AD 296 789

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

LINCOLN LABORATORY

37 G - 1

A THEOREM ON FOURIER INTEGRALS AND AN APPLICATION TO THE THEORY OF MEASUREMENT IN QUANTUM MECHANICS

Harry E. Moses

5 February 1963

The work reported in this document was performed at Lincoln Laboratory, a center for research operated by Massachusetts Institute of Technology, with the joint support of the U.S. Army, Navy and Air Force under Air Force Contract AF 19(628)-500.

LEXINGTON MASSACHUSETTS

A THEOREM ON FOURIER INTEGRALS AND AN APPLICATION TO THE THEORY OF MEASUREMENT IN QUANTUM MECHANICS

BY: Harry E. Moses

LINCOLN LABORATORY
MASSACHUSETTS INSTITUTE OF TECHNOLOGY
LEXINGTON 73, MASSACHUSETTS

ABSTRACT

It is shown that a complex function of one real variable is determined uniquely up to a constant phase factor by its absolute value and the absolute values of Fourier transforms of the function and certain related functions.

An application is given to the determination of the state function in quantum mechanics from the measurements of certain probability distributions of position and momentum.

A THEOREM ON FOURIER INTEGRALS AND AN APPLICATION TO THE THEORY OF MEASUREMENT IN QUANTUM MECHANICS

BY: Harry E. Mosés

LINCOLN LABORATORY
MASSACHUSETTS INSTITUTE OF TECHNOLOGY
LEXINGTON 73, MASSACHUSETTS

I INTRODUCTION AND SUMMARY

Let us consider the class of differentiable complex functions of one real variable x which are in $L^2(-\infty, \infty)$ and whose derivatives are also in $L^2(-\infty, \infty)$.

For every function f(x) of this class and for every interval $\mu = (A, B)$ on the x-axis, we define functions $f(x, \mu)$ by

$$f(x,\mu) = f(x) \quad \text{if} \quad x \in \mu$$
$$= 0 \quad \text{if} \quad x \notin \mu \quad .$$

Of course, if μ is the entire x axis, $f(x,\mu) = f(x)$. Let us define $\hat{f}(x,\mu)$ as the Fourier transform of $f(x,\mu)$, i.e.

$$\hat{f}(x,\mu) = \frac{1}{\sqrt{2\pi}} \int_A^B f(y,\mu)\, e^{iyx}\, dy$$

$$f(x,\mu) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \hat{f}(y,\mu)\, e^{-iyx}\, dy.$$

We shall prove the following theorem about functions of this class.

A necessary and sufficient condition that two functions $f_1(x)$ and $f_2(x)$ are related by

$$f_1(x) = \omega f_2(x)$$

where ω is a complex scalar of unit modulus is that

$$|f_1(x)| = |f_2(x)|$$

and $|\hat{f}_1(x,\mu)| = |\hat{f}_2(x,\mu)|$

for all intervals μ.

It should be mentioned that in references 1 and 2 certain related problems are considered. Roughly speaking, a class of functions f(x) is considered which vanish outside an interval. It is required for all such functions that the absolute value of the Fourier transforms are equal. The relationship of two functions of this class is then given in terms of the zeros of the Fourier transforms. The proof involves the analytic combination of the Fourier transforms.

The theorem which we state was motivated by an attempt to find sets of measurements which would identify uniquely a given state in quantum mechanics. We shall discuss the relationship of the theorem to this problem in Sec. 3.

2. PROOF OF THE THEOREM

The theorem is easily proved. The necessity is obvious. Hence we shall prove only the sufficiency.

From the conditions of the theorem it follows that

$$f_1(x) = \beta(x)\, f_2(x) \tag{1}$$

where $\beta(x)$ is a function of unit modulus. We write

$$\beta(x) = e^{i\varphi(x)}. \tag{2}$$

In (2) $\varphi(x)$ is a real function of x. It is our objective to prove that

$$\varphi'(x) = 0 \tag{3}$$

for all x, where the prime means derivative with respect to x.

For a fixed interval μ it follows from the conditions that

$$\int_{-\infty}^{+\infty} x\hat{f}_1^{\,*}(x)\, \hat{f}_1(x)\, dx = \int_{-\infty}^{-\infty} x\hat{f}_2^{\,*}(x)\, \hat{f}_2(x)\, dx \tag{4}$$

Expressed in terms of the Fourier transforms we have

$$\int_A^B f_1^{\,*}(x)\, f_1'(x)\, dx = \int_A^B f_2^{\,*}(x)\, f_2'(x)\, dx. \tag{5}$$

Now, from equation (1) and (2)

$$f'_1(x) = i\varphi'(x) f_1(x) + \beta(x) f'_2(x). \tag{6}$$

On substituting (6) in the left hand side of (5) we find

$$\int_A^B \varphi'(x) \left| f_1(x) \right|^2 dx = 0. \tag{7}$$

Equation (7) holds for every interval $\mu = (A, B)$. If $\varphi'(x)$ is not identically zero, there must be an interval within μ where $\varphi'(x)$ is positive. Then since (7) applies to the smaller interval, and we are led to a contradiction.

3. APPLICATION TO QUANTUM MECHANICS

In classical mechanics the specification of all dynamical variables at a particular time gives the state at that time. A particularly simple system consists of a particle moving in one-dimension. A specification of the position and momentum of the particle determines the state, since all other dynamical variables can be expressed in terms of them.

In quantum mechanics the interference of measurements precludes such a simple definition of state. In quantum mechanics one introduces rays of vectors of unit length in Hilbert space and one associates a state with each such a ray.

(Two unit vectors $|\Theta_1\rangle$ and $|\Theta_2\rangle$ belong to the same ray if a scalar ω of unit modulus exists such that $|\Theta_1\rangle = \omega|\Theta_2\rangle$).

A natural question which arises is whether we can conceive of a series of measurements whose results can be put into a one-to-one correspondence with a ray of vectors and hence a state. Such measurements would then be said to "determine a state." We cannot give such a set generally. However, in the case of a non-relativistic particle moving in one dimension we can give a set of measurements. Our results can be generalized to particles moving in several dimensions but breaks down in field theories.

Briefly, our theorem leads to the following for the one-dimensional case. Assume that the same state can be prepared repeatedly.

We then can measure the probability density of finding a particle at x. This density is given by $|\langle x|\Theta\rangle|^2$ where $|\Theta\rangle$ is a member of the ray. Let us now conduct a second set of experiments. One takes an interval $\mu = (A, B)$ on the x-axis and repeatedly tests whether a particle is in that interval. Whenever the particle is in that interval one measures the momentum of the particle. In this way one obtains the conditional probability density that a particle is in a given interval and has a certain momentum. This density is given by

$$\frac{|\langle p|\Theta\mu\rangle|^2}{\langle\Theta\mu|\Theta\mu\rangle}$$

where the state $|\Theta_\mu\rangle$ is given by

$$\begin{aligned} \langle x|\Theta\mu\rangle &= \langle x|\Theta\rangle \qquad && x \in \mu \\ &= 0 && x \notin \mu \end{aligned}$$

The second experiment is to be done for all intervals on the x axis. From our theorem, these two sets of measurements determine the ray of unit vectors $\omega|\Theta\rangle$ uniquely.

This result suggests a more abstract formalism which we as yet have not been able to prove explictly.

Let $\{A_i\}$ be an irreducible set of dynamical variables. Then we conjecture that a state can be associated with probability density measurements in the following. We first measure the probability density associated with the operator A_1.

We then measure the probability density of A_2 if first a measurement of A_1 yields an eigenvalue in a given interval of the spectrum of A_1. We measure the probability density of A_3 if a measurement of A_1 was first found to yield a value in a given interval of its spectrum and then a measurement of A_2 yields a value in a given interval of the spectrum of the latter operator. One continues this process until all the dynamical variables A_i have been exhausted.

REFERENCES

1. E. J. Akutowicz, Trans. Amer. Math. Soc., 83, (1956) 179-182

2. E. J. Akutowicz, Proc. Amer. Math. Soc., 8, (1957) 234-238.

Distribution List for Group Report 37G-1 by Harry E. Moses

All Division Heads

All Group Leaders

All Members of Group 37

H. E. Moses (4 copies)

Dr. R. Prosser, Lincoln Laboratory

Prof. F. Villars, MIT

Prof. G. C. Rota, MIT

Prof. J. B. Keller	Institute of Mathematical Sciences 25 Waverly Place, New York City
Prof. W. Magnus	Institute of Mathematical Sciences 25 Waverly Place New York City
Prof. A. S. Wightman	Princeton University Princeton, New Jersey
Prof. E. P. Wegner	Princeton University Princeton, New Jersey
Prof. J. S. Lomont	Windsor Tower Hotel Tudor City, New York