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Conditional Means and Covariances  
of Normal Variables with  
Singular Covariance Matrix

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CONDITIONAL MEANS AND COVARIANCES OF NORMAL  
VARIABLES WITH SINGULAR COVARIANCE MATRIX

by

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Let  $(\xi, \eta)$  be a partitioned zero mean normal random vector with covariance matrix  $\begin{pmatrix} A & B' \\ B & C \end{pmatrix}$  where  $\text{cov}(\xi) = A$  and  $\text{cov}(\eta) = C$ , with  $\text{cov}(\ )$  meaning the covariance matrix of the random vector enclosed. The usual way of getting the conditional mean and covariance matrix of  $\xi$ , given that  $\eta = \beta$ , is to divide the joint density of  $\xi$  and  $\eta$  by that of  $\eta$ . The following is an alternative method which is more general, in that it does not require that  $\xi$  and  $\eta$  have a joint density, or even that  $\eta$  have a density.

This is the result we want to prove: The conditional mean and covariance of  $\xi$ , given that  $\eta = \beta$ , are

$$E(\xi|\eta = \beta) = \beta C^+ B, \quad \text{cov}(\xi|\eta = \beta) = A - B' C^+ B$$

where  $C^+$  is the pseudoinverse of  $C$ , that is, if  $C = T'T$  with  $T$   $r \times m$  of rank  $r$ , then  $C^+ = T'(TT')^{-2}T$ . If  $C^{-1}$  exists, then  $C^+ = C^{-1}$ . The pseudoinverse of a symmetric matrix, although perhaps not under that name, is well known and has been used in statistics for some time. For a recent discussion and references, see [1].

Let  $E = T'(TT')^{-1}T = C^+C = CC^+$  be the projector of the row space of  $C$ . Note that  $CE = C$ , and hence for any matrix  $F$  whose rows are in the row space of  $C$ ,  $FE = F$ . In particular,  $B'$  in the covariance matrix above satisfies  $B'E = B'$ , since the rows of  $B'$  are in the row space of  $C$ . (The general covariance matrix

may be assumed to have the form  $\begin{pmatrix} S' \\ U' \end{pmatrix} (SU) = \begin{pmatrix} S'S & S'U \\ U'S & U'U \end{pmatrix}$ , and the row space of  $S'U$  lies in the row space of  $U$ , which has the same row space as  $U'U$ .)

We will derive the formulas for conditional mean and covariance of  $\xi$ , given  $\eta = \beta$ , by representing  $\xi$  in such a way that it is obvious what conditioning on  $\eta$  means. We need only the fact that the sum of two normal random vectors is normal, and that if  $\eta$  has covariance  $C$ , then  $\eta M$  has covariance  $M'CM$ . Let  $\zeta$  be a zero mean normal random vector which is independent of  $\eta$  and which has covariance  $A - B'C^+B$ . (This is a valid covariance matrix, for example, that of  $\xi - \eta C^+B$ .) Then  $\xi = \zeta + \eta C^+B$  is our representation for  $\xi$ , since the covariance matrix of  $(\xi, \eta) = (\zeta, \eta) \begin{pmatrix} I & 0 \\ C^+B & I \end{pmatrix}$  is

$$\begin{pmatrix} I & B'C^+ \\ 0 & I \end{pmatrix} \begin{pmatrix} A - B'C^+B & 0 \\ 0 & C \end{pmatrix} \begin{pmatrix} I & 0 \\ C^+B & I \end{pmatrix} = \begin{pmatrix} A & B' \\ B & C \end{pmatrix}.$$

Since  $\zeta$  and  $\eta$  are independent, it is obvious that the conditional mean of  $\zeta + \eta C^+B$ , given that  $\eta = \beta$ , is  $\beta C^+B$ , and that the conditional covariance is that of  $\zeta$ . Hence our general result: If  $(\xi, \eta)$  is a zero mean normal vector with  $\text{cov}(\xi, \eta) = \begin{pmatrix} A & B' \\ B & C \end{pmatrix}$ ,  $\text{cov}(\xi) = A$ , and  $\text{cov}(\eta) = C$ , then the expected value and covariance of  $\xi$ , given that  $\eta = \beta$ , are

$$E(\xi | \eta = \beta) = \beta C^+B, \quad \text{cov}(\xi | \eta = \beta) = A - B'C^+B$$

where  $C^+$  is the pseudoinverse of  $C$ , that is,  $C^+ = C^{-1}$  if  $C^{-1}$  exists, otherwise, if  $C = T'T$  with  $T$   $r \times m$  of rank  $r$ , then  $C^+ = T'(TT')^{-2}T$ .

## REFERENCE

- [1] Greville, T. N. E. The pseudoinverse of a rectangular or singular matrix and its application to the solution of systems of linear equations. SIAM Review, Vol. 1, No. 1, January 1959, p. 38-43.