8665-6002-RU-000

A DYNAMICAL THEORY OF BARYONS

by

H. C. Corben

SPACE TECHNOLOGY LABORATORIES, INC.
a subsidiary of Thompson Ramo Wooldridge Inc.
ONE SPACE PARK • REDONDO BEACH, CALIFORNIA

8665-6002-RU-000

A DYNAMICAL THEORY OF BARYONS

H. C. Corben

December, 1963

Research Supported by the STL Company Independent Research Program and by Contract Nonr-3769(00) NR 013-110

QUANTUM PHYSICS LABORATORY
Physical Research Division

Space Technology Laboratories, Inc.
One Space Park
Redondo Beach, California

ABSTRACT

The simplest relativistic wave equations for a particle which in the classical limit possesses moments of inertia about more than one axis are Dirac and Kemmer-Duffin equations containing extra terms which cause these equations to describe a variety of spin states. The classical field theory of such wave equations is developed and the generalized Dirac equation for particles of spin $\frac{1}{2}$ and $\frac{3}{2}$ is examined in detail. It is found that with the choice of only one parameter, which merely determines the scale, this equation not only correctly describes the spin and charge states of the particles and resonances Ξ^-, Ξ^0, N, P, N^{**}, Y_0^{**}; it also yields their masses correct to better than 2%. In addition, with the same choice of this parameter, the theory has so far yielded the correct masses, to the same accuracy, for the resonances N^{***}, Y_1^{**}, Y_0^{***}, N^{****}, giving their spins as 5/2, 5/2, 7/2 and 9/2 respectively. The $\Xi^- - \Xi^0$ and N - P mass differences have the correct sign but are several times their observed values. Choice of one other parameter to give the correct N - P mass difference would lead to even better agreement with experiment for the other states, but would also lead to proton and neutron isobars lying 20 MeV above the ground state.

CONTENTS

1. Introduction

Each level of theoretical effort, from Newtonian physics to relativity to relativistic quantum field theory, is solidly based on the discipline which preceded it. The Correspondence Principle shows us how each quantum theory has its classical point-particle limit, and how each second-quantized theory has its classical field-theoretic limit. It is reasonable to expect, then, that a dynamical theory of elementary particles and nuclear forces will also have its roots in classical mechanics and quantum mechanics.

For some years there has existed, within the approximation of classical relativistic particle mechanics, a theory of the dynamics of a spinning particle which possesses moments of inertia about more than one axis.[1] As a consequence of the equations of motion for such a particle, it was found that the mass of the particle is not required to be a constant of the motion and that the intrinsic spin angular momentum is the sum of two vectors, along and perpendicular to the angular velocity[2]. For the case of a pure gyroscope, for which the moments of inertia about all axes normal to the spin axis are zero, the theory reduced to the classical limit of the Dirac and Kemmer-Duffin theories, in so far as it is possible to distinguish spin and quantum effects in going

to this limit.

More recently, the quantum theory corresponding to the more general case was formulated[3]. The essential features of the more general classical case were shown to be retained in the quantum theory. The variable mass of the classical particle theory became an operator in the corresponding quantum theory, and, as in the classical case, the spin became the sum of two operators, one of which is the usual spin operator. Thus the generalized Dirac equation, for example, now includes an extra term and may describe a particle of spin other than $\frac{1}{2}$.

In Ref. 3 the laws of conservation of momentum and angular momentum were shown to lead to an expression for the mass operator, so that we obtained the relativistic wave equation

$$(i \epsilon_\mu P_\mu + Mc) = 0 \qquad (1.1)$$

where $P_\mu = p_\mu - \frac{e}{c} A_\mu$

$$M = m + m_o \epsilon_{\mu\nu} \lambda_{\mu\nu} + m' \epsilon_\mu \lambda_\mu \qquad (1.2)$$

and m, m_o, m' are arbitrary parameters.
The spin of the particle is now

$$S_{\mu\nu} = - i\hbar (\epsilon_{\mu\nu} + \lambda_{\mu\nu}) \qquad (1.3)$$

where

$$\left.\begin{aligned}(\epsilon_{\mu\nu}\,,\epsilon_\sigma) &= \epsilon_\mu\,\delta_{\nu\sigma} - \epsilon_\nu\,\delta_{\mu\sigma}\\ (\lambda_{\mu\nu}\,,\lambda_\sigma) &= \lambda_\mu\,\delta_{\nu\sigma} - \lambda_\nu\,\delta_{\mu\sigma}\end{aligned}\right\} \tag{1.4}$$

$$\left.\begin{aligned}(\epsilon_{\mu\nu}\,,\epsilon_{\sigma\tau}) &= -\left[\epsilon_{\mu\sigma}\delta_{\nu\tau} + \epsilon_{\nu\tau}\delta_{\mu\sigma} - \epsilon_{\mu\tau}\,\delta_{\nu\sigma} - \epsilon_{\nu\sigma}\delta_{\mu\tau}\right]\\ (\lambda_{\mu\nu}\,,\lambda_{\sigma\tau}) &= -\left[\lambda_{\mu\sigma}\delta_{\nu\tau} + \lambda_{\nu\tau}\delta_{\mu\sigma} - \lambda_{\mu\tau}\,\delta_{\nu\sigma} - \lambda_{\nu\sigma}\delta_{\mu\tau}\right]\end{aligned}\right\} \tag{1.5}$$

$$(\epsilon_\mu\,,\,\lambda_\nu) = 0\;;\quad (\epsilon_{\mu\nu}\,,\,\lambda_{\sigma\tau}) = 0$$

Hence $(S_{\mu\nu}\,,M) = 0$.

For $m' = 0$, the theory exhibits a detailed correspondence with the classical theory. The classical point-particle equations relating the spin $S_{\mu\nu}$, angular velocity $\omega_{\mu\nu}$ and mass M are[1,2]

$$\left.\begin{aligned}\dot{S}_{\mu\nu} &= Ic(\dot{\omega}_{\mu\nu} + \dot{\Omega}_{\mu\nu}) = -(v_\mu P_\nu - v_\nu P_\mu)\\ \dot{x}_\mu &= v_\mu\,.\quad (v_\mu v^\mu = -1)\\ \dot{\Omega}_{\mu\nu} &= -\frac{K}{Ic}\,(\omega_{\mu\sigma}\,\ddot{\omega}_{\sigma\nu} - \omega_{\nu\sigma}\,\ddot{\omega}_{\sigma\mu})\\ \dot{M} &= \frac{K}{2I^2}\,\ddot{\omega}_{\mu\nu}\,\dot{S}_{\mu\nu}\end{aligned}\right\} \tag{1.6}$$

$$M = -\frac{K}{4c}\,\omega_{\mu\nu}\,\ddot{\omega}_{\mu\nu} + m \tag{1.7}$$

These may be compared with the similar equations derived from (1.1) for $m' = 0$:

$$\left.\begin{aligned}
\dot{S}_{\mu\nu} &= -i\hbar(\dot{\epsilon}_{\mu\nu} + \dot{\lambda}_{\mu\nu}) = -i(\epsilon_\mu P_\nu - \epsilon_\nu P_\mu) \\
\dot{x}_\mu &= v_\mu = i\,\epsilon_\mu \\
\dot{\lambda}_{\mu\nu} &= -\frac{2im_o c}{\hbar}\left[\epsilon_{\mu\sigma}\lambda_{\sigma\nu} - \epsilon_{\nu\sigma}\lambda_{\sigma\mu}\right] \\
\dot{M} &= \frac{im_o c}{\hbar}\,\lambda_{\mu\nu}\,\dot{S}_{\mu\nu}
\end{aligned}\right\} \qquad (1.8)$$

$$M = m_o\,\epsilon_{\mu\nu}\,\lambda_{\mu\nu} + m \qquad (1.9)$$

If we write

$$\begin{aligned}
Ic\;\omega_{\mu\nu} &= -i\hbar\,\epsilon_{\mu\nu} \\
Ic\,\Omega_{\mu\nu} &= -i\hbar\,\lambda_{\mu\nu} \\
\hbar K\,\dot{\omega}_{\mu\nu} &= 2i\,m_o\,c^2\,I\,\lambda_{\mu\nu}
\end{aligned}$$

we note that each of equations (1.6) becomes formally identical with the corresponding equation of (1.9) and that (1.7) assumes the form of (1.9) apart from a factor 2.

Equations (1.5) are satisfied if $\epsilon_{\mu\nu}$ and $\lambda_{\mu\nu}$ assume the form of either of the operators

$$\frac{1}{4}\,\gamma_{\mu\nu} = \frac{1}{4}\,(\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu) \qquad (1.10)$$

or

$$\beta_{\mu\nu} = \beta_\mu\beta_\nu - \beta_\nu\beta_\mu \qquad (1.11)$$

where γ_μ, β_μ respectively satisfy the Dirac and Kemmer-Duffin commutation relations:

$$\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 2\,\delta_{\mu\nu}$$

$$\beta_\mu\beta_\nu\,\beta_\sigma + \beta_\sigma\beta_\nu\,\beta_\mu = \beta_\mu\,\delta_{\nu\sigma} + \beta_\sigma\,\delta_{\mu\nu}.$$

Equations (1.4) are then satisfied if ϵ_μ, λ_μ are proportional to either γ_μ or β_μ.

If $\epsilon_{\mu\nu}$ is given by (1.10) and ϵ_μ is a constant times γ_μ, equation (1.1) becomes a generalization of the Dirac equation to describe a particle of spin given by (1.3)

$$S_{\mu\nu} = -\frac{i\hbar}{4}(\gamma_{\mu\nu} + 4\lambda_{\mu\nu}) \tag{1.12}$$

On the other hand, if $\epsilon_{\mu\nu}$ is given by (1.11) and ϵ_μ is a constant times β_μ, Equation (1.1) becomes a generalization of the Kemmer equation to describe a particle of spin

$$S_{\mu\nu} = -i\hbar(\beta_{\mu\nu} + \lambda_{\mu\nu}) \tag{1.13}$$

If the $\lambda_{\mu\nu}$ are also of the form (1.10) or (1.11) (in a different space, since they commute with $\epsilon_{\mu\nu}$) the generalized Dirac equation will then describe particles of spin 0, $\frac{1}{2}$, 1, $\frac{3}{2}$, and the generalized Kemmer equation will yield all values of the spin up to 2. More general forms for $\lambda_{\mu\nu}$ lead to particles of higher spin.

In this paper we first develop the classical field theory of the generalized Dirac and Kemmer equations derived from Equation (1.1) when M is any hermitean operator which commutes with P_μ but not with ϵ_μ. We then consider in detail the particle states of spin $\frac{1}{2}$ and $\frac{3}{2}$ obtained from the special case of the generalized Dirac equation when we set $\lambda_\mu = \beta_\mu$, $\lambda_{\mu\nu} = \beta_{\mu\nu}$ in equation (1.1):

$$(i\epsilon_\mu P_\mu + mc + m_o c\ \epsilon_{\mu\nu}\beta_{\mu\nu} + m'c\ \epsilon_\mu\beta_\mu)\psi = 0 \quad (1.14)$$

with

$$\epsilon_\mu = \tfrac{1}{2}\gamma_\mu,\ \epsilon_{\mu\nu} = \tfrac{1}{4}\gamma_{\mu\nu},\ \epsilon_\mu\epsilon_\mu = 1. \quad \text{(cf. (1.6),(1.8))}$$

In general, this equation leads to eight distinct eigenvalues for the rest-energy of particles of spin $\frac{1}{2}$ and four such eigenvalues for particles of spin $\frac{3}{2}$, but for the special case $m' = 0$, which exhibits a closer correspondence with the classical point-particle theory , there are four mass eigenvalues for spin $\frac{1}{2}$ and two for spin $\frac{3}{2}$.

For $m' = o$, $\eta_5 = \eta_1\eta_2\eta_3\eta_4$ commutes with the Hamiltonian ($\eta_\mu = 2\beta_\mu^2 - 1$) and it is found that neutral particles are characterized by $\eta_5 = -1$, charged particles by $\eta_5 = +1$, the charge density being $\frac{e_o}{2}\psi^*\eta_4(1+\eta_5)\psi$.

For $m' = o$, $m_o = -\frac{1}{8}m$, Equation (1.14) becomes

$$\left[i\ \epsilon_\mu P_\mu + mc\ (1 - \tfrac{1}{4}\sum_{\mu<\nu}\epsilon_{\mu\nu}\beta_{\mu\nu})\right]\psi = 0 \qquad (1.15)$$

It is shown in this paper that for the choice $m = 1297\ m_e$ of the single parameter m, Equation (1.15) with $\epsilon_\mu = \frac{1}{2}\gamma_\mu$ not only describes correctly the charges and spins of the hyperons and resonances Ξ^-, Ξ^o, N, P, N^{**}, Y_o^{**}, it also yields values for their masses which are accurate to better than $2^o/o$. The neutron described by Equation (1.15) is found to be heavier than the proton, and the Ξ^- heavier than the Ξ^o, although the magnitudes of these mass differences are several times the observed values. Particles

resembling the Λ Σ Y_1^* Y_0^* are not described by this special case of Equation (1.1) and a study of the other fermions and bosons given by Equation (1.1) is in progress.

While perhaps one should not expect greater accuracy from a classical field theory, the case in which m' is a small imaginary quantity has also been investigated. The choice $m' = 24\ i\ m_e$, coupled with the values $m = 1297\ m_e$, $m_o = -\frac{1}{8} m$ as before, not only gives the correct values for both m_p and m_n and their difference, it also materially improves the agreement with experiment for the masses of other particles described by this equation. Such a non-zero value for m' would split the proton-antiproton state giving an excited level of the proton (and its corresponding antiproton) lying approximately 20 MeV above the ground state. An excited neutron state lying at approximately the same height above the ground state is also predicted by the case $m' = 24\ i\ m_e$ together with some fine structure for the Ξ^o, N^{**} and Y_o^{**} states. The term proportional to m' leads to exchange forces between the neutron and proton states, which otherwise would remain uncoupled.

In Sec. 4 it is shown that Equation (1.15) also leads to approximately correct energies for the resonances N^{***}, Y_1^{**}, Y_o^{***}, N^{****}, and to the correct spin in the one case (N^{***}) where it is known.

2. Field theory of generalized Dirac and Kemmer equations

We first consider the equation

$$(\gamma_\mu \partial_\mu + \kappa)\psi = 0 \tag{2.1}$$

where γ_μ are the Dirac operators and κ is an operator which commutes with ∂_μ, but not with γ_μ. We define

$$\psi^+ = i\,\psi^* \gamma_4 \chi$$

where χ is an operator possessing the properties

$$(\chi, \partial_\mu) = 0; \quad (\chi, \gamma_\mu) = 0; \quad (\gamma_4\chi, \kappa) = 0 \tag{2.2}$$

It then follows that, if κ is hermitean*

$$\partial_\mu \psi^+ \gamma_\mu - \psi^+ \kappa = 0 \tag{2.3}$$

so that we may define a conserved density

$$s_\mu = \psi^+ \gamma_\mu \psi \tag{2.4}$$

The energy momentum tensor

$$T_{\mu\nu} = -\frac{i\hbar c}{2}(\psi^+ \gamma_\nu \partial_\mu \psi - \partial_\mu \psi^+ \gamma_\nu \psi) \tag{2.5}$$

satisfies

$$\partial_\nu T_{\mu\nu} = 0$$

as in the constant mass case, but the symmetrized tensor

$$\tilde{T}_{\mu\nu} = \frac{1}{2}(T_{\mu\nu} + T_{\nu\mu}) \tag{2.6}$$

is now no longer conserved. If, however, we introduce the tensor

---

* The case in which κ has a small anti-hermitean part must be treated separately.

$$\Sigma_{\mu\nu} = -\Sigma_{\nu\mu} = \frac{i\hbar c}{4}\left[\psi^{+}(\gamma_\nu \gamma_\mu \kappa + \kappa \gamma_\mu \gamma_\nu)\psi - 2\delta_{\mu\nu}\psi^{+}\kappa\psi\right] \tag{2.7}$$

it is found that $\partial_\nu \Theta_{\mu\nu} = 0$ where

$$\Theta_{\mu\nu} = \tilde{T}_{\mu\nu} + \Sigma_{\mu\nu}$$

$$= T_{\mu\nu} + \frac{i\hbar c}{8}\partial_\rho\left[\psi^{+}(\gamma_\rho\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu\gamma_\rho)\psi\right] \tag{2.8}$$

The tensor $\Sigma_{\mu\nu}$ vanishes for κ = const., but here it is non-zero and antisymmetric.

The usual angular momentum of the Dirac theory

$$P_{ik} = -\frac{1}{c}\int (x_i \Theta_{k4} - x_k \Theta_{i4})\,dV$$

$$= -\hbar \int (x_i \psi^{+}\gamma_4 \partial_k \psi - x_k \psi^{+}\gamma_4 \partial_i \psi)\,dV \tag{2.9}$$

$$-\frac{\hbar}{4}\int \psi^{+}\gamma_4\gamma_{ik}\psi\, dV \quad (\gamma_{ik} = \gamma_i\gamma_k - \gamma_k\gamma_i)$$

is no longer conserved, since now the tensor $\Theta_{\mu\nu}$ is not symmetrical:

$$\frac{dP_{ik}}{dt} = \int (\Theta_{ki} - \Theta_{ik})\, dV$$

$$= \frac{i\hbar c}{4}\int \psi^{+}(\gamma_{ik}\kappa)\psi\, dV$$

However,

$$J_{ik} = P_{ik} - \hbar \Lambda_{ik} \tag{2.10}$$

is then conserved if

$$\frac{d\Lambda_{ik}}{dt} = \frac{ic}{4}\int \psi^{+}(\gamma_{ik}\kappa)\psi\, dV.$$

Writing $\Lambda_{ik} = \int \psi^{+}\gamma_4\lambda_{ik}\psi\, dV$

where λ_{ik} commutes with γ_j, we then have

$$\frac{d\Lambda_{ik}}{dt} = - ic \int \psi^+ (\lambda_{ik}, \kappa)\, \psi \; dV$$

so that we reuire that

$$\int \psi^+ \left(\tfrac{1}{4}\gamma_{ik} + \lambda_{ik}, \kappa\right) \psi \; dV = 0 \qquad (2.11)$$

This condition is satisfied if $(\frac{1}{4}\gamma_{ik} + \lambda_{ik})$ commutes with κ. The conserved angular momentum J_{ik} is obtained from (1.9) by replacing γ_{ik} in the last term by $\gamma_{ik} + 4\lambda_{ik}$. The equation then describes a particle of spin (c.f. (1.12))

$$S_{ik} = \tfrac{\hbar}{4} \int \psi^+ \gamma_4 (\gamma_{ik} + 4\lambda_{ik})\, \psi \; dV. \qquad (2.12)$$

If $\lambda_{\mu\nu}$ commutes with γ_σ, the antisymmetrical part of $\Theta_{\mu\nu}$ may now be expressed as a divergence:

$$\mathcal{L}_{\mu\nu} = - \frac{i\hbar c}{8} \psi^+ (\gamma_{\mu\nu}, \kappa)\, \psi$$

$$= - \frac{i\hbar c}{2} \partial_\rho \left[\psi^+ \gamma_\rho \lambda_{\mu\nu} \psi \right]$$

Further $\Theta_{44} = \tilde{T}_{44} = T_{44} = - W(\psi^* \chi \psi)$ (2.13)

for an eigenstate ψ of $i\hbar \frac{\partial}{\partial t}$ belonging to the eigenvalue W.

The conditions

$$(\lambda_{\mu\nu}, \gamma_\sigma) = 0 \quad \left(\tfrac{1}{4}\gamma_{\mu\nu} + \lambda_{\mu\nu}, \kappa\right) = 0 \qquad (2.14)$$

are those used in Ref. 3.

The generalized Kemmer equation[4]

$$(\beta_\mu \partial_\mu + \kappa)\, \psi = 0 \qquad (2.15)$$

may be developed in a similar manner. Here κ is an operator which commutes with ∂_μ but not with β_μ. We

now define

$$\psi^{+} = i\,\psi^{*}\eta_4 X$$

where $\eta_4 = 2\beta_4^2 - 1$ and X possesses properties similar to those of Equation (2.2): $(X,\beta_\mu) = 0$, $(\eta_4 X, \kappa) = 0$.

It then follows that

$$\partial_\mu \psi^{+}\beta_\mu - \psi^{+}\kappa = 0 \qquad (2.16)$$

and that

$$s_\mu = \psi^{+}\beta_\mu\psi$$

is conserved. We then have

$$\beta_\nu\beta_\mu\kappa\,\partial_\nu\psi = \kappa\,\partial_\mu\psi$$

$$\partial_\nu\psi^{+}\kappa\beta_\mu\beta_\nu = \partial_\mu\psi^{+}\kappa$$

so that

$$\partial_\mu(\psi^{+}\kappa\psi) = \psi^{+}\beta_\nu\beta_\mu\kappa\partial_\nu\psi + \partial_\nu\psi^{+}\kappa\beta_\mu\beta_\nu\psi$$

In this case we define

$$T_{\mu\nu} = -\frac{ihc}{2}\left[\psi^{+}\beta_\nu\partial_\mu\psi - \partial_\mu\psi^{+}\beta_\nu\psi\right]$$

$$\Theta_{\mu\nu} = ihc\left[\psi^{+}(\beta_\nu\beta_\mu\kappa + \kappa\beta_\mu\beta_\nu)\psi - \delta_{\mu\nu}\psi^{+}\kappa\psi\right] \qquad (2.17)$$

so that

$$\partial_\nu T_{\mu\nu} = \partial_\nu\Theta_{\mu\nu} = 0$$

and $$\Theta_{\mu\nu} = T_{\mu\nu} + \frac{ihc}{2}\partial_\rho\left[\psi^{+}(\beta_\rho\beta_\mu\beta_\nu - \beta_\nu\beta_\mu\beta_\rho)\psi\right] \qquad (2.18)$$

$\Theta_{\mu\nu}$ is symmetrical only in the case in which κ is a c-number, so that we define

$$J_{ik} = -\frac{1}{c}\int (x_i\Theta_{k4} - x_k\Theta_{i4})\,dV - \hbar\,\Lambda_{ik}$$

where

$$\Lambda_{ik} = \int \psi^+ \beta_4 \lambda_{ik} \psi \, dV$$

and $(\lambda_{\mu\nu}, \beta_\rho) = 0$. Thus J_{ik} is conserved if

$$\int \psi^+ (\beta_{ik} + \lambda_{ik}, \kappa) \psi \, dV = 0 \tag{2.19}$$

the spin of the particle being (c.f. (1.13))

$$S_{ik} = \hbar \int \psi^+ \beta_4 (\beta_{ik} + \lambda_{ik}) \psi \, dV \tag{2.20}$$

We may now define a symmetrical energy-momentum tensor which differs from $\theta_{\mu\nu}$ only by a divergence:

$$\tau_{\mu\nu} = \frac{i\hbar c}{2}\Big[\psi^+ \Big\{ (\beta_\mu \beta_\nu + \beta_\nu \beta_\mu)\kappa + \kappa (\beta_\mu \beta_\nu + \beta_\nu \beta_\mu) \Big\} \psi - 2 \delta_{\mu\nu} \psi^+ \kappa \psi \Big] \tag{2.21}$$

$$= \theta_{\mu\nu} + \frac{i\hbar c}{2} \psi^+ (\beta_{\mu\nu}, \kappa) \psi$$

$$= \theta_{\mu\nu} - \frac{i\hbar c}{2} \psi^+ (\lambda_{\mu\nu}, \kappa) \psi$$

$$= \theta_{\mu\nu} + \frac{i\hbar c}{2} \partial_\rho (\psi^+ \beta_\rho \lambda_{\mu\nu} \psi) \tag{2.22}$$

Further

$$\tau_{44} = \theta_{44} = \frac{i\hbar c}{2} \psi^+ (\eta_4 \kappa + \kappa \eta_4) \psi$$

$$= -\frac{\hbar c}{2} \psi^\chi (\chi \kappa + \kappa \chi) \psi \tag{2.23}$$

$$\rightarrow -mc^2 \psi^\chi \psi \quad \text{for } \chi = 1, \ \kappa = \frac{mc}{\hbar} = \text{const.}$$

3. The generalized Dirac equation

If one of the ϵ_μ, λ_ν is a set of Dirac operators $\frac{1}{2}\gamma_\mu$ and the other is a set of Kemmer-Duffin operators β_ν, the resulting equation describes fermions with spin tensor given by

$$S_{\mu\nu} = -i\hbar\left[\beta_{\mu\nu} + \tfrac{1}{4}\gamma_{\mu\nu}\right]$$

Although the mass operator is the same in each case, the operators multiplying the p_μ are different, and we therefore obtain two distinct equations describing particles of spin $\frac{1}{2}$ and $\frac{3}{2}$. In this paper we consider only one of these (Eq.(1.14)). We use the notation $\gamma_i = \rho_2\underset{\sim}{\sigma}$ $(i = 1,2,3)$ $\gamma_4 = \rho_3$ and

$$\left.\begin{aligned} \underset{\sim}{\Sigma} &= -i\,(\beta_{23}, \beta_{31}, \beta_{12}) \\ \underset{\sim}{\lambda} &= -i\,(\beta_{14}, \beta_{24}, \beta_{34}) \\ \underset{\sim}{\beta} &= (\beta_1, \beta_2, \beta_3) \end{aligned}\right\} \qquad (3.1)$$

so that the spin of the particle is

$$S = \hbar\left(\tfrac{1}{2}\,\underset{\sim}{\sigma} + \underset{\sim}{\Sigma}\right) \qquad (3.2)$$

The spin $\frac{1}{2}$ states are therefore characterized by

$$\underset{\sim}{\sigma}\cdot\underset{\sim}{\Sigma} = -2 \quad (\Sigma^2 = 2) \quad (\downarrow, \uparrow) \qquad (3.3)$$

or by

$$\underset{\sim}{\sigma}\cdot\underset{\sim}{\Sigma} = 0 \quad (\Sigma^2 = 0) \quad (\uparrow, ..) \qquad (3.4)$$

The spin $\frac{3}{2}$ states are similarly characterized by

$$\underset{\sim}{\sigma}\cdot\underset{\sim}{\Sigma} = 1 \quad (\Sigma^2 = 2) \quad (\uparrow, \uparrow) \qquad (3.5)$$

If in Equation (1.14) m, m_o, m' are real parameters,

the conserved density (2.4) may be written

$$s_\mu = - ic \psi^* \gamma_4 \eta_4 \gamma_\mu \psi \tag{3.6}$$

where $\eta_4 = 2\beta_4^2 - 1$, so that η_4 commutes with β_4 and anticommutes with β_1, β_2, β_3. For the special case $m' = 0$, however, we note that $\eta_5 = \eta_1 \eta_2 \eta_3 \eta_4$ commutes with the Hamiltonian so that in this case we may define another conserved four-vector j_μ which we identify with the charge-current density:

$$j_\mu = - \frac{e_o c}{2} \psi^+ (1 + \eta_5) \gamma_\mu \psi \tag{3.7}$$

where $\psi^+ = i \psi^* \gamma_4 \eta_4$

The charge density is therefore, for $m' = 0$,

$$\rho = - i j_{4/c} = - \frac{e_o}{2} \psi^* \eta_4 (1 + \eta_5) \psi \tag{3.8}$$

In the 5 x 5 representation of the β_μ, η_5 is diagonal with the value -1 for the first four elements and +1 for the fifth, while in the 10 x 10 representation it is +1 for the first six elements and -1 for the others.

We note that for m' imaginary the four-vector $\psi^+ \eta_5 \gamma_\mu \psi$ is strictly conserved, and that for small imaginary m' the currents (3.6) (3.7) are separately conserved only approximately.

In the rest system of the particle ($\underset{\sim}{p} = 0$) the energy operator according to Equation (1.14) is given by

$$\frac{W}{c^2} \psi = \left[2\rho_3 m - 2m_o (\rho_3 \underset{\sim}{\sigma} \cdot \underset{\sim}{\xi} + i\rho_2 \underset{\sim}{\sigma} \cdot \underset{\sim}{\lambda}) - m' (i\rho_1 \underset{\sim}{\sigma} \cdot \underset{\sim}{\beta} - \beta_4) \right] \psi \tag{3.9}$$

The eigenvalues of W for the case m' = 0 have been computed in Ref. 5 (although their physical interpretation in that reference is incorrect). More generally, we now write

$$\frac{W - 2mc^2}{2m_o c^2} = \alpha, \quad \frac{W+2mc^2}{2m_o c^2} = \beta, \quad \frac{m'}{2m_o} = \epsilon$$

For the 5 x 5 representation[4] of the β_μ, the 20-component spinor ψ decomposes into four separate 5-component spinors, e.g.

$$\begin{bmatrix} \alpha & 1 & -i & 1 & i\epsilon \\ -1 & \beta & -i & 1 & i\epsilon \\ -i & i & \beta & i & -\epsilon \\ -1 & 1 & -i & \beta & i\epsilon \\ -i\epsilon & i\epsilon & \epsilon & i\epsilon & \alpha \end{bmatrix} \begin{bmatrix} \psi_1 \\ \psi_{17} \\ \psi_{18} \\ \psi_{14} \\ \psi_5 \end{bmatrix} = 0 \qquad (3.10)$$

In this sub-space

$$\underset{\sim}{\sigma}\cdot\underset{\sim}{\xi} = \begin{pmatrix} 0 & 0, & 0, & 0, & 0 \\ 0, & 0, & i, & -1, & 0 \\ 0, & -i, & 0, & -i, & 0 \\ 0, & -1 & i, & 0, & 0 \\ 0, & 0, & 0, & 0, & 0 \end{pmatrix} \qquad S_z = \tfrac{1}{2}\hbar \begin{pmatrix} 1, & 0, & 0, & 0, & 0 \\ 0, & -1, & -2i, & 0, & 0 \\ 0, & 2i, & -1, & 0, & 0 \\ 0, & 0, & 0, & 1, & 0 \\ 0, & 0, & 0, & 0, & 1 \end{pmatrix}$$

(3.11)

The characteristic equations for the other spinors are obtained by replacing α by $-\beta$, and β by $-\alpha$ (i.e. changing the sign of W) and by reversing the spin direction.

Three solutions of (3.10) for states of spin $\frac{1}{2}$

($\underset{\sim}{\sigma}.\underset{\sim}{\xi} = -2$ or 0) are

$$\Psi_{\frac{1}{2}}^{\uparrow} = \begin{pmatrix} \alpha(\beta - 1) \\ \alpha(\alpha + 1) \\ i\alpha(\alpha + 1) \\ \alpha(\alpha + 1) \\ i\epsilon(\beta - 3\alpha - 4) \end{pmatrix} \tag{3.12}$$

with

$$\alpha(\alpha\beta + 2\alpha + 3) = \epsilon^2(\beta - 3\alpha - 4) \tag{3.13}$$

These are also eigenstates of S_z belonging to the eigenvalue $\frac{1}{2}\hbar$.

For m' = 0, Equation (3.13) breaks up into three states

$$\begin{aligned} &\alpha = 0, \quad W = 2mc^2, \quad \eta_5 = 1, \quad \eta_4 = 1 \\ &\alpha^3 + 2\alpha + 3 = 0, \quad \eta_5 = -1. \end{aligned} \tag{3.14}$$

According to (3.8), the first of these is negatively charged and has a mass $2m = 2594m_e$, and we identify it with the Ξ^- hyperon. The other two particles of Equation (3.14) are neutral ($\eta_5 = -1$) and we identify them with the neutron and Ξ^0. (see Table I).

Equation (3.11) also gives rise to two states of spin 3/2 ($\underset{\sim}{\sigma}.\underset{\sim}{\xi} = 1$):

$$\Psi_{3/2,\,-3/2} = \begin{pmatrix} 0 \\ 1 \\ -i \\ 0 \\ 0 \end{pmatrix}; \quad \Psi_{3/2,\,\frac{1}{2}} = \begin{pmatrix} 0 \\ 1 \\ 1 \\ -2 \\ 0 \end{pmatrix} \tag{3.15}$$

where the second suffix on Ψ refers to the eigenvalue of S_z. These states have the same mass:

$$\beta = 1, \quad W = 2(m_o - m)c^2 \qquad (\uparrow, \uparrow) \tag{3.16}$$

with $\eta_4 = -1$, $\eta_5 = -1$. We identify this particle with the Y_o^{**} resonance.

For the values of m_o,m given in Equation (1.15) the masses of these particles assume the values given in Table I.

In the 10 x 10 representation of the β_μ, the spinor Ψ has 40 components, decomposing into four spinors of ten components each, e.g.

$$\begin{vmatrix}
\alpha & -i & 1 & 0 & 1 & i & i\epsilon & 0 & 0 & -i\epsilon \\
i & \alpha & -i & -1 & 0 & 1 & 0 & i\epsilon & 0 & -\epsilon \\
1 & i & \alpha & i & -1 & 0 & 0 & 0 & i\epsilon & i\epsilon \\
0 & 1 & i & \beta & i & -1 & 0 & -i\epsilon & \epsilon & 0 \\
-1 & 0 & 1 & -i & \beta & i & i\epsilon & 0 & -i\epsilon & 0 \\
i & -1 & 0 & -1 & -i & \beta & \epsilon & i\epsilon & 0 & 0 \\
-i\epsilon & 0 & 0 & 0 & i\epsilon & -\epsilon & \alpha & -i & 1 & 1 \\
0 & -i\epsilon & 0 & -i\epsilon & 0 & i\epsilon & i & \alpha & -i & -i \\
0 & 0 & -i\epsilon & -\epsilon & -i\epsilon & 0 & 1 & i & \alpha & -1 \\
-i\epsilon & \epsilon & i\epsilon & 0 & 0 & 0 & -1 & -i & 1 & \beta
\end{vmatrix}
\begin{vmatrix}
\psi_1 \\ \psi_2 \\ \psi_{13} \\ \psi_{24} \\ \psi_{25} \\ \psi_{36} \\ \psi_7 \\ \psi_8 \\ \psi_{19} \\ \psi_{40}
\end{vmatrix}
= 0 \tag{3.17}$$

In this sub-space

$$\underset{\sim}{\sigma}.\underset{\sim}{\xi} = \begin{pmatrix} x & 0 & 0 & . \\ 0 & x & 0 & . \\ 0 & 0 & x & . \\ . & . & . & . \end{pmatrix}; \qquad S_z = \tfrac{1}{2}\hbar \begin{pmatrix} y & 0 & 0 & . \\ 0 & y & 0 & . \\ 0 & 0 & y & . \\ . & . & . & -1 \end{pmatrix}$$

with

$$x = \begin{pmatrix} 0 & -i & 1 \\ i & 0 & -i \\ 1 & i & 0 \end{pmatrix} \qquad y = \begin{pmatrix} 1, & -2i, & 0 \\ 2i, & 1, & 0 \\ 0, & 0, & -1 \end{pmatrix}$$

the dots representing a single row and column.

Four solutions of (3.17) for states of spin $\frac{1}{2}$ ($\underset{\sim}{\sigma}.\underset{\sim}{\xi} = -2$ or 0) are given by

$$\begin{bmatrix} \psi \\ -i\psi \\ -\psi \\ \phi \\ -i\phi \\ -\phi \\ \chi \\ -i\chi \\ -\chi \\ \lambda \end{bmatrix}$$

with

$$\begin{aligned} (\alpha-2)\psi - 2i\phi + i\epsilon\chi - i\epsilon\lambda &= 0 \\ -2i\psi + (\beta+2)\phi - 2\epsilon\chi &= 0 \\ -i\epsilon\psi + 2\epsilon\phi + (\alpha-2)\chi + \lambda &= 0 \\ -3i\epsilon\psi - 3\chi + \beta\lambda &= 0 \end{aligned}$$

so that

$$(\alpha\beta+2\alpha-2\beta)(\alpha\beta-2\beta+3) + \epsilon^2(7\alpha\beta-\beta^2+6\alpha-2\beta-24) + 12\epsilon^4 = 0 \qquad (3.18)$$

$(\downarrow,\uparrow)$ $(\uparrow,.)$

For m' = 0, Equation (3.18) separates into four states, two of which (Ξ^0 and N) also appeared in the (5 x 5) representation (Equation (3.14)):

$$\left.\begin{aligned} \alpha\beta - 2\beta + 3 = 0, \quad \eta_5 = -1 \\ \alpha\beta + 2\alpha - 2\beta = 0, \quad \eta_5 = +1 \end{aligned}\right\} \qquad (3.19)$$

The first of Equations (3.19) is obtained from the second of Equations (3.14) by replacing α by $-\beta$ and β by $-\alpha$, i.e. by changing the sign of the energy. The second of Equations (3.19) now represents charged ($\eta_5 = 1$) particle and anti-particle states

$$W = \pm 2c^2 \sqrt{m^2 + 4mm_o}$$

which, with the same value as before for m and m_o have the mass of the proton i.e. $\pm \sqrt{2}m = 1835\, m_e$.

The corresponding spin $\frac{3}{2}$ states are described by the following six solutions of the same equation (3.17) for the case $\underset{\sim}{\sigma}.\underset{\sim}{\xi} = 1$:

$$\Psi_{\frac{3}{2},\frac{3}{2}} = \begin{vmatrix} \psi \\ i\psi \\ 0 \\ \phi \\ i\phi \\ 0 \\ \chi \\ i\chi \\ 0 \\ 0 \end{vmatrix} \qquad \Psi_{3/2,-\frac{1}{2}} = \begin{vmatrix} \psi \\ -i\psi \\ 2\psi \\ \phi \\ -i\phi \\ 2\phi \\ \chi \\ -i\chi \\ 2\chi \\ 0 \end{vmatrix}$$

with

$$(\alpha+1)\psi + i\phi + i\epsilon\chi = 0$$
$$i\psi + (\beta-1)\phi + \epsilon\chi = 0$$
$$-i\epsilon\psi - \epsilon\phi + (\alpha+1)\chi = 0$$

so that

$$(\beta-2)\phi = -i(\alpha+2)\psi \ , \quad (\alpha+1)\ (\beta-2)\chi = -i\epsilon(\alpha-\beta+4)\psi$$

and

$$(\alpha+1)\ (\alpha\beta+\beta-\alpha) + \epsilon^2(\alpha-\beta+4) = 0 \quad (\uparrow,\uparrow) \qquad (3.20)$$

For $m' = 0$ the spin $\frac{3}{2}$ solution corresponding to the proton state (second of Equations (3.19)) is ($\chi = 0$)

$$\alpha\beta + \beta-\alpha = 0, \quad \eta_5 = +1$$

again representing charged particle anti-particle states

$$W = \pm 2c^2 \ \sqrt{m^2 - 2mm_o}$$

Again with the same values of m and m_o, this leads to a spin $\frac{3}{2}$ excited proton state at 2900 m_e. The corresponding neutral state from Equation (3.20) is given by $\alpha = -1$, for which $\psi = 0$, $\phi = 0$ so that $\eta_5 = -1$. Its mass is therefore $2(m-m_o) = 2918m_e$. We are therefore led to identify the solutions of Equation (3.20) with the N^{**} resonance.

Finally, for the 1 x 1 representation of the β_μ ($\beta_\mu = 0$) Equation (3.9) describes a particle of spin $\frac{1}{2}$, mass $2m = 2594m_e$, ($\eta_5 = 1$). According to (3.8) its charge is positive. It could transform to the other states only through an interaction which could not be expressed in

terms of the three inequivalent irreducible representations of the β_μ considered here.

In general, then, apart from the sign of W, there are eight distinct values of the rest-energy given by Equation (3.9) for states of spin $\frac{1}{2}$. These are solutions of the equations

$$W = 2m \tag{3.21}$$

$$W^3 + 2(2m_o - m)W^2 + 2\left[6m_o^2 - 8mm_o - 2m^2 + m'^2\right]W + 8m(m-m_o)(m+3m_o) - 8m'^2(m-m_o) = 0 \tag{3.22}$$

$$W^4 - 4m_oW^3 + 2\left[6m_o^2 - 12mm_o - 4m^2 + 3m'^2\right]W^2 + 4\left[4m_om^2 + 16m\,m_o^2 - mm'^2 + 2m_om'^2\right]W + 16\,m(m+4m_o)(m^2+2mm_o - 3m_o^2) - 32m'^2(m^2+mm_o + 3m_o^2) + 12m'^4 = 0 \tag{3.23}$$

For the case $m' = 0$, these equations reduce simply to four distinct eigenvalues:

$$\left.\begin{aligned} W &= 2m \\ W &= -2m_o \pm 2\sqrt{m^2+2mm_o-2m_o^2} \end{aligned}\right\} (3.21),(3.22)$$

$$\left.\begin{aligned} W &= 2m_o \pm 2\sqrt{m^2+2mm_o-2m_o^2} \\ W &= \pm 2\sqrt{m(m+4m_o)} \end{aligned}\right\} (3.23)$$

$$\Big\} (3.24)$$

Similarly, for states of spin 3/2, there are in general four eigenvalues, which are given by

$$W = 2(m_o - m) \tag{3.25}$$

$$W^3 + 2(m_o-m)\,W^2 + 4m(2m_o-m)W + 4(m-2m_o)\,(2m^2-2mm_o-m'^2) = 0 \tag{3.26}$$

For $m' = 0$, these reduce to two distinct values

$$W = 2(m_o - m) \qquad (3.25)$$

$$\left.\begin{aligned} W &= -2(m_o - m) \\ W &= \pm\, 2\sqrt{m(m-2m_o)} \end{aligned}\right\} (3.26)$$

All of these eigenvalues for the case $m' = 0$ are given by the formula

$$W = m_o(S_2 - S_1) \pm \sqrt{[\,m_o(S_1+S_2) - 2m\,]^2 - 4m_o^2(\sigma.\lambda)^2} \tag{3.27}$$

where S_1 and S_2 are the eigenvalues of $\underset{\sim}{\sigma}.\underset{\sim}{\xi}$:

Equation Number	Components	S_1	S_2	$(\underset{\sim}{\sigma}.\underset{\sim}{\lambda})^2$
(3.12)	First four	-2	0	3
(3.12)	Fifth	0	0	0
(3.15)	All non-zero Components	1	1	0
(3.18)	First six	-2	-2	4
(3.18)	Last four	-2	0	3
(3.20)	First six	1	1	1
(3.20)	Last four	1	1	0

While the calculated masses agree with experimental observation to better than 2%, the mass differences

N - P and $\Xi^- - \Xi^0$ are not accurately described by the case m' = 0. We therefore suppose that m' is a small imaginary quantity, the magnitude of which we adjust to give the correct neutron-proton mass difference. Writing $m_o = -am$ ($a = \frac{1}{8}$) as before, $m' = -i\delta m$ ($\delta << 1$) and neglecting terms of higher order than δ^2, we find that Equation (3.23) becomes, with $W = 2mc^2x$

$$\left[x^2 - \frac{\delta^2}{2a^2}(2a+1)x + 4a-1 - \frac{\delta^2}{2a^2}(1-4a^2)\right]$$

$$\left[x^2 + (2a + \frac{\delta^2}{2a^2}(2a+1))x + (3a^2+2a-1) - \frac{\delta^2}{2a^2}(3a^2-2a-1)\right] = 0$$

The masses of the N, Ξ^0, P states are therefore

$$\left.\begin{aligned} M_N &= (1.44558 - 79.51\ \delta^2)\ m \\ M'_{\Xi^0} &= (1.94558 + 0.49\ \delta^2)\ m \\ M_p, M_p' &= \left[1.41421 + \delta^2(42.43 \pm 40.0)\right] m \end{aligned}\right\} \qquad (3.28)$$

We therefore obtain <u>two</u> proton states, and the mass difference between the neutron and the lower of these is

$$\delta m = (.03137 - 81.94\ \delta^2)\ m$$

This has the experimental value of 2.53 m_e if $\delta^2 = 3.59 \times 10^{-4}$, and the lower proton state then has the correct experimental value of 1836.1 m_e if as before m = 1297 m_e. The theory would then predict the existence of an excited proton state lying 37.2 m_e = 19 MeV above the ground state.

The mass m'_{Ξ^0} of the Ξ^0 particle given by (3.28) is 2524 m_e, but this is not the Ξ^0 which is coupled

to the Ξ^-, for the evaluation of which we return to Equation (3.22). We obtain from this equation

$$m_{\Xi^-} = (2 - \frac{80\,\delta^2}{3})\,m = 2581\ m_e$$

$$m_{\Xi^0} = (1.94558 + 27.68\,\delta^2)m = 2537\ m_e$$

$$m'_N = (1.44558 + 1.01\,\delta^2)\,m = 1875\ m_e$$

giving an excited neutron state also lying 19 MeV above the ground state. However, the value of m_{Ξ^0} computed here is 1°/o too low, so that the magnitude $m'_N - m_N$ may also be in error and may change by a large fraction when the radiative corrections are taken into account.

The neutral and charged spin 3/2 resonances given by Equations (3.25) (3.26) now split as follows:

$$Y_o^{**} \qquad 2.250\ m = 2918\ m_e \qquad \text{(Eq.3.25)}$$

$$N^{**0} \qquad 2.222\ m = 2882\ m_e$$

$$N^{**+-} \qquad \left\{ \begin{array}{l} 2.265\ m = 2941\ m_e \\ 2.236\ m = 2888\ m_e \end{array} \right\} \qquad \text{(Eq.3.26)}$$

These values lie below the observed spin 3/2 resonances by one or two percent.

Equation Number	Spin	η_4	η_5	Charge	Particle	Characteristic Equation ($m'=0$)	Calculated mass $(a=\frac{1}{8})m'=0$	Calculated mass $a=\frac{1}{8}$ $m'=24im_e$	Observed mass
(3.13)	$\frac{1}{2}$	1	1	-	Ξ^-	$\alpha=0$	2594	2581	2579
	$\frac{1}{2}$	$\pm$	-1	0	Ξ^0, $\bar{\Xi}^0$	$\alpha\beta+2\alpha+3=0$	2524	2537	2565
	$\frac{1}{2}$	$\pm$	-1	0	N', $\bar{N}'$		1876	1876	
(3.16)	$\frac{3}{2}$	-1	-1	0	Y_o^{**}	$\beta=1$	2918	2918	2975
(3.18)	$\frac{1}{2}$	$\pm$	-1	0	Ξ'^0, $\bar{\Xi}'^0$	$\alpha\beta-2\beta+3=0$	2524	2524	
	$\frac{1}{2}$	$\pm$	-1	0	$\bar{N}$,N		1876	1838.6	1838.6
	$\frac{1}{2}$	$\pm$	1	$\pm$	P,$\bar{P}$	$\alpha\beta+2\alpha-2\beta=0$	1835	1836.1	1836.1
	$\frac{1}{2}$	$\pm$	1	$\pm$	$\bar{P}'$,P'		1835	1873.3	
(3.20)	$\frac{3}{2}$	$\pm$	1	$\pm$	N^{**+-}	$\alpha\beta+\beta-\alpha=0$	2900	2941	2955
	$\frac{3}{2}$	$\pm$	1	$\pm$			2900	2888	
	$\frac{3}{2}$	$\pm$	-1	0	N^{**0}	$\alpha = -1$	2918	2882	
(3.21)	$\frac{1}{2}$	-1	1	+		$\alpha = 0$	2594	2594	

TABLE I. Spin, charge and mass states of Equation (3.9) for $m = 1297m_e$, $m_o = -\frac{1}{8}m$, $m' = 0$ or $24\ i\ m_e$. The observed and calculated masses are given in terms of the unit m_e. States which are not eigenstates of η_4 are denoted by $\pm$.

4. Generalization to higher spin states

To allow for the possibility of resonances with higher values of spin and charge (although states of spin $\frac{1}{2}$ and $\frac{3}{2}$ also appear in such a generalization) we replace $\beta_{\mu\nu}$ in Equation (1.15) by the sum of n commuting β operators so that the spin of the particle becomes ($\epsilon_\mu = \frac{1}{2}\gamma_\mu$)

$$S_{\mu\nu} = -i\hbar\left[\tfrac{1}{4}\gamma_{\mu\nu} + \sum_{i=1}^{n} \beta_{\mu\nu}^{(i)}\right]$$

or
$$\underset{\sim}{S} = \hbar\left[\tfrac{1}{2}\,\underset{\sim}{\sigma} + \underset{\sim}{\Sigma}\right] \qquad (4.1)$$

where $\underset{\sim}{\Sigma} = \underset{\sim}{\Sigma}^{(1)} + \underset{\sim}{\Sigma}^{(2)} + \ldots + \underset{\sim}{\Sigma}^{(n)}$.

If we also write

$$\underset{\sim}{\lambda} = \underset{\sim}{\lambda}^{(1)} + \underset{\sim}{\lambda}^{(2)} + \ldots + \underset{\sim}{\lambda}^{(n)}$$

the eigenvalue equation for the rest-energy of the particle assumes the same form as before (Eq. 3.9), $m'=0$, $m_o = -\frac{1}{8}m$)

$$\frac{W}{2mc^2}\psi = \left[\rho_3 + \tfrac{1}{8}\left(\rho_3\,\underset{\sim}{\sigma}\cdot\underset{\sim}{\Sigma} + i\rho_2\underset{\sim}{\sigma}\cdot\underset{\sim}{\lambda}\right)\right]\psi \qquad (4.2)$$

with $m = 1297\, m_e$.

We now consider the quantity

$$J_\mu = (-1)^n\, ie_o c\, \psi^* \eta_4^{(1)}\eta_4^{(2)}\ldots\eta_4^{(n)}\gamma_4\left(1 + \sum_{i=1}^{n}\Gamma^{(i)}\right)\gamma_\mu\psi \qquad (4.3)$$

where

$$\Gamma^{(i)} = \tfrac{1}{2}\left(\eta_5^{(i)} - 1\right)$$

In addition to being conserved, this four-vector has the

property that, when the last set of β_μ vanishes ($\beta_{\mu 2}^{(n)} = 0$, $\eta_4^{(n)} = -1$, $\eta_5^{(n)} = 1$, $\Gamma^{(n)} = 0$) it reduces to that obtained from Equation (4.3) by replacing n by n-1. For n = 1, the expression for j_μ becomes

$$j_\mu = -\frac{ie_0c}{2}\psi^* \eta_4^{(1)}\gamma_4 (1+\eta_5^{(1)})\gamma_\mu \psi$$

which is the current density adopted in the above study of the case n = 1 (Equation (3.7)).

Since the eigenvalues of $\eta_4^{(1)}$ are ± 1, and those of $\Gamma^{(1)}$ are 0, -1, Equation (4.3) describes charge states for which the charge ranges from (n-1) e_0 to $-e_0$, or from e_0 to $-(n-1)e_0$, so that the maximum isotopic spin which a particle can have is n/2. From (4.1), the maximum spin which a particle can have is $n + \frac{1}{2}$.

Since

$$\underset{\sim}{\Sigma} \times \underset{\sim}{\Sigma} = \underset{\sim}{\lambda} \times \underset{\sim}{\lambda} = i\underset{\sim}{\Sigma}$$

as before, we find that

$$\underset{\sim}{\sigma}\cdot\underset{\sim}{\Sigma} = \frac{1}{2}\left(-1 \pm \sqrt{1+4\Sigma^2}\right)$$

so that the maximum spin value $J_m = n + \frac{1}{2}$ is characterized by the values

$$\Sigma^2 = n(n+1) = J_m^2 - \frac{1}{4}, \quad \underset{\sim}{\sigma}\cdot\underset{\sim}{\Sigma} = n = J_m - \frac{1}{2}$$

Lower spin values $J = n + \frac{1}{2} - r$ (r an integer $1 \leq r \leq n$) are characterized by the values

$$\left.\begin{aligned} \Sigma^2 &= (n-r)(n-r+1) = J^2 - \tfrac{1}{4}, \quad \underset{\sim}{\sigma}\cdot\underset{\sim}{\Sigma} = n-r = J - \tfrac{1}{2} \\ \Sigma^2 &= (n-r+1)(n-r+2) = (J+\tfrac{1}{2})(J+3/2), \quad \underset{\sim}{\sigma}\cdot\underset{\sim}{\Sigma} = -(n-r+2) = -(J+3/2 \end{aligned}\right\}$$

In the case in which the spin assumes its maximum possible value for given n ($J_m = n + \frac{1}{2}$) the wave function of the particle is an eigenstate of $\underset{\sim}{\sigma}.\underset{\sim}{\xi}$ and this allows us to obtain from Equation (4.3) a simple expression for the rest-energy:

$$\frac{W}{2mc^2} = \pm \sqrt{(\frac{15}{16} + \frac{J_m}{8})^2 - \frac{1}{64}(\sigma.\lambda)^2} \qquad (4.4)$$

For the case n = 1, $J_m = \frac{3}{2}$, this quantity becomes

$$\frac{W}{2mc^2} = \pm \frac{1}{8} \sqrt{81 - (\sigma.\lambda)^2}$$

which for $(\underset{\sim}{\sigma}.\underset{\sim}{\lambda})^2 = 0$ led above to the Y_o^{**} resonance (Equation 3.16)) and the neutral component of the N^{**} resonance (Equation (3.20)) while for $(\underset{\sim}{\sigma}.\underset{\sim}{\lambda})^2 = 1$, it led to the charged components of the N^{**} resonance (Equation (3.20)) all of spin 3/2.

In general, Equation (4.4) is not very sensitive to the value of $(\underset{\sim}{\sigma}.\underset{\sim}{\lambda})^2$, and the states of highest mass given by this equation ($(\underset{\sim}{\sigma}.\underset{\sim}{\lambda})^2 = 0$) with the same value of m as used previously, have rest-energies

$$W = (2432 + 324.3\ J_m)\ m_e c^2. \qquad (4.5)$$

as listed in Table 2.

Theory	n	1	2	3	4	5
	spin	3/2	5/2	7/2	9/2	11/2
	Mass (m_e)	2918	3243	3567	3891	4215
Experiment	Resonance	Y_o^{**}	N^{***}	Y_o^{***}	N^{****}	
	spin	3/2	5/2	$\geq 3/2$	?	
	Mass (m_e)	2975	3290	3552	3800	

TABLE 2. Highest mass and spin values possible for given values of n, and comparison with experimentally observed resonances.

The theory therefore predicts spin values for the particles and resonances listed below:

Particle	Energy (MeV)	Spin
Ξ	1320	$\frac{1}{2}$
Y_1^{**}	1685	$\frac{5}{2}$
Y_o^{***}	1815	$\frac{7}{2}$
N^{****}	1922	$\frac{9}{2}$

Another Y resonance is expected at 1940 MeV, with spin $\frac{9}{2}$, and further N and Y resonances with spin $\frac{11}{2}$, $\frac{13}{2}$ etc. at intervals of 160 MeV until their line widths cause them to become experimentally indistinguishable. The neutral component of the spin $\frac{3}{2}$ N^{**} resonance at 1512 MeV is

predicted to lie 10 MeV above the charged component, but radiative corrections could materially change this value. Predictions concerning the isotopic spins of these higher resonances require further analysis, and the validity of the whole theory will depend on the charge spin and mass eigenvalues presently being derived for the other states described by Equation (1), and on the calculated selection rules and transition probabilities between these states.

I wish to thank the United States Office of Naval research and the Independent Research program at Space Technology Laboratories for support, and the Department of Mathematics, Imperial College, London, for hospitality, during the development of this work.

References

1. Bhabha, H. J. and Corben, H.C.
Proc. Roy. Soc. (London) A 178, 273 (1941)
Shanmugadhasan, S., Proc. Camb. Phil. Soc. 43 106 (1947)
2. Corben, H. C. Proc. Nat. Acad. Sci. (U.S.A.) 48 387 (1962).
3. Corben, H. C. Proc. Nat. Acad. Sci. (U.S.A.) 48 1559 (1962)
4. Kemmer, N., Proc. Roy. Soc. (London) A 173 91 (1939).
5. Corben, H. C. Proc. Nat. Acad. Sci. (U.S.A.) 48 1746 (1962).