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MEMORANDUM
RM-3148-PR
DECEMBER 1962

NOTE ON A HYPERSONIC TAIL-SHOCK PROBLEM

R. E. Meyer

PREPARED FOR:

UNITED STATES AIR FORCE PROJECT RAND

The RAND Corporation

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PREFACE

This Memorandum presents a preliminary idealized analysis of a part of the flow field close behind a hypersonic projectile. As such, it is of interest primarily to those engaged in hypersonic-wake research. This work is part of a continuing study of the physical and electromagnetic characteristics of hypersonic trails in the atmosphere.

An analysis, by the same author, of a related problem was published as RM-2929-PR, Note on an Axially Symmetric Expansion Behind a Body in Hypersonic Flow.

SUMMARY

For the purpose of studying the structure of the inviscid gas flow near the narrow neck of the wake behind a projectile, the formation of a shock by a steady gas stream converging toward an axis of symmetry is investigated. The flow is assumed to be a conical field with vertex at the point where the inner boundary of the stream first reaches the axis and where the shock starts.

A detailed treatment is given for the case where the flow downstream of the shock is uniform, axial, and of large Mach number M, so that the semivertex angle τ of the shock is small, but $M\tau$ is not. It is also assumed that the shock is at least fairly strong. An additional small parameter then appears, and a uniform first-order approximation with respect to it is derived. The streamlines are found to be nearly straight and parallel, but the gas is found to suffer a precompression, upstream of the shock, such that the inner boundary of the inviscid gas stream is a vacuum line. Similarly, vacuum is approached asymptotically on all streamlines with distance upstream from the shock.

It is shown that the result concerning the occurrence of vacuum must be expected to be independent of all but one assumption--that the gas impinges on the axis at a non-zero angle. Accordingly, tail-shock formation in two-dimensional and axially symmetrical flows must be expected to differ in significant, still-unknown ways.

CONTENTS

I. INTRODUCTION*

When a body of revolution with a blunt base flies at constant, supersonic speed, the wake is known to form a neck from which a tail-shock issues (Fig. 1).

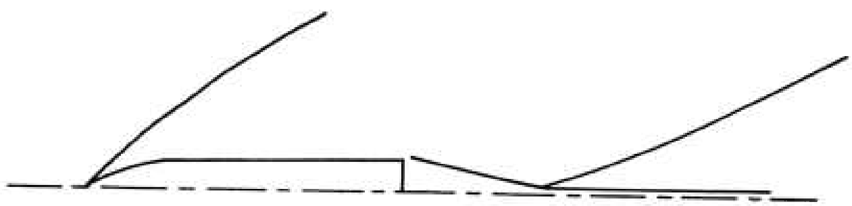

Fig. 1

If the boundary layer is thin, the neck is very narrow, especially in the axially symmetrical case, and the shock forms at a radius that is small compared with both the radius of the base and the radius of curvature of the shock. The flow near the shock and just outside the region in which viscosity plays a strong, direct role may therefore possess a structure similar to that obtained in the limiting case of zero neck and boundary-layer thickness.** In this case, the shock would be

*The author is indebted to Prof. J. D. Cole for drawing his attention, after completion of the work reported here, to a note by A. A. Nikol'skii on the boat-tail problem. Since this note is untranslated and not easily accessible, its main argument is reproduced in Sec. VI. Nikol'skii's argument complements the work reported here, so as to leave no escape from the conclusion that axially symmetrical tail-shock formation is not well understood.

**At high Mach numbers, the wake does not look thin, but if Prandtl's approximation concerning the momentum balance in the wake is relevant, then the limit of infinite Reynolds number should remain similarly relevant.

expected to form on the axis at a point embedded in a region in which the radius takes values that are large compared with the mean free path but only small compared with any other lengths representative of the flow. Accordingly, the existence of a continuum flow with conical symmetry (Fig. 2) should be expected in this region. This flow will be studied in the following.

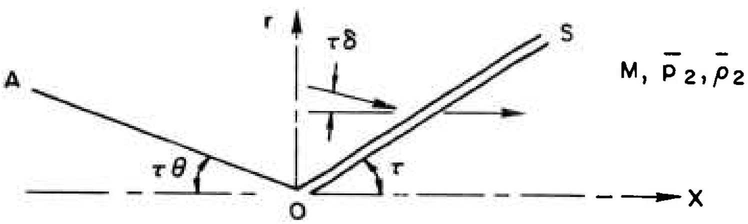

Fig. 2

It might be anticipated that a significant difference exists, just as in the analogous nose-shock problem, between the cases of two-dimensional and axially symmetrical flow. In the former case, the solution would be expected to consist simply of a plane shock flanked by two regions of uniform flow. In the axially symmetrical case, on the other hand, a nontrivival conical solution may exist such that the shock deflection angle $\tau\delta$ (Fig. 2) differs from the approach angle $\tau\theta$. A distinction then arises between the ratio $\bar{p}_2/\bar{p}_1$ of the limiting pressures on the two sides of the shock and the ratio $\bar{p}_2/\bar{p}_a$ of the pressure $\bar{p}_2$ just downstream of the shock to the pressure $\bar{p}_a$ on the boundary OA (Fig. 2) of the incident stream. These are respectively the shock-pressure and the approach-pressure ratios. In order to determine the latter in terms of the approach angle, an explicit solution will first be derived for a restricted range of circumstances. In Sec. V, it will then be shown that the main results are independent of most of these restrictions.

II. GOVERNING EQUATIONS

For the quantitative treatment, we assume that the flow downstream of the shock is purely axial. Since the conical assumption implies uniformity of the entropy except for a jump across the shock, it follows that the flow downstream of the shock is uniform. For hypersonic flight speeds, the shock angle τ may be taken to be a small parameter. It will also be assumed that the Mach number M downstream of the shock is large, so that $M\tau$ is not small. The problem then falls within the hypersonic small-disturbance theory.[1]

Accordingly, let $\bar{x}$, $\bar{r}$, $\bar{p}$, $\bar{\rho}$, $\bar{u}$, and $\bar{v}$ denote respectively the axial distance, radius, pressure, density, and axial and radial velocity components, and

$$\bar{x} = x, \ \bar{r} = \tau r, \ \bar{u} = \bar{u}_2(1 + \tau^2 u), \ \bar{v} = \tau \bar{u}_2 v(r,x)$$
$$\bar{p} = \bar{p}_2 \gamma M^2 \tau^2 p(r,x), \ \bar{\rho} = \bar{\rho}_2 \rho(r,x) \qquad (1)$$

where the subscript 2 refers to the uniform flow downstream of the shock. Then as $\tau \to 0$, the first approximations to the equations of continuity and momentum become[2]

$$\frac{\partial r\rho}{\partial x} + \frac{\partial r\rho v}{\partial r} = 0 \qquad (2)$$

$$\frac{\partial v}{\partial x} + \frac{v\partial v}{\partial r} + \frac{1}{\rho}\ \frac{\partial p}{\partial r} = 0 \qquad (3)$$

The former can be satisfied by the introduction of a stream function ψ such that

$$\frac{\partial \psi}{\partial r} = r\rho, \qquad \frac{\partial \psi}{\partial x} = -r\rho v \qquad (4)$$

and, since viscous effects are neglected

$$\frac{p}{\rho^\gamma} = \omega(\psi) \qquad (5)$$

where γ is the ratio of the specific heats and is assumed to be constant.

Let M_1 denote the Mach number just upstream of the shock, and let $\sigma = M_1^2 \sin^2 (\tau + \tau\delta)$, where $\tau\delta$ is the stream deflection across the shock (Fig. 2). The density and pressure ratios across the shock are

$$\frac{\bar{\rho}_2}{\bar{\rho}_1} = \frac{1}{R\sigma} , \quad \frac{\bar{p}_2}{\bar{p}_1} = P\sigma$$

and for a perfect gas[3]

$$R = \frac{2 + (\gamma - 1)\sigma}{(\gamma + 1)\sigma} , \quad P = \frac{2\gamma\sigma - (\gamma + 1)}{\gamma + 1} \tag{6}$$

Let the subscript 1 denote conditions just upstream of the shock. Then, by Eq. (1)

$$\rho_1 = R, \; p_1 = \frac{1}{\gamma M^2 \tau^2 P} , \quad \omega_1 = \frac{1}{\gamma M^2 \tau^2 P R^\gamma} \tag{7}$$

The continuity of normal mass-flow rate and tangential velocity across the shock implies (Fig. 2)

$$\tan \tau = R \tan (\tau + \tau\delta)$$

and so to the first order

$$v_1 = -\delta = - \frac{1 - R}{R} \tag{8}$$

The final boundary condition required is the continuity of ψ across the shock.

The assumption of conical symmetry implies that ψ has the form*

$$\psi = \frac{1}{2} r^2 F(\eta), \quad \eta = \frac{x}{r} \tag{9}$$

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*To avoid confusion, note that $\psi = x^2 f(r/x)$ in Ref. 1 (Eq. (13b)) is misprinted. Since the line $x = 0$ falls in the nonuniform flow region of the tail-shock problem, the use of η is more convenient here.

and by Eqs. (3) and (5), F must satisfy

$$F^2 \frac{d^2F}{d\eta^2} - \frac{F}{2}\left(\frac{dF}{d\eta}\right)^2 = \gamma\omega_1\eta\left(\eta\frac{d^2F}{d\eta^2} - \frac{dF}{d\eta}\right)\left(F - \frac{\eta}{2}\frac{dF}{d\eta}\right)^{\gamma+1} \tag{10}$$

upstream of the shock, where $\omega = \text{const} = \omega_1$, and the boundary conditions at the shock are

$$F(1) = 1, \quad F'(1) = 2(1 - R) \tag{11}$$

by Eqs. (1), (8), and (9). The mathematical problem is to integrate Eq. (10) from $\eta = 1$ towards smaller values of η in order to find the value $\eta = -\theta$ for which $F = 0$, and to compute the pressure. The analogous problem for the nose shock was solved numerically in Ref. 1. For the tail shock, however, an analytical approach will be found more suitable. Since it is the interval of F, not of η, which is known a priori

$$F = \frac{2\psi}{r^2}$$

will be regarded as the independent variable, and

$$G(F) = R + (1 - R)\eta \tag{12}$$

as the dependent variable. The system of Eqs. (10) and (11) then transforms into

$$F^2G'' + \frac{1}{2}FG' = \frac{\gamma\omega_1(G - R)}{(1 - R)^2}\left[(G - R)G'' + G'^2\right]\left[F - \frac{G - R}{2G'}\right]^{\gamma+1} \tag{13}$$

$$G(1) = 1, \quad G'(1) = \frac{1}{2} \tag{14}$$

where primes denote differentiation with respect to F.

III. STRONG-SHOCK APPROXIMATION

The solution of Eqs. (13) and (14) is seen to depend on the two parameters $R(\sigma)$ and $\omega_1(\sigma)$. At hypersonic flight speeds, the experimental evidence indicates tail shocks which possess appreciable pressure ratios near the axis. The shock strength $\sigma - 1 = M_1^2 \sin^2(\tau + \tau\delta) - 1$ might therefore be expected to be large for a significant range of flight conditions. The limit of R as $1/\sigma \to 0$ depends on the chemical state of the gas and might be small, but for a perfect gas with $\gamma = 1.4$, $R(\infty) = 1/6$; accordingly, $R(\infty)$ will not be regarded here as a small parameter. On the other hand, $P^{-1} \to 0$ as $1/\sigma \to 0$, and so $\omega_1 \to 0$, corresponding to the well-known fact that the entropy upstream of a shock becomes negligible compared with that downstream, in the limit of infinite shock strength. Moreover, the relevant parameter will not be ω_1 but

$$\mu = \frac{\gamma\omega_1 R^{\gamma+2}}{(1 - R)^2} = \frac{R^2}{M^2\tau^2 P(1 - R)^2} \tag{15}$$

and, for example, if $\gamma = 1.4$ (1.2) and shock strength $\sigma - 1 = 2$ only, corresponding to a shock pressure ratio of 3.33 (3.2), the value of $\gamma M^2\tau^2\mu$ is only 0.27 (0.19). The assumption that $\mu \ll 1$ should therefore be proper for an important range of hypersonic flight conditions.

Hence, let $\omega_1 \to 0$. Then by Eq. (3), v = const on the streamlines as well as on the conical rays, so $v \equiv$ const upstream of the shock, just as in the two-dimensional case, and $\theta = \delta = (1 - R)/R$. However, this approximation yields no information on the approach pressure beyond $\overline{p}_a \ll \overline{p}_2$. By Eqs. (4), (8), and (9), moreover, $\rho = RF^{1/2}$, indicating vacuum at the approach line AO (Fig. 2).

Now consider the problem of obtaining a higher approximation. The first approximation, corresponding to $\omega_1 = 0$ and $v \equiv$ const, is

$$G_1(F) = F^{1/2} \tag{16}$$

If the perturbation problem were a regular one, the second approximation,

$G_2(F)$, would be obtained from Eqs. (13) and (14) by substituting G_1 on the right-hand side of Eq. (13); i.e., solving

$$FG_2'' + \frac{1}{2} G_2' = \frac{\mu}{4} (F^{1/2} - R)F^{(\gamma/2)-2} \tag{17}$$

with $G_2(1) = 1$, $G_2'(1) = 1/2$, which yields

$$G_2(F) = F^{1/2} + \frac{\mu}{2-\gamma}\left[\left(\frac{2-\gamma}{3-\gamma} R - 1\right)\left(1 - F^{1/2}\right) + \frac{1}{\gamma - 1}\left(1 - F^{(\gamma-1)/2}\right) + \frac{R}{3-\gamma}\left(1 - F^{(\gamma/2)-1}\right)\right] \tag{18}$$

But as $F \to 0$, the right-hand side of Eq. (17) tends to ∞ if $\mu \neq 0$, and hence, while Eq. (18) provides an accurate description of the flow in most of the region between AO and OS (Fig. 2), there yet remains a narrow "boundary" sector adjacent to the approach line AO in which Eq. (16) fails to furnish even a first approximation for arbitrarily small $\mu > 0$.

The root of the difficulty is easily understandable from Eqs. (3) and (5), which, if q and s denote respectively the velocity magnitude and arc length on the streamlines, may be combined to

$$q \frac{\partial v}{\partial s} = - \frac{\omega_1 \partial \rho^\gamma}{\rho \partial y}$$

The nondimensional pressure gradient represented by the right-hand side is singular at O (Fig. 3) but small everywhere else, so v changes very little if any chosen streamline is followed upstream from the shock over a finite distance (Fig. 3).

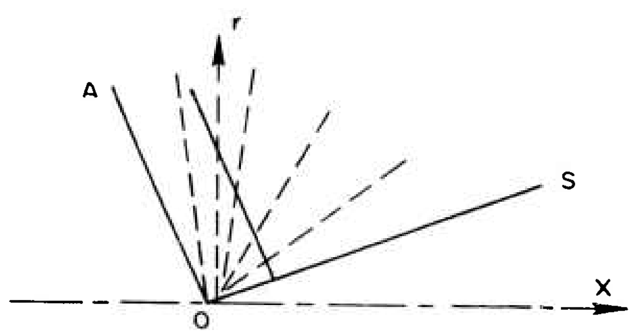

Fig. 3

But to reach very small values of the conical variable $F = 2\psi/r^2$ requires following the streamline over a very large distance, over which the effect of the small pressure gradient might accumulate significantly.

IV. THE BOUNDARY SECTOR

The results of the preceding section show that as long as $F = 2\psi/r^2$ is not small, the first approximation is given by Eq. (13) with the right-hand side deleted, so that $G = G_1(F) = F^{1/2}$. But in the boundary sector where F and hence also G are sufficiently small, Eq. (13) must reduce to a different, approximate equation in which μ plays an essential role.

Note that this sector near the boundary F = 0 must contain a transition sector for which Eq. (16), while inaccurate, yet provides the correct relative orders of magnitude. In this transition sector, therefore, $|F| \ll 1$, $|G| \ll R$, and

$$|FG'| \ll R \tag{19}$$

so that

$$\rho = F - \frac{G - R}{2G'} \approx \frac{R}{H}\ , \quad H = 2G' \tag{20}$$

and Eq. (13) must reduce to

$$(F^2H^{\gamma+1} - \mu R)H' = -\frac{1}{2}H^2(FH^\gamma + \mu) \tag{21}$$

The transition sector begins where F is small enough for μ to be significant in Eq. (21); by Eq. (19), that must be when $F^2H^{\gamma+1} = O(\mu)$, while still $FH^\gamma \gg \mu$. Hence, the limiting form of Eq. (13) must become

$$(F^2H^{\gamma+1} - \mu R)H' = -\frac{1}{2}FH^{\gamma+2} \tag{22}$$

whence

$$H^4F^2 = \frac{4\mu RH^{3-\gamma}}{(3 - \gamma)} + \text{const} \tag{23}$$

and const = 1 in order that Eq. (23) be identical with Eq. (16) in the "regular" sector. Hence

$$F^2 = \frac{1}{H^4} + \frac{4\mu R}{3 - \gamma H^{\gamma+1}} \tag{24}$$

in the transition sector. The regular sector, then, corresponds to

$$F >> \mu^{2/(3-\gamma)}, \quad H << \frac{1}{\mu^{1/(3-\gamma)}}$$

and the transition sector corresponds to

$$F = O(H^{-2}) = O(\mu^{2/(3-\gamma)}) \tag{25}$$

In the range expressed in Eq. (25), $FH = O(\mu^{1/(3-\gamma)}) << R$; i.e., Eq. (19) remains valid for $1 < \gamma < 3$. By Eq. (22), moreover

$$H^{(\gamma+1)/2} \frac{dG}{dH} = \frac{\mu R - F^2 H^{(\gamma+1)}}{FH^{(\gamma+1)/2}} = O(\mu^{1/2})\, H^{(1-\gamma)/2}$$

$$= O(\mu^{(\gamma-1)/(6-2\gamma)}) \tag{26}$$

in the range of Eq. (25), and so the corresponding range of G is << R, and hence still $|G| << R$. Note also that $F^2H^{\gamma+1}$ decreases as F does, by Eqs. (24) and (26), while H increases. But Eq. (24) shows that $F^2H^{\gamma+1}$ possesses a lower bound in the transition sector

$$F^2H^{\gamma+1} \geq \frac{4\mu R}{3 - \gamma}$$

and so in Eq. (21), $F^2H^{\gamma+1} - \mu R \geq (\gamma + 1)\mu R/(3 - \gamma)$. Hence, however small the fraction of $\mu^{2/(3-\gamma)}$ to which F may have decreased, $FH^{\gamma} >> \mu$ and Eq. (22) remains the first approximation to Eq. (13)! Accordingly, <u>Eq. (24) is the uniform first-order approximation to the solution $dG/dF = H/2$ of Eqs. (13) and (14) for $1 \geq F \geq 0$</u>.

The sector of Eq. (25) thus describes the transition from the "regular" sector, where Eq. (16) is the first approximation, to the boundary sector, $F << \mu^{2/(3-\gamma)}$, where the first approximation is

$$F^2H^{\gamma+1} = \frac{4\mu R}{3 - \gamma} \tag{27}$$

It follows from Eq. (26) that $G(0) \ll 1$, and the first approximation to the approach angle remains

$$\theta = \frac{1 - R}{R}$$

The first approximations for the nondimensional density and pressure in the boundary sector $F \ll \mu^{2/(3-\gamma)}$ are

$$\rho = \frac{R}{H} = R^{\gamma k}\left[\frac{(3 - \gamma)F^2}{4\mu}\right]^k$$

$$p = \omega_1\left(\frac{R}{H}\right)^\gamma = \frac{(1 - R)^2\mu^k}{\gamma}\left[\frac{(3 - \gamma)F^2}{4}\right]^{\gamma k}\frac{1}{R^{(3\gamma+2)k}}$$

$$k = \frac{1}{\gamma + 1}$$

On the approach line AO (Fig. 2), the density and pressure vanish, not because the shock strength is infinite, but because the conical solution implies a precompression along the streamlines from vacuum at infinity upstream, to the density $\bar{\rho}_1$ and pressure $\bar{p}_1$ at the shock.

V. A QUALITATIVE ARGUMENT

It may be instructive to review the problem solved in the preceding sections in the light of a simpler, but only qualitative, argument which illuminates especially the role played by the assumption of conical symmetry. This assumption states that ρ and v are to be functions of only a single independent, conical variable and implies that the entropy measure ω is constant upstream of the shock. In the following, the independent variable will be chosen to be, not $\eta = x/r$, but the arc length σ on an arbitrarily chosen streamline measured upstream from the shock in multiples of some suitable reference length. Then, if $q = (1 + v^2)^{1/2}$ represents the velocity magnitude

$$\frac{\partial\rho}{\partial x} = \frac{-q}{1 - v\eta} \frac{d\rho}{d\sigma}, \qquad \frac{\partial\rho}{\partial r} \; \frac{\eta q}{1 - v\eta} \frac{d\rho}{d\sigma}$$

The same relations hold for the derivatives of v. The momentum and energy equations, Eqs. (3) and (5), may therefore be combined to obtain

$$\frac{dv}{d\sigma} = \gamma\omega\rho^{\gamma-2} \frac{\eta}{1 - v\eta} \frac{d\rho}{d\sigma} \tag{28}$$

and the continuity equation, Eq. (2), may be written with the help of Eq. (28) as

$$\frac{q}{\rho} \frac{d\rho}{d\sigma} = \frac{v}{rD} \tag{29}$$

$$D = 1 - \gamma\omega\rho^{\gamma-1} \frac{\eta^2}{(1 - v\eta)^2} \tag{30}$$

Equation (29) may be used in turn to cast Eq. (28) into the form

$$\frac{q}{v} \frac{dv}{d\sigma} = \gamma\omega\rho^{\gamma-1} \frac{\eta}{1 - v\eta} \frac{1}{rD} \tag{31}$$

Observe that $1 - v\eta = 0$ on the approach line AO (Fig. 2) and hence that $1 - v\eta \to 0$ as $\sigma \to \infty$ on our streamline, far upstream of the shock. Similarly, $1/r \to 0$ as $\sigma \to \infty$.

We now introduce the strong-shock assumption $\omega << 1$, with R not small. Since $\eta = 1$ and $v = -(1 - R)/R$ at the shock, by Eq. (8), Eq. (31) shows $v(\sigma)$ to be approximately constant as long as $1 - v\eta = O(1)$. To this approximation, therefore, the streamline is straight, and $\eta/(1 - v\eta)$ is a linearly decreasing function of σ, while r is a linearly increasing function of σ; therefore, Eq. (29) shows ρ to decrease in inverse proportion to some positive power of r as the streamline is followed upstream from the shock.

Sufficiently far upstream, however, we must anticipate that σ will become so large* that

$$\omega\rho^{\gamma-1}\sigma^2 = O(1)$$

This relation defines a "transition" segment of the streamline, on which $\omega\rho^{\gamma-1}\sigma << 1$ still, so that v remains approximately constant, by Eq. (31), while the law of dependence of ρ on σ changes appreciably, by Eq. (29). When this new trend only begins to take effect, D must still be positive, and since $v < 0$, ρ must still decrease with increasing σ.

As the streamline is followed farther upstream, there are therefore just two possibilities. One is that a stage is reached where $\omega\rho^{\gamma-1}\sigma^2 >> 1$; that must certainly occur if ρ is bounded away from zero on the streamline. But before this stage can be reached, a value of σ must occur at which $D = 0$ and $d\rho/d\sigma$ and $dv/d\sigma$ are singular, by Eqs. (29) and (31). Since σ represents also a conical variable, the singularity is a limit line, and the solution is multivalued. A solution of this kind, but with expansion in the stream direction, is envisaged by Reyn.[4]

The second alternative is that D remains positive, and hence $\rho \to 0$ as $\sigma \to \infty$ upstream.

It should be observed that the argument just given is independent of the assumption of uniform, axial flow downstream of the shock. Whatever that flow, if the assumption of conical symmetry is made, the shock must be straight, and if its inclination $\tau << 1$ (Fig. 2) and $M_2\tau$ is not

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*The quantitative discussion of Sec. III confirms that $\rho^{\gamma-1}/(1 - v\eta)^2$ increases with σ when $1 - v\eta = O(1)$.

small, where M_2 is the Mach number just downstream of the shock, then Eqs. (2), (3), and (5) are valid, and hence also Eqs. (29) and (31). The argument then remains quite unchanged, unless the highly implausible assumption be made that the gas enters the shock at a glancing angle $\ll \tau$ (so that $1 - v_1 \ll 1$). Again, since only the product $\omega\rho^{\gamma-1}$ appears on the right-hand side of Eqs. (29) and (31), the argument is unchanged should the density ratio $R = \bar{\rho}_1/\bar{\rho}_2 = \rho_1$ across the shock be small, even though $v \neq O(1)$ in this case. In fact, the argument is based essentially on the structure of the denominator D; therefore, it is also independent of the strong-shock assumption. The value of D at the shock is

$$D_1 = 1 - \frac{\gamma p_1}{\rho_1(1 - v_1)^2} = 1 \frac{1}{\left[\tau(1 - v_1)M_1\right]^2}$$

by Eqs. (5) and (1), and if the plausible assumption be made that the gas is still converging towards the axis when it enters the shock (so that $v_1 < 0$), then the condition $M_1\tau \geq 1$ is sufficient to insure $D_1 > 0$. It follows from Eq. (29) that a compression precedes the shock, and if v remains negative as the streamlines are followed upstream a limit line $D = 0$ can be avoided only if $\rho \to 0$ as $(1 - v\eta) \to 0$ far upstream.

VI. NIKOL'SKII'S ARGUMENT

A quite different argument[2] shows that the hypersonic assumption is also unnecessary and that not even the conical assumption is essential. The characteristic equations of steady, axially symmetrical flow of inviscid gas are[3]

$$d\theta \mp \frac{\cot \mu dp}{q^2 \rho} \mp \frac{\sin \theta \sin \mu}{\sin (\theta \mp \mu)} \frac{dr}{r} = 0 \quad \text{on} \quad \frac{dr}{dx} = \tan (\theta \mp \mu) \qquad (32)$$

where θ is the stream direction, μ is the Mach angle, and where ordinary, not hypersonically distorted, variables are used. More explicitly, for a perfect gas with constant specific heat, Eq. (32) may be written[5]

$$d\theta \pm d\omega \pm \frac{\lambda}{c_v} dS \mp \sin \theta \sin \mu \operatorname{cosec} (\theta \mp \mu) \frac{dr}{r} = 0$$

$$\text{on} \ \frac{dr}{dx} = \tan (\theta \mp \mu) \qquad (33)$$

where $\gamma(\gamma - 1)\lambda = \sin \mu \cos \mu$, S is the specific entropy, and

$$\omega(\mu) \int_{\pi/2}^{\mu} \frac{\cot \mu'}{q} \left(\frac{dq}{d\mu'} \right) d\mu'$$

is the Prandtl angle, which is a monotonic, continuously differentiable function of μ with $\omega(0) = (n - 1)\pi/2$, $n^2 = (\gamma + 1)/(\gamma - 1)$.

Consider now the flow past a body of revolution with boat tail (Fig. 4), the family of Mach lines C'E' sloping down to the body surface, and, in particular, the limiting member CE of this family which meets the body surface at the tip of the tail. If the flow is not totally degenerate, the respective differences between the values of θ, ω, and S at the points C and E must be finite. On the other hand if Eq. (32) or (33) is integrated along the Mach line from C to E, the term proportional to r^{-1} furnishes a divergent contribution, unless $\theta = 0$ or $\mu = 0$ at E. But the meridian slope of the body is negative at E if the tail is not cusped.

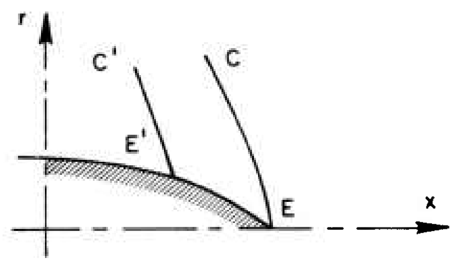

Fig. 4

Nikol'skii excludes the possibility that $\mu = 0$ $(1/M = 0)$ and concludes that the paradox must be resolved by the occurrence of subsonic flow near the tip of the boat tail. In support of this, he shows that the gas flowing along the body surface must undergo an appreciable compression as the radius becomes small on the tail.

Nonetheless, such a conclusion does not follow. The relevant problem is not that of the flow of inviscid gas past the geometrically prescribed body surface but that of the limit of real-gas flow past the body as viscosity tends to zero.

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