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TECHNICAL REPORT 2508

VARIABILITY OF LETHAL AREA (U)

SYLVAIN EHRENFELD

FEBRUARY 1959

FELTMAN RESEARCH AND ENGINEERING LABORATORIES
PICATINNY ARSENAL
DOVER, N. J.

ORDNANCE PROJECT TS5-5001
DEPT. OF THE ARMY PROJECT 5A04-21-004

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by

Sylvain Ehrenfeld

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Approved:

A. L. DORFMAN
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ABSTRACT

An investigation was made of ~~This report investigates~~ the variability of lethal area as a chance quantity, in repeated use, as well as the variability of estimates that can be made of mean lethal area by conventional methods. Mathematical statements for the standard deviation of lethal area as a chance quantity, ~~and~~ of the standard deviations for estimates of mean lethal area, also as a chance quantity, are derived in terms of the mean and standard deviations of such factors as: P_{HK} ; the average presented area of a human target; the number of hits per square foot and the breakdown of such hits into mass groups; the initial velocity and the mass-to-presented-area ratio of fragments; and the terrain limitation. By neglecting all such variations except the variation in P_{HK}, the effect of this factor alone, as a function of burst height, is illustrated (by numerical examples). In addition, a method is given for comparing the lethal areas of two warheads.

CONCLUSIONS

Numerical examples were computed to estimate the adequacy of available P_{HK} data for the controlled fragmentation case. The computations indicate that the two-standard-deviation percentage error of the estimate of the mean lethal area can be as much as 70% due to P_{HK} alone.

The two-standard-deviation percentage error of the estimated mean lethal area is smaller than the two-standard-deviation percentage error of lethal area as a chance quantity. It is indicated, therefore, that the variability of lethal area as a chance quantity is quite extensive. Therefore, it seems that not much reliance can be placed upon the concept of lethal area for predicting the number of troops incapacitated in any one case, from knowing the mean lethal area of a weapon; i.e. not much reliance can be placed on absolute values. It does not necessarily follow, however, that the concept of mean lethal area is of no value in comparing weapons. Computations are contemplated for future reports to determine the effect of the vari[illegible] percentage errors when comparing different weapons for both controlled and uncontrolled fragmentation cases.

INTRODUCTION

One of the most important problems in Ordnance is to measure the effectiveness of weapons. For this purpose, the concept of "lethal area" is currently used. The idea underlying this concept is that when the lethal area is multiplied by the density (assumed to be uniform) of troops on the ground, the expected number of incapacitated troops is obtained. It will be observed, however, that in practice the number of such incapacitations will differ each time the weapon is used, no matter how carefully an attempt is made to hold conditions fixed. Accordingly, the number of incapacitated troops may be regarded as a chance quantity (random variable); it follows that lethal area will have a corresponding meaning.

A critical examination of relevant literature indicates that the term "lethal area" as commonly used is actually meant to be the mean value of the chance quantity just mentioned. (This mean value must be estimated from experimental data.) This observation leads immediately to two important questions heretofore neglected. These questions are: (1) What is the variability of lethal area as a chance quantity when the weapon is used repeatedly? (2) How adequate is the available data used in estimating said mean? This is the first of a series of reports concerned with examining these questions.

In the present report the basic equations necessary to answer the questions raised are derived. These basic equations include the cases of controlled and uncontrolled fragmentation.

The basic equations derived include the variations of:

(a) Terrain limitation (R)
(b) Fragment density (ρ)
(c) Probability of incapacitation (P_{Hk})
(d) Presented area of a human target (S)
(e) Initial velocity of fragments (V_o)
(f) Mass-to-average-presented-area ratio of a fragment (m/a).

The examples given in this report deal with controlled fragmentation and include the variations due to (b), (c), and (e). The examples furthermore are concerned with the second question raised previously.

An important use of lethal area is in comparing one weapon with another. It would be desirable to know how much larger one lethal area should be than another lethal area in order to say with confidence that one weapon is more effective than another. In the present report, a method for testing the difference between the lethal areas of two weapons is derived.

SOME PERTINENT CONCEPTS IN PROBABILITY AND STATISTICS

In performing certain experiments a number of times, it may not be possible to predict a particular outcome. For example, in tossing a coin, the outcome of a head or a tail cannot be predicted with certainty on any given toss. However, the results of extensive sequences of such experiments have often shown certain statistical regularity, as in the averages of the results, and in the long-run stability of frequency ratios. Such experiments are usually called random experiments. In the mathematical theory constructed to serve such situations, the fact of statistical regularity is idealized by assuming the existence of mathematical probabilities as conceptual counterparts of the frequency ratios, and expected values as counterparts of averages.

Suppose a random experiment is repeated a number of times under uniformly controlled conditions. Suppose furthermore that the outcome of an experiment is described by a number X (a measurement, reading face of a rolled die, etc.). The value which the outcome X takes on may vary for each particular experiment. If X can take on a finite number of values, say $x_1, \ldots, x_k$ (1, . . . , 6 in the rolling of the die) with probabilities $P_1, \ldots, P_k$ ($0 \leq P_i \leq 1$; $\Sigma P_i = 1$) then X is called a discrete random variable, with probability distribution ($P_1, \ldots, P_k$).

If X can take on a continuous set of values such that:

$$\text{Prob } \{ x \leq X \leq x + dx \} = f(x)\, dx$$

where $f(x) \geq 0$ and $\int_{-\infty}^{\infty} f(x)\, dx = 1$

then X is called a continuous random variable with density function f (x). It should be noted that any measurable function of a random variable, $G(X_1, \ldots X_m)$, is again a random variable.

Standard Deviation, Expected Value and Variance, Covariance and Correlation

(A) Discrete case: Suppose X is a discrete random variable which can take on values $x_1, \ldots, x_k$ with probabilities $P_1, \ldots, P_k$ respectively.

Def A1

$$m_x = E(X) = x_1 P_1 + \ldots + x_k P_k$$

m_X is called the expected value of X

Def A2

$$\sigma_X^2 = V(X) = \sum_i (X_i - m_X)^2 \; P_i = E[(X - m_X)^2]$$

V(X) is usually called the variance of X. The standard deviation of X is defined by

$$\sigma_X = \sqrt{V(X)}$$

Let X and Y be two discrete random variables.

Def A3

$$\mathrm{Cov}\,(X,Y) = \sum_i \sum_j \,(X_i - m_X)(Y_j - m_Y)\ \mathrm{Prob}\,\{X = x_i\,;\ Y = y_j\}$$

Cov (X,Y) is usually called the covariance between X and Y.

Def A4

$$\rho(X,Y) = \frac{\mathrm{Cov}\,(X,Y)}{\sigma_X \sigma_Y}$$

ρ (X,Y) is usually called the correlation between X and Y.

(B) Continuous Case: Suppose X is a continuous random variable with density function f(x). Then:

Def B1

$$m_X = E(X) = \int_{-\infty}^{\infty} x\, f(x)\, dx$$

Def B2

$$\sigma_X^2 = V(X) = \int_{-\infty}^{\infty} (X - m_X)^2 f(x)\, dx$$

The expected value, m_X, and the variance, V(X), of a random variable X are numbers describing some aspect of the density function of X. The mean, m_X, describes the central tendency; V (X) describes the variation or spread of the distribution (density).

Let X and Y be two continuous random variables.

Def B3

$$Cov(X,Y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (X - m_X)(Y - m_Y) P(X, Y)\, dx\, dy = E[(X - m_X)(Y - m_Y)]$$

where P (X,Y) is the joint distribution function of X and Y.

Def B4

$$\rho(X,Y) = \frac{Cov(X,Y)}{\sigma_X \sigma_Y}$$

Sampling

Consider a random experiment, E, with associated random variable X (with m_X, σ_X^2) and density function f (x). Suppose the random experiment is repeated n times and the values of X observed are $x_1, \ldots, x_n$. $(x_1, \ldots, x_n)$ is then called a sample of size n from population (density) f (x). Any function of $(x_1, \ldots, x_n)$, $F(x_1, \ldots, x_n)$ is called a sample function. F can be considered as a random variable associated with the experiment (repeating E n times). In terms of an appropriate choice, F may sometimes be used to estimate certain properties of the density function f (x).

Example

Let X have a density function f (x) with mean m_X. Let $F(x_1, \ldots, x_n) = \frac{x_1 + \ldots + x_n}{n} = \bar{x}$

$\bar{x}$ is usually called the average of the sample $(x_1, \ldots, x_n)$.

The average $\bar{x}$ is often used to estimate m_X. One property of $\bar{x}$ is:

$$E\{\bar{x}\} = m_X$$

$\bar{x}$ therefore is called an unbiased estimate of m_X.

The study of statistics consists partly of studying the distribution of F for various F's with the view to drawing inferences about f(x), such as its mean variance.

Testing Hypothesis

Consider two random variables, X and Y with means m_X and m_Y, respectively. It is often desirable to deduce from data about X and Y whether $m_X > m_Y$. A criterion which can be used to do this is the following:

$$m_X > m_Y$$

when $\bar{x} - \bar{y} \geq k\,\sigma_{\bar{X} - \bar{Y}}$ (k chosen appropriately) where $\sigma_{\bar{X} - \bar{Y}}$ is the standard deviation of the random variable $\bar{X} - \bar{Y}$, and $\bar{X}$ and $\bar{Y}$ are unbiased estimates of m_X and m_Y.

The value of k is chosen such that a given value is obtained for the probability of deciding that $m_X > m_Y$ when actually $m_X = m_Y$. This probability is known as the type I error. Two useful lemmas (Ref 1, pp 183 - 184; Ref 3, p 198) to aid in choosing k when there is little information about the distribution of $\bar{X} - \bar{Y}$ are the following:

Lemma 1

Consider a continuous random variable X with mean m and standard deviation σ, then:

$$\text{Prob}\,|X - m \geq k\,\sigma| \leq \frac{1}{1 + k^2}$$

Lemma 2

Let the value of S be defined as follows:

$$S = \frac{m - x_0}{\sigma}$$

where x_0 is the mode of the distribution function of the random variable X. Then:

$$\text{Prob}\left[\,|X - m| \geq k\sigma\right] \leq \frac{4}{9k^2}\left\{\frac{k^2(1+S^2)}{(k-|S|)^2}\right\}$$

Assuming only continuity for the distribution function of X, the type I error for k = 2 is less than 20%, for k = 3, it is less than about 10%. This can be seen from Lemma 1. For moderate values of |S|, the above estimates can be improved. It can often be assumed that the mode equals the mean, for which case, S = o, and when k = 2 the type I error is less than 10%.

The choice of k is determined from economic considerations. If the type I error proved costly, then k would be chosen large, and vice versa.

USEFUL THEOREMS

(a) Properties of expected value

Suppose $X_1, \ldots, X_m$ are m random variables. Then:

$$E[X_1 + \ldots + X_m] = \sum_i E(X_i) \qquad (1)$$

$$E[aX] = aE(X) \qquad (2)$$

$$E(a) = a \qquad (3)$$

where a is any constant.

From (1), (2), and (3), it is concluded:

$$E\left[\sum_i a_i X_i\right] = \sum_i a_i E(X_i) \qquad (4)$$

Suppose there are 2 independent random variables, X and Y.

Then: $$E(XY) = E(X)\,E(Y) \qquad (5)$$

(b) Properties of variance

If $X_1, \ldots, X_m$ are m independent random variables, then:

$$V(X_1 + \ldots + X_m) = \sum_i V(X_i) \qquad (1)$$

$$V(aX) = a^2 V(X) \qquad (2)$$

$$V(a) = 0 \qquad (3)$$

where a is any constant.

From (1), (2), and (3), it is concluded:

$$V\left[\sum_i a_i X_i\right] = \sum_i a_i^2 V(X_i) \qquad (4)$$

(c) Properties of covariance

(1) When X and Y are two independent random variables, then:

$$\text{Cov}(X,Y) = 0$$

(2) If $X_1, \ldots, X_m$ and $Y_1, \ldots, Y_k$ are two sets of random variables, then:

$$\text{Cov}\left[\sum_i a_i X_i, \sum_j b_j Y_j\right] = \sum_i \sum_j a_i b_j \, \text{Cov}(X_i, Y_j)$$

where a_i and b_j are any constants.

(3) When X and Y are two random variables, then:

$$V(X - Y) = V(X) + V(Y) - 2\,\text{Cov}(X, Y)$$

and

$$V(X + Y) = V(X) + V(Y) + 2\,\text{Cov}(X,Y)$$

The properties in (a), (b), and (c) are stated without proof and can be found in many texts on probability theory (Refs 1,2, and 3).

Theorem 2-1

Let X be a variable with mean m and variance σ_X^2.

Consider a function of X, G(X): G is again a random variable. The first approximation to the expected value and variance of G can be gotten from the following:

$$m_G = E[G(X)] \doteq G(m)$$

$$\sigma_G^2 = V[G(X)] \doteq \left(\frac{dG}{dX}\right)^2_{X=m} \sigma_X^2$$

Proof: Let G (X) be expanded in a Taylor series around m, up to and including the linear term. Thus:

$$G(X) \doteq G(m) + \left(\frac{dG}{dX}\right)_{X=m}(X-m)$$

Thus:

$$E[G(X)] \doteq G(m) + \left(\frac{dG}{dX}\right)_{X=m} E[X-m) = G(m)$$

Since:

$$E[X-m] = E(X) - m = 0$$

Furthermore:

$$V[G(X)] = E[(G(X) - E(G))^2]$$

$$(G(X) - E(G))^2 \doteq \left(\frac{dG}{dX}\right)^2_{X=m} (X-m)^2$$

and thus:

$$\sigma_G^2 = E[(G(X) - E(G))^2] \doteq \left(\frac{dG}{dX}\right)^2_{X=m} E(X-m)^2 = \left(\frac{dG}{dX}\right)^2_{X=m} \sigma_X^2$$

Better approximations to E[G(X)] and V[G(X)] can be obtained by going further in the expansion.

Example:

$$G(X) \cong G(m) + \left(\frac{dG}{dX}\right)_{X=m}(X-m) + \frac{1}{2}\left(\frac{d^2G}{dX^2}\right)_{X=m}(X-m)^2$$

Thus:

$$E[G(X)] \cong G(m) + \frac{1}{2}\left(\frac{d^2G}{dX^2}\right)_{X=m}\sigma_X^2$$

In the application of the above theorem, the extent of possible errors due to neglecting part of the Taylor expansion should be investigated. This has been done for the special application in this report (Appendix A, p 26).

Theorem 2.2

Let X and Y be independent random variables with means m_X and m_Y and variances σ_X^2 and σ_Y^2, respectively.

Then:

$$V[XY] = \sigma_X^2\sigma_Y^2 + m_X^2\sigma_Y^2 + m_Y^2\sigma_X^2$$

Proof: From the independence assumption:

$$E[XY] = m_X m_Y$$

By definition:

$$V[XY] = E[(XY - m_X m_Y)^2]$$

Thus:

$$V[XY] = E[X^2Y^2 - 2m_X m_Y XY + m_X^2 m_Y^2]$$

$$= E[X^2Y^2] - 2m_X m_Y m_X m_Y + m_X^2 m_Y^2$$

$$= E[X^2Y^2] - m_X^2 m_Y^2$$

Furthermore, from independence:

$$E(X^2Y^2) = E(X^2)E(Y^2)$$

Now:

$$\sigma_X^2 = E(X - m)^2 = E(X^2) - m_X^2$$

Thus:

$$E(X^2) = \sigma_X^2 + m_X^2$$

Similarly:

$$E(Y^2) = \sigma_Y^2 + m_Y^2$$

Combining:

$$V(XY) = E(X^2)\,E(Y^2) - m_X^2 m_Y^2$$

$$= (\sigma_X^2 + m_X^2)(\sigma_Y^2 + m_Y^2) - m_X^2 m_Y^2$$

$$= \sigma_X^2 \sigma_Y^2 + m_X^2 \sigma_Y^2 + m_Y^2 \sigma_X^2$$

APPLICATION TO LETHALITY CALCULATIONS

The Concept of Lethal Area

It was indicated in the introduction that lethal area is a number, A_L, such that when a uniform density, μ, of troops are situated on the terrain, then $N_I = \mu A_L$ is the number of troops incapacitated.[1] Since experience has shown that N_I is a random variable, A_L has a corresponding meaning.

Consider a random variable, $Y(\theta, r)$, associated with a point (θ, r) situated on the terrain. The random variable, $Y(\theta, r)$, is defined as follows:

$Y(\theta,r) = 1$, when the target at (θ,r) is incapacitated;

$Y(\theta,r) = 0$ otherwise.

[1]The term "incapacitation" is used in a particular sense (Ref 4).

Let the probability that Y is equal to 1 be P_k, (prob (Y = 1) = P_k) and the probability that Y is equal to 0 be $1 - P_k$(prob (Y = 0) = $1 - P_k$).

Let the terrain be divided into areas equal to ΔA (ΔA chosen to be approximately equal to the area of an average target).

The number of troops incapacitated, N_I, on the terrain is the following:

$$N_I = \mu \sum_i Y_i \Delta A$$

where the summation is taken over all ΔA on terrain. Thus, lethal area is defined as a random variable as follows:

$$A_L = \sum_i Y_i \Delta A$$

It follows that:

$$E(A_L) = \sum_i E(Y_i) \Delta A = \sum_i P_k \Delta A \doteq \int\int_{\text{Terrain}} P_k (\theta, r) r dr d\theta$$

and

$$V(A_L) = \sum_i V(Y_i) (\Delta A)^2 = (\Delta A)^2 \sum_i V(Y_i)$$

It is known that:

$$V(Y_i) = P_{k_i} (1 - P_{k_i})$$

and thus

$$V(A_L) = (\Delta A)^2 \sum_i P_{k_i} (1 - P_{k_i})$$

The value of P_k can be computed[1] from the following, (assuming controlled fragmentation):

$$P_k \doteq 1 - e^{-\varpi S \varpi \rho \varpi P_{Hk}}$$

where S, ρ, P_{Hk} are random variables, and

[1] Approximation may not be adequate when $\sigma^2_{S\rho P_{Hk}}$ is relatively large. See page 15

$$E(S) = m_S$$

$$E(\rho) = m_\rho$$

$$E(P_{Hk}) = m_{P_{Hk}}$$

The interpretation of S, ρ, P_{Hk} is as follows:

S is the presented area of target at (θ, r)

ρ is the density of fragments at (θ, r)

P_{Hk} is the probability that a hit will result in an incapacitation at (θ, r).

Lemma 1 (controlled fragmentation)

$$P_k = 1 - e^{-m_S m_\rho m_{P_{Hk}}}$$

Proof:

Let there be n fragments hitting a target situated at (θ, r), with a hit-disablement probability P_{Hk}, and let n_L be the number of those n fragments which are lethal.

Then:

$$n_L = X_1 + \ldots + X_n$$

where

$$X_i = 1 \text{ with probability } P_{Hk};$$
$$= 0 \text{ with probability } 1 - P_{Hk}$$

$X_i = 1$ when the i^{th} fragment is lethal, and 0 otherwise.

The target at (θ, r) is incapacitated if one or more lethal fragments hits it (i.e., $Y = 1$).

$$\text{Prob}\{Y = 1\} = \text{Prob}\{N_i \geq 1\} = 1 - \text{Prob}\{N_i = 0\}.$$

From a theorem on conditional probability it is known that:

$$\text{Prob}\{N_i = 0\} = E\{\text{Prob}(N_i = 0 | n, P_{Hk})\}$$

where prob $(N_i = 0 \mid n, P_{Hk})$ means the probability that $N_i = 0$, when there are n fragments, and a target with P_{Hk}.

Now: $N_i = 0$ if, and only if, all n X's are zero.

Assuming that the fragments are independent with respect to their lethality,[1] it is concluded:

$$\text{Prob}\{N_i = 0 | n, P_{Hk}\} = \text{Prob}\{X_1 = 0, X_2 = 0, \ldots, X_n = 0 | n, P_{Hk}\} = (1 - P_{Hk})^n \cong e^{-nP_{Hk}}$$

and

$$E[\text{Prob}(N_i = 0 | n, P_{Hk})] = E[e^{-nP_{Hk}}] \cong e^{-E(n)E(P_{Hk})}$$

(theorem 2.1)

Now, since

$$n = S\rho; \; E(n) = E(S)E(\rho) = m_s m_\rho$$

also, $E(P_{Hk}) = m_{P_{Hk}}$

Combining terms:

$$P_k = \text{Prob}\{Y = 1\} \cong 1 - e^{-m_s m_\rho m_{P_{Hk}}}$$

which proves the lemma.

It should be noted that two approximations were involved in the derived expression of P_k above. These are

(1) $(1 - P_{Hk})^n \cong e^{-nP_{Hk}}$

(2) $E[e^{-nP_{Hk}}] \cong e^{-E(n)E(P_{Hk})}$

The approximation in (2) can be improved by including further terms in the Taylor expansion of $e^{-nP_{Hk}}$ around $E(nP_{Hk})$.

---

[1] This amounts to disregarding the complications of cumulative damage.

For example:

$$E(e^{-aP_{Hk}}) \approx \frac{e^{-E(a)E(P_{Hk})}}{2}\left(2+\sigma^2_{aP_{Hk}}\right)$$

(example, page 10)

Thus:

$$P_k \approx 1 - \frac{e^{-m_S m_\rho m_{P_{Hk}}}}{2}\left(2+\sigma^2_{aP_{Hk}}\right)$$

It should be noted that when $\sigma^2_{aP_{Hk}}$ is comparatively large, the approximation $P_k = 1 - e^{-m_S m_\rho m_{P_{Hk}}}$ may be inadequate.

A similar proof can be given for the case of uncontrolled fragmentation: that is,

$$P_k = 1 - e^{-m_S \sum_i m_{\rho_{m_i}} m_{P_{Hkm_i}}}$$

where: ρ_{m_i} is the density of fragments at (θ,r) with mass m_i,

and P_{Hkm_i} is the probability that a hit by a fragment with mass m_i will incapacitate a target at (θ,r).

Suppose there are available unbiased estimates of m_S, $m_{\rho_{m_i}}$, $m_{P_{Hkm_i}}$, denoted by $\bar{S}$, $\bar{\rho}_{m_i}$, $\bar{P}_{Hkm_i}$. Let the variances of $\bar{S}$, $\bar{\rho}_{m_i}$, $\bar{P}_{Hkm_i}$, be denoted by σ_S^2, $\sigma^2_{\rho_{m_i}}$, $\sigma^2_{P_{Hkm_i}}$, respectively, (or $V(\bar{S})$, $V(\bar{\rho}_{m_i})$, $V(\bar{P}_{Hkm_i})$)

Let:

$$\bar{P}_k = 1 - e^{-\bar{S}\sum_i \bar{\rho}_{m_i} \bar{P}_{Hkm_i}}$$

Then:

$$E(\bar{P}_k) \approx 1 - e^{-m_S \sum m_{\rho_{m_i}} m_{P_{Hkm_i}}} \approx P_k$$

(theorem 2.1)

Thus $\bar{P}_k$ is an approximately unbiased estimate of P_k. The previous notation can be remembered as follows: *The* – *(bar) denotes unbiased estimates of expected values.*

A question of great interest in connection with $\bar{P}_k$ is the following: What is the standard deviation, $\sigma_{\bar{P}_k}$, of $\bar{P}_k$? (The value of $\sigma_{\bar{P}_k}$ gives a method for estimating how close the value $\bar{P}_k$ is to $P_k = E(Y)$).

In the literature on lethal area there is often a confusion between $\bar{P}_k$ and P_k. $\bar{P}_k$ is usually called the kill probability; actually it is only an unbiased estimate of P_k.[1]

Computation of Variance of $\bar{P}_k$ (θ, r)

Theorem 3.1

$$\sigma^2_{\bar{P}_k} = \bar{V}(\bar{P}_k) \doteq T\, e^{-2\bar{S}\sum_i \bar{p}_{m_i} \bar{P}_{Hkm_i}} = T(1 - \bar{P}_k)^2$$

where

$$T = [\bar{V}(\bar{S}) + \bar{S}^2] [\sum_i (\bar{V}(\bar{p}_{m_i}) \bar{V}(\bar{P}_{Hkm_i}) + \bar{p}^2_{m_i} \bar{V}(\bar{P}_{Hkm_i}) + \bar{P}^2_{Hkm_i} \bar{V}(\bar{p}_{m_i}))$$

$$+ \bar{V}(\bar{S}) [\sum_i \bar{p}_{m_i} \bar{P}_{Hkm_i}]^2]$$

Proof:

Let the expression for $\bar{P}_k$ (θ, r) be rewritten as $\bar{P}_k = 1 - e^{-Z}$

where:

$$Z = \bar{S} \times \sum_i \bar{p}_{m_i} \bar{P}_{Hkm_i}$$

Therefore: $E(\bar{P}_k) \doteq 1 - e^{m\,Z} \doteq P_k$ (theorem 2.1)

The variance of $\bar{P}_k$ is now calculated approximately as:

$$V(\bar{P}_k) \doteq e^{-2m\,Z}\, V(\bar{Z})$$ (theorem 2.1)

[1] Expected values are fixed numbers for a given population, while unbiased estimates are random variables.

$V(\bar{Z})$ is calculated as follows:

Let $X = \bar{S}$ and $Y = \sum_i \bar{\rho}_{m_i} \bar{P}_{Hkm_i}$

Then: $V(\bar{Z}) = V(XY)$

Thus: $V(\bar{Z}) = V(X)\, V(Y) + m_X^2 V(Y) + m_Y^2 V(X)$ (theorem 2.2)

Now: $V(X) = V(\bar{S})$

and $V(Y) = V[\sum_i \bar{\rho}_{m_i} \bar{P}_{Hkm_i}] = \sum_i V(\bar{\rho}_{m_i} \bar{P}_{Hkm_i})$ (property b(1), p 8)

It is in turn computed that:

$$V(\bar{\rho}_{m_i} \bar{P}_{Hkm_i}) = V(\bar{\rho}_{m_i})\, V(\bar{P}_{Hkm_i}) + m^2_{\rho_{m_i}} V(\bar{P}_{Hkm_i}) + m^2_{P_{Hkm_i}} V(\bar{\rho}_{m_i})$$

Combining the various expressions:

$$V(\bar{P}_k) \doteq e^{-2mZ} [V(\bar{S})(\sum_i V(\bar{\rho}_{m_i} \bar{P}_{Hkm_i})) + m_S^2 (\sum V(\bar{\rho}_{m_i} \bar{P}_{Hkm_i})) + V(\bar{S}) (\sum_i m_{\rho_{m_i}} m_{P_{Hkm_i}})^2]$$

which in turn becomes

$$V(\bar{P}_k) \doteq e^{-2mZ} \{(\sum_i V(\bar{\rho}_{m_i} \bar{P}_{Hkm_i}))(V(\bar{S}) + m_S^2) + V(\bar{S}) (\sum_i m_{\rho_{m_i}} m_{P_{Hkm_i}})^2\}$$

Substituting for $V(\bar{\rho}_{m_i} \bar{P}_{Hkm_i})$

$$V(\bar{P}_k) \doteq e^{-2mZ} [\sum_i (V(\bar{\rho}_{m_i})\, V(\bar{P}_{Hkm_i}) + m^2_{\rho_{m_i}} V(\bar{P}_{Hkm_i}) + m^2_{P_{Hkm_i}} V(\bar{\rho}_{m_i}))$$

$$\times [V(\bar{S}) + m_S^2] + V(\bar{S}) (\sum_i m_{\rho_{m_i}} m_{P_{Hkm_i}})^2]$$

To get an approximate unbiased estimate of $V(\bar{P}_k)$ substitute the unbiased estimates for the various quantities, and the result as quoted in the statement of the theorem follows directly.

An expression for the approximate estimated variance of $\bar{P}_k$ was derived as a function of $\bar{V}(\bar{\rho}_{m_i})$, $\bar{V}(\bar{S})$, $\bar{V}(\bar{P}_{Hkm_i})$. $\bar{V}(\bar{\rho}_{m_i})$ and $\bar{V}(\bar{S})$ can be computed from direct experimental data. The value of $\bar{V}(\bar{P}_{Hkm_i})$ cannot be computed directly from data as it stands.

$\bar{P}_{Hkm_i}$ depends on the value of $m_i V/a$ (momentum per presented area for a fragment of mass m_i). $m_i V/a$ can be considered as a random variable for which data is available. For a fixed value of $m_i V/a$, $\bar{P}_{Hkm_i}$ is a random variable for which data is available.

Computation of Variance of $\bar{P}_{Hk}$

To compute the variance of $\bar{P}_{Hkm_i}$, a theorem in conditional probabilities is used.

Theorem 3.2 For any random variables, X and Z,

$$V(X) = E[V(X|Z)] + V[E(X|Z)] \qquad \text{(Ref 7, No. 2, p 326)}$$

where $V(X|Z)$ is the conditional variance of X for fixed Z and $E(X|Z)$ is the conditional expected value of X for fixed Z. $V(X|Z)$ and $E(X|Z)$ can be considered as random variables since they depend on the value of Z.

Theorem 3.2 can be applied to computing $V(\bar{P}_{Hk})$ as follows:

Let: $X = \bar{P}_{Hk}(\bar{Z})$

and $Z = (\frac{m}{a})V$

Furthermore denote:

$$\frac{m}{a} = u \quad \text{and let } V = V_o e^{-\frac{kat}{m}} = V_o e^{-\frac{kt}{u}}$$

Then: $\bar{Z} = \bar{u}\ \bar{V}$

Theorem 3.3

$$V[\bar{P}_{Hk}(\bar{Z})] = \frac{m_{P_{Hk}(m_Z)}[1 - m_{P_{Hk}(m_Z)}]}{n} \left\{ 1 + \frac{(n-1)[m_{y(m_Z)}]^2 \sigma^2_{\bar{Z}}}{m_{P_{Hk}(m_Z)}[1 - m_{P_{Hk}(m_Z)}]} \right\}$$

where:

$$\sigma^2_{\bar{Z}} = e^{-\frac{2kt}{m_u}} \left\{ \left[\left(\frac{kt}{m_u}\right)^2 \sigma^2_{\bar{u}} (\sigma^2_{\bar{V}_o} + m^2_{V_o}) + \sigma^2_{\bar{V}_o} \right] [\sigma^2_{\bar{u}} + m_u^2] \right\} + m_V^2 \sigma^2_{\bar{u}}$$

$m_{y(m_Z)}$ is the mean slope of $\bar{P}_{Hk}(\bar{Z})$ at m_Z.

To get the estimated $V[\bar{P}_{Hk}(\bar{Z})]$, namely $\tilde{V}[\bar{P}_{Hk}(\bar{Z})]$, substitute bars for means in the above formula.

Proof:

From the theorem of conditional variance:

$$V[\bar{P}_{Hk}(\bar{Z})] = E[V(\bar{P}_{Hk}(\bar{Z}) \mid \bar{Z})] + V[E(\bar{P}_{Hk}(\bar{Z}) \mid \bar{Z})]$$

Now:

$$V[\bar{P}_{Hk}(\bar{Z}) \mid \bar{Z}] = \frac{m_{P_{Hk}}(\bar{Z})[1 - m_{P_{Hk}}(\bar{Z})]}{n}$$

This follows from the fact that $P_{Hk}(\bar{Z})$ for fixed $\bar{Z}$ was gotten from binomial data with sample size n (Ref 4).

Furthermore:

$$E[V(\bar{P}_{Hk}(\bar{Z}) \mid \bar{Z})] = E\left\{\frac{m_{P_{Hk}}(\bar{Z})[1 - m_{P_{Hk}}(\bar{Z})]}{n}\right\}$$

$$= \frac{m_{P_{Hk}}(m_Z)[1 - m_{P_{Hk}}(m_Z)]}{n} + \left\{\frac{m''_{P_{Hk}}(m_Z) - 2(m_{P_{Hk}}(m_Z))m''_{P_{Hk}}(m_Z)}{2n}\right.$$

$$\left. - \frac{2(m'_{P_{Hk}}(m_Z))^2}{2n}\right\}\sigma^2_{\bar{Z}}$$

(by including second order term in procedure illustrated by theorem 2.1)

Now:

$$E[\bar{P}_{Hk}(\bar{Z}) \mid \bar{Z}] = m_{P_{Hk}}(\bar{Z})$$

and

$$V[m_{P_{Hk}}(\bar{Z})] \doteq [m'_{P_{Hk}}(m_Z)]^2 \sigma^2_{\bar{Z}} \qquad \text{(theorem 2.1)}$$

By assuming that $m''_{P_{Hk}}(m_Z) = 0$ and combining, there results:

$$V[\bar{P}_{Hk}(\bar{Z})] = \frac{m_{P_{Hk}}(m_Z)\,[1-m_{P_{Hk}}(m_Z)]}{n} + (\tfrac{n-1}{n})(m'_{P_{Hk}(m_Z)})^2\,\sigma^2_{\bar{Z}}$$

Letting $m'_{P_{Hk}(m_Z)} = m_{y(m_Z)}$, it can be concluded:

$$V[\bar{P}_{Hk}(\bar{Z})] = \frac{m_{P_{Hk}}(m_Z)\,[1-m_{P_{Hk}}(m_Z)]}{n}\left\{1+\frac{(n-1)(m_{y(m_Z)})^2\sigma^2_{\bar{Z}}}{m_{P_{Hk}}(m_Z)\,[1-m_{P_{Hk}}(m_Z)]}\right\}$$

The $\sigma^2_{\bar{Z}}$ is computed as follows:

$$\sigma^2_{\bar{Z}} = V(\bar{Z}) = V(\bar{u}\bar{V}) = \sigma^2_{\bar{u}}\sigma^2_{\bar{V}} + \sigma^2_{\bar{u}}m^2_V + \sigma^2_{\bar{V}}m^2_u = \sigma^2_{\bar{V}}(\sigma^2_{\bar{u}} + m^2_u) + m^2_V\sigma^2_{\bar{u}}$$

Now:

$$\bar{V} = \bar{V}_0 e^{-k(\frac{\bar{a}}{\bar{u}})r} = \bar{V}_0 e^{-\frac{kr}{(\bar{u}/\bar{a})}} = \bar{V}_0 e^{-\frac{kr}{\bar{u}}}$$

Let:

$$e^{-\frac{kr}{\bar{u}}} = t;\ \text{then } \bar{V} = \bar{V}_0 t$$

Thus:

$$\sigma^2_{\bar{V}} = (\frac{kr}{m_u})^2 e^{-\frac{2kr}{m_u}}\,\sigma^2_{\bar{u}}(\sigma^2_{\bar{V}_0} + m^2_{V_0}) + \sigma^2_{\bar{V}_0} e^{-\frac{2kr}{m_u}}$$

Now:

$$\sigma_t^2 \cong e^{-\frac{2kr}{m_u}}(\frac{kr}{m_u})^2\sigma^2_{\bar{u}} \qquad \text{(theorem 2.1)}$$

Also:

$$m_t \cong e^{-\frac{kr}{m_u}}$$

Thus:

$$\sigma^2_{\bar{V}_0} \cong (\frac{kr}{m_u})^2 e^{-\frac{2kr}{m_u}}\sigma^2_{\bar{u}}(\sigma^2_{\bar{V}_0} + m^2_{V_0}) + \sigma^2_{\bar{V}_0} e^{-\frac{2kr}{m_u}} = e^{-\frac{2kr}{m_u}}[(\frac{kr}{m_u})^2\sigma^2_{\bar{u}}(\sigma^2_{\bar{V}_0} + m^2_{V_0}) + \sigma^2_{\bar{V}_0}]$$

Combining, it can be concluded:

$$\sigma_{\bar{Z}}^2 \doteq e^{\frac{-2kr}{m_u}} \left[\left(\frac{kr}{m_u}\right)^2 \sigma_d^2 \left(\sigma_{V_0}^2 + m_{V_0}^2\right) + m_{V_0}^2 \right] \left[\sigma_u^2 + m_u^2 \right] + m_V^2 \sigma_u^2$$

Computation of the Variance of Lethal Area When $P_k(\theta, r)$ Depends Only on r

1. In general, mean lethal area, m_{A_L}, can be expressed as:

$$m_{A_L} = \int_0^{2\pi} \int_0^R P_k(\theta, r)\, r\,dr\,d\theta$$

where R denotes the terrain limitation.

When P_k does not depend on θ

$$m_{A_L} = 2\pi \int_0^R P_k(r)\, r\,dr$$

Let:

$$\bar{A}_L = 2\pi \int_0^R \bar{P}_k(r)\, r\,dr$$

Then $\bar{A}_L$ is an unbiased estimate of m_{A_L}. (property a(4), p 7).

Theorem 3.4

$$\sigma_{\bar{A}_L}^2 \doteq 4\pi^2 h^2 \left[r_1^2 \sigma_{\bar{P}_k(r_1)}^2 + \ldots + r_{n-1}^2 \sigma_{\bar{P}_k(r_{n-1})}^2 + \frac{r_n^2}{4} \sigma_{\bar{P}_k(r_n)}^2 \right] = 4\pi^2 h^2 \bar{W}$$

where: $r_j = jh,\ j = 0, \ldots, n;\ nh = R$

Proof:

By the trapezoidal rule for numerical integration:

$$\bar{A}_L \doteq 2\pi h \left[r_1 \bar{P}_k(r_1) + \ldots + r_{n-1} \bar{P}_k(r_{n-1}) \right] + \frac{2\pi h}{2} \left[r_0 \bar{P}_k(r_0) + r_n \bar{P}_k(r_n) \right]$$

Thus, by rules on variance:

$$\sigma_{\bar{A}_L}^2 = V(\bar{A}_L) \doteq 4\pi^2 h^2 \left[r_1^2 \sigma_{\bar{P}_k(r_1)}^2 + \ldots + r_{n-1}^2 \sigma_{\bar{P}_k(r_{n-1})}^2 \right] + \pi^2 h^2 \left[r_n^2 \sigma_{\bar{P}_k(r_n)}^2 \right]$$

To get the desired result, note that $r_0 = 0$, and the unbiased estimate for $\sigma^2_{\bar{A}_L}$ namely, $\bar{\sigma}^2_{\bar{A}_L}$ is gotten by putting bars over the σ's .

2. In section 1 it was assumed that R (the terrain limitation) was a fixed number. In reality, however, R can only be estimated to be typical of an assumed tactical situation (Ref 6). In this section, therefore, R will be considered as a random variable, with standard deviation σ_R. Thus $\sigma^2_{\bar{A}_L}$ must be re-estimated to include the effect of the error in using R instead of the true mean of R.

Theorem 3.5

$$\bar{\sigma}^2_{\bar{A}_L} = 4\pi^2 \bar{h}^2 \bar{\bar{W}} (1 + \alpha^2_{\bar{R}}) + \bar{A}_L^2 \alpha^2_{\bar{R}}$$

where,

$$\bar{h} = \bar{R}/n$$

$$\alpha_{\bar{R}} = \bar{\sigma}_{\bar{R}}/\bar{R}$$

$$\bar{W} = r_1^2 \bar{\sigma}^2_{\bar{P}_k(r_1)} + \dots + r^2_{n-1} \bar{\sigma}^2_{\bar{P}_k(r_{n-1})} + \frac{r_n^2}{4} \bar{\sigma}^2_{\bar{P}_k(r_n)}$$

Proof.

$$\bar{A}_L = 2\pi \bar{h} \,[r_1 \bar{P}_k(r_1) + \dots + r_{n-1} \bar{P}_k(r_{n-1})] + \pi \bar{h} \,[r_n \bar{P}_k(r_n)]$$

$$= 2\pi \bar{h} [r_1 \bar{P}_k(r_1) + \dots + r_{n-1} \bar{P}_k(r_{n-1}) + \frac{r_n \bar{P}_k(r_n)}{2}]$$

If the quantity in the bracket is denoted by $\bar{Z}$ it is computed:

$$\bar{V}(\bar{A}_L) = \bar{V}(2\pi \bar{h} \bar{Z}) = 4\pi^2 [\bar{\sigma}^2_Z \bar{\sigma}^2_{\bar{h}} + \bar{h}^2 \bar{\sigma}^2_Z + \bar{Z}^2 \bar{\sigma}^2_{\bar{h}}] \qquad \text{(theorem 2.1)}$$

$$= 4\pi^2 \bar{h}^2 \sigma^2_{\bar{Z}} [1 + (\frac{\bar{\sigma}_{\bar{h}}}{\bar{h}})^2] + 4\pi^2 \bar{h}^2 \bar{Z}^2 (\frac{\bar{\sigma}_{\bar{h}}}{\bar{h}})^2$$

Since: $\bar{h} = \bar{R}/n$

it is computed

$\sigma_{\bar{h}} = \sigma_{\bar{R}/n}$ (Property b (2), p 8)

Thus: $\sigma_{\bar{h}}/\bar{h} = \sigma_{\bar{R}}/\bar{R}$

The desired result is obtained when it is noticed that:

$$\bar{\sigma}_{\bar{Z}}^2 = \bar{W}_1 4a^2 \bar{h}^2 \bar{Z}^2 = \bar{A}_L^2 \text{ and } \sigma_{\bar{h}}/\bar{h} = \sigma_{\bar{R}}/\bar{R}$$

Comparison of Two Lethal Areas (Controlled Fragmentation and Normal Approach)

An important use of lethal area calculations is the comparison of one weapon with another. It would be desirable to know whether it could be said with confidence that one weapon is more effective than another. In this section a method for such comparison is given. If the same criterion of incapacitation is used in the comparison, the two lethal areas computed will be dependent. The dependence arises from the fact that the same P_{Hk} data is used for both calculations.

Assume two weapons, with mean lethal areas M_1 and M_2 respectively. M_1 and M_2 are unknown quantities, which are estimated by the calculated values $\bar{A}_L^{(1)}$ and $\bar{A}_L^{(2)}$. How much larger should $\bar{A}_L^{(1)}$ be than $\bar{A}_L^{(2)}$ to make it possible to say with confidence that $M_1 > M_2$? The test which will be used is the k standard deviation criterion. Let $\sigma_{\bar{A}_L}^{(1)}$ and $\sigma_{\bar{A}_L}^{(2)}$ denote the estimated standard deviations of $\bar{A}_L^{(1)}$ and $\bar{A}_L^{(2)}$ respectively, and cov $(\bar{A}_L^{(1)}, \bar{A}_L^{(2)})$ denote the covariance between $\bar{A}_L^{(1)}$ and $\bar{A}_L^{(2)}$ (see page 4 for definition of covariance). It is known that the estimated standard deviation of $\bar{A}_L^{(1)} - \bar{A}_L^{(2)}$ is

$$\left(\sigma^2_{\bar{A}_L^{(1)}} + \sigma^2_{\bar{A}_L^{(2)}} - 2\,\mathrm{Cov}(\bar{A}_L^{(1)}, \bar{A}_L^{(2)})\right)^{1/2}$$

(property 5 (c), p 8). The k standard deviation criterion states:

M_1 is significantly greater than M_2 when:

$$\bar{A}_L^{(1)} - \bar{A}_L^{(2)} \geq k(\hat{\sigma}^2_{\bar{A}_L^{(1)}} + \hat{\sigma}^2_{\bar{A}_L^{(2)}} - 2\,\overline{\mathrm{Cov}}\,(\bar{A}_L^{(1)}, \bar{A}_L^{(2)}))^{1/2}$$

The formula and proof expressing $\overline{\mathrm{cov}}\,(\bar{A}_L^{(1)}, \bar{A}_L^{(2)})$ are given in Appendix II. It should be noted that the estimated standard deviation of $\bar{A}_L^{(1)} - \bar{A}_L^{(2)}$ is an approximately unbiased estimate. The substitution of the unbiased estimate of $\sigma_{A_L^{(1)} - A_L^{(2)}}$ for $\sigma_{\bar{A}_L^{(1)} - \bar{A}_L^{(2)}}$ will change the probability of type I error as described on p. 6 (Testing Hypothesis). It seems reasonable that further information about the distribution function of $\bar{A}_L^{(1)} - \bar{A}_L^{(2)}$ should be obtained to get a more accurate probability statement for type I error.

In future reports the above questions as well as the nonnormal approach and the uncontrolled fragmentation case will be studied.

The comparison of two weapons by means of the difference $M_1 - M_2$ may not have sufficient intuitive appeal. A convincing case might be made for considering the relative values of $\sigma_{A_L^{(1)}}$ and $\sigma_{A_L^{(2)}}$ in addition, since these reflect the reliability of the weapons. A more intuitively appealing criterion for comparing weapons may possibly be:

$$\frac{M_1}{\sigma_{A_L}^{(1)}} - \frac{M_2}{\sigma_{A_L}^{(2)}}$$

Some Simplified Examples

The examples computed in this report are for the controlled fragmentation case. A normal approach to the ground was assumed for simplicity (p_k depends only on r).

The example assumed the following:

5-inch diameter spherical warheads with zero terminal velocity

20.6-grain cubical fragments

230 fragments (total)

3340 fps initial fragment velocity (V_o)

The wound ballistics data for the 5-minute disablement criterion was taken from Reference 4.

$\bar{A}_L$, $\sigma_{\bar{A}_L}$ were computed for the variation of $\bar{p}_{11k}$ alone, as well as with assumed values for $\bar{\sigma}_{\bar{p}}/\bar{p}$ and $\bar{\sigma}_{V_o}/\bar{V}_o$. Values of 11% and 2%, assumed

for these functions, are believed typical for a warhead of this size based upon data available for the T38E6 hand grenade (Ref 8). The variations due to $\bar{S}$ and terrain limitation were assumed as zero. The basic formulas simplify appreciably for this case. The computations were made for three different burst heights: h = 5, 10, and 20 feet. The results of these computations are shown in Table 1 and Figure 1 (pp 35 and 36) (See Ref 5 and 6 for method of calculating $\bar{S}$, $\bar{p}$)

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APPENDIX A

Extent of Error in Use of Taylor Expansion

In the discussion of the approximate variance of $\overline{P}_k$ which appears in this report (p 12), only the linear terms of the Taylor expansion were used. In this appendix, the extent of possible error in this procedure is investigated.

Some preliminary statements in probability theory will be required.

Let X be a random variable with mean m, and denote:

$$a_r = E\{(X-m)^r\}$$

$$\mu_r = E\{|X-m|^r\}$$

For $r = 2$; $a_2 = \mu_2 = \sigma^2$; also, when r is an even integer, $a_r = \mu_r$.

The following will be stated without proof:

Lemma 1 $\quad |a_r| \le \mu_r$ $\quad$ (See Ref 3, p 264)

Lemma 2 $\quad \mu_r^2 \le \mu_2^r = \sigma^{2r}$ $\quad$ (See Ref 3, p 267)

With these lemmas, the main result can be proved.

Theorem 4.1 Let X be a random variable with mean m and variance σ_x^2. Then:

$$V\{1-e^{-x}\} = e^{-2m}\sigma_x^2 + \Delta$$

with

$$|\Delta| \le 2\{e^{-2m}(e^{\sigma_x} - 1 - \sigma_x)^2 + e^{-2m}\sigma_x(e^{\sigma_x} - 1 - \sigma_x)\}$$

Proof:

Consider any function of the random variable X such as G(X). Expand G around m in a Taylor expansion.

Thus:

$$|\Delta| \leq 2 \sum_{r=2}^{\infty} \sum_{s=2}^{\infty} \frac{|M_r|\,|M_s|}{r!\;s!} \sigma_x^r \sigma_x^s + 2|M_1| \sum_{r=2}^{\infty} \frac{|M_r|}{r!} \sigma_x^{r+1} \leq 2\left(\sum_{r=2}^{\infty} \frac{|M_r|}{r!} \sigma_x^r\right)^2 + 2|M_1| \sigma_x \sum_{r=2}^{\infty} \frac{|M_r|}{r!} \sigma_x^r$$

Let $|M_r| \leq M$ for $r \geq 2$

Then:

$$|\Delta| \leq 2M^2 \left(\sum_{r=2}^{\infty} \frac{\sigma_x^r}{r!}\right)^2 + 2M\sigma_x \left(\frac{\sigma_x^r}{r!}\right) = 2M^2A^2 + 2MA$$

Since

$$e^{\sigma_x} = 1 + \sigma_x + \sum_{r=2}^{\infty} \frac{\sigma_x^r}{r!}$$

and

$$A = \sum_{r=2}^{\infty} \frac{\sigma_x^r}{r!}$$

A can be computed as follows:

$$A = e^{\sigma_x} - 1 - \sigma_x$$

Consider the case where:

$$G(X) = 1 - e^{-x}$$

In general:

$$G^{(r)}(m) = (-1)^{r+1} e^{-m}$$

Thus:

$$|G^{(r)}(m)| = e^{-m}$$

Thus: M can be taken as equal to e^{-m}

Thus:

$$|\Delta| \leq 2\{e^{-2m}(e^{\sigma_x} - 1 - \sigma_x)^2 + \sigma_x e^{-m}(e^{\sigma_x} - 1 - \sigma_x)\}$$

which is the result as stated in the theorem.

APPENDIX B

Computation of Cov $(\bar{A}_L^{(1)}, \bar{A}_L^{(2)})$ (Controlled Fragmentation and Normal Approach)

It is known that:

$$\bar{A}_L^{(1)} = \sum_i a_i \bar{P}_{k_i}^{(1)}$$

and

$$\bar{A}_L^{(2)} = \sum_j b_j \bar{P}_{k_j}^{(2)}$$

where

$$a_i = 2\pi h^{(1)} r_i^{(1)} \quad , i = 1, \ldots, n^{(1)} - 1$$

$$a_{n^{(1)}} = \pi h^{(1)} r_{n^{(1)}}^{(1)}$$

$$b_j = 2\pi h^{(2)} r_j^{(2)} \quad , j = 1, \ldots, n^{(2)} - 1$$

$$b_{n^{(2)}} = \pi h^{(2)} r_{n^{(2)}}^{(2)}$$

Now:

$$\text{Cov}\,(\bar{A}_L^{(1)}\,\bar{A}_L^{(2)}) = \sum_i \sum_j a_i b_j \,\text{cov}\,(\bar{P}_{k_i}^{(1)}, \bar{P}_{k_j}^{(2)})$$

$\bar{P}_{k_i}^{(1)}$ and $\bar{P}_{k_j}^{(2)}$ can be written as follows:

$$\bar{P}_{k_i}^{(1)} = 1 - e^{-\bar{S}_i^{(1)} \bar{\rho}_i^{(1)} \bar{P}_{Hk_i}^{(1)}}$$

$$\bar{P}_{k_j}^{(2)} = 1 - e^{-\bar{S}_j^{(2)} \bar{\rho}_j^{(2)} \bar{P}_{Hk_j}^{(2)}}$$

Let:

$$\bar{S}_i^{(1)} \bar{\rho}_i^{(1)} = X$$

$$S_j^{(2)} \bar{\rho}_j^{(2)} = Y$$

$$\bar{P}_{(k)}^{(1)} = u$$

$$\bar{P}_{(k)}^{(2)} = v$$

Thus: $\bar{P}_{k_i}^{(1)} = 1 - e^{-Xu}$

$$\bar{P}_{k_j}^{(2)} = 1 - e^{-Yv}$$

Now:

$$E(\bar{P}_{k_i}^{(1)}) = E(1 - e^{-Xu}) \doteq (1 - e^{-m_x m_u} - \frac{e^{-m_x m_u}}{2} \sigma_{xu}^2 = 1 - \frac{e^{-m_x m_u}}{2} (2 + \sigma_{xu}^2)$$

Similarly:

$$E(\bar{P}_{k_j}^{(2)}) \doteq 1 - \frac{e^{-m_y m_v}}{2} (2 + \sigma_{yv}^2)$$

Thus:

$$\bar{P}_{k_i}^{(1)} - E(\bar{P}_{k_i}^{(1)}) \doteq \frac{e^{-m_x m_u}}{2} (2 + \sigma_{xu}^2) - e^{-Xu}$$

$$\bar{P}_{k_j}^{(2)} - E(\bar{P}_{k_j}^{(2)}) \doteq \frac{e^{-m_y m_v}}{2} (2 + \sigma_{yv}^2) - e^{-Yv}$$

Let:

$$A = \frac{e^{-m_x m_u}}{2} (2 + \sigma_{xu}^2)$$

$$B = \frac{e^{-m_y \mu_y}}{2} (2 + \sigma_{yy}^2)$$

$$R = e^{-X x}$$

$$S = e^{-Y y}$$

Now:

$$\bar{P}_{k_j}^{(1)} - E(\bar{P}_{k_j}^{(1)}) = A - R$$

$$\bar{P}_{k_j}^{(2)} - E(\bar{P}_{k_j}^{(2)}) = B - S$$

From the definition of covariance:

$$\text{Cov}(\bar{P}_{k_i}^{(1)}, \bar{P}_{k_j}^{(2)}) = E(A - R)(B - S) = AB - E(BR) - E(AS) - E(RS)$$

Now:

$$E(BR) = \frac{e^{-m_y \mu_y}}{2} (2 + \sigma_{yy}^2) \left[\frac{2e^{-m_x \mu_x}}{2} + \frac{e^{-m_x \mu_x}}{2} \sigma_{xx}^2 \right] = \frac{e^{-(m_x \mu_x + m_y \mu_y)}}{4} (2 + \sigma_{xx}^2)(2 + \sigma_{yy}^2)$$

Similarly:

$$E(AS) = \frac{e^{-(m_x \mu_x + m_y \mu_y)}}{4} (2 + \sigma_{xx}^2)(2 + \sigma_{yy}^2)$$

Finally:

$$E(RS) = \frac{4\, e^{-(m_x \mu_x + m_y \mu_y)}}{4} + \frac{2\, e^{-(m_x \mu_x + m_y \mu_y)}}{4} \sigma_{xx+yy}^2$$

$$= \frac{e^{-(m_x \mu_x + m_y \mu_y)}}{4} \left[4 + 2\sigma_{xx+yy}^2 \right]$$

Combining, it is concluded that:

$$\text{Cov}(\bar{P}_{k_j}^{(1)}, \bar{P}_{k_j}^{(2)}) = \frac{e^{-(m_x \mu_x + m_y \mu_y)}}{4} \left[4 + 2\sigma_{xx+yy}^2 - (2 + \sigma_{xx}^2)(2 + \sigma_{yy}^2) \right]$$

Now:

$$\sigma^2_{Xu+Yv} = \sigma_{Xu}^2 + \sigma_{Yv}^2 + 2\,\mathrm{cov}\,(Xu, Yv)$$

Thus:

$$\mathrm{Cov}\,(\bar{P}_{k_i}^{(1)}, \bar{P}_{k_j}^{(2)}) = \frac{e^{-(m_x m_u + m_y m_v)}}{4} [\,4 + 2\,\sigma_{Xu}^2 + 2\,\sigma_{Yv}^2 +$$

$$4\,\mathrm{cov}\,(Xu, Yv) - 4 - 2\sigma_{Xu}^2 - 2\sigma_{Yv}^2 - \sigma_{Xu}^2\,\sigma_{Yv}^2\,]$$

$$= \frac{e^{-(m_x m_u + m_y m_v)}}{4} [\,4\,\mathrm{cov}\,(Xu, Yv) - \sigma_{Xu}^2\,\sigma_{Yv}^2\,]$$

Since X and Y, X and u, X and v, Y and u, Y and v are independent,

$$\mathrm{Cov}\,(Xu, Yv) = \rho_{uv}\,\sigma_u\sigma_v\,m_x m_y$$

Thus:

$$\mathrm{Cov}\,(\bar{P}_{k_i}^{(1)}, \bar{P}_{k_j}^{(2)}) = \frac{e^{-(m_x m_u + m_y m_v)}}{4} [\,4\rho_{uv}\,\sigma_u\sigma_v m_x m_y - \sigma_{Xu}^2\,\sigma_{Yv}^2\,]$$

Letting the correlation $\rho_{uv} = 1$, it is concluded that:

$$\mathrm{Cov}\,(\bar{P}_{k_i}^{(1)}, \bar{P}_{k_j}^{(2)}) = (e^{-m_x m_u} m_x \sigma_u)(e^{-m_y m_v} m_y \sigma_v)$$

$$-\left(\frac{e^{-m_x m_u}}{2}\,\sigma_{Xu}^2\right)\left(\frac{e^{-m_y m_v}}{2}\,\sigma_{Yv}^2\right)$$

To get the estimated covariance put in bars for true values and it is concluded that:

$$\overline{\mathrm{Cov}}\,(\bar{P}_{k_i}^{(1)}, \bar{P}_{k_j}^{(2)}) = (e^{-\overline{Xu}}\,\bar{X}\,\bar{\sigma}_u)(e^{-\overline{Yv}}\,\bar{Y}\,\bar{\sigma}_v) - \left(\frac{e^{-\overline{Xu}}}{2}\,\bar{\sigma}_{Xu}^2\right)\left(\frac{e^{-\overline{Yv}}}{2}\,\bar{\sigma}_{Yv}^2\right)$$

The desired result can be obtained by remembering that:

$$\sigma_{Xu}^2 = \sigma_X^2\,(\sigma_u^2 + m_u^2) + m_X^2\,\sigma_u^2$$

$$\sigma_{yw}^2 = \sigma_y^2(\sigma_w^2 + m_w^2) + m_y^2\sigma_w^2$$

Siece:

$$X = \bar{S}_i^{(1)}\bar{\rho}_i^{(1)}$$

$$Y = \bar{S}_j^{(2)}\bar{\rho}_j^{(2)}$$

$$\sigma_x^2 = (\bar{S}_i^{(1)}\bar{\rho}_i^{(1)})^2 (e^2_{\bar{S}_i(1)}\, e^2_{\bar{\rho}_i(1)} + e^2_{\bar{S}_i(1)} + e^2_{\bar{\rho}_i(1)})$$

$$\sigma_y^2 = (\bar{S}_j^{(2)}\bar{\rho}_j^{(2)})^2 (e^2_{\bar{S}_j(2)}\, e^2_{\bar{\rho}_j(2)} + e^2_{\bar{S}_j(2)} + e^2_{\bar{\rho}_j(2)})$$

$$e_{\bar{S}} = \sigma_{\bar{S}}/\bar{S} \quad e_{\bar{\rho}} = \sigma_{\bar{\rho}}/\bar{P}$$

Further:

$$u = \bar{P}_{Hk_i}^{(1)}$$

$$v = \bar{P}_{Hk_j}^{(2)}$$

$$\bar{\sigma}_u^2 = \frac{\bar{P}_{Hk_i}^{(1)}(1 - \bar{P}_{Hk_i}^{(1)})}{n} = \bar{\sigma}_v^2$$

when mv/a for $P_{k_i}^{(1)}$ and $\bar{P}_{k_i}^{(2)}$ are the same.

$$X\,\bar{\sigma}_u\, e^{-Xu} = (1 - \bar{P}_{k_i}^{(1)})\,\bar{S}_i^{(1)}\bar{\rho}_i^{(1)}\,\frac{\bar{P}_{Hk_i}^{(1)}(1 - \bar{P}_{Hk_i}^{(1)})}{n} = t_i^{(1)}$$

$$Y\,\bar{\sigma}_v\, e^{-Yv} = (1 - \bar{P}_{k_j}^{(2)})\,\bar{S}_j^{(2)}\bar{\rho}_j^{(2)}\,\frac{\bar{P}_{Hk_j}^{(2)}(1 - \bar{P}_{Hk_j}^{(2)})}{n} = t_j^{(2)}$$

$$\frac{e^{-Xu}}{2}\bar{\sigma}_{xu} = \frac{(1 - \bar{P}_{k_i}^{(1)})}{2}\left\{\left(\frac{\bar{P}_{Hk_i}^{(1)}(1 - \bar{P}_{Hk_i}^{(1)})}{n} + (\bar{P}_{Hk_i}^{(1)})^2\right)(\bar{S}_i^{(1)}\bar{\rho}_i^{(1)})^2\right.$$

$$\left.(\sigma^2_{\bar{S}_i(1)}\sigma^2_{\bar{\rho}_i(1)} + \sigma^2_{\bar{\rho}_i(1)} + \sigma^2_{\bar{S}_i(1)}) + (\bar{S}_i^{(1)}\bar{\rho}_i^{(1)})^2\,\frac{P_{Hk_i}^{(1)}(1 - P_{Hk_i}^{(1)})}{2}\right\} = \beta_i^{(1)}$$

A similar expression is gotten for $\beta_j^{(2)}$.

Thus:

$$\overline{\text{Cov}}\,(\bar{A}_L^{(1)}, \bar{A}_L^{(2)}) \doteq (\sum_i a_i\, t_i^{(1)})\,(\sum_j b_j t_j^{(2)}) - (\sum_i a_i \beta_i^{(1)})(\sum_j b_j \beta_j^{(2)})$$

where the summation is taken over values of i and j with the same mv/a.

TABLE I

Example of Lethal Area Calculations

h	$\frac{2\sigma_{\bar V_e}}{\bar V_e}$	$\frac{2\sigma_{\bar\rho}}{\bar\rho}$	$\bar A_L$	$\frac{2\sigma_{\bar A_L}}{\bar A_L}$
5	0	0	352.9	45.2%
10	0	0	424.5	52.1%
20	0	0	316.7	71.1%
5	4%	22%	352.9	46.2%
10	4%	22%	424.5	52.9%
20	4%	22%	316.7	73.5%

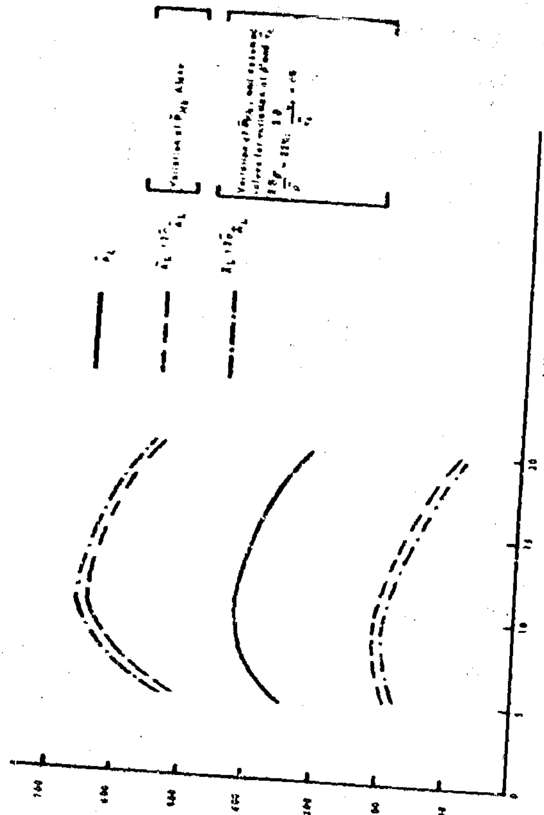

Fig 1   Mean Lethal Area vs Burst Height

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