t} \\ -i \frac{\partial}{\partial z} B_- = \frac{1}{c} \frac{\partial E_-}{\partial t} + \frac{4\pi e}{c} \int g_- du \end{cases}$$

The plasma oscillations are the same as in the zero field case, since electrons undergoing longitudinal vibrations ( $\|\vec{B}_o$ ) do not experience any $\vec{u} \times \vec{B}_o$ force.

The normal modes may be exhibited by reproducing the zero field treatment of chapter II, in fact, the remaining equations in this section will be seen to be almost identical with corresponding equations in chapter II.

Fourier analysis in the time variable yields (the z - dependence and the ± subscript is suppressed)

$$\begin{pmatrix} u & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \frac{\partial}{\partial z} \begin{pmatrix} g(u) \\ E \\ B \end{pmatrix} = \begin{pmatrix} i(\omega \pm \Omega) & \frac{ne}{m} F(u) & 0 \\ 0 & 0 & \pm \frac{\omega}{c} \\ \mp \frac{4\pi e i}{c} \int & \mp \frac{\omega}{c} & 0 \end{pmatrix} \begin{pmatrix} g(u) \\ E \\ B \end{pmatrix} \tag{3.108}$$

or:

$$\rho \frac{\partial}{\partial z} \Psi = H \Psi \tag{3.109}$$

The corresponding adjoint equation is:

$$\frac{\partial}{\partial z} \begin{pmatrix} g^{\dagger}(u) & E^{\dagger} & B^{\dagger} \end{pmatrix} \begin{pmatrix} u & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} g^{\dagger}(u) & E^{\dagger} & B^{\dagger} \end{pmatrix} \begin{pmatrix} i(\omega \pm \Omega) & \frac{ne}{m} \int F & 0 \\ 0 & 0 & \pm \frac{\omega}{c} \\ \mp \frac{4\pi e i}{c} & \mp \frac{\omega}{c} & 0 \end{pmatrix} \tag{3.110}$$

or:

$$\frac{\partial \Psi^{\dagger}}{\partial z} \rho = \Psi^{\dagger} H^{\dagger} \tag{3.111}$$

The solutions of (3.108) and (3.110) with space behavior of the form $e^{i\,\omega/\nu\, z}$ will satisfy the orthogonality relation (2.15). In this case, discrete solutions will exist whenever $\nu_j$ is a root of

$$\Lambda(\nu) \equiv c^2 - \nu^2 - \frac{\omega_p^2}{\omega^2} \nu^3 \int_{-\infty}^{\infty} \frac{F(u)\,du}{u - \alpha\nu} = 0 \tag{3.112}$$

where

$$\alpha = 1 \pm \frac{\Omega}{\omega} \tag{3.113}$$

The continuum includes the entire real axis. The solutions to (3.108) are

$$B_\nu = \mp \frac{4\pi e \nu c}{\omega}$$
$$E_\nu = \frac{4\pi i e \nu^2}{\omega} \tag{3.114}$$
$$g_\nu(u) = \frac{\omega_p^2}{\omega} \nu^3 \frac{F(u)}{u - \alpha\nu}$$

for the discrete modes. For the continuum modes

$$g_\nu(u) = \frac{\omega_p^2}{\omega^2} \nu^3 P \frac{F(u)}{u-\alpha\nu} + \lambda(\nu)\delta(u-\alpha\nu)$$
$$\lambda(\nu) = \frac{1}{2}\left[\Lambda^+(\nu) + \Lambda^-(\nu)\right] \tag{3.115}$$

The adjoint solutions are

$$B_\nu^\dagger = \mp \frac{n e \nu^2 c}{m\omega}$$
$$E_\nu^\dagger = \frac{n e \nu c^2}{m i \omega} \tag{3.116}$$
$$g_\nu^\dagger(u) = \frac{\omega_p^2}{\omega^2} \nu^3 \frac{1}{u-\alpha\nu}$$

for the discrete modes, with

$$g_\nu^\dagger(u) = \frac{\omega_p^2}{\omega^2} \nu^3 P \frac{1}{u-\alpha\nu} + \tilde{\lambda}(\nu)\delta(u-\alpha\nu)$$
$$\tilde{\lambda}(\nu) F(\alpha\nu) = \lambda(\nu) \tag{3.117}$$

for the continium modes. By direct calculation, we find

$$N(\nu) = \frac{\alpha}{|\alpha|}\nu \frac{\lambda^2(\nu) + \pi^2\left(\frac{\omega_p^2}{\omega^2}\nu^3\right)^2 F^2(\alpha\nu)}{F(\alpha\nu)} \tag{3.118}$$

which may be written in the form (cf. (2.85))

$$\frac{1}{N(\nu)} = \frac{1}{2\pi i \frac{\omega_p^2}{\omega^2}\nu^4}\left\{\frac{1}{\Lambda^+(\nu)} - \frac{1}{\Lambda^-(\nu)}\right\} \tag{3.119}$$

The discrete spectrum also is modified by the presence of the magnetic field. Consider first the "+" or "ordinary" modes.

$$\Lambda(\nu) = c^2 - \nu^2 - \frac{\omega_p^2}{\omega^2}\nu^3 \int_{-\infty}^{\infty} \frac{F(u)\,du}{u - \nu(1+\frac{\Omega}{\omega})} \tag{3.120}$$

At zero temperature ( $F(u) = \delta(u)$ ) we find

$$\nu^2\left[1 - \frac{\omega_p^2}{\omega(\omega+\Omega)}\right] = c^2 \tag{3.121}$$

which yields

a) no complex roots: two real roots, when $\omega > (\omega_p^2 + (\Omega/2)^2)^{1/2} - \Omega/2$

b) two complex roots at $\nu = \pm i\,\nu_{0+}$ when $\omega < (\omega_p^2 + (\Omega/2)^2)^{1/2} \quad \Omega/2$

with

$$\nu_{0+} = \left(\frac{\omega_p^2}{\omega(\omega+\Omega)} - 1\right)^{-\frac{1}{2}} c \tag{3.122}$$

At finite temperatures, the situation is similar. There will be $\left\{\begin{matrix}0\\2\end{matrix}\right\}$ discrete modes when $\omega \gtrless (\omega_p^2 + (\Omega/2)^2)^{1/2} - \Omega/2$. The real roots have wandered off the real axis to the other sheet of the double-valued function $\Lambda(\nu)$. In other words, if we consider the expression (3.120) for $\Lambda(\nu)$ with $\nu$ in the upper half plane, we will find no zeroes, but if we analytically continue it into the lower half plane, we find (among others) two zeroes which, at low temperatures at any rate, are near the real axis. Of course, these zeroes do not represent actual discrete modes because the expression (3.120) for $\Lambda(\nu)$ does not represent its analytic continuation across the real axis. This is no different from the case when there is no magnetic field.

The "-" or "extraordinary" modes are somewhat different. At zero temperature

$$\nu^2\left(1 - \frac{\omega_p^2}{\omega(\omega-\Omega)}\right) = c^2 \tag{3.123}$$

I.e.

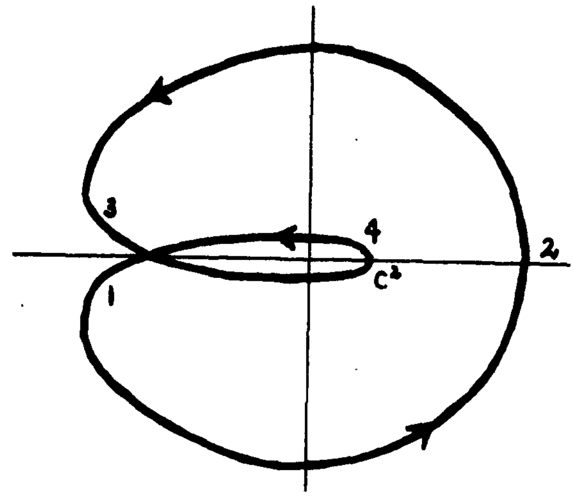
a) no complex roots: two real roots, when $\omega > (\omega_p^2 + (\Omega/2)^2)^{1/2} + \Omega/2$

b) two complex roots at $\nu = \pm i\, \nu_{0-}$ when $\Omega < \omega < (\omega_p^2 + (\Omega/2)^2)^{1/2} + \Omega/2$ with

$$\nu_{0-} = \left(\frac{\omega_p^2}{\omega(\omega-\Omega)} - 1\right)^{-1/2} c \tag{3.124}$$

c) two real roots when $\omega < \Omega$

In cases a) and b), the situation is the same as for the ordinary mode. There will be 0 and 2 discrete modes for cases a) and b) respectively. In case c), something new develops. The real zeroes of the zero-temperature $\Lambda(\nu)$ become complex at finite temperatures and give rise to four complex roots, two each in the upper and lower half planes, of the dispersion relation $\Lambda(\nu) = 0$ with corresponding discrete modes. In fact, if we were to apply the argument principle to obtain the number of roots in (say) the upper half plane, we would find that the curve corresponding to figure 2, p. 44, would look like




which encircles the origin twice, so $\Delta \arg \Lambda = 4\pi$. The discrete roots are symmetrically located at $\pm \nu_e$, $\pm \nu_e^*$, and at low temperatures, near the zero-temperature values

$$\pm \nu_{e0} = \pm \left(1 + \frac{\omega_p^2}{\omega(\omega+\Omega)}\right)^{-1/2} c \tag{3.125}$$

We find

$$\nu_e \cong \nu_{e0} + i\gamma \qquad |\gamma| \ll \nu_{e0} \tag{3.126}$$

with

$$\gamma \cong \frac{\sqrt{\pi}}{2} \frac{\omega_p^2}{\omega^2} \frac{\nu_{e0}}{1 + \frac{\omega_p^2}{\omega(\omega+\Omega)}} x e^{-x^2} \tag{3.127}$$

$$x = \left[\frac{2\Theta}{mc^2}\left(1 + \frac{\omega_p^2}{\omega^2}\right)\right]^{-1/2} \tag{3.128}$$

The presence of these discrete modes has led to some confusion in the literature.[20] For example, in the transmission of electromagnetic radiation through a half-space (in the extraordinary mode with $\omega < \Omega$), the spatial dependence of the amplitude far from the boundary will be given by the slightly damped discrete modes. In the paper cited above, this behavior is incorrectly attributed to Landau damping, a phenomenon discussed in the next chapter.

# Chapter IV. OTHER APPLICATIONS

## 1. Initial Value Problems

In the initial value problem, the boundary values specified are the values of all field quantities for all points in space at a fixed initial time, $t_o$. It is desired to determine the complete behavior of the system for later times. There is, then, a basic difference between the initial value problem and boundary value problems of the kind considered previously, where the data was specified for all times at a finite number of space points. The nature of the initial value problem boundary conditions requires that the problem be Fourier analyzed in the space variable rather than in the time variable as previously (cf eq. (2.1)). Also, as a rule, in the initial value problem, the plasma is taken to be an infinite medium.* Nevertheless, in terms of the resulting equations to be solved, these differences are purely formal. Actually, the singular eigenfunction method we have been using has been applied more often to the initial value problem than to other types of boundary value problems. In fact, the initial value problem was the subject of van Kampen's paper[21] in which singular normal modes were introduced.

## 2. The Initial Value for Transverse Modes

This is by far the most often treated problem in plasma oscillations. The discussion in this section is included as background material, and contains

*Montgomery and Gorman[22] have shown that in certain cases, slab geometry may be treated by using the method of images to convert the slab problem into an infinite medium problem.

only an outline of the results in the quoted papers.

Consider the spatial Fourier decomposition of eq. (1.47a), i.e. let

$$g_z(z,u,t) = u\,g(u,t)\,e^{ikz}$$
$$E_z(z,t) = E(t)\,e^{ikz} \tag{4.1}$$

Then

$$\frac{\partial}{\partial t} g(u,t) + iku g = -\frac{ne}{m} E(t) F'(u) \tag{4.2}$$

and

$$ikE = 4\pi\rho = 4\pi e \int_{-\infty}^{\infty} g\, du \tag{4.3}$$

The coupled equations (4.2, 3) were first solved by Landau[16] using the Laplace transform. The result for the Laplace transform of the electric field is

$$\mathcal{L}\left[E(t)\right] \equiv \int_0^{\infty} e^{-pt} E(t)dt = \frac{\dfrac{m}{ne}\displaystyle\int_{-\infty}^{\infty} \frac{g(u,o)du}{u + p/ik}}{1 - \dfrac{\omega_p^2}{k^2}\displaystyle\int_{-\infty}^{\infty} \frac{F'(u)du}{u+p/ik}} \tag{4.4}$$

Van Kampen[21] utilized the normal mode approach. Under the <u>ansatz</u> that the solutions have time dependence $e^{-ik\nu t}$ eqs. (4.2, 3) reduce to

$$(u-\nu)\,g_\nu(u) = \frac{\omega_p^2}{k^2} F'(u) \int_{-\infty}^{\infty} g_\nu(u)du$$
$$E_\nu = \frac{4\pi e}{ik} \int_{-\infty}^{\infty} g_\nu(u)du \tag{4.5}$$

The normal modes are then

$$g_\nu(u) = \frac{\omega_p^2}{k^2} P \frac{F'(u)}{u-\nu} + \lambda(\nu)\delta(u-\nu) \quad \text{(all real } \nu) \tag{4.6}$$

with

$$\lambda(\nu) = \frac{1}{2}\left(\Lambda^+(\nu) + \Lambda^-(\nu)\right) = 1 - \frac{\omega_p^2}{k^2} P \int_{-\infty}^{\infty} \frac{F'(u)du}{u-\nu} \tag{4.7}$$

Discrete modes would also exist for those $\nu$ which are roots of

$$\Lambda(\nu) \equiv 1 - \frac{\omega_p^2}{k^2} \int_{-\infty}^{\infty} \frac{F'(u)du}{u-\nu} = 0 \tag{4.8}$$

but for the case when $\mathcal{F}_o(\vec{u})$ is Maxwellian, the discrete spectrum is empty. Note that the denominator in eq. (4.4) is just $\Lambda(\frac{ip}{k})$.

Van Kampen proved that the $g_\nu(u)$ formed a complete set, by means of the singular integral equation technique discussed in chapter II. He proved completeness by constructing the solution to the integral equation implied by

$$g(u,o) = \int_{-\infty}^{\infty} A(\nu) g_\nu(u) du \tag{4.9}$$

where $g(u,o)$ is an arbitrary function, (the initial perturbation in this instance). The solution is

$$A(\nu) = \frac{G^+(\nu)}{\Lambda^+(\nu)} - \frac{G^-(\nu)}{\Lambda^-(\nu)} \tag{4.10}$$

where

$$G(z) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{g(u,o)du}{u-z} \tag{4.11}$$

The solution to the initial value problem is then simply

$$E(t) = \frac{4\pi e}{ik}\int_{-\infty}^{\infty} A(\nu) e^{-ik\nu t} d\nu \tag{4.12}$$

Case[14] placed the normal mode treatment on a more formal basis. He showed that the $g_\nu(u)$ are orthogonal to the solutions of the adjoint equation:

$$(u-\nu) g_\nu^\dagger(u) = \frac{\omega_p^2}{k^2}\int_{-\infty}^{\infty} F'(u) g_\nu^\dagger(u) du \tag{4.13}$$

That is

$$\int_{-\infty}^{\infty} g_{\nu'}^\dagger(u) g_\nu(u) du = N(\nu)\delta(\nu-\nu') \tag{4.14}$$

The adjoint solutions are

$$g_\nu^\dagger(u) = \frac{\omega_p^2}{k^2} P\frac{1}{u-\nu} + \tilde{\lambda}(\nu)\delta(u-\nu) \tag{4.15}$$

with

$$\tilde{\lambda}(\nu) = \frac{\lambda(\nu)}{F'(\nu)} \tag{4.16}$$

Then an arbitrary initial perturbation $g(u,o)$ may be expanded in terms of the $g_\nu(u)$ with the expansion coefficient given by

$$A(\nu) = \frac{1}{N(\nu)}\int_{-\infty}^{\infty} g_\nu^\dagger(u) g(u,o) du \tag{4.17}$$

The normalization coefficient is found to be

$$N(\nu) = \frac{\left[\pi \frac{\omega_p^2}{k^2} F'(\nu)\right]^2 + \lambda^2(\nu)}{F'(\nu)} \tag{4.18}$$

which may be written

$$\frac{1}{N(\nu)} = \frac{1}{2\pi i \frac{\omega_p^2}{k^2}} \left\{ \frac{1}{\Lambda^+(\nu)} - \frac{1}{\Lambda^-(\nu)} \right\} \tag{4.19}$$

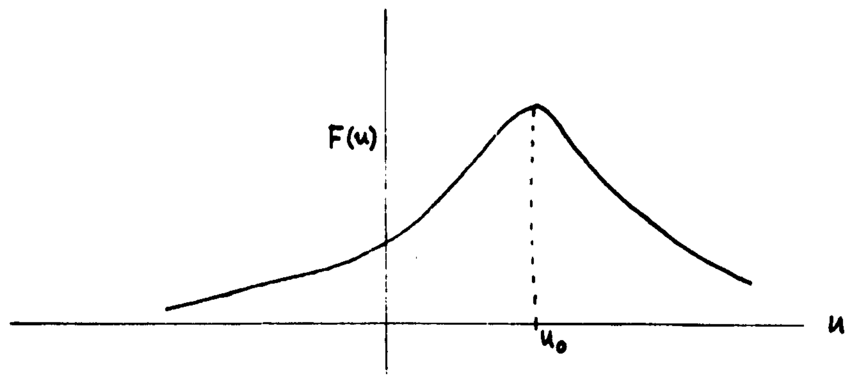
Case also pointed out that in view of the Plemelj formulae, these apparently different approaches to the initial value problem are actually equivalent. That is, it follows almost by inspection, that the normal mode expansion (4.12) for $E(t)$ $t > 0$ and the Laplace inverse of (4.4) are actually the same contour integral taken along two equivalent paths.

This last result is quite general, and is valid even when the steady-state distribution function $\mathcal{F}_o(\vec{u})$ gives rise to discrete modes.

## 3. Stability and Landau Damping

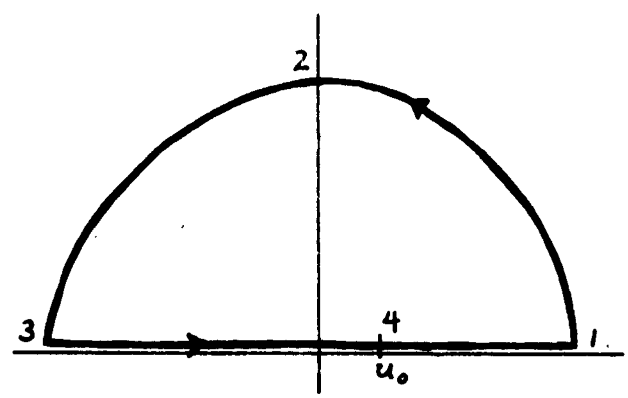
It is clear that if there is a complex root $\nu_o$ of the secular equation (4.8), the system will be unstable, since (at least for some initial perturbations) the electric field will contain a term $e^{-ik\nu_o t}$ which grows exponentially either for $k$ or $-k$. It has been pointed out that for Maxwellian $\mathcal{F}_o(\vec{u})$ there are no discrete roots. In fact, it is simple to show that the discrete spectrum will be empty if $F(u)$ represents any "single-humped" distribution. I.e.

$$F'(u) \lesseqgtr 0 \qquad u \gtrless u_o \tag{4.20}$$




Single-humped Distribution

We prove this by applying the argument principle to $\Lambda(z)$ as $z$ traverses the contour C




The contour C

Along the large semicircle

$$\Lambda(z) \approx 1 \tag{4.21}$$

Along the real axis

$$\Lambda^+(x) = 1 - \frac{\omega_p^2}{k^2} P \int_{-\infty}^{\infty} \frac{F'(u)du}{u - x} - \pi i \frac{\omega_p^2}{k^2} F'(x) \tag{4.22}$$

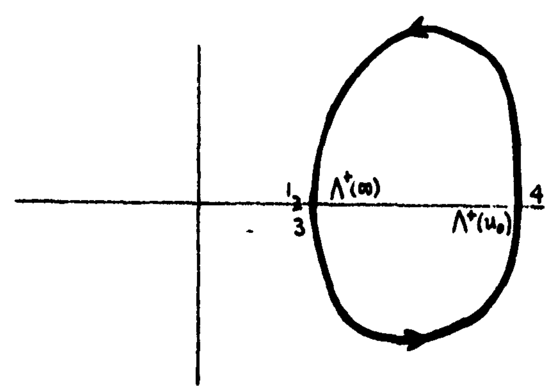
and since F (u) is single-humped

$$\mathcal{Im}\, \Lambda^+(x) \gtrless 0 \qquad x \gtrless u_0 \tag{4.23}$$

Thus the image of the path $\begin{bmatrix} 3 \rightarrow 4 \\ 4 \rightarrow 1 \end{bmatrix}$ lies entirely $\begin{bmatrix}\text{below}\\ \text{above}\end{bmatrix}$ the real axis. The image of the path 3 - 4 - 1 will then cross the real axis only once, at the point $\Lambda^+(u_o)$. But

$$\Lambda^+(u_0) = 1 - \frac{\omega_p^2}{k^2} \int \frac{F'(u)du}{u - u_0} > 1 \tag{4.24}$$

since $\frac{F'(u)}{u - u_o}$ is negative definite. Thus the image of the contour C looks something like




which does not encircle the origin, and there are no discrete roots.

When there are no discrete modes, the system will be stable. This follows from the normal mode representation (4.12) of the electric field. For any g (u, o), the function G (z) will be analytic in the cut plane, and if (as we will assume) g (u, o) is reasonably well behaved, the boundary values $G^{\pm}(\nu)$ will be non-singular. Thus, for $t > 0$

$$E(t) = \frac{4\pi e}{ik}\int_{-\infty}^{\infty} \frac{G^{+}(\nu)}{\Lambda^{+}(\nu)}\, e^{-ik\nu t}\, d\nu \tag{4.25}$$

(the term involving $\frac{G^{-}(\nu)}{\Lambda^{-}(\nu)}$ gives no contribution, since $\frac{G^{-}}{\Lambda^{-}}$ is the boundary value of a function analytic throughout the lower half plane).

The asymptotic behavior ($t \to \infty$) of E (t) is governed by the singularity nearest the real axis of the analytic continuation of $\frac{G^{+}}{\Lambda^{+}}$ into the lower half plane. Landau showed that if $\mathcal{F}_0(\vec{u})$ is Maxwellian, the integral expression (4.8) with $\mathfrak{Im}\, z > 0$ when continued into the lower half plane, has a zero (for small k) given by

$$\omega_o = \omega_{or} + i\,\omega_{oi} \tag{4.26}$$

where

$$\omega_p \omega_{or} \cong \omega_p^2 + 3k^2 \frac{\Theta}{m} \tag{4.27}$$

and

$$\omega_{oi} \cong -\, \omega_p^4 \sqrt{\frac{\pi}{8}} \left(\frac{k^2 \Theta}{m}\right)^{-3/2} e^{-\frac{\omega_p^2 m}{2k^2\Theta}}$$

Jackson[23] has shown that a slightly better approximation is given by

$$\omega_{or}^2 \cong \omega_p^2 + 3k^2 \frac{\Theta}{m}$$

$$\omega_{oi} \cong -\omega_p^4 \sqrt{\frac{\pi}{8}} \left(\frac{k^2 \Theta}{m}\right)^{-3/2} e^{-\frac{\omega_{or}^2 m}{2k^2 \Theta}} \tag{4.28}$$

At any rate, a non-singular initial disturbance gives rise to an electric field which after long times undergoes damped oscillations $\sim e^{-i\omega_{or}t}\, e^{\omega_{oi}t}$. This "Landau" damping may (in keeping with the van Kampen singular normal mode picture) be thought of as the result of the phase-mixing of an initial disturbance composed of oscillations which propagate with a continuous spectrum of velocities.

## 4. Landau Damping of Transverse Modes

The initial value problem for transverse modes may be treated in the same manner. The discrete modes of the system would have space-time behavior $\sim e^{ikz}\, e^{-ik\nu_j t}$, where the $\nu_j$ are the zeroes of the dispersion function $\Lambda(\nu)$. In this case, however,

$$\Lambda(\nu) = c^2 - \nu^2 - \frac{\omega_p^2}{k^2}\nu \int_{-\infty}^{\infty} \frac{F(u)\,du}{u-\nu} \tag{4.29}$$

with fixed, real $k$ and complex $\nu = \omega/k$, and it is easy to show that there are no discrete modes. Hence, for positive [negative] $k$ the long-time behavior of the system is of the form $\sim e^{ikz}\, e^{-ik\nu_o t}$, where $\nu_o$ is the zero lying nearest the real axis of the analytic continuation of $\Lambda^+(\nu)$ $[\Lambda^-(\nu)]$ into the lower [upper] half plane.

We note that the analytic continuation of (4.29) from the upper half plane into the lower half plane is given by

$$\Lambda_{cont}(\nu) = c^2 - \nu^2 - \frac{\omega_p^2}{k^2}\nu\int_{-\infty}^{\infty}\frac{F(u)du}{u-\nu} - \frac{\omega_p^2}{k^2}2\pi i\,\nu F(\nu) \tag{4.30}$$

Then, if we assume that $k\,\nu_0$ lies near to its zero-temperature value of $(\omega_p^2 + k^2c^2)^{1/2}$, we may compute $k\,\nu_0$ by means of successive approximations. In particular, the attenuation decrement $\mathrm{Im}\,(k\,\nu_0)$ is found to be

$$\mathrm{Im}\,(k\nu_0) \cong -\,\omega_p\left(\frac{m\frac{\omega_p^2}{k^2}}{2\pi\Theta}\right)^{1/2} e^{-\frac{m}{2\Theta}\left(\frac{\omega_p^2}{k^2}+c^2\right)} \tag{4.31}$$

## 5. Longitudinal Oscillations: Fixed Frequency Modes

The longitudinal oscillations, like the tranverse oscillations, may be analyzed into fixed frequency modes:

$$\begin{aligned} g_z(z,u,t) &= u\,g(z,u)\,e^{-i\omega t} \\ E_z(z,t) &= E(z)\,e^{-i\omega t} \end{aligned} \tag{4.32}$$

The equations corresponding to (4.2-3) are

$$\begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix}\frac{\partial}{\partial z}\begin{pmatrix} g(z,u) \\ E(z) \end{pmatrix} = \begin{pmatrix} i\omega & -\frac{ne}{m}F'(u) \\ 4\pi e\int & 0 \end{pmatrix}\begin{pmatrix} g \\ E \end{pmatrix} \tag{4.33}$$

or

$$\rho\frac{\partial}{\partial z}\Psi = H\Psi \tag{4.34}$$

The adjoint equation is

$$\frac{\partial}{\partial z}\Psi^\dagger\rho = \Psi^\dagger H^\dagger \tag{4.35}$$

where

$$H^{\dagger} = \begin{pmatrix} i\omega & -\frac{ne}{m}\int_{F'} \\ 4\pi e & 0 \end{pmatrix} \tag{4.36}$$

Here we must modify the procedure used in the transverse mode case. Zero is an eigenvalue of the operator H. I.e.

$$H\Psi_0 = 0 = \Psi_0^{\dagger} H^{\dagger} \tag{4.37}$$

with

$$\Psi_0 = \begin{pmatrix} \frac{ne}{mi\omega} F'(u) \\ 1 \end{pmatrix} \qquad \Psi_0^{\dagger} = \left( \frac{4\pi e i}{\omega} \quad 1 \right) \qquad (\Psi_0^{\dagger}, \rho \Psi_0) = 1 - \frac{\omega_p^2}{\omega^2} \tag{4.38}$$

Thus the set of solutions of (4.34) contains the spatially uniform solution $\psi_0$ as well as the modes obtained by the separation $\psi = \psi_\nu \; e^{-i\omega z/\nu}$. These last solutions are obtained in exactly the same manner as the transverse modes derived in chapter II.

The continuum modes are

$$\Psi_\nu \begin{cases} g_\nu(u) = \frac{\omega_p^2}{\omega^2} \nu^2 P \frac{F'(u)}{u-\nu} + \lambda(\nu)\delta(u-\nu) \\ E_\nu = \frac{4\pi e \nu}{i\omega} \end{cases} \tag{4.39}$$

$$\Psi_\nu^{\dagger} \begin{cases} g_\nu^{\dagger}(u) = \frac{\omega_p^2}{\omega^2} \nu^2 P \frac{1}{u-\nu} + \tilde{\lambda}(\nu)\delta(u-\nu) \\ E_\nu^{\dagger} = \frac{nei\nu}{m\omega} \end{cases} \tag{4.40}$$

with

$$\lambda(\nu) = \tilde{\lambda}(\nu)\, F'(\nu) = 1 - \frac{\omega_p^2}{\omega^2} \nu^2 P \int_{-\infty}^{\infty} \frac{F'(u)\,du}{u - \nu} \tag{4.41}$$

When $\omega > \omega_p$, there are no roots of

$$\Lambda(z) \equiv 1 - \frac{\omega_p^2}{\omega^2} z^2 \int_{-\infty}^{\infty} \frac{F'(u)\,du}{u - z} \tag{4.42}$$

and when $\omega < \omega_p$, there are two imaginary roots $\pm i\, \nu_o$ and corresponding discrete modes. Note that

$$\Lambda(\infty) = 1 - \frac{\omega_p^2}{\omega^2} \tag{4.43}$$

As before, we have the orthogonality relation

$$\left(\psi^{\dagger}_{\nu'}, \rho\, \psi_{\nu}\right) = N(\nu)\, \delta(\nu - \nu') \tag{4.44}$$

where

$$N(\nu) = \nu \frac{\left[\pi \frac{\omega_p^2}{\omega^2} \nu^2 F'(\nu)\right]^2 + \lambda^2(\nu)}{F'(\nu)} \tag{4.45}$$

which may be written

$$\frac{1}{N(\nu)} = \frac{1}{2\pi i \frac{\omega_p^2}{\omega^2} \nu^3} \left\{ \frac{1}{\Lambda^+(\nu)} - \frac{1}{\Lambda^-(\nu)} \right\} \tag{4.46}$$

Of course, the discrete modes, if any, are orthogonal to the continuum modes and to each other.

The set $\{\psi_\nu\}$ (discrete and continuum) is complete when supplemented by the spatially independent solution $\psi_0$. The completeness proof is straightforward. We attempt to expand a state function $\Psi$ in terms of the continuum modes.

$$\Psi = \int_{-\infty}^{\infty} A(\nu)\,\psi_\nu\, d\nu \tag{4.47}$$

or

$$g(u) = \frac{\omega_p^2}{\omega} F'(u)\, P\int_{-\infty}^{\infty} \frac{\nu^2 A(\nu)\,d\nu}{u-\nu} + \lambda(u) A(u) \tag{4.48}$$

$$E = \frac{4\pi e}{i\omega}\int_{-\infty}^{\infty} \nu A(\nu)\,d\nu \tag{4.49}$$

Combining:

$$u\left( g(u) + \frac{nie}{m\omega} F'(u)\right) = \Lambda^+(u)\,\mathcal{a}^+(u) - \Lambda^-(u)\,\mathcal{a}^-(u) \tag{4.50}$$

where we have defined

$$\mathcal{a}(z) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{\nu A(\nu)\,d\nu}{\nu - z} \tag{4.51}$$

$\mathcal{a}(z)$ is, by assumption, analytic in the cut plane, vanishing at least as fast as $1/|z|$ as $|z| \to \infty$. In fact,

$$\lim_{|z|\to\infty} -z\,\mathcal{a}(z) = \lim_{|z|\to\infty} \frac{1}{2\pi i}\int \frac{\nu A(\nu)\,d\nu}{1 - \nu/z} = \frac{1}{2\pi i}\,\frac{i\omega}{4\pi e} E \tag{4.52}$$

which is a subsidiary condition that $\mathcal{a}(z)$ must satisfy in order that $A(\nu)$

given by

$$\nu A(\nu) = a^{+}(\nu) - a^{-}(\nu) \tag{4.53}$$

shall be a solution of (4.47).

The solution of (4.50) is clearly

$$a(z) = \frac{1}{2\pi i} \frac{1}{\Lambda(z)} \int_{-\infty}^{\infty} \frac{u\left(g(u) + \frac{nie}{m\omega} E F'(u)\right) du}{u - z} \tag{4.54}$$

which is, in fact, analytic in the cut plane provided that

$$\left(\psi^{\dagger}_{\pm i\nu_0}, \rho\, \Psi\right) = 0 \qquad \text{(cf. eq. (2.79))} \tag{4.55}$$

The subsidiary condition (4.52) becomes

$$\frac{i\omega}{4\pi e} E = \frac{1}{1 - \frac{\omega_p^2}{\omega^2}} \int_{-\infty}^{\infty} u\, du \left(g(u) + \frac{nie}{m\omega} E F'(u)\right)$$

This may be written, using (4.38)

$$\left(\psi^{\dagger}_{0}, \rho\, \Psi\right) = 0 \tag{4.56}$$

Thus, any function which is orthogonal to $\psi_0^{\dagger}$ as well as to the $\psi^{\dagger}_{\pm i\nu_0}$ may be expanded in terms of the continuum $\{\psi_\nu\}$. This completes the proof, since an arbitrary function $\Psi$ may always be written

$$\Psi = \Psi' + \frac{(\psi_0^{\dagger}, \rho \Psi)}{(\psi_0^{\dagger}, \rho \psi_0)} \psi_0 + \sum_{\pm} \frac{(\psi^{\dagger}_{\pm i\nu_0}, \rho \Psi)}{(\psi^{\dagger}_{\pm i\nu_0}, \rho\, \psi_{\pm i\nu_0})} \psi_{\pm i\nu_0} \tag{4.57}$$

where $\Psi'$ is orthogonal to $\psi_0^{\dagger}$ and to the $\psi^{\dagger}_{\pm i\nu_0}$.

## 6. Boundary Value Problems (Longitudinal Oscillations)

By virtue of the completeness proof of the last section, we may use the representation

$$\Psi(z,u) = \int_{-\infty}^{\infty} A(\nu)\, e^{i\frac{\omega}{\nu}z}\, \Psi_\nu \, d\nu + A_0 \Psi_0 \tag{4.58}$$

and apply the boundary conditions appropriate to the problem under consideration to evaluate the expansion coefficients. We have assumed $\omega > \omega_p$ only to avoid the necessity of carrying the discrete terms along in the calculation.

One consideration may be applied to the distribution function at the start. Regardless of the nature of the reflecting surface, there will be no net current carried by the electrons across a plasma boundary. Hence, at a boundary $z = a$, $\int_{-\infty}^{\infty} u\, g\,(a,\, u)\, du$ must vanish, which implies

$$A_0 = \frac{(\psi_0^\dagger,\, \rho\, \Psi(z=a))}{(\psi_0^\dagger,\, \rho\, \psi_0)} = \frac{1}{1-\frac{\omega_p^2}{\omega^2}} \Big\{ \underbrace{\int_{-\infty}^{\infty} g_0^\dagger(u)\, u\, g(a,u)\, du}_{=\,0} + E_0^\dagger E(a) \Big\}$$

$$= \frac{E(a)}{1-\omega_p^2/\omega^2} \tag{4.59}$$

On the other hand, if the plasma fills a half-space, then

$$A_0 = E(\infty) \tag{4.60}$$

which follows from

$$E(z) = \int_{-\infty}^{\infty} A(\nu)\, E_\nu\, e^{i\frac{\omega}{\nu}z}\, d\nu + A_0 \tag{4.61}$$

since the integral term vanishes as $z \longrightarrow \infty$, by virtue of the Riemann-Lebesque lemma. Thus, the familiar result

$$E(\infty) = \frac{E(0)}{1 - \frac{\omega_p^2}{\omega^2}} \tag{4.62}$$

for a half-space with boundary at $z = 0$.

Now consider the boundary condition for the electron distribution. If the boundary is perfectly reflecting

$$g(a, u) = g(a, -u) \tag{4.63}$$

This condition, together with the relations

$$\begin{aligned} g_\nu(-u) &= g_{-\nu}(u) \quad \text{(assuming } F(u) = F(-u)\text{)} \\ g_0(u) &= -g_0(-u) \\ E_\nu &= -E_{-\nu} \end{aligned} \tag{4.64}$$

yields

$$\int_{-\infty}^{\infty} \left[A(\nu) e^{i\frac{\omega}{\nu}a} - A(\nu) e^{-i\frac{\omega}{\nu}a}\right] \psi_\nu \, d\nu + A_0 \psi_0 = \begin{pmatrix} 0 \\ 2E(a) \end{pmatrix} \tag{4.65}$$

which is of the form of an expansion of the function $\begin{pmatrix} 0 \\ 2E(a) \end{pmatrix}$ in terms of the $\psi_\nu$ and $\psi_0$, and the expansion coefficients may be found by means of the orthogonality relations:

$$A(\nu) e^{i\frac{\omega}{\nu}a} - A(-\nu) e^{-i\frac{\omega}{\nu}a} = 2E(a) \frac{E_\nu^\dagger}{N(\nu)} \tag{4.65}$$

Thus, if the electrons reflect specularly at every boundary, we have immediately:

a) For the half-space, boundary at $z = 0$

$$A(\nu) - A(-\nu) = 2E(0)\,\frac{nei}{m\omega}\,\frac{\nu}{N(\nu)}$$

$$A(\nu) = 0 \qquad (\nu < 0) \qquad \text{(outgoing wave condition)}$$

$$A_0 = E(\infty) = \frac{E(0)}{1 - \omega_p^2/\omega^2} \tag{4.66}$$

which yields

$$E(z)\,e^{-i\omega t} = e^{-i\omega t}\left\{\int_{-\infty}^{\infty} A(\nu) E_\nu\, e^{i\frac{\omega}{\nu}z}\,d\nu + A_0\right\}$$

$$= e^{-i\omega t}\left\{\int_0^{\infty} \frac{2E(0)}{N(\nu)}\,\frac{\omega_p^2}{\omega^2}\,\nu^2 e^{i\frac{\omega}{\nu}z}\,d\nu + \frac{E(0)}{1+\omega_p^2/\omega^2}\right\} \tag{4.67}$$

$$= E(0)\,e^{-i\omega t}\left\{\frac{1}{\pi i}\int_0^{\infty}\frac{1}{\nu}\left[\frac{1}{\Lambda^+(\nu)} - \frac{1}{\Lambda^-(\nu)}\right] e^{i\frac{\omega}{\nu}z}\,d\nu + \frac{1}{1+\omega_p^2/\omega^2}\right\}$$

b) **For the slab**

$$A_0 = \frac{E(0)}{1 - \frac{\omega_p^2}{\omega^2}} = \frac{E(a)}{1 - \frac{\omega_p^2}{\omega^2}}$$

$$A(\nu) - A(-\nu) = A(\nu)e^{i\frac{\omega}{\nu}a} - A(-\nu)e^{-i\frac{\omega}{\nu}a} = 2E(0)\,\frac{nei}{m\omega}\,\frac{\nu}{N(\nu)} \tag{4.68}$$

Thus,

$$A(\nu) = \frac{1 - e^{i\frac{\omega}{\nu}a}}{1 - e^{2i\frac{\omega}{\nu}a}}\; 2E(0)\,\frac{nei}{m\omega}\,\frac{\nu}{N(\nu)} \tag{4.69}$$

and

$$E(z)e^{-i\omega t} = E(0)e^{-i\omega t}\left\{2\int_{-\infty}^{\infty}\frac{1 - e^{i\frac{\omega}{\nu}a}}{1 - e^{2i\frac{\omega}{\nu}a}}\, e^{i\frac{\omega}{\nu}z}\,\frac{\omega_p^2}{\omega^2}\,\frac{\nu^2}{N(\nu)}\,d\nu + \frac{1}{1 - \frac{\omega_p^2}{\omega^2}}\right\} \tag{4.70}$$

Of particular interest is the impedence $Z$ of this "plasma capacitor," defined by:

$$Z\,J = \int_0^a E(z)\,dz \tag{4.71}$$

where J is the net current flowing into the condenser, and is given entirely by the displacement current at $z = 0$, since there is no net electron transport across the boundary, i.e.

$$\frac{J}{A} = \frac{i\omega}{4\pi} E(0) \tag{4.72}$$

Hence

$$Z = \frac{1}{i\omega C_0} \frac{\int_0^a E(z)dz}{E(0)\,a} \equiv \frac{1}{i\omega C_0} \frac{1}{\epsilon_{eff}} \tag{4.73}$$

with

$$C_0 = \frac{A}{4\pi a} \tag{4.74}$$

the capacitance of the condenser without the plasma.

$\epsilon_{eff}$, the "effective dielectric coefficient" may be written in terms of the function $\Lambda(z)$, since

$$\int_0^a E(z)dz = E(0)\left\{\frac{a}{1-\frac{\omega_p^2}{\omega^2}} + 2\int_{-\infty}^{\infty} \frac{\nu}{i\omega} \frac{1-\cos\frac{\omega a}{\nu}}{-i\sin\frac{\omega a}{\nu}} \frac{\omega_p^2}{\omega^2} \frac{\nu^2}{N(\nu)} d\nu\right\} \tag{4.75}$$

where the singularities in the integrand are to be avoided by ascribing to $\omega$ a small positive part. Thus, we may apply (4.46) and write the result as a contour integral, which may be evaluated in a residue series:

$$\epsilon_{eff} = \frac{1}{1-\frac{\omega_p^2}{\omega^2}} + \frac{4}{\pi^2}\sum_{odd\ n} \frac{1}{n^2}\left\{\frac{1}{\Lambda^+\left(\frac{\omega a}{n\pi}\right)} - \frac{1}{1-\frac{\omega_p^2}{\omega^2}}\right\} \tag{4.76}$$

Note at at zero temperature, the term in braces vanishes identically.

Finally, we may treat the half-space with diffuse reflecting boundary. In this case

$$g(0,u) = C E(0) F(u) \qquad u > 0 \tag{4.77}$$

which represents the fact that the wall remits the electrons in a thermal distribution at the same temperature as the plasma. C is a normalization constant to be determined. (4.77) and (4.59), along with the outgoing wave condition $A(\nu) = 0$ $(\nu < 0)$, imply

$$E(0)\,G(u) \equiv E(0)\left\{CF(u) - \frac{g_0(u)}{1-\frac{\omega_p^2}{\omega^2}}\right\} = \int_0^\infty A(\nu)\, g_\nu(u)\, d\nu \qquad u>0 \tag{4.78}$$

which amounts to a half-range expansion of the left hand side in terms of half of the continuum modes. This is of the same form as the expansion treated in Chapter III, sec. 6 , and we only sketch the results. We define

$$a(z) = \frac{1}{2\pi i}\int_0^\infty \frac{\nu A(\nu)\, d\nu}{\nu - z} \tag{4.79}$$

Then (4.78) combined with (4.59) and (4.61) gives

$$E(0)\, u\left[CF(u) - g_0(u)\right] = a^+(u)\Lambda^+(u) - a^-(u)\Lambda^-(u) \qquad u>0 \tag{4.80}$$

$a(z)$ is to be analytic in the cut plane (cut from 0 to $\infty$ along the real axis). $\Lambda(z)$ does not have this property, but we may write

$$\frac{\Lambda^+(u)}{\Lambda^-(u)} = \frac{X^+(u)}{X^-(u)} \qquad u>0 \tag{4.81}$$

where $X(z)$ is analytic and non-vanishing in the cut plane. An appropriate $X(z)$ is

$$X(z) = e^{\Gamma(z)} = e^{\frac{1}{2\pi i}\int_0^\infty \frac{\ln\left(\Lambda^+(u)/\Lambda^-(u)\right)}{u-z}\, du} \tag{4.82}$$

Note that

$$\lim_{z\to\infty} X(z) = 1 \tag{4.83}$$

In terms of the X function, (4.80) becomes

$$E(0)u\left[CF(u)-g_0(u)\right]\frac{X^-(u)}{\Lambda^-(u)} = X^+(u)a^+(u) - X^-(u)a^-(u) \qquad u>0 \tag{4.84}$$

Thus J (z) defined by

$$J(z) \equiv X(z)a(z) - \frac{1}{2\pi i}E(0)\int_0^\infty \frac{u\left[CF(u)-g_0(u)\right]X^-(u)}{(u-z)\quad \Lambda^-(u)}\,du \tag{4.85}$$

is an entire function. But since we require that $a(z)$ vanishes as $z \to \infty$, J (z) must also vanish at $\infty$. Hence

$$J(z) = 0 \tag{4.86}$$

and

$$a(z) = E(0)\frac{1}{X(z)}\frac{1}{2\pi i}\int_0^\infty \frac{X^-(u)}{\Lambda^-(u)}\frac{\left[CF(u)-g_0(u)\right]u}{u-z}\,du \tag{4.87}$$

with the desired expansion coefficient A ( $\nu$) given by

$$\nu A(\nu) = a^+(\nu) - a^-(\nu) \tag{4.88}$$

The normalization constant C may be obtained by letting $z \to \infty$ in (4.87)

$$\frac{i\omega}{4\pi e}\left[1 - \frac{1}{1-\frac{\omega_p^2}{\omega^2}}\right] = \int_0^\infty \frac{X^-(u)}{\Lambda^-(u)}\left[CF(u)-g_0(u)\right]u\,du \tag{4.89}$$

The integrals in (4.89) may be simplified through the use of the identity[18]

$$X(z) - 1 = \frac{1}{2\pi i}\int_0^\infty \frac{X^+(u)-X^-(u)}{u-z}\,du \tag{4.90}$$

$$= \frac{1}{2\pi i}\int_0^\infty \frac{X^-(u)\left[\Lambda^+(u)-\Lambda^-(u)\right]}{\Lambda^-(u)\,(u-z)}\,du = -\frac{\omega_p^2}{\omega^2}\int_0^\infty \frac{X^-(u)\,u^2F(u)}{\Lambda^-(u)(u-z)}\,du$$

which follows from the fact that $X(z) - 1$ is analytic in the cut plane, vanishing as $|z| \longrightarrow \infty$. For our purposes, we need only

$$X(0) - 1 = -\frac{\omega_p^2}{\omega^2}\int_0^\infty \frac{X^-(u)}{\Lambda^-(u)}\, u F'(u) du \tag{4.91}$$

$$X'(0) = -\frac{\omega_p^2}{\omega^2}\int_0^\infty \frac{X^-(u)}{\Lambda^-(u)} F'(u) du \tag{4.92}$$

Then, using the definition of $g_0(u)$ (4.38) and the fact that

$$u F(u) = -\frac{\Theta}{m} F'(u) \tag{4.93}$$

we may obtain the normalization constant

$$C = \frac{ne}{i\omega\Theta}\left[\frac{1}{1-\frac{\omega_p^2}{\omega^2}} - X(0)\right]\frac{1}{X'(0)} \tag{4.94}$$

which could be further simplified slightly since we know (cf. eq. (3.67))

$$X(0) = \left(1 - \frac{\omega_p^2}{\omega^2}\right)^{-1/2} \tag{4.95}$$

## 7. Conclusion - General Considerations

We have carried out the eigenfunction expansion and its application to boundary value problems in plasma oscillations. It is seen that the equations treated contain a continuous spectrum of allowed values with associated singular normal modes. In other, more fundamental respects - completeness and orthogonality to the accompanying normal modes of the adjoint problem - the eigenfunction expansion for plasma problems does not differ from the eigenfunction expansions that arise in all branches of physics. Thus the main effort in this dissertation has been the attempt to place boundary value problems in plasma oscillations on an equal footing with other, perhaps more familiar, boundary value problems.

The existence of a continuous spectrum with singular normal modes is not peculiar to the plasma problem. Actually it is a characteristic outgrowth of the $\frac{D}{Dt}$ or "streaming" operator that appears in the equations governing a wide variety of transport phenomena. Along this line, Case and Dyson[24] have presented the continuum modes for the Euler equations of hydrodynamics (linearized about a linear velocity profile). Also, Cercignani[25] has obtained the solutions to certain boundary value problems in gas dynamics through an expansion in the (singular) normal modes of the linearized Boltzmann equation for a gas after approximating the scattering kernel by a kernel of finite rank. Finally, Case[26] has provided a comprehensive singular-normal-mode treatment of one-velocity neutron transport theory, and his results have been applied and generalized.[27-30]

In Case's treatment, the time-dependent homogeneous neutron transport equation with plane symmetry and isotropic scattering

$$\frac{\partial}{\partial t}\psi + \mu\frac{\partial}{\partial x}\psi + \psi = \frac{c}{2}\int_{-1}^{1}\psi\, d\mu \tag{4.96}$$

under the <u>ansatz</u>

$$\psi(x,\mu,t) = e^{ikx}e^{-(1+i\alpha k)t}\varphi_{\alpha,k}(\mu) \tag{4.97}$$

takes on the familiar form

$$(\alpha-\mu)\varphi_{\alpha,k}(\mu) = \frac{ic}{2k}\int_{-1}^{1}\varphi_{\alpha,k}(\mu)d\mu \tag{4.98}$$

Distances are measured in unit of the absorption mean free path, $\lambda_a$, time in units of $\lambda_a/v$ and c is the average number of neutrons emerging from a scattering event.

The solutions to (4.98) are

a) for any k, a continuum $-1 < \alpha < 1$

$$\varphi_{\alpha,k}(\mu) = \frac{ic}{2k} P \frac{1}{\alpha-\mu} + \lambda_k(\alpha)\delta(\mu-\alpha)$$

$$\lambda_k(\alpha) = 1 + \frac{ic}{2k} P\int_{-1}^{1} \frac{d\mu}{\mu-\alpha} = \frac{1}{2}\left[\Lambda_k^+(\alpha) + \Lambda_k^-(\alpha)\right]$$

$$\Lambda_k(\alpha) = 1 + \frac{ic}{2k}\int_{-1}^{1}\frac{d\mu}{\mu-\alpha} = 1 - \frac{ic}{k}\tanh^{-1}\alpha \tag{4.99}$$

b) for $|k| < c\pi/2$ one discrete mode

$$\varphi_{*,k}(\mu) = \frac{ic}{2k}\frac{1}{\alpha_o-\mu} \tag{4.100}$$

with $\alpha_o$ the root of $\Lambda(\alpha_o) = 0$ or,

$$\frac{1}{\alpha_o} = \tanh\left(\frac{k}{c}\right) \tag{4.101}$$

The orthogonality relations are

$$\int_{-1}^{1} \varphi_{\alpha,k}(\mu)\varphi_{\alpha',k}(\mu)d\mu = N_{\alpha,k}\,\delta(\alpha-\alpha')$$

$$N_{\alpha,k} = \lambda_k^2(\alpha) - \frac{c^2\pi^2}{4k^2}$$

$$\int_{-1}^{1}\left[\varphi_{*,k}(\mu)\right]^2 d\mu \equiv N_{*,k} = \frac{c^2}{2k^2(1-\alpha_o^2)} \tag{4.102}$$

The normal mode expansion may be used to obtain the general time dependent Green's function - the neutron distribution due to a source

$q = \delta(t - t_o)\ \delta(x - x_o)\ \delta(\mu - \mu_o)$. For our purposes, it suffices to state a less general result: the total neutron density $\rho(x,t) = \int_{-1}^{1} \psi\, d\mu$ produced by an isotropic plane source $q = \delta(t - t_o)\ \delta(x - x_o)$ is given simply by

$$\rho_{pl}(x-x_o, t-t_o) = \frac{e^{-(t-t_o)}}{2\pi} \int_{-\infty}^{\infty} e^{ik(x-x_o)}\, dk \sum_{\alpha} \frac{e^{-i\alpha k(t-t_o)}}{N_{\alpha,k}} \tag{4.103}$$

(here $\sum_\alpha$ means adding the discrete term to an integral over the continuum).

There is an interesting connection between this neutron transport problem and the transport problem of the gaseous discharge.[31] Let $\psi(v, v_z, z, t)$ be the distribution function for electrons in an electric field diffusing among gas molecules. The electric field is in the $z$ direction and plane symmetry and elastic isotropic scattering is assumed. Then $\psi$ satisfies

$$\frac{\partial}{\partial t}\psi + v_z \frac{\partial}{\partial z}\psi + \frac{eE}{m}\frac{\partial}{\partial v_z}\psi + v\sigma_a(v)\psi$$
$$= \frac{v\sigma_s(v)}{4\pi}\int \psi\, d\hat{\Omega}_v + Q \tag{4.104}$$

where $\sigma_a$ and $\sigma_s$ are the absorption and scattering cross-sections. If, finally, we assume that the cross-sections are proportional to $1/v$ and choose units in terms of the mean free time $(v\,\sigma_a)^{-1}$ and the basic acceleration $eE/m$, we get

$$\frac{\partial}{\partial t}\psi + v_z\frac{\partial}{\partial z}\psi + \frac{\partial}{\partial v_z}\psi = \frac{c}{4\pi}\int \psi\, d\hat{\Omega}_v + q \tag{4.105}$$

where $c = \sigma_s/\sigma_a$.

We seek a Green's function solution $q = \delta(v)\ \delta(z)\ \delta(t)$ corresponding to the injection of a pulse of electrons at rest. Since the solution will contain a factor $\delta(v - \sqrt{2z})$, by conservation of energy; no information is lost by integrating (4.105) over $z$ to obtain

$$\frac{\partial}{\partial t}\Phi(v, v_z, t) + \frac{\partial}{\partial v_z}\Phi + \Phi = \frac{c}{4\pi}\int \Phi \, d\hat{\Omega}_v + q' \tag{4.106}$$

where

$$\Phi(v, v_z, t) = \int_{-\infty}^{\infty}\Psi(v, v_z, z, t)\,dz \qquad q' = \delta(\vec{v})\delta(t) \qquad \Psi = \frac{1}{v}\Phi\,\delta(v - \sqrt{2z}) \tag{4.107}$$

Now if we use instead of $v$, $v_z$, the velocity variables $v$ and $\mu = v_z/v$ the transport equation takes on the form

$$\frac{\partial}{\partial t}\Phi(v, \mu, t) + \mu\frac{\partial}{\partial v}\Phi + \frac{1-\mu^2}{v}\frac{\partial}{\partial \mu}\Phi = \frac{c}{2}\int_{-1}^{1}\Phi\,d\mu + q' \tag{4.108}$$

which is identical, under the transcription $v \longrightarrow r$ to the neutron transport equation in spherical geometry. Thus, the solution $\Phi$ corresponding to a point source in velocity space is obtained immediately by transcription from the neutron transport point source Green's function. Thus, the results of the eigenfunction expansion in neutron transport may be applied directly to the gaseous discharge problem.

# BIBLIOGRAPHY


1. Langmuir, I. and Tonks, L., Phys. Rev. 33, 195 (1929).

2. Langmuir, I., Proc. Nat. Acad. Sci. 14, 627 (1926).

3. Mimno, H. R., Revs. Mod. Phys. 9, 1 (1937).

4. Ford, G. W., Electrodynamics in Low Density Plasmas (Space Technology Labs.) STL/TR-60-P001, 2-202184.

5. Chapman, S. and Cowling, T.G., The Mathematical Theory of Non-Uniform Gases (Oxford, 1958).

6. Власов, А. Журн. Экспер. и Теор. Физ. 8, 291 (1938) (Vlasov, A.) (Jour. Exptl. and Theo. Phys.) (AEC tr-2729).

7. Harris, E. G., On a Self-Consistent Field Method for a Completely Ionized Gas (Naval Research Lab.) NRL-4944.

8. Rostocker, N. and Rosenbluth, M., Physics of Fluids 3, 1 (1960).

9. Боголюбов, Н. Н. Проблемы Динамической Теории в Статистической Физике (Москва, ОГИЗ Гостехиздат, 1946) (Bogoliubov, N.N.) (Problems in the Dynamic Theory of Statistical Physics) (Moscow, 1946) (AEC tr - 3852).

10. Guernsey, R. L., Dissertation, The University of Michigan (1960).

11. Prigogine, I. and Balescu, R., Physica 25, 281, 302 (1959).

12. Balescu, R., Physics of Fluids 3, 52 (1960).

13. Lighthill, M., Introduction to Fourier Analysis and Generalized Functions (Cambridge, 1958).

14. Case, K. M., Annals of Physics 8, 349 (1959).

15. Muskhelishvili, N., Singular Integral Equations (P. Noordhoff N.V. Groningen, Holland, 1953).

16. Landau, L., Journal of Physics (USSR) 10, 25 (1946).

17. Fried, B. and Conte, S. D., The Plasma Dispersion Function (New York, Academic Press 1961).

18. Aamodt, R. E. and Case, K. M., Useful Identities for Half-Space Problems in Linear Transport Theory (preprint).

19. Rostocker, N. and Simon, A., Fokker-Planck Coefficients for a Plasma Including Cyclotron Radiation (General Atomic Labs.) GA-2480 (1961).

20. Scarf, F. L., Landau Damping and the Attenuation of Whistlers (Space Technology Labs.) (AFBMD-TR-61-33).

21. van Kampen, N. G., Physica 21, 949 (1955).

22. Montgomery, D. and Gorman, D., Physics of Fluids 5, 947 (1962).

23. Jackson, J. D., Jour. Nuclear Energy C. 1, 171 (1960).

24. Case, K. M. and Dyson, F., Physics of Fluids 3, 143 (1960).

25. Cercignani, I., Annals of Physics (to be published).

26. Case, K. M., Annals of Physics 9, 1 (1960).

27. Mika, J., Nuclear Science and Engineering 4, 415 (1961).

28. Mitsis, G. J., Transport Solutions to the One-Dimensional Critical Problem. (Argonne National Lab.) ANL-6459.

29. Zelazny, R., Transport Theory of Neutrons in Heavy Gas (Case Inst. of Tech.) Plasma Research Technical Report A-8.

30. Corngold, N. and Michael, P., Nuc. Sci. and Engin. (to be pub.).

31. Shure, F., A Soluble Model for Gaseous Discharge (General Atomic Labs.) GAMD-967.