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TECHNICAL REPORT: MATHEMATICS

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THE DUAL SIMPLEX METHOD AND ITS APPLICATION TO THE SYNTHESIS OF MINIMAL WEIGHTS

by

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SUNNYVALE, CALIFORNIA

NOTICE

FOREWORD

This study was performed by Dr. Sze-Tsen Hu, Professor of Mathematics, University of California at Los Angeles, while acting as consultant to the Electronic Sciences Laboratory, Lockheed Missiles & Space Company, Sunnyvale, California.

ABSTRACT

This report describes an algorithm for applying the dual simplex method in linear programming to the problem of determining whether or not a given switching function is linearly separable. Further, where separability is possible, a description is given of the application of the dual simplex method to finding the most economic system of weights and threshold.

A detailed elementary exposition of the dual simplex method is given in Sections 2 through 4. A completely worked out illustrative numerical example is presented in Section 5. In Sections 6 through 8, the dual simplex method is applied to the problem of determining minimal weights and threshold; this is illustrated by a numerical example which reduces to the example given in Section 5.

CONTENTS

Section 1
INTRODUCTION

Linear programming was first applied to the general synthesis problem of linear separability of switching functions by R. C. Minnick (Ref. 1). Using Minnick's method for a switching function of n variables, it becomes necessary to solve 2^n equations in $2(n + 1 + 2^n)$ nonnegative variables in such a way that the cost function will be minimized (Ref. 1, p. 11). For $n = 7$, which is the lowest dimension of the problem not yet solved, it is necessary to solve 128 equations in 272 nonnegative variables for each of the 2^{128} switching functions of seven variables. This is by far too large a task for even the best electronic computers to accomplish.

In a joint paper of S. Muroga, I. Toda, and S. Takasu (Ref. 2), which appeared two months following Ref. 1, linear programming was also applied to the general synthesis problem. Unlike Minnick, the authors of Ref. 2 introduced a reduction process which reduced the number of inequalities by applying the deletion law in Boolean algebra. This method is considerably simpler than that of Minnick but not simple enough for actual computation.

In an earlier report by the author (Ref. 3), linear programming was applied to the Willis synthesis method (Ref. 4) for finding canonical separating systems; the canonical partial order was fully used and, hence, the number of inequalities reduced to a minimum. The solution of the resulting system of inequalities for any given regular switching function of n variables depends completely on that of a system of $n + 2$ linear equations in accordance with a well-known theorem on linear inequalities. The problem of solving this system of $n + 2$ linear equations is made into a canonical minimum problem of linear programming by introducing one more variable. Then, the usual simplex method is applied to this canonical minimum problem.

In another of the author's reports (Ref. 5), the problem of finding the most economic separating systems was formulated in the form of a standard minimum problem of linear programming. It was proved in Ref. 5 that this standard minimum problem has optimal solutions if and only if it is feasible. However, the usual method in linear programming for solving a standard minimum problem consists of two major steps: first, one must determine whether or not the problem is feasible and, if so, find a feasible solution of the problem; second, using the feasible solution obtained in the first step, one can find an optimal solution of the problem by means of the usual simplex method. The first step can be accomplished by the method given in Ref. 3 which involves another application of the simplex method. It appears that one has to use the simplex method twice to get a set of minimal weights and threshold. Fortunately, this apparent complication is by no means inevitable. In fact, in a recent report by the author (Ref. 6), an algorithm was described for solving this problem by applying the usual simplex method on its dual maximum problem.

The objective of the present report is to give a simpler direct method of solving the minimization problem by making use of the dual simplex method of C. E. Lemke (Ref. 7). The minimization problem is always dually feasible no matter whether the given regular switching function is linearly separable or not. Hence, the problem can be solved by applying the dual simplex once, not twice.

For the convenience of the reader, a detailed exposition of the dual simplex method in the form suitable for our application is given in Sections 2 through 4. An illustrative numerical example is presented in Section 5. In Sections 6 and 7, we will review briefly the preliminaries and formulate the synthesis and minimization problem. The solution of the problem by means of the dual simplex method is given and illustrated in Section 8.

Section 2
STANDARD PROBLEM IN LINEAR PROGRAMMING

A standard problem in linear programming is to find nonnegative real numbers which minimize (or maximize) a given linear function subject to a given system of linear inequalities. Since a maximum problem is reduced to a minimum problem simply by multiplying the given linear function with -1, we will describe and study the standard minimum problem only.

For this purpose, let us consider a given linear function

$$a_{o,o} + \sum_{j=1}^{q} a_{o,j}\, t_j \tag{2.1}$$

of q variables $t_1, \cdots, t_q$, where $a_{o,o}, a_{o,1}, \cdots, a_{o,q}$ are given real numbers. On the other hand, let

$$a_{i,o} + \sum_{j=1}^{q} a_{i,j}\, t_j \geqq 0 \tag{2.2}$$

where $i = 1, 2, \cdots, p$, be a given system of p inequalities in the same variables $t_1, \cdots, t_q$ with given real coefficients $a_{i,j}$ and constant terms $a_{i,o}$. Then, the <u>standard minimum problem</u> in linear programming is the problem of finding <u>nonnegative real numbers</u> $t_1, \cdots, t_q$ <u>which minimize the given linear function</u> (2.1) <u>subject to the system</u> (2.2) <u>of</u> p <u>linear inequalities</u>.

Now, let y denote the given linear function (2.1) and let x_i , $(i = 1, 2, \cdots, p)$, denote the linear function on the left side of the inequality (2.2). Consider $y, x_1, \cdots, x_p$ also as variables. Then, the standard minimum problem described above can be restated as follows:

Problem: Find the minimum value of the variable y subject to the following $p + 1$ linear equations:

$$y = a_{o,o} + \sum_{j=1}^{q} a_{o,j}\, t_j \tag{2.3}$$

$$x_i = a_{i,o} + \sum_{j=1}^{q} a_{i,j}\, t_j \tag{2.4}$$

where $i = 1, 2, \cdots, p$, and the condition that

$$x_i \geqq 0, \qquad (i = 1, 2, \cdots, p) \,, \tag{2.5}$$

$$t_j \geqq 0, \qquad (j = 1, 2, \cdots, q) \,. \tag{2.6}$$

Let A denote the $p + 1$ by $q + 1$ matrix

$$A = \left\| \, a_{i,j} \, \right\| \tag{2.7}$$

where $0 \leqq i \leqq p$ and $0 \leqq j \leqq q$.

A column β in A is said to be (lexicographically) positive,

$$\beta \geqq 0$$

provided that the first nonzero entry of β , counting from top down, is positive. A column β in A is said to be (lexicographically) greater than another column γ in A provided that

$$\beta - \gamma > 0 \,.$$

Hereafter, we will omit the word "lexicographically" when there is no danger of misunderstanding.

The matrix A is said to be dually feasible in case the q columns

$$\alpha_j = \left(a_{o,j}, a_{1,j}, \cdots, a_{p,j}\right) \geqq 0$$

where $j = 1, 2, \cdots, q$. Note that the leading column α_o in A is not required to be positive or zero for the dual feasibility of A . In particular, A is dually feasible if

$$a_{o,j} > 0 , \qquad (j = 1, 2, \cdots, q) .$$

If the matrix A is dually feasible, then our standard minimum problem is also said to be dually feasible.

A trial solution of our problem is obtained by setting

$$t_j = 0 , \qquad (j = 1, 2, \cdots, q)$$

in the $p + 1$ linear equations (2.3) and (2.4). Hence, the trial solution of our problem is given by

$$y = a_{o,o}$$

$$x_i = a_{i,o} , \qquad (i = 1, 2, \cdots, p)$$

$$t_j = 0 , \qquad (j = 1, 2, \cdots, q) .$$

If A is dually feasible, then $y = a_{o,o}$ is its minimum value for all nonnegative values of the q variables t_j , $(j = 1, 2, \cdots, q)$. However, unless

$$a_{i,o} \geqq 0 , \qquad (i = 1, 2, \cdots, p) ,$$

the trial solution obtained above is not a solution of our problem because the condition (2.5) is not satisfied. In fact, we have the following:

Theorem 2.1. The trial solution

$$y = a_{o,o}, \qquad x_i = a_{i,o}, \qquad t_j = 0$$

where $i = 1, 2, \cdots, p$ and $j = 1, 2, \cdots, q$, is an optimal solution of the standard minimum problem if and only if A is dually feasible and

$$a_{i,o} \geqq 0, \qquad (i = 1, 2, \cdots, p).$$

Section 3
DUAL SIMPLEX METHOD

The dual simplex method for a dually feasible, standard minimum problem with matrix

$$A = \| a_{i,j} \| \qquad (3.1)\ \&\ (2.7)$$

where $0 \leqq i \leqq p$ and $0 \leqq j \leqq q$, is a finite process of transforming the given problem into equivalent standard minimum problems with matrices

$$A_o,\ A_1,\ A_2,\ \cdots,\ A_r$$

respectively described so that the trial solution of the last problem with matrix A_r is an optimal solution of the problem.

To insure that all variables remain nonnegative in the final trial solution, we adjoin to the system (2.3) and (2.4) of linear equations in Section 2 the identical equations

$$t_j = t_j\ , \qquad (j = 1,\ 2,\ \cdots,\ q)\ . \qquad (3.2)$$

The coefficients and constant terms of the right members of the system (2.3), (2.4) and (3.2) form the following matrix:

$$A_o = \left\| \begin{matrix} a_{o,o}, & a_{o,1}, & a_{o,2}, & \cdots, & a_{o,q} \\ a_{1,o}, & a_{1,1}, & a_{1,2}, & \cdots, & a_{1,q} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{p,o}, & a_{p,1}, & a_{p,2}, & \cdots, & a_{p,q} \\ 0, & 1, & 0, & \cdots, & 0 \\ 0, & 0, & 1, & \cdots, & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0, & 0, & 0, & \cdots, & 1 \end{matrix} \right\| .$$

In more compact notation, we have

$$A_o = \left\| \begin{matrix} & A \\ 0 & I \end{matrix} \right\| \tag{3.3}$$

where A denotes the $p + 1$ by $q + 1$ matrix (3.1), 0 denotes the q by 1 matrix which consists of q zeroes, and I denotes the q by q unit matrix. Therefore, A^o is a $p + q + 1$ by $q + 1$ matrix. Let us label the rows of A_o by the integers $i = 0, 1, \cdots, p + q$ and the columns of A_o by the integers $j = 0, 1, \cdots, q$. Denote by

$$a_{i,j}, \qquad (0 \leq i \leq p + q, \ 0 \leq j \leq q)$$

the element of A_o at the i-th row and the j-th column.

On the left member of the identical equation (3.2), let us use the symbol

$$x_{p+j}, \qquad (1 \leqq j \leqq q)$$

for the variable t_j. Then, our standard minimum problem is equivalent to <u>the standard minimum problem with matrix</u> A^o <u>which is to find the minimum value of the variable</u> y <u>subject to the following</u> $p + q + 1$ <u>linear equations:</u>

$$y = a_{o,o} + \sum_{j=1}^{q} a_{o,j}\, t_j \tag{3.4}$$

$$x_i = a_{i,o} + \sum_{j=1}^{q} a_{i,j}\, t_j \tag{3.5}$$

where $i = 1, 2, \cdots, p+q$ and the condition that

$$x_i \geqq 0 , \qquad (i = 1, 2, \cdots, p+q) . \tag{3.6}$$

Since the given problem A is assumed to be dually feasible and since A_o is obtained by adjoining q nonnegative rows to A , it is clear that A_o is also dually feasible.

The variables $t_1, t_2, \cdots, t_q$ on the right members of (3.4) and (3.5) are traditionally called the nonbasic variables of the problem A_o . By the definition of the new variables x_{p+j} , $(1 \leqq j \leqq q)$, introduced above, we know that the nonbasic variables $t_1, \cdots, t_q$ also appear on the left members of (3.5) as q of the $p+q$ variables $x_1, \cdots, x_{p+q}$. Because of this fact, the condition (3.6) implies that $t_j \geqq 0$ for every $j = 1, 2, \cdots, q$.

Now, let k be a given positive integer and assume that our standard minimum problem A has been reduced to an equivalent standard minimum problem with matrix

$$A_{k-1} = \| b_{i,j} \|$$

$(0 \leqq i \leqq p+q,\ 0 \leqq j \leqq q)$, which is to find the minimum value of the variable y subject to the following $p+q+1$ linear equations:

$$y = b_{o,o} + \sum_{j=1}^{q} b_{o,j}\, u_j \tag{3.7}$$

$$x_i = b_{i,o} + \sum_{j=1}^{q} b_{i,j}\, u_j \tag{3.8}$$

where $i = 1, 2, \cdots, p+q$ and the condition

$$x_i \geqq 0\ , \qquad (i = 1, 2, \cdots, p+q)\ . \tag{3.9}$$

Furthermore, we assume that A_{k-1} is dually feasible and that the nonbasic variables $u_1, \cdots, u_q$ on the right members of (3.7) and (3.8) also appear on the left members of (3.8) as q of the $p+q$ variables $x_1, \cdots, x_{p+q}$. Because of this, the condition (3.9) implies that $u_j \geqq 0$ for every $j = 1, 2, \cdots, q$.

If $b_{i,o} \geqq 0$ for every $i = 1, 2, \cdots, p+q$, then the trial solution

$$y = b_{o,o} \tag{3.10}$$

$$x_i = b_{i,o}\ , \qquad (i = 1, 2, \cdots, p+q)\ , \tag{3.11}$$

is an optimal solution of the standard minimum problem with matrix A_{k-1} according to Theorem 2.1. Since the standard minimum problem with matrix A_{k-1} is equivalent to our original standard minimum problem with matrix A, it follows that the trial solution given by (3.10) and (3.11) is also an optimal solution of our original problem. In this case, we have obtained an optimal solution of our original problem and hence the problem is solved affirmatively.

Now let us assume that A_{k-1} is dually feasible but not all of the $b_{i,o}$'s, $(1 \leqq i \leqq p+q)$, are nonnegative. Choose some $i = i_o$, $(1 \leqq i_o \leqq p+q)$, with

$$b_{i_o,o} \leqq 0\ . \tag{3.12}$$

Consider the other elements $b_{i_o,j}$ of the row in A_{k-1} labelled by i_o. If

$$b_{i_o,j} \leqq 0\ , \qquad (j = 1, 2, \cdots, q) \tag{3.13}$$

then the negative value $x_{i_o} = b_{i_o, o}$ is the maximal possible value of x_{i_o} subject to the condition

$$u_j \geqq 0 , \qquad (j = 1, 2, \cdots, q) . \tag{3.14}$$

In this case, the standard minimum problem with matrix A_{k-1} has no optimal solution; in fact, the system (3.8) of p + q linear equations has no nonnegative solution in $x_1, \cdots, x_{p+q}$. Since the standard minimum problem with matrix A_{k-1} is equivalent to our original standard minimum problem with matrix A , it follows that <u>our original problem has no optimal solution if there exists an integer i_o such that (3.12) and (3.13) both hold</u>. In this case, the system (2.4) of p linear equations fails to have nonnegative solution in $x_1, \cdots, x_p$ and $t_1, \cdots, t_j$. Hence, the problem is solved negatively in this case.

Next, let us assume that (3.13) is false. Consider the columns β_j of A_{k-1} with

$$b_{i_o, j} > 0 \tag{3.15}$$

and select from these a column β_{j_o} of A_{k-1} such that

$$\frac{\beta_{j_o}}{b_{i_o, j_o}} \leqq \frac{\beta_j}{b_{i_o, j}} \tag{3.16}$$

in the lexicographical order defined in Section 2 for all j satisfying (3.15). In particular, we have

$$b_{i_o, j_o} > 0 . \tag{3.17}$$

The fundamental operation used to derive A_k from A_{k-1} is the <u>Garissian elimination,</u> traditionally called <u>pivoting (on rows)</u> in the theory of linear programming. Once the integers i_o and j_o have been chosen as above, the element b_{i_o, j_o} is called the <u>pivot element.</u>

The Garissian elimination with b_{i_o, j_o} as the pivot element can be described as follows. We first solve the i_o-th equation of the system (3.8) for u_{j_o} in terms of x_{i_o} and the remaining u_j's; in other words, we write the i_o-th equation of (3.8) in the following form:

$$u_{j_o} = \frac{1}{b_{i_o, j_o}} \left(- b_{i_o, o} - \sum_{j \neq j_o} b_{i_o, j} \, u_j + x_{i_o} \right) . \tag{3.18}$$

Then substitute u_{j_o} in each equation of (3.7) and (3.8) by the right member of (3.18). Thus, the variables y and $x_1, \cdots, x_{p+q}$ are expressed as linear functions of the q variables

$$u_1, \cdots, u_{j_o-1}, x_{i_o}, u_{j_o+1}, \cdots, u_q .$$

For the sake of neatness, let us denote these q variables by $v_1, \cdots, v_q$; in other words, let

$$v_j = \begin{cases} u_j , & (\text{if } j \neq j_o) \\ x_{i_o} , & (\text{if } j = j_o) . \end{cases}$$

The result of the Garissian elimination with b_{i_o, j_o} as pivot element is that the equations (3.7) and (3.8) are transformed into an equivalent system consisting of $p + q + 1$ linear equations of the following form:

$$y = c_{o,o} + \sum_{j=1}^{q} c_{o,j} \, v_j \tag{3.19}$$

$$x_i = c_{i,o} + \sum_{j=1}^{q} c_{i,j} \, v_j \tag{3.20}$$

where $i = 1, 2, \cdots, p+q$.

Since the new nonbasic variables $v_1, \cdots, v_q$ are the variables $u_1, \cdots, u_{j_o-1}, x_{i_o}$, $u_{j_o+1}, \cdots, u_q$, they also appear on the left members of (3.20) as q of the p + q variables $x_1, \cdots, x_{p+q}$. Let

$$A_k = \left\| c_{i,j} \right\|$$

$(0 \leqq i \leqq p + q,\ 0 \leqq j \leqq q)$ denote the matrix of constant terms and coefficients on the right members of (3.19) and (3.20).

Since (3.19) and (3.20) are derived by substituting (3.18) into (3.7) and (3.8), the elements $c_{i,j}$ of A_k can be easily computed from those of A_{k-1} as follows: For each $j = 0, 1, \cdots, q$, let β_j and γ_j denote the j-th column in A_{k-1} and A_k respectively, that is to say,

$$\beta_j = \left(b_{o,j},\ b_{1,j},\ \cdots,\ b_{p+q,j}\right)$$

$$\gamma_j = \left(c_{o,j},\ c_{1,j},\ \cdots,\ c_{p+q,j}\right) .$$

Then, one can easily verify that

$$\gamma_j = \begin{cases} \beta_j - \dfrac{b_{i_o,j}}{b_{i_o,j_o}} \beta_{j_o}, & (\text{if } j \neq j_o) \\[2ex] \dfrac{1}{b_{i_o,j_o}} \beta_{j_o}, & (\text{if } j = j_o) . \end{cases}$$

In words, the matrix A_k can be obtained from the matrix A_{k-1} as follows: First, divide the j_o-th column β_{j_o} of A_{k-1} by the pivot element b_{i_o,j_o}; second, subtract from every other column β_j, $j \neq j_o$, this new j_o-th column multiplied by $b_{i_o,j}$. The result is the matrix A_k.

Note that, in the resulting matrix A_k, we have $c_{i_o,j_o} = 1$ and $c_{i_o,j} = 0$ for every $j \neq j_o$. Therefore, the second step in the preceding paragraph can be interpreted as to subtract from every column β_j, $j \neq j_o$, a multiple of the new j_o-th column so that its element $c_{i_o,j}$ in the i_o-th row vanishes.

Having determined the columns of the new matrix A_k, we will prove that this matrix A_k is still dually feasible. For this purpose, let j be an arbitrary integer satisfying $1 \leqq j \leqq q$. We will prove that $\gamma_j \geqq 0$ lexicographically.

To do this, let us first consider the case $j = j_o$. In this case, since

$$\beta_{j_o} > 0 , \qquad b_{i_o,j_o} > 0$$

by the dual feasibility of A_{k-1} and (3.17), we have

$$\gamma_{j_o} = \frac{1}{b_{i_o,j_o}} \left(\beta_{j_o}\right) > 0 .$$

Next, assume that $j \neq j_o$. Then we have

$$\gamma_j = \beta_j - \frac{b_{i_o,j}}{b_{i_o,j_o}} \beta_{j_o} \tag{3.21}$$

$$= b_{i_o,j} \left(\frac{\beta_j}{b_{i_o,j}} - \frac{\beta_{j_o}}{b_{i_o,j_o}} \right) . \tag{3.22}$$

If $b_{i_o,j} \leqq 0$, then it follows from (3.21) that

$$\gamma_j \geqq \beta_j \geqq 0 .$$

If $b_{i_o,j} > 0$, then (3.16) holds. Hence, by (3.22), we have

$$\gamma_j \geqq 0 .$$

This completes the proof that A_k is dually feasible.

Thus, our given dually feasible standard minimum problem with matrix A has been reduced to <u>an equivalent dually feasible minimum problem with matrix</u> $A_k = \| c_{i,j} \|$, $(0 \leqq i \leqq p+q,\ 0 \leqq j \leqq q)$, <u>which is to find the minimum value of the variable</u> y <u>subject to the</u> $p+q+1$ <u>linear equations</u> (3.19) and (3.20) <u>and the condition</u> (3.9). Since the new nonbasic variables $v_1, \cdots, v_q$ also appear on the left members of (3.20) as q of the $p+q$ variables $x_1, \cdots, x_{p+q}$, we have

$$v_j \geqq 0 , \qquad (j = 1, 2, \cdots, q) . \tag{3.23}$$

This completes the inductive construction of the sequence of equivalent dually feasible minimum problems with matrices

$$A_o, A_1, \cdots, A_{k-1}, A_k, \cdots . \tag{3.24}$$

It remains to prove the finiteness of the sequence (3.24). For this purpose, let us consider the leading column γ_o in the matrix A_k. By the dual feasibility of A_{k-1} and (3.17), we have

$$\beta_{j_o} > 0 . \tag{3.25}$$

On the other hand, it follows from (3.12) and (3.17) that

$$-\frac{b_{i_o,o}}{b_{i_o,j_o}} > 0 . \tag{3.26}$$

By (3.25) and (3.26), we have

$$\gamma_o = \beta_o - \frac{b_{i_o,o}}{b_{i_o,j_o}} \beta_{j_o} > \beta_o . \tag{3.27}$$

Hence, as k increases, the leading column in A_k increases strictly in the lexicographical order. Since there is only a finite number of possible sets of q nonbasic variables chosen from the $p+q$ variables $x_1, \cdots, x_{p+q}$ and any choice uniquely determines the leading column, the process must stop at some A_k.

This can happen either if there is no negative element in the leading column γ_o of A_k, or if there is a negative element

$$c_{i_o, o} < 0$$

in the leading column γ_o of A_k but no positive element in the i_o-th row of A_k; i.e.,

$$c_{i_o, j} \leqq 0 , \qquad (0 \leqq j \leqq q) .$$

In the first case, the trial solution

$$y = c_{o, o}$$

$$x_i = c_{i, o} , \qquad (1 \leqq i \leqq p + q)$$

of the problem with matrix A_k is an optimal solution of our given standard minimum problem with matrix A. Hence, in this case, the problem is solved affirmatively.

In the second case, the negative value $c_{i_o, o}$ is the largest value of the variable x_{i_o} and consequently our standard minimum problem with matrix A has no nonnegative solution. Hence, in this case, the problem is solved negatively.

Section 4
DUAL SIMPLEX TABLEAUX

For computational purpose, the dual simplex method, which is described and proved in Section 3, can be worked exclusively on the matrices involved.

The given standard minimum problem is represented by the matrix

$$A = \| a_{i,j} \| , \qquad (0 \leqq i \leqq p,\ 0 \leqq j \leqq q) .$$

The dual simplex method is essentially a finite process of deriving from A a sequence of $p + q + 1$ by $q + 1$ matrices:

$$A_o, A_1, \cdots, A_r .$$

The initial matrix A_o is given by

$$A_o = \left\| \begin{matrix} & A \\ 0 & I \end{matrix} \right\|$$

as described in Section 3. In the actual computation, we will write A_o in the form of the following tableau T_o which will be called the <u>initial</u> <u>tableau:</u>

(T_o)

	Const.	t_1	t_2	$\cdots$	t_q
y	$a_{o,o}$	$a_{o,1}$	$a_{o,2}$	$\cdots$	$a_{o,q}$
x_1	$a_{1,o}$	$a_{1,1}$	$a_{1,2}$	$\cdots$	$a_{1,q}$
x_2	$a_{2,o}$	$a_{2,1}$	$a_{2,2}$	$\cdots$	$a_{2,q}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$
x_p	$a_{p,o}$	$a_{p,1}$	$a_{p,2}$	$\cdots$	$a_{p,q}$
x_{p+1}	0	1	0	$\cdots$	0
x_{p+2}	0	0	1	$\cdots$	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$		0
x_{p+q}	0	0	0	$\cdots$	1

For each $k = 1, 2, \cdots, r$, the matrix A_k is represented by a similar tableau T_k in which the columns are headed by const., $v_1, v_2, \cdots, v_q$, the new nonbasic variables. The operation of deriving T_k from T_{k-1} is described as follows:

Consider the tableau T_{k-1} which has been constructed by the preceding step:

(T_{k-1})

	Const.	u_1	u_2	$\cdots$	u_q
y	$b_{o,o}$	$b_{o,1}$	$b_{o,2}$	$\cdots$	$b_{o,q}$
x_1	$b_{1,o}$	$b_{1,1}$	$b_{1,2}$	$\cdots$	$b_{1,q}$
x_2	$b_{2,o}$	$b_{2,1}$	$b_{2,2}$	$\cdots$	$b_{2,q}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$
x_p	$b_{p,o}$	$b_{p,1}$	$b_{p,2}$	$\cdots$	$b_{p,q}$
x_{p+1}	$b_{p+1,o}$	$b_{p+1,1}$	$b_{p+1,2}$	$\cdots$	$b_{p+1,q}$
x_{p+2}	$b_{p+2,o}$	$b_{p+2,1}$	$b_{p+2,2}$	$\cdots$	$b_{p+2,q}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$
x_{p+q}	$b_{p+q,o}$	$b_{p+q,1}$	$b_{p+q,2}$	$\cdots$	$b_{p+q,q}$

If the constant column in T_{k-1} contains no negative entry, then it gives an optimal solution of the given standard minimum problem. Precisely, the minimum value of the linear function

$$y = a_{o,o} + \sum_{j=1}^{q} a_{o,j} t_j$$

is $b_{o,o}$ reached when

$$t_j = x_{p+j} = b_{p+j,o}$$

for every $j = 1, 2, \ldots, q$.

Otherwise, there is at least one entry in the constant column of T_{k-1} which is negative. Choose an i_o with

$$b_{i_o,o} < 0 .$$

Then consider the row of T_{k-1} which is headed by x_{i_o} . If this row contains no positive entry, then our given problem has no solution. Otherwise, consider the columns β_j of T_{k-1} such that

$$b_{i_o,j} > 0$$

and select from these a column β_{j_o} of T_{k-1} such that

$$\frac{\beta_{j_o}}{b_{i_o,j_o}} \leqq \frac{\beta_j}{b_{i_o,j}}$$

for every j satisfying $b_{i_o,j} > 0$. Thus, we have obtained the pivot element b_{i_o,j_o} . In numerical computations, the pivot element is usually marked by a circle around the numerical entry in the tableau T_{k-1} .

Having chosen the pivot element b_{i_o,j_o} in the tableau T_{k-1}, we can construct the next tableau T_k as follows:

First, change the nonbasic variable u_{j_o} into x_{i_o}. Thus, we obtain the nonbasic variables $v_1, \cdots, v_q$ given by

$$v_j = \begin{cases} u_j , & (\text{if } j \neq j_o) \\ x_{i_o} , & (\text{if } j = j_o) . \end{cases}$$

Next, the column γ_{j_o} in T_k headed by $v_{j_o} = x_{i_o}$ is obtained by dividing β_{j_o} by the pivot element b_{i_o,j_o}; in symbols,

$$\gamma_{j_o} = \frac{\beta_{j_o}}{b_{i_o,j_o}} .$$

Next, the constant column γ_o in T_k is obtained from β_o by subtracting $b_{i_o,o}\,\gamma_{j_o}$; in symbols,

$$\gamma_o = \beta_o - b_{i_o,o}\,\gamma_{j_o} .$$

Since $b_{i_o,o}$ is negative, we actually have

$$\gamma_o = \beta_o + \left(- b_{i_o,o}\right)\gamma_{j_o} .$$

Finally, the column γ_j in T_k headed by $v_j = u_j$, $(j \neq j_o)$, is obtained from β_j by subtracting $b_{i_o,j}\,\gamma_{j_o}$; in symbols,

$$\gamma_j = \beta_j - b_{i_o,j}\,\gamma_{j_o} .$$

This completes the construction of T_k.

Section 5
AN ILLUSTRATIVE EXAMPLE

In the present section, let us study the standard minimum problem of finding non-negative real numbers

$$t_1, t_2, t_3, t_4, t_5, t_6$$

which minimize a given linear function

$$y = t_1 + t_2 + t_3 + t_4 + t_5 + t_6$$

subject to a system of 10 linear inequalities

$$\begin{aligned}
t_6 - t_5 &\geqq 0 \\
t_6 - t_1 - t_4 &\geqq 0 \\
t_6 - t_2 - t_3 &\geqq 0 \\
t_1 + t_2 + t_3 - t_6 - 1 &\geqq 0 \\
t_2 + t_4 - t_6 - 1 &\geqq 0 \\
t_1 + t_5 - t_6 - 1 &\geqq 0 \\
t_2 - t_1 &\geqq 0 \\
t_3 - t_2 &\geqq 0 \\
t_4 - t_3 &\geqq 0 \\
t_5 - t_4 &\geqq 0 .
\end{aligned}$$

Introduce slack variables x_i, $(1 \leqq i \leqq 10)$, for the left members of the inequalities. Thus, the given problem is reduced to find the minimum value of the variable y subject to the following eleven equations:

$$
\begin{aligned}
y &= t_1 + t_2 + t_3 + t_4 + t_5 + t_6 \\
x_1 &= t_6 - t_5 \\
x_2 &= t_6 - t_1 - t_4 \\
x_3 &= t_6 - t_2 - t_3 \\
x_4 &= t_1 + t_2 + t_3 - t_6 - 1 \\
x_5 &= t_2 + t_4 - t_6 - 1 \\
x_6 &= t_1 + t_5 - t_6 - 1 \\
x_7 &= t_2 - t_1 \\
x_8 &= t_3 - t_2 \\
x_9 &= t_4 - t_3 \\
x_{10} &= t_5 - t_4
\end{aligned}
$$

and the condition that

$$
\begin{aligned}
x_i &\geqq 0, \qquad (1 \leqq i \leqq 10) \\
t_j &\geqq 0, \qquad (1 \leqq j \leqq 6)\ .
\end{aligned}
$$

The matrix A of the right members of these equations is as follows:

$$
A = \begin{Vmatrix}
0 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & -1 & 1 \\
0 & -1 & 0 & 0 & -1 & 0 & 1 \\
0 & 0 & -1 & -1 & 0 & 0 & 1 \\
-1 & 1 & 1 & 1 & 0 & 0 & -1 \\
-1 & 0 & 1 & 0 & 1 & 0 & -1 \\
-1 & 1 & 0 & 0 & 0 & 1 & -1 \\
0 & -1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 1 & 0
\end{Vmatrix}
$$

Since all coefficients of the linear function y are positive, the problem is dually feasible. Let

$$x_{10+j} = t_j , \qquad (1 \leqq j \leqq 6) .$$

Then we obtain the initial tableau T_o as follows:

(T_o)

	Const.	t_1	t_2	t_3	t_4	t_5	t_6
y	0	1	1	1	1	1	1
x_1	0	0	0	0	0	-1	1
x_2	0	-1	0	0	-1	0	1
x_3	0	0	-1	-1	0	0	1
x_4	-1	1	1	1	0	0	-1
x_5	-1	0	1	0	1	0	-1
x_6	-1	1	0	0	0	1	-1
x_7	0	-1	1	0	0	0	0
x_8	0	0	-1	1	0	0	0
x_9	0	0	0	-1	1	0	0
x_{10}	0	0	0	0	-1	1	0
t_1	0	1	0	0	0	0	0
t_2	0	0	1	0	0	0	0
t_3	0	0	0	1	0	0	0
t_4	0	0	0	0	1	0	0
t_5	0	0	0	0	0	1	0
t_6	0	0	0	0	0	0	1

The constant column contains 3 negative entries in the rows headed by x_4, x_5 and x_6. Each of these rows contains some positive entries. Let us choose the row headed by x_4. This row contains 3 positive entries in the columns headed by t_1, t_2, and t_3. Of these 3 columns, the one headed by t_1 is the least in the lexicographical order and hence (3.16) is satisfied. Therefore, we have to choose the element 1 located in the row headed by x_4 and in the column headed by t_1 as the pivot element for the construction of the next tableau T_1.

Thus, we indicate the pivot element in T_o by a circle around it. Then, by following the process of constructing T_k from T_{k-1} described in Section 4, we obtain our next tableau T_1, listed as follows:

(T_1)

	Const.	x_4	t_2	t_3	t_4	t_5	t_6
y	1	1	0	0	1	1	2
x_1	0	0	0	0	0	-1	1
x_2	-1	-1	1	1	-1	0	0
x_3	0	0	-1	-1	0	0	1
x_4	0	1	0	0	0	0	0
x_5	-1	0	1	0	1	0	-1
x_6	0	1	-1	-1	0	1	0
x_7	-1	-1	2	1	0	0	-1
x_8	0	0	-1	1	0	0	0
x_9	0	0	0	-1	1	0	0
x_{10}	0	0	0	0	-1	1	0
t_1	1	1	-1	-1	0	0	1
t_2	0	0	1	0	0	0	0
t_3	0	0	0	1	0	0	0
t_4	0	0	0	0	1	0	0
t_5	0	0	0	0	0	1	0
t_6	0	0	0	0	0	0	1

The constant column of T_1 contains 3 negative entries in the rows headed by x_2, x_5 and x_7. Let us choose x_2. Then, by (3.16), the pivot element is in the column headed by t_3. Hence, we obtain the next tableau T_2 as follows:

(T_2)

	Const.	x_4	t_2	x_2	t_4	t_5	t_6
y	1	1	0	0	1	1	2
x_1	0	0	0	0	0	-1	1
x_2	0	0	0	1	0	0	0
x_3	-1	-1	0	-1	-1	0	1
x_4	0	1	0	0	0	0	0
x_5	-1	0	1	0	1	0	-1
x_6	-1	0	0	-1	-1	1	0
x_7	0	0	1	1	1	0	-1
x_8	1	1	-2	1	1	0	0
x_9	-1	-1	1	-1	0	0	0
x_{10}	0	0	0	0	-1	1	0
t_1	0	0	0	-1	-1	0	1
t_2	0	0	1	0	0	0	0
t_3	1	1	-1	1	1	0	0
t_4	0	0	0	0	1	0	0
t_5	0	0	0	0	0	1	0
t_6	0	0	0	0	0	0	1

The constant column of T_2 contains 4 negative entries in the rows headed by x_3, x_5, x_6 and x_9. Let us choose x_3. Then the pivot element is in the column headed by t_6. Thus, we obtain:

(T_3)

	Const.	x_4	t_2	x_2	t_4	t_5	x_3
y	3	3	0	2	3	1	2
x_1	1	1	0	1	1	-1	1
x_2	0	0	0	1	0	0	0
x_3	0	0	0	0	0	0	1
x_4	0	1	0	0	0	0	0
x_5	-2	-1	1	-1	0	0	-1
x_6	-1	0	0	-1	-1	1	0
x_7	-1	-1	1	0	0	0	-1
x_8	1	1	-2	1	1	0	0
x_9	-1	-1	1	-1	0	0	0
x_{10}	0	0	0	0	-1	1	0
t_1	1	1	0	0	0	0	1
t_2	0	0	1	0	0	0	0
t_3	1	1	-1	1	1	0	0
t_4	0	0	0	0	1	0	0
t_5	0	0	0	0	0	1	0
t_6	1	1	0	1	1	0	1

The constant column of T_3 contains 4 negative entries in the rows headed by x_5, x_6, x_7, and x_9. Let us choose x_5. Then the pivot element is in the column headed by t_2. Thus we obtain:

(T_4)

	Const.	x_4	x_5	x_2	t_4	t_5	x_3
y	3	3	0	2	3	1	2
x_1	1	1	0	1	1	-1	1
x_2	0	0	0	1	0	0	0
x_3	0	0	0	0	0	0	1
x_4	0	1	0	0	0	0	0
x_5	0	0	1	0	0	0	0
x_6	-1	0	0	-1	-1	1	0
x_7	1	0	1	1	0	0	0
x_8	-3	-1	-2	-1	1	0	-2
x_9	1	0	1	0	0	0	1
x_{10}	0	0	0	0	-1	1	0
t_1	1	1	0	0	0	0	1
t_2	2	1	1	1	0	0	1
t_3	-1	0	-1	0	1	0	-1
t_4	0	0	0	0	1	0	0
t_5	0	0	0	0	0	1	0
t_6	1	1	0	1	1	0	1

The constant column of T_4 contains 3 negative entries in the rows headed by x_6, x_8 and t_3. Let us choose x_6. Then the pivot element is in the column headed by t_5. Thus we obtain:

(T_5)

	Const.	x_4	x_5	x_2	t_4	x_6	x_3
y	4	3	0	3	4	1	2
x_1	0	1	0	0	0	-1	1
x_2	0	0	0	1	0	0	0
x_3	0	0	0	0	0	0	1
x_4	0	1	0	0	0	0	0
x_5	0	0	1	0	0	0	0
x_6	0	0	0	0	0	1	0
x_7	1	0	1	1	0	0	0
x_8	-3	-1	-2	-1	1	0	-2
x_9	1	0	1	0	0	0	1
x_{10}	1	0	0	1	0	1	0
t_1	1	1	0	0	0	0	1
t_2	2	1	1	1	0	0	1
t_3	-1	0	-1	0	1	0	-1
t_4	0	0	0	0	1	0	0
t_5	1	0	0	1	1	1	0
t_6	1	1	0	1	1	0	1

The constant column of T_5 contains 2 negative entries in the rows headed by x_8 and t_3. Let us choose x_8. Then the pivot element is in the column headed by t_4. Thus we obtain:

(T_6)

	Const.	x_4	x_5	x_2	x_8	x_6	x_3
y	16	7	8	7	4	1	10
x_1	0	1	0	0	0	-1	1
x_2	0	0	0	1	0	0	0
x_3	0	0	0	0	0	0	1
x_4	0	1	0	0	0	0	0
x_5	0	0	1	0	0	0	0
x_6	0	0	0	0	0	1	0
x_7	1	0	1	1	0	0	0
x_8	0	0	0	0	1	0	0
x_9	1	0	1	0	0	0	1
x_{10}	1	0	0	1	0	1	0
t_1	1	1	0	0	0	0	1
t_2	2	1	1	1	0	0	1
t_3	2	1	1	1	1	0	1
t_4	3	1	2	1	1	0	2
t_5	4	1	2	2	1	1	2
t_6	4	2	2	2	1	0	3

Since the constant column of T_6 contains no negative entry, we obtain an optimal solution of our problem; namely, the minimum value of the linear function

$$y = t_1 + t_2 + t_3 + t_4 + t_5 + t_6$$

subject to the given constraints is

$$y = 16$$

reached when

$$t_1 = 1$$
$$t_2 = 2$$
$$t_3 = 2$$
$$t_4 = 3$$
$$t_5 = 4$$
$$t_6 = 4 \; .$$

This completes the illustrative example.

In the preceding example, we have always chosen the first negative entry, counting from top down next to the y row, in the constant column of the tableaux. For definiteness, one can make this a rule in practice especially when programming to electronic computers.

Section 6
SWITCHING FUNCTIONS

Let Q denote the set which consists of the two integers 0 and 1. For any given integer $n \geqq 1$, consider the Cartesian power

$$Q^n = Q \times \ldots \times Q$$

which is the Cartesian product of n copies of Q. Thus, the elements of Q^n are the 2^n ordered n-tuples

$$\left(x_1, x_2, \ldots, x_n\right)$$

where the k-th coordinate x_k is in Q for every $k = 1, 2, \ldots, n$. Hereafter, Q^n will be called the n-cube and its 2^n elements will be called its points.

By a switching function (or truth function or Boolean function) of n variables, we mean any subset F of the n-cube Q^n. Since Q^n has 2^n points, there are 2^{2^n} different switching functions of n variables.

A switching function F of n variables is said to be linearly separable provided that there exist $n+1$ real numbers $w_1, w_2, \ldots, w_n, w_{n+1}$ such that, for every point $x = (x_1, \ldots, x_n)$ in Q^n, we have $x \epsilon F$ if and only if

$$w_1 x_1 + w_2 x_2 + \ldots + w_n x_n \leqq w_{n+1} .$$

The set $w = (w_1, w_2, \ldots, w_n, w_{n+1})$ is called a separating system of F; the real numbers $w_1, w_2, \ldots, w_n$ are called the weights, and the real number w_{n+1} is called the threshold. By taking the threshold w_{n+1} as small as possible while the weights $w_1, \ldots, w_n$ are held fixed, we may assume that, in case F is not empty, there exists a point $x = (x_1, \ldots, x_n)$ in F such that

$$w_1 x_1 + w_2 x_2 + \ldots + w_n x_n = w_{n+1} .$$

Consider the complement $F' = Q^n - F$ of F. For every point $y = (y_1, \ldots, y_n)$ in F', we have

$$w_1 y_1 + w_2 y_2 + \ldots + w_n y_n > w_{n+1} .$$

Let M denote the minimal value of

$$w_1 y_1 + w_2 y_2 + \ldots + w_n y_n - w_{n+1}$$

for all points $y = (y_1, \ldots, y_n)$ in F' in case F' is not empty. This positive real number M is called the <u>margin</u> of the separating system W (Ref. 8, p. 6). A separating system $W = (w_1, \ldots, w_{n+1})$ of F is said to be <u>normal</u> provided that $M = 1$. Every separating system W of F can be <u>normalized</u> by dividing each w_i, $(i = 1, 2, \ldots, n+1)$, by the margin M of W. In particular, every linearly separable switching function F has a normal separating system.

Let $W = (w_1, \ldots, w_{n+1})$ be any normal separating system of a given linearly separable switching function F of n variables. Then, for every point

$$x = \left(x_1, \ldots, x_n\right) \in Q^n$$

we have:

$$w_1 x_1 + \ldots + w_n x_n \leqq w_{n+1}, \qquad (\text{if } x \in F)$$

$$w_1 x_1 + \ldots + w_n x_n \geqq w_{n+1} + 1, \ (\text{if } x \in F') .$$

By a <u>canonical switching function</u> of n variables, we mean a linearly separable switching function F of n variables which admits a separating system $W = (w_1, \ldots, w_n, w_{n+1})$ satisfying

$$0 \leqq w_1 \leqq w_2 \leqq \cdots \leqq w_n .$$

In words, the weights $w_1, w_2, \ldots, w_n$ in W are nonnegative and nondecreasing.

It is well-known (Refs. 9 and 10) that every linearly separable switching function F of n variables can be reduced to a unique canonical switching function by permuting and complementing a number of the variables.

By a <u>weight function</u> of n variables, we mean a homogenous linear function

$$w : R^n \to R$$

on the n-dimensional Euclidean space R^n with real values. Precisely, there are n real numbers $w_1, \ldots, w_n$ such that, for an arbitrary point $x = (x_1, \ldots, x_n)$ of R^n, we have

$$w(x) = w_1 x_1 + \ldots + w_n x_n .$$

The real numbers $w_1, \ldots, w_n$ are called the <u>coefficients</u> of the weight function w.

A weight function $w : R^n \to R$ with coefficients $w_1, \ldots, w_n$ is said to be <u>canonical</u> provided

$$0 \leqq w_1 \leqq w_2 \leqq \cdots \leqq w_n .$$

By means of the canonical weight functions of n variables, we can define a partial order in the n-cube Q^n as follows: Let $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ be any two points of Q^n; then we define $x \leqq y$ if and only if $w(x) \leqq w(y)$ for every canonical weight function w of variables. This partial order in Q^n is called the <u>canonical partial order</u> (Ref. 10).

Using the canonical partial order $\leqq$ in the n-cube, we can define the regular switching functions as follows: a switching function F of n variables is said to be regular if and only if it satisfies the regularity condition:

$$\text{If } x \epsilon F \text{ and } y \leqq x, \text{ then } y \epsilon F .$$

Obviously, every canonical switching function of n variables is regular. In Ref. 11, it was proved that every regular switching function of $n \leqq 5$ variables is canonical; in Ref. 12, an example is given which shows that not every regular switching function is canonical.

Section 7
SYNTHESIS AND MINIMIZATION

Let F be an arbitrary switching function of n variables. The synthesis problem for the linear separability of F is to determine whether or not F is linearly separable and to find a separating system $(w_1, \ldots, w_n, w_{n+1})$ for F in case F in linearly separable.

Among various synthesis methods for linear separability introduced in the literature, the one given by D. G. Willis (Ref. 4), turns out to be the most convenient because it involves as few linear inequalities as possible. In Ref. 4, the synthesis problem for the linear separability of arbitrary switching functions of n variables was reduced to that of the regular switching functions of n variables. Indeed, it remains to determine whether or not a given regular switching function F is linearly separable and to find a canonical separating system $(w_1, \ldots, w_n, w_{n+1})$ for F in case F is linearly separable and hence canonical.

For the convenience of the reader, we describe briefly the Willis synthesis method as follows:

Let F be an arbitrary nonempty regular switching function of n variables as defined in Section 6. Let L denote the set of all maximal points of F with respect to the canonical partial order in the n-cube Q^n; and let M denote the set of all minimal points of the complement $F' = Q^n - F$ with respect to the same canonical partial order in Q^n. Let

$$a_i = \left(a_{i1}, \ldots, a_{in}\right), \qquad \left(i = 1, 2, \ldots, \lambda\right)$$

be the points of L and

$$b_j = \left(b_{j1}, \ldots, b_{jn}\right), \quad \left(j = 1, 2, \ldots, \mu \right)$$

be the points of M. Consider the following system of linear inequalities:

$$\left.\begin{aligned} &\sum_{k=1}^{n} a_{ik} w_k \leqq w_{n+1}, \ (i = 1, 2, \ldots, \lambda) \\ &\sum_{k=1}^{n} b_{jk} w_k \geqq w_{n+1} + 1, \ (j = 1, 2, \ldots, \mu) \\ &0 \leqq w_1 \leqq w_2 \leqq \cdots \leqq w_n \end{aligned}\right\} \quad . \tag{7.1}$$

Then, the Willis synthesis theorem states that the given regular switching function F is linearly separable if and only if the system (7. 1) of linear inequalities has a solution (and hence an integral solution) in $w_1, \ldots, w_n$ and w_{n+1}.

In Ref. 13, the system (7. 1) was solved by Fan's principle of bounding solutions. An elimination process for solving the system (7. 1) was formulated in Ref. 14, and methods of successive approximation were applied to solve (7. 1) in Ref. 15. Finally, the simplex method in linear programming was used in finding a solution of the system (7. 1) in Ref. 3.

The next problem is naturally the <u>minimization problem</u> which is to find the most economical solution of the system (7. 1) in case the given regular switching function F is linearly separable. In other words, the minimization problem is to find a solution

$$\left(w_1, \ldots, w_n, w_{n+1} \right)$$

of the system (7. 1) which makes some cost function

$$\phi\left(w_1, \ldots, w_n, w_{n+1}\right)$$

minimal. The precise formulation of the problem is as follows:

First, let us pick the cost function ϕ . Assume that the cost of realizing the w_i, $(i = 1, 2, \ldots, n, n+1)$, is proportional to the magnitude of w_i . Under this assumption, the cost function ϕ will be a homogeneous linear function

$$\phi\left(w_1, \ldots, w_{n+1}\right) = \sum_{i=1}^{n+1} \gamma_i w_i \tag{7.2}$$

where the coefficients $\gamma_1, \ldots, \gamma_{n+1}$ are nonnegative real numbers. In the literature, (see Ref. 16), two different cost functions have been studied; one of these is defined by $\gamma_i = 1$ for all $i = 1, 2, \ldots, n+1$ and the other is given by $\gamma_i = 1$ for all $i \leqq n$ and $\gamma_{n+1} = 0$.

Having fixed the cost function ϕ by (7.2), the minimization problem for the given regular switching function F is that of finding a canonical normal separating system which minimizes ϕ ; in other words, nonnegative real numbers

$$w_1, w_2, \ldots, w_n, w_{n+1}$$

are to be determined which minimize the cost function (7.2) and satisfy the system (7.1) of linear inequalities.

Next, let us write the system (7.1) of linear inequalities in the form of (2.2). Thus, the first λ linear inequalities become

$$-\sum_{k=1}^{n} a_{ik} w_k + w_{n+1} \geqq 0, \quad (i = 1, 2, \ldots, \lambda) . \tag{7.3}$$

The next μ linear inequalities of (7.1) become

$$-1 + \sum_{k=1}^{n} b_{jk} w_k - w_{n+1} \geqq 0, \quad (j = 1, 2, \ldots, \mu) . \tag{7.4}$$

Under the condition that the variables $w_1, \ldots, w_{n+1}$ are nonnegative, the remaining inequalities in (7. 1) are equivalent to following n-1 linear inequalities:

$$w_k - w_{k-1} \geqq 0, \qquad (k = 2, 3, \ldots, n). \tag{7.5}$$

Consequently, the minimization problem for the given regular switching function F is a standard minimization problem in linear programming of finding $q = n + 1$ non-negative real numbers

$$\left(w_1, w_2, \ldots, w_n, w_{n+1}\right)$$

which minimize the linear cost function (7. 2) and also satisfy the system of $p = \lambda + \mu + n - 1$ linear inequalities (7.3), (7. 4), and (7. 5).

Section 8
SOLUTION OF THE MINIMIZATION PROBLEM

In Section 7, we have seen that the minimization problem for a given regular switching function F is a standard minimization problem in linear programming. In the present section, we are concerned with the methods of solving this problem. For this purpose, we assume that the first n coefficients in the cost function (7.2) are positive; in symbols,

$$\gamma_i > 0, \quad (i = 1, 2, \ldots, n). \tag{8.1}$$

Economically, this means that it costs something to realize the weights $w_1, \ldots, w_n$ physically. For the two different cost functions studied in Ref. 16, $\gamma_i = 1$ for all $i = 1, 2, \ldots, n$. Hence, the condition (8.1) is rather reasonable.

Let us briefly review some standard terminology in linear programming. By a feasible solution of the minimization problem for F, we mean a set of n+1 non-negative real numbers

$$\left(w_1, w_2, \ldots, w_n, w_{n+1}\right)$$

which satisfy the linear inequalities (7.3), (7.4), and (7.5). By an optimal solution of the problem, we mean a feasible solution which minimizes the cost function (7.2).

The minimization problem for F is said to be feasible provided that it has a feasible solution. Hence, the problem is feasible if and only if the given regular switching function F is linearly separable.

Furthermore, for the nontrivial case where F is nonempty and different from the whole n-cube, it was proved in Ref. 5 that the minimization problem for F has optimal solutions if and only if it is feasible. Hence every linearly separable regular switching function F of n variables has a minimal canonical normal separating system.

As to methods of solving the minimization problem for a given regular switching function F of n variables, we certainly think of the usual simplex method in linear programming. This method splits into two parts of work. The first part is the search of a feasible solution of the problem as in Ref. 3; and the second part is the application of the simplex technique to find an optimal solution starting from the feasible solution found in the first part. Because of this, the usual simplex method is rather inefficient for this particular minimization problem.

On the other hand, the dual simplex method described in Section 3 suits especially well for solving our particular standard minimization problem. Indeed, under the reasonable assumption (8. 1), the minimization problem for any nonempty regular switching function F of n variables is always dually feasible as defined in Section 2. Hence, we can apply the dual simplex method to our problem right away.

As an illustrative example, let us consider the regular switching function

$$F = 531$$

of five variables in the notation introduced in Refs. 12 and 17. Precisely, F consists of the following ten points of Q^5:

$$\begin{array}{ll} (0,\ 0,\ 0,\ 0,\ 0) & (0,\ 0,\ 0,\ 0,\ 1) \\ (1,\ 0,\ 0,\ 0,\ 0) & (1,\ 1,\ 0,\ 0,\ 0) \\ (0,\ 1,\ 0,\ 0,\ 0) & (1,\ 0,\ 1,\ 0,\ 0) \\ (0,\ 0,\ 1,\ 0,\ 0) & (1,\ 0,\ 0,\ 1,\ 0) \\ (0,\ 0,\ 0,\ 1,\ 0) & (0,\ 1,\ 1,\ 0,\ 0)\ . \end{array}$$

The maximal points of F can be read from its label 531 in accordance with Ref. 12. In fact, F has three maximal points, namely,

$$\begin{array}{l} (0,\ 0,\ 0,\ 0,\ 1) \\ (1,\ 0,\ 0,\ 1,\ 0) \\ (0,\ 1,\ 1,\ 0,\ 0)\ . \end{array}$$

To find the minimal points of the complement F' of F, we may first find the dual F^* of F defined in Ref. 18. According to the table at the end of Ref. 18, the dual of $F = 531$ is

$$F^* = 54321/32/1$$

in the notation of Ref. 17. By the table given at the end of Ref. 12, the regular switching function F^* has three maximal points, namely,

$$(0,\ 0,\ 0,\ 1,\ 1)$$
$$(1,\ 0,\ 1,\ 0,\ 1)$$
$$(0,\ 1,\ 1,\ 1,\ 0)\ .$$

By Theorem 5.1 in Ref. 18, it follows immediately that the complement F' of F has three minimal points, namely,

$$(1,\ 1,\ 1,\ 0,\ 0)$$
$$(0,\ 1,\ 0,\ 1,\ 0)$$
$$(1,\ 0,\ 0,\ 0,\ 1)\ .$$

Having obtained the maximal points of F and the minimal points of its complement F', we can exhibit the system of inequalities (7.3), (7.4) and (7.5) in the form of the following ten linear inequalities:

$$\begin{aligned}
w_6 - w_5 &\geqq 0 \\
w_6 - w_1 - w_4 &\geqq 0 \\
w_6 - w_2 - w_5 &\geqq 0 \\
w_1 + w_2 + w_3 - w_6 - 1 &\geqq 0 \\
w_2 + w_4 - w_6 - 1 &\geqq 0 \\
w_1 + w_5 - w_6 - 1 &\geqq 0 \\
w_2 - w_1 &\geqq 0 \\
w_3 - w_2 &\geqq 0 \\
w_4 - w_3 &\geqq 0 \\
w_5 - w_4 &\geqq 0\ .
\end{aligned}$$

Next, for the definiteness, let us choose the cost function to be the linear function

$$y = w_1 + w_2 + w_3 + w_4 + w_5 + w_6 .$$

Thus, we obtain exactly the illustrative example in Section 5 as our standard minimum problem. Hence, the given regular switching function $F = 531$ has a minimal normal canonical separating system

$$\left(w_1, w_2, w_3, w_4, w_5, w_6\right)$$

with

$$w_1 = 1$$
$$w_2 = 2$$
$$w_3 = 2$$
$$w_4 = 3$$
$$w_5 = 4$$
$$w_6 = 4 .$$

Section 9
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