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George Marsaglia

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CONDITIONAL MEANS AND COVARIANCES OF NORMAL VARIABLES WITH SINGULAR COVARIANCE MATRIX

by

George Marsaglia

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Let (ξ,η) be a partitioned zero mean normal random vector with covariance matrix $\begin{pmatrix} A & B' \\ B & C \end{pmatrix}$ where $\operatorname{cov}(\xi) = A$ and $\operatorname{cov}(\eta) = C$, with cov( ) meaning the covariance matrix of the random vector enclosed. The usual way of getting the conditional mean and covariance matrix of ξ, given that $\eta = \beta$, is to divide the joint density of ξ and η by that of η. The following is an alternative method which is more general, in that it does not require that ξ and η have a joint density, or even that η have a density.

This is the result we want to prove: The conditional mean and covariance of ξ, given that $\eta = \beta$, are

$$E(\xi|\eta = \beta) = \beta C^{+}B, \qquad \operatorname{cov}(\xi|\eta = \beta) = A - B'C^{+}B$$

where C^{+} is the pseudoinverse of C, that is, if $C = T'T$ with T $r \times m$ of rank r, then $C^{+} = T'(TT')^{-2}T$. If C^{-1} exists, then $C^{+} = C^{-1}$. The pseudoinverse of a symmetric matrix, although perhaps not under that name, is well known and has been used in statistics for some time. For a recent discussion and references, see [1].

Let $E = T'(TT')^{-1}T = C^{+}C = CC^{+}$ be the projector of the row space of C. Note that $CE = C$, and hence for any matrix F whose rows are in the row space of C, $FE = F$. In particular, B' in the covariance matrix above satisfies $B'E = B'$, since the rows of B' are in the row space of C. (The general covariance matrix may be assumed to have the form $\binom{S'}{U'}(SU) = \begin{pmatrix} S'S & S'U \\ U'S & U'U \end{pmatrix}$, and the row space of $S'U$ lies in the row space of U, which has the same row space as $U'U$.)

We will derive the formulas for conditional mean and covariance of ξ, given $\eta = \beta$, by representing ξ in such a way that it is obvious what conditioning on η means. We need only the fact that the sum of two normal random vectors is normal, and that if η has covariance C, then ηM has covariance $M'CM$. Let ζ be a zero mean normal random vector which is independent of η and which has covariance $A - B'C^+B$. (This is a valid covariance matrix, for example, that of $\xi - \eta C^+B$.) Then $\xi = \zeta + \eta C^+B$ is our representation for ξ, since the covariance matrix of $(\xi,\eta) = (\zeta,\eta)\begin{pmatrix} I & 0 \\ C^+B & I \end{pmatrix}$ is

$$\begin{pmatrix} I & B'C^+ \\ 0 & I \end{pmatrix}\begin{pmatrix} A - B'C^+B & 0 \\ 0 & C \end{pmatrix}\begin{pmatrix} I & 0 \\ C^+B & I \end{pmatrix} = \begin{pmatrix} A & B' \\ B & C \end{pmatrix} .$$

Since ζ and η are independent, it is obvious that the conditional mean of $\zeta + \eta C^+B$, given that $\eta = \beta$, is βC^+B, and that the conditional covariance is that of ζ. Hence our general result: <u>If</u> (ξ,η) <u>is a zero mean normal vector with</u> $\mathrm{cov}(\xi,\eta) = \begin{pmatrix} A & B' \\ B & C \end{pmatrix}$, $\mathrm{cov}(\xi) = A$, <u>and</u> $\mathrm{cov}(\eta) = C$, <u>then the expected value and covariance of</u> ξ, <u>given that</u> $\eta = \beta$, <u>are</u>

$$E(\xi|\eta = \beta) = \beta C^+B, \quad \mathrm{cov}(\xi|\eta = \beta) = A - B'C^+B$$

<u>where</u> C^+ <u>is the pseudoinverse of</u> C, <u>that is</u>, $C^+ = C^{-1}$ <u>if</u> C^{-1} <u>exists</u>, <u>otherwise</u>, <u>if</u> $C = T'T$ <u>with</u> T $r \times m$ <u>of rank</u> r, <u>then</u> $C^+ = T'(TT')^{-2}T$.

REFERENCE

[1] Greville, T. N. E. The pseudoinverse of a rectangular or singular matrix and its application to the solution of systems of linear equations. SIAM Review, Vol. 1, No. 1, January 1959, p. 38-43.