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Air Force Office of Scientific Research
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# APPLICATION OF VARIATIONAL EQUATION OF MOTION TO THE NONLINEAR VIBRATION ANALYSIS OF HOMOGENEOUS AND LAYERED PLATES AND SHELLS[1]

Yi-Yuan Yu[2]

## ABSTRACT

An integrated procedure is presented for applying the variational equation of motion to the approximate analysis of nonlinear vibrations of homogeneous and layered plates and shells involving large deflections. The procedure consists of a sequence of variational approximations. The first of these involves an approximation in the thickness direction and yields a system of equations of motion and boundary conditions for the plate or shell. Subsequent variational approximations with respect to the remaining space coordinates and time, wherever needed, lead to a solution to the nonlinear vibration problem. The procedure is illustrated by a study of the nonlinear free vibrations of homogeneous and sandwich cylindrical shells, and it appears to be applicable to still many other homogeneous and composite elastic systems.

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# INTRODUCTION

In solving certain equilibrium and vibration problems in the linear or nonlinear theory of elasticity, two distinct and unrelated steps are often taken. In the case of a plate or shell problem, for instance, an approximate system of equations that governs the problem is usually derived first by the use of one of a wide variety of available methods. Then, when the system of equations deduced cannot be solved exactly, a wide variety of methods is again available for obtaining an approximate solution of the equations. The method used for deriving the approximate equations and that used for solving the equations usually bear no relation to each other.

One of the main purposes of this paper is to advocate an integrated approximate procedure of solving a large class of problems in the linear or nonlinear theory of elasticity, and in particular, problems of plates and shells of the layered as well as homogeneous type of construction, solely on the basis of the variational equation of motion. It is an integrated procedure in that the aforementioned two steps are no longer unrelated to each other. In fact, the procedure consists of a sequence of variational approximations with respect to the space and time coordinates, carried out in relation to the differential equations and/or the boundary conditions. Although not much originality can be claimed on the variational approach, the treatment does make the fullest systematic use of the variational equation of motion. Besides, it not only integrates some of the variational approximations which have been known only as unrelated individual procedures, but it also reveals the possibility of having more general variational approximations.

The variational equation of motion in the theory of elasticity is a direct consequence of Hamilton's principle and applies to both linear and nonlinear cases. The usual formulation of the equation, as given in Love's book [1], contains only the volume and surface integrals with respect to the space coordinates. In this paper we shall further include the integration with respect to time as a necessary part in the formulation of the variational equation of motion. Without the additional time integration, variational approximations can only be performed with respect to the space coordinates.

In what follows the proposed procedure is first outlined. Then, as illustrations and as problems of interest by themselves, approximate systems of nonlinear equations of motion and boundary conditions of homogeneous and sandwich cylindrical shells are derived and subsequently solved for the cases of axially symmetrical vibrations of closed shells with immovable hinged edges. Results for the nonlinear frequencies are finally discussed.

## The Integrated Variational Approach

On the basis of the linear or nonlinear variational equation of motion in a form as has just been specified, the solutions to a large class of problems in the theory of elasticity, and in particular problems of homogeneous and layered plates and shells, may be obtained by carrying out a sequence of successive variational approximations. In the first of these approximations, and in the case of plate and shell problems, for instance, the dependence of the displacements on the thickness coordinate is assumed, and integration is carried out with respect to this coordinate. The first variational approximation thus constitutes essentially the process of deriving plate or shell equations or other approximate equations. However, even in such a process in this first step, the variational equation of motion does not appear to have been fully made use of before, since, until recently, only the volume integral, and only that in the linear variational equation of motion (without the time integration), has been employed in the derivation of linear differential equations of motion of plates [2-7] and shells [8]. The surface integral in either the linear or nonlinear variational equation of motion is believed to have been used for the first time in the recent derivation of the appropriate boundary conditions in reference 9 for sandwich plates. Although the boundary conditions (as well as the equations of motion) may also be derived by other means, the integrated treatment on the basis of the variation equation of motion has the advantage of being simple and straightforward, and it permits the surface traction terms that appear in the boundary conditions to be incorporated immediately in the equations of motion, which is particularly desirable in nonlinear cases.

Sometimes even the approximate system of differential equations and boundary conditions cannot be solved exactly for a given equilibrium or vibration problem. A well-known approximate procedure named usually after Galerkin may often be used, in which displacements are assumed such as to satisfy exactly the boundary conditions but not the differential equations. An alternative approximate procedure is that due to Trefftz [10] in which the assumed displacements satisfy exactly the differential equations but not the boundary conditions. A natural generalization of the two appears to be one in which the assumed displacements satisfy exactly some of the differential equations and boundary conditions and are made to satisfy the remaining differential equations and boundary conditions approximately in the variational sense. The use of the generalized procedure remains to be explored, but, together with its above two variants, the procedure clearly may be considered, and in effect is, a second variational approximation in the solution of an elasticity problem based on the variational equation of motion. In equilibrium problems this is also the final variational approximation that is needed.

For problems of vibration, and, in particular, nonlinear vibration, a third and final variational approximation with respect to time is often useful. It is this last step that needs the integration with respect to time which has been included as part of the variational equation of motion. The approximation consists of essentially another application of the Galerkin procedure. The successful use of it in solving nonlinear vibration problems involving single-degree-of-freedom systems has been demonstrated by Klotter [11], who prefers to call it the Ritz Procedure. In reference 11 it is also mentioned that the same procedure may be applied to nonlinear vibration analysis of two-degrees-of-freedom systems. Further applicability of the procedure to composite continuous systems has been demonstrated in reference 9 where nonlinear vibrations of sandwich plates are discussed.

Thus, by the use of the procedure just outlined, we are enabled to derive the approximate solution to an elasticity problem from a unified point of view and in an integrated manner, solely on the basis of the variational equation of motion, although much that is involved may have been well-known as isolated individual procedures. In the remaining part of this paper, the nonlinear vibrations of homogeneous and sandwich cylindrical shells will be investigated by means of the proposed procedure, with a system of nonlinear equations of motion of cylindrical shells derived in the first step. The effect of thickness-shear deformation is included. The problem may be considered as an extension of the previous one of nonlinear vibration of sandwich plates [9], and the results are also reducible to some of the linear results obtained previously for homogeneous [12] and sandwich cylindrical shells [8].

## Nonlinear Equations of Homogeneous and Sandwich Cylindrical Shells

Equations of the sandwich cylindrical shell will be derived first. Those of the homogeneous shell will then be obtainable as a special limiting case by putting equal to zero the thickness of the face layers of the sandwich shell. The cylindrical coordinates $x$, $s = a\theta$ and $r$ are chosen to be in the longitudinal, circumferential, and radial directions, respectively, of the shell, whose middle surface has the radius $a$. The middle surface is further designated as $z = o$ so that the relation between the two variables $r$ and $z$ is given by

$$r = a + z$$

In the z-direction the thicknesses of the inner face, core, and outer face layers of the shell extend from $-h$ to $-h_1$, $-h_1$ to $h_1$, and $h_1$ to $h$, respectively. In the x-direction the shell extends from $x = o$ to $x = \ell$. In the s-direction the shell is a closed one in the specific vibration problems to be discussed, but the equations derived will also be applicable to open shells. In particular, the appropriate boundary conditions for an open edge $s$=constant as well as those for $x$=constant will be formulated.

In the case of small deformations and small angles of rotation, small in a sense such as specified in Novoshilov's book [13], the variational equation of motion in the nonlinear theory of elasticity may be written in cylindrical coordinates for the sandwich shell as follows:

$$
\begin{aligned}
&\int_{t_0}^{t_1} dt \sum_{i=1}^{3} \iiint \Big\{ \Big[ \frac{\partial \sigma_{xi}^{*}}{\partial x} + \frac{a}{r}\frac{\partial \tau_{sxi}^{*}}{\partial s} + \frac{\partial \tau_{rxi}^{*}}{\partial r} + \frac{\tau_{rxi}^{*}}{r} - \rho_i \ddot{u}_i \Big] \delta u_i \\
&\qquad + \Big[ \frac{1}{r}\frac{\partial \sigma_{si}^{*}}{\partial s} + \frac{\partial \tau_{xsi}^{*}}{\partial x} + \frac{\partial \tau_{rsi}^{*}}{\partial r} + \frac{2\tau_{rsi}^{*}}{r} - \rho_i \ddot{v}_i \Big] \delta v_i \\
&\qquad + \Big[ \frac{\partial \sigma_{ri}^{*}}{\partial r} + \frac{\sigma_{ri}^{*} - \sigma_{si}^{*}}{r} + \frac{\partial \tau_{xri}^{*}}{\partial x} + \frac{a}{r}\frac{\partial \tau_{sri}^{*}}{\partial s} - \rho_i \ddot{w}_i \Big] \delta w_i \Big\} dV_i \\
&- \int_{t_0}^{t_1} dt \sum_{i=1}^{3} \iint \Big\{ \Big[ \sigma_{xi}^{*} \cos(\nu,x) + \tau_{sxi}^{*} \cos(\nu,s) + \tau_{rxi}^{*} \cos(\nu,r) - \bar{p}_{xi} \Big] \delta u_i \\
&\qquad + \Big[ \tau_{xsi}^{*} \cos(\nu,x) + \sigma_{si}^{*} \cos(\nu,s) + \tau_{rsi}^{*} \cos(\nu,r) - \bar{p}_{si} \Big] \delta v_i \\
&\qquad + \Big[ \tau_{xri}^{*} \cos(\nu,x) + \tau_{sri}^{*} \cos(\nu,s) + \sigma_{ri}^{*} \cos(\nu,r) - \bar{p}_{ri} \Big] \delta w_i \Big\} dS_i \\
&= 0 \qquad (1)
\end{aligned}
$$

where $t_0$ and $t_1$ are two arbitrary moments of time and

$$
\begin{aligned}
\sigma_{xi}^{*} &= \sigma_{xi} - \tau_{xsi}\,\omega_{ri} + \tau_{xri}\,\omega_{si} \\
\tau_{xsi}^{*} &= \tau_{xsi} + \sigma_{xi}\,\omega_{ri} - \tau_{xri}\,\omega_{xi} \\
\tau_{xri}^{*} &= \tau_{xri} + \tau_{xsi}\,\omega_{xi} - \sigma_{xi}\,\omega_{si} \\
\tau_{sxi}^{*} &= \tau_{sxi} - \sigma_{si}\,\omega_{ri} + \tau_{sri}\,\omega_{si} \\
\sigma_{si}^{*} &= \sigma_{si} + \tau_{sxi}\,\omega_{ri} - \tau_{sri}\,\omega_{xi} \\
\tau_{sri}^{*} &= \tau_{sri} + \sigma_{si}\,\omega_{xi} - \tau_{sxi}\,\omega_{si} \\
\tau_{rxi}^{*} &= \tau_{rxi} - \tau_{rsi}\,\omega_{ri} + \sigma_{ri}\,\omega_{xi} \\
\tau_{rsi}^{*} &= \tau_{rsi} + \tau_{rxi}\,\omega_{ri} - \sigma_{ri}\,\omega_{si} \\
\sigma_{ri}^{*} &= \sigma_{ri} + \tau_{rsi}\,\omega_{xi} - \tau_{rxi}\,\omega_{si}
\end{aligned}
\qquad (2)
$$

In addition, the subscript $i = 1, 2,$ or $3$ refers to the core, inner or outer face layer of the sandwich, $\sigma_{xi}$, $\tau_{xsi}$, $\tau_{xri}, \dots$ are the stresses, $\omega_{xi}$, $\omega_{si}$, $\omega_{ri}$ the angles of rotation about the x, s, r directions, $\bar{p}_{xi}$, $\bar{p}_{si}$, $\bar{p}_{ri}$ the prescribed surface tractions in these directions, $\nu$ refers to the external normal direction, and $\rho_i$ is the density. While the volume integral in the equation is to cover the volume of each of the three layers, the surface integral will cover only those portions of the surfaces of the layer on which tractions, but not displacements, are prescribed. The equations of motion and the appropriate boundary conditions of the sandwich cylindrical shell will be derived from Eq. (1) by carrying out explicitly the integration with respect to z. We note here that this will involve exactly the same manipulation as in the corresponding linear case if the angles of rotation are independent of z, which as we shall see will be assumed.

As in reference 6 the flexural rigidities of the face layers will be neglected, and the displacements are assumed in the form

$$\begin{aligned} u_1 &= u + z\psi & u_2, u_3 &= u \mp h_1 \psi \\ v_1 &= v + z\varphi & v_2, v_3 &= v \mp h_1 \varphi \\ & & w_1 = w_2 = w_3 = w & \end{aligned} \tag{3}$$

The angles of rotation $\omega_{x1}$ and $\omega_{s1}$ in the core are then

$$\omega_{x1} = \frac{1}{2}\left(\frac{a}{r}\frac{\partial w}{\partial s} - \varphi - \frac{v + z\varphi}{r}\right), \qquad \omega_{s1} = -\frac{1}{2}\left(\frac{\partial w}{\partial x} - \psi\right)$$

Since the face layers have been taken as membranes, their angles of rotation about the x- and s-directions are

$$\omega_{x2} = \omega_{x3} = \frac{\partial w}{\partial s}, \qquad \omega_{s2} = \omega_{s3} = -\frac{\partial w}{\partial x}$$

The angles of rotation $\omega_{ri}$ about an axis normal to the shell middle surface are in general much smaller than $\omega_{xi}$ and $\omega_{si}$ about axes lying in the surface and are assumed to be negligible. Among all the angles of rotation, therefore, only $\omega_{x1}$ depends on z. To simplify the formulation of the problem, this z-dependence is suppressed and $\omega_{x1}$ is assumed to take the simplified form

$$\omega_{x1} = \frac{1}{2}\left(\frac{\partial w}{\partial s} - \varphi - \frac{v}{a}\right)$$

Since the original z-dependence of $\omega_{x1}$ decreases with the thickness-to-radius ratio of the shell and $\omega_{x1}$ effects only the nonlinear terms, the simplifying assumption should not introduce much inaccuracy so long as the shell is thin and the nonlinearity small. It is also noted that the assumption will not at all affect symmetrical vibrations which are to be discussed later in this paper. To summarize, the angles of rotation now take the form

$$\omega_{x1} = \frac{1}{2}\left(\frac{\partial w}{\partial s} - \varphi - \frac{v}{a}\right), \qquad \omega_{s1} = -\frac{1}{2}\left(\frac{\partial w}{\partial x} - \psi\right)$$

$$\omega_{x2} = \omega_{x3} = \frac{\partial w}{\partial s}, \qquad \omega_{s2} = \omega_{s3} = -\frac{\partial w}{\partial x} \tag{4}$$

$$\omega_{r1} = \omega_{r2} = \omega_{r3} = 0$$

When Eqs. (3) are substituted in Eq. (1) and integration is carried out with respect to z, the volume integrals in Eq. (1) yield a double integral, involving the values of the stresses $\tau_{rxi}^*$, $\tau_{rsi}^*$, $\tau_{ri}^*$ at the curved boundaries of the shell. The surface integrals in Eq. (1) yield three parts after integration. The first part is the result of integration over the curved surfaces of the shell, which is also in the form of a double integral. When only tractions are prescribed on the curved surfaces, the two double integrals deduced from the volume and the surface integrals may be combined, and, by equating to zero the coefficients of $\delta u$, $\delta v$, $\delta w$, $\delta\psi$, $\delta\varphi$ in the resulting integrand, the following stress equations of motion of the shell are obtained:

$$\frac{\partial N_x^*}{\partial x}+\frac{\partial N_{sx}^*}{\partial s}+\bar{p}_x^+\left(1+\frac{h}{a}\right)+\bar{p}_x^-\left(1-\frac{h}{a}\right)-2(\rho_1 h_1+\rho_2 h_2)\ddot{u}-\rho_1\frac{2h_1^3}{3a}\ddot{\psi}=0$$

$$\frac{\partial N_{xs}^*}{\partial x}+\frac{\partial N_s^*}{\partial s}+\frac{Q_{rs}^*}{a}+\bar{p}_s^+\left(1+\frac{h}{a}\right)+\bar{p}_s^-\left(1-\frac{h}{a}\right)-2(\rho_1 h_1+\rho_2 h_2)\ddot{v}-\rho_1\frac{2h_1^3}{3a}\ddot{\varphi}=0$$

$$\frac{\partial Q_{xr}^*}{\partial x}+\frac{\partial Q_{sr}^*}{\partial s}-\frac{N_s^*}{a}+\bar{p}_r^+\left(1+\frac{h}{a}\right)+\bar{p}_r^-\left(1-\frac{h}{a}\right)-2(\rho_1 h_1+\rho_2 h_2)\ddot{w}=0 \tag{5}$$

$$\frac{\partial}{\partial x}\left[M_{x1}^*+h_1(N_{x3}^*-N_{x2}^*)\right]+\frac{\partial}{\partial s}\left[M_{sx1}^*+h_1(N_{sx3}^*-N_{sx2}^*)\right]-Q_{rx1}^*$$
$$+\bar{p}_x^+h_1\left(1+\frac{h}{a}\right)-\bar{p}_x^-h_1\left(1-\frac{h}{a}\right)-\rho_1\frac{2h_1^3}{3a}\ddot{u}-\left(\rho_1\frac{2h_1^3}{3}+2\rho_2 h_1^2h_2\right)\ddot{\psi}=0$$

$$\frac{\partial}{\partial x}\left[M_{xs1}^*+h_1(N_{xs3}^*-N_{xs2}^*)\right]+\frac{\partial}{\partial s}\left[M_{s1}^*+h_1(N_{s3}^*-N_{s2}^*)\right]-Q_{rs1}^*+\frac{h}{a}(Q_{rs3}^*-Q_{rs2}^*)$$
$$+\bar{p}_s^+h_1\left(1+\frac{h}{a}\right)-\bar{p}_s^-h_1\left(1-\frac{h}{a}\right)-\rho_1\frac{2h_1^3}{3a}\ddot{v}-\left(\rho_1\frac{2h_1^3}{3}+2\rho_2 h_1^2h_2\right)\ddot{\varphi}=0$$

where $N_x^* = N_{x1}^* + N_{x2}^* + N_{x3}^*$ ...

Eqs. (5) contain only the surface tractions $\bar{p}_x^+$, $\bar{p}_x^-$, ... but no longer the boundary values of the stresses $\tau_{xri}^*$, $\tau_{rsi}^*$, $\sigma_{ri}^*$. If the two double integrals deduced from Eq. (1) are left uncombined, the integrand of one will yield shell equations which still contain the latter stress values, and that of the other will yield the stress or displacement boundary conditions for the curved surfaces of the shell.

The remaining two parts derived from the surface integrals of Eq. (1) are results of integration over sections across the thickness of the shell, and they are in the form of line integrals along the edges x=constant and s=constant. From these line integrals, the following appropriate boundary conditions are obtained:

Along an edge x = constant,

$$N_x^* = \int_{-h}^{h} \bar{p}_x \left(1+\frac{z}{a}\right) dz \qquad \text{or } u = \bar{u}$$

$$N_{xs}^* = \int_{-h}^{h} \bar{p}_s \left(1+\frac{z}{a}\right) dz \qquad \text{or } v = \bar{v}$$

$$Q_{xr}^* = \int_{-h}^{h} \bar{p}_r \left(1+\frac{z}{a}\right) dz \qquad \text{or } w = \bar{w}$$

$$M_{x1}^* + h_1(N_{x3}^* - N_{x2}^*) = \int_{-h_1}^{h_1} \bar{p}_x z \left(1+\frac{z}{a}\right) dz + h_1 \int_{-h_1}^{h_1} \bar{p}_x \left(1+\frac{z}{a}\right) dz$$

$$- h_1 \int_{-h}^{-h_1} \bar{p}_x \left(1+\frac{z}{a}\right) dz \qquad \text{or } \psi = \bar{\psi} \tag{6}$$

$$M_{xs1}^* + h_1(N_{xs3}^* - N_{xs2}^*) = \int_{-h_1}^{h_1} p_s z \left(1+\frac{z}{a}\right) dz + h_1 \int_{h_1}^{h} \bar{p}_s \left(1+\frac{z}{a}\right) dz$$

$$- h_1 \int_{-h}^{-h_1} \bar{p}_s \left(1+\frac{z}{a}\right) dz \qquad \text{or } \varphi = \bar{\varphi}$$

Aong an edge s = constant (for an open shell).

$$N_{sx}^* = \int_{-h}^{h} \bar{p}_x dz \qquad \text{or } u = \bar{u}$$

$$N_s^* = \int_{-h}^{h} \bar{p}_s dz \qquad \text{or } v = \bar{v}$$

$$Q_{rs}^* = \int_{-h}^{h} p_r dz \qquad \text{or } w = \bar{w}$$

$$M_{sx1}^* + h_1(N_{sx3}^* - N_{sx2}^*) = \int_{-h_1}^{h_1} \bar{p}_x z dz + h_1 \int_{h_1}^{h} \bar{p}_x dz - h_1 \int_{-h}^{-h_1} \bar{p}_x dz \qquad \text{or } \psi = \bar{\psi} \tag{7}$$

$$M_{s1}^* + h_1(N_{s3}^* - N_{s2}^*) = \int_{-h_1}^{h_1} \bar{p}_s z dz + h_1 \int_{h_1}^{h} \bar{p}_s dz - h_1 \int_{-h}^{-h_1} \bar{p}_s dz \qquad \text{or } \varphi = \bar{\varphi}$$

In Eqs. (5) to (7) overbars in general denote prescribed quantitites. Thus $\bar{p}_x^+$, $\bar{p}_s^+$, $\bar{p}_r^+$ are the prescribed tractions at the outer boundary $z = +h$, $\bar{p}_x^-$, $\bar{p}_s^-$, $\bar{p}_r^-$ those at the inner boundary $z = -h$ and $\bar{p}_x, \bar{p}_s, \bar{p}_r$ those across the thickness of the shell. Likewise, $\bar{u}$, $\bar{v}$, $\bar{w}$, $\bar{\psi}$, $\bar{\varphi}$ are the prescribed values of the shell displacements. The shell-stresses introduced in these equations are defined by

$$
\begin{aligned}
(M_{xi}^{*}, M_{xsi}^{*}) &= \int (\sigma_{xi}^{*}, \tau_{xsi}^{*})\left(1+\frac{z}{a}\right) z\, dz \\
(M_{si}^{*}, M_{sxi}^{*}) &= \int (\sigma_{si}^{*}, \tau_{sxi}^{*})\, z\, dz \\
(N_{xi}^{*}, N_{xsi}^{*}) &= \int (\sigma_{xi}^{*}, \tau_{xsi}^{*})\left(1+\frac{z}{a}\right) dz \\
(N_{si}^{*}, N_{sxi}^{*}) &= \int (\sigma_{si}^{*}, \tau_{sxi}^{*})\, dz \\
(Q_{xri}^{*}, Q_{rxi}^{*}) &= \int (\tau_{xri}^{*}, \tau_{rxi}^{*})\left(1+\frac{z}{a}\right) dz \\
(Q_{sri}^{*}, Q_{rsi}^{*}) &= \int (\tau_{sri}^{*}, \tau_{rsi}^{*})\, dz
\end{aligned}
\tag{8}
$$

where the integrations are understood to cover the thickness of the layer.

The stress-strain-displacement relations of the shell will next be formulated. Under the assumptions of small $\omega_{xi}$ and $\omega_{si}$ and zero $\omega_{ri}$ the strain-displacement relations in the nonlinear theory of elasticity are [13]

$$
\begin{aligned}
\epsilon_{xi} &= \frac{\partial u_i}{\partial x} + \frac{1}{2}\omega_{si}^2 \\
\epsilon_{si} &= \frac{a}{r}\frac{\partial v_i}{\partial s} + \frac{w_i}{r} + \frac{1}{2}\omega_{xi}^2 \\
\epsilon_{xsi} &= \frac{a}{r}\frac{\partial u_i}{\partial s} + \frac{\partial v_i}{\partial x} - \omega_{xi}\omega_{si} \\
\epsilon_{xri} &= \frac{\partial u_i}{\partial r} + \frac{\partial w_i}{\partial x} \\
\epsilon_{sri} &= \frac{\partial v_i}{\partial r} - \frac{v_i}{r} + \frac{a}{r}\frac{\partial w_i}{\partial s}
\end{aligned}
$$

which yield, by virtue of Eqs. (3),

$$
\begin{aligned}
\epsilon_{x1} &= \frac{\partial u}{\partial x} + z\frac{\partial \psi}{\partial x} + \frac{1}{2}\omega_{s1}^{2} \\
\epsilon_{s1} &= \frac{a}{r}\left(\frac{\partial v}{\partial s} + z\frac{\partial \varphi}{\partial s}\right) + \frac{w}{r} + \frac{1}{2}\omega_{x1}^{2} \\
\epsilon_{xs1} &= \frac{a}{r}\left(\frac{\partial u}{\partial s} + z\frac{\partial \psi}{\partial s}\right) + \frac{\partial v}{\partial x} + z\frac{\partial \varphi}{\partial x} - \omega_{x1}\omega_{s1} \\
\epsilon_{xr1} &= \psi + \frac{\partial w}{\partial x} \\
\epsilon_{sr1} &= \varphi - \frac{v + z\varphi}{r} + \frac{a}{r}\frac{\partial w}{\partial s} \\
\epsilon_{x2}, \epsilon_{x3} &= \frac{\partial u}{\partial x} \mp h_1\frac{\partial \psi}{\partial x} + \frac{1}{2}\omega_{s2}^{2} \\
\epsilon_{s2}\ \ \epsilon_{s3} &= \frac{a}{r}\left(\frac{\partial v}{\partial s} \mp h_1\frac{\partial \varphi}{\partial s}\right) + \frac{w}{r} + \frac{1}{2}\omega_{x2}^{2} \\
\epsilon_{xs2}\ \ \epsilon_{xs3} &\ \ \frac{a}{r}\left(\frac{\partial u}{\partial s} \mp h_1\frac{\partial \psi}{\partial s}\right) + \frac{\partial v}{\partial x} \mp h_1\frac{\partial \varphi}{\partial x} - \omega_{x2}\omega_{s2}
\end{aligned}
\tag{9}
$$

where $\omega_{x1}$, $\omega_{s1}$, $\omega_{x2}$, $\omega_{s2}$ are given in Eqs. (4)

The stress-strain relations for the materials of the various layers are

$$
\begin{aligned}
\sigma_{xi} &= \mathcal{E}_i\left(\epsilon_{xi} + \nu_i\,\epsilon_{si}\right) \\
\sigma_{si} &= \mathcal{E}_i\left(\epsilon_{si} + \nu_i\,\epsilon_{xi}\right) \\
\tau_{xsi} &= \mu_i\,\epsilon_{xsi} \\
\tau_{xrl} &= \mu_{zl}\,\epsilon_{xrl} \\
\tau_{srl} &= \mu_{zl}\,\epsilon_{srl}
\end{aligned}
\tag{10}
$$

where $\mathcal{E}_i = E_i/(1-\nu_i^2)$, $\mu_i = E_i/2(1+\nu_i)$ . $E_i$ is Young's modulus, $\nu_i$ Poisson's ratio, and $\mu_{zl}$ an additional shear modulus of the orthotropic core. Successive substitution of Eqs. (9) in Eqs. (10), (2), and (8) yields

$$
\begin{aligned}
N_{x1}^{*} &= N_{x1} + Q_{x1}\,\omega_{s1} & M_{x1}^{*} &= M_{x1} + Q_{x1}\,\omega_{s1}\,\frac{h_1^2}{3a} \\
N_{s1}^{*} &= N_{s1} - Q_{s1}\,\omega_{x1} & M_{s1}^{*} &= M_{s1} + Q_{s1}\,\omega_{x1}\,\frac{h_1^2}{3a} \\
N_{sx1}^{*} &= N_{sx1} + Q_{s1}\,\omega_{s1} & M_{sx1}^{*} &= M_{sx1} - Q_{s1}\,\omega_{s1}\,\frac{h_1^2}{3a} \\
N_{xs1}^{*} &= N_{xs1} - Q_{x1}\,\omega_{x1} & M_{xs1}^{*} &= M_{xs1} - Q_{x1}\,\omega_{x1}\,\frac{h_1^2}{3a} \qquad (11) \\
Q_{xr1}^{*} &= Q_{x1} + N_{xs1}\,\omega_{x1} - N_{x1}\,\omega_{s1} & Q_{rx1}^{*} &= Q_{x1} \\
Q_{sr1}^{*} &= Q_{s1} + N_{s1}\,\omega_{x1} - N_{sx1}\,\omega_{s1} & Q_{rs1}^{*} &= Q_{s1}
\end{aligned}
$$

**where**

$$N_{x1} = \int \sigma_{x1}\left(1+\frac{z}{a}\right) dz = 2E_1 h_1\left[\frac{\partial u}{\partial x} + \nu_1\left(\frac{\partial v}{\partial s} + \frac{w}{a}\right) + \frac{h_1^2}{3a}\frac{\partial \psi}{\partial x} + \frac{1}{2}\omega_{s1}^2 - \frac{\nu_1}{2}\omega_{x1}^2\right]$$

$$N_{s1} = \int \sigma_{s1}\, dz = 2E_1 h_1\left[\frac{\partial v}{\partial s} + \frac{w}{a} + \nu_1\frac{\partial u}{\partial x} - \frac{h_1^2}{3a}\frac{\partial \varphi}{\partial s} + \frac{1}{2}\omega_{x1}^2 + \frac{\nu_1}{2}\omega_{s1}^2\right]$$

$$N_{xs1} = \int \tau_{xs1}\left(1+\frac{z}{a}\right) dz = 2\mu_1 h_1\left[\frac{\partial u}{\partial s} + \frac{\partial v}{\partial x} + \frac{h_1^2}{3a}\frac{\partial \varphi}{\partial x} - \omega_{x1}\,\omega_{s1}\right]$$

$$N_{sx1} = \int \tau_{sx1}\, dz = 2\mu_1 h_1\left[\frac{\partial u}{\partial s} + \frac{\partial v}{\partial x} - \frac{h_1^2}{3a}\frac{\partial \psi}{\partial s} - \omega_{x1}\,\omega_{s1}\right]$$

$$Q_{x1} = \int \tau_{xr1}\left(1+\frac{z}{a}\right) dz = 2\chi_1\mu_{z1}h_1\left(\psi + \frac{\partial w}{\partial x}\right)$$

$$Q_{s1} = \int \tau_{sr1}\, dz = 2\chi_1\mu_{z1}h_1\left(\varphi + \frac{\partial w}{\partial s} - \frac{v}{a}\right) \qquad (12)$$

$$M_{x1} = \int \sigma_{x1}\left(1+\frac{z}{a}\right) z\, dz = E_1\frac{2h_1^3}{3}\left(\frac{\partial \psi}{\partial x} + \nu_1\frac{\partial \varphi}{\partial s} + \frac{1}{a}\frac{\partial u}{\partial x} + \frac{\omega_{s1}^2}{2a} + \frac{\nu_1\omega_{x1}^2}{2a}\right)$$

$$M_{s1} = \int \sigma_{s1}\, z\, dz = E_1\frac{2h_1^3}{3}\left(\frac{\partial \varphi}{\partial s} + \nu_1\frac{\partial \psi}{\partial x} - \frac{1}{a}\frac{\partial v}{\partial s} - \frac{w}{a^2}\right)$$

$$M_{xs1} = \int \tau_{xs1}\left(1+\frac{z}{a}\right) z\, dz = \mu_1\frac{2h_1^3}{3}\left(\frac{\partial \psi}{\partial s} + \frac{\partial \varphi}{\partial x} + \frac{1}{a}\frac{\partial v}{\partial x} - \frac{1}{a}\omega_{x1}\,\omega_{s1}\right)$$

$$M_{sx1} = \int \tau_{sx1}\, z\, dz = \mu_1\frac{2h_1^3}{3}\left(\frac{\partial \psi}{\partial s} + \frac{\partial \varphi}{\partial x} - \frac{1}{a}\frac{\partial u}{\partial s}\right)$$

$$
\begin{aligned}
N_{x2}, N_{x3} &= \int (\sigma_{x2}, \sigma_{x3})\left(1+\frac{z}{a}\right) dz \\
&= E_2 h_2\Big[\left(1 \mp \frac{h}{a}\right)\left(\frac{\partial u}{\partial x} \mp h_1\frac{\partial \psi}{\partial x} + \frac{1}{2}\omega_{s2}^2 + \frac{\nu_2}{2}\omega_{x2}^2\right) \\
&\qquad + \nu_2\left(\frac{\partial v}{\partial s} \mp h_1\frac{\partial \varphi}{\partial s} + \frac{w}{a}\right)\Big]
\end{aligned}
$$

$$N_{s2}, N_{s3} = \int (\sigma_{s2}, \sigma_{s3})\, dz = E_2 h_2 \Big[ \left(1 \pm \frac{h}{a}\right)\left(\frac{\partial v}{\partial s} + \frac{w}{a} \mp h_1 \frac{\partial \varphi}{\partial s}\right) + \frac{1}{2}\omega_{x2}^2 + \nu_2 \left( \frac{\partial u}{\partial x} \mp h_1 \frac{\partial \psi}{\partial x} + \frac{1}{2}\omega_{s2}^2 \right) \Big] \tag{12}$$

$$N_{xs2}, N_{xs3} = \int (\tau_{xs2}, \tau_{xs3}) \left(1 + \frac{z}{a}\right) dz = \mu_2 h_2 \Big[ \left(1 \mp \frac{h}{a}\right)\left(\frac{\partial v}{\partial x} \mp h_1 \frac{\partial \varphi}{\partial x} - \omega_{x2}\omega_{s2}\right) + \frac{\partial u}{\partial s} \mp h_1 \frac{\partial \psi}{\partial s} \Big]$$

$$N_{sx2}, N_{sx3} = \int (\tau_{sx2}, \tau_{sx3})\, dz = \mu_2 h_2 \Big[ \left(1 \pm \frac{h}{a}\right)\left(\frac{\partial u}{\partial s} \mp h_1 \frac{\partial \psi}{\partial s}\right) + \frac{\partial v}{\partial x} \mp h_1 \frac{\partial \varphi}{\partial x} - \omega_{x2}\omega_{s2} \Big]$$

$$Q_{x2} = Q_{x3} = Q_{s2} = Q_{s3} = 0$$

As those in Eqs. (8), the limits of the integrations in Eqs.(12) cover the thickness of the layer. In carrying out these integrations the thin-shell assumption of $h^2/3a^2 \ll 1$ has been made use of wherever applicable. In the expressions of $Q_{x1}$ and $Q_{s1}$ a shear coefficient $\kappa_1$ has been inserted in the same manner as in the linear case [14, 15, 9] and may also be determined similarly. The transverse shear forces $Q_{x2}$, $Q_{s2}$, $Q_{x3}$, $Q_{s3}$ are zero because the face layers have been assumed to be membranes.

Substitution of Eqs. (11), and (12) in Eqs. (5) leads immediately to the displacement equations of motion, although the results will not be recorded here.

The system of Eqs. (5), (6), (7), (11), and (12) may be readily reduced to previous results for simpler special cases. Thus, by letting the radius a equal to infinity, the nonlinear equations of sandwich plates of reference 9 are obtained. On the other hand, if the nonlinear effect is suppressed, we arrive at one of the simpler systems of linear equations of sandwich cylindrical shells of reference 8, which are further reducible to the equations of homogeneous shells of reference 12.

## Modification and Simplification of Equations

As in reference 5, a second system of equations of somewhat better accuracy may be derived by assuming, instead of Eqs. (3),

$$u_1 = u + z\psi \qquad u_2, u_3 = u \mp (h_1 + h_2/2)\,\psi$$

$$v_1 = v + z\varphi \qquad v_2, v_3 = v \mp (h_1 + h_2/2)\,\varphi$$

$$w_1 = w_2 = w_3 = w$$

The results are similar to Eqs. (5), (6), (7), (11), and (12). In fact, they are obtainable from these equations by replacing $h_1$ by $h_1 + h_2/2$ in the last two of Eqs. (5) (except the $h_1$ in $2h_1^3/3$), in the last two of Eqs. (6) and those of Eqs. (7) (except the $h_1$ in the limits of the integration), and in the expressions of $N_{x2}$, $N_{x3}$, $N_{s2}$, $N_{s3}$, $N_{xs2}$, $N_{xs3}$, $N_{sx2}$, $N_{sx3}$ in equations (12). With the modifications made, we shall designate the newly obtained second system of equations as

Eqs. (5), (6), (7), (11), (12) modified

$$(5'),\ (6'),\ (7'),\ (11'),\ (12')$$

As may be seen from Eqs. (11) and (12), by having taken into consideration the rotations $\omega_{x1}$ and $\omega_{s1}$ in the core, not only the membrane forces $N_{x1}$, $N_{s1}$, $N_{xs1}$, and $N_{sx1}$ themselves are affected, but also they are augmented by the transverse shear forces $Q_{x1}$ and $Q_{s1}$. Conversely, the transverse shear forces are also augmented by the membrane forces. Since for relatively low frequencies the motion of the shell is predominantly transverse in nature, the contribution of $Q_{x1}$ and $Q_{s1}$ to the membrane forces should be of less importance than the contribution of $N_{x1}$, $N_{s1}$, $N_{xs1}$, and $N_{sx1}$ to the transverse shear forces. The contribution of $Q_{x1}$ and $Q_{s1}$ to the moments is also small, for, according to Eqs. (11), they are multiplied by both a small angle of rotation $\omega_{x1}$ or $\omega_{s1}$ and a small factor $h_1^2/3a$. As a simplification, the contributions of $Q_{x1}$ and $Q_{s1}$ to the membrane forces and moments are neglected in Eqs. (11). If we are primarily interested in low frequencies for which the motion is predominently transverse, the inertia terms involving $\ddot{u}$, $\ddot{v}$, $\ddot{\psi}$ and $\ddot{\varphi}$ become of much less importance than the transverse inertia term involving $\ddot{w}$ and may also be neglected. Incorporating both types of simplifications, we shall designate the results obtained from the first system of equations by

Eqs. (5), (6), (7), (11), (12) simplified
(5a), (6a), (7a), (11a), (12a)

The second system of equations may similarly be simplified and denoted by

Eqs. (5'), (6'), (7'), (11'), (12') simplified
(5a'), (6a'), (7a'), (11a'), (12a')

For sandwich shells with soft cores the membrane forces and moments in the core may further be neglected, that is, we may put

$$N_{x1} = N_{s1} = N_{xs1} = N_{sx1} = M_{x1} = M_{s1} = M_{xs1} = M_{sx1} = 0 \qquad (13)$$

Contrary to this, we have in the case of homogeneous shells, for which $h_2 = 0$,

$$N_{x2} = N_{s2} = N_{xs2} = N_{sx2} = 0 \qquad (14)$$

In the axially symmetrical case with zero surface tractions, Eqs. (5a') become

$$\frac{\partial}{\partial x}\left(N_{x1} + N_{x2} + N_{x3}\right) = 0$$

$$\frac{\partial}{\partial x}\left[M_{x1} + \left(h_1 + \frac{h_2}{2}\right)\left(N_{x3} - N_{x2}\right)\right] - Q_{x1} = 0$$

$$\frac{\partial}{\partial x}\left[Q_{x1} - N_{x1}\omega_{s1} - \left(N_{x2} + N_{x3}\right)\omega_{s2}\right] \qquad (15)$$

$$- \frac{1}{a}\left(N_{s1} + N_{s2} + N_{s3}\right) = 2\left(\rho_1 h_1 + \rho_2 h_2\right)\dot{w}$$

where

$$M_{x1} = E_1 \frac{2h_1^3}{3}\left(\psi' + \frac{1}{a}u' + \frac{1}{2a}\omega_{s1}^2\right)$$

$$Q_{x1} = 2\kappa_1 \mu_{z1} h_1 (\psi + w')$$

$$N_{x1} = 2E_1 h_1 \left(u' + \frac{\nu_1}{a} w + \frac{h_1^2}{3a}\psi' + \frac{1}{2}\omega_{s1}^2\right)$$

$$N_{s1} = 2E_1 h_1 \left(\frac{w}{a} + \nu_1 u' + \frac{\nu_1}{2}\omega_{s1}^2\right)$$

$$N_{x2} + N_{x3} = 2E_2 h_2 \left[u' + \frac{h}{a}\left(h_1 + \frac{h_2}{2}\right)\psi' + \frac{\nu_2}{2} w + \frac{1}{2}\omega_{s2}^2\right]$$

$$N_{x3} - N_{x2} = 2E_2 h_2 \left[\left(h_1 + \frac{h_2}{2}\right)\psi' + \frac{h}{a}u' + \frac{h}{2a}\omega_{s2}^2\right] \qquad (16)$$

$$N_{s2} + N_{s3} = 2E_2 h_2 \left(\frac{w}{a} + \nu_2 u' + \frac{\nu_2}{2}\omega_{s2}^2\right)$$

with primes denoting differentiation with respect to x. The final simplifications given by Eqs. (1[illegible]) and (14) will be incorporated in Eqs. (15) in the next sections in which axially symmetrical vibrations of sandwich and homogeneous cylindrical shells are discussed.

## Nonlinear Vibrations of Homogeneous and Sandwich Cylindrical Shells

It should be emphasized at this point that each of the above systems of shell equations is essentially the result of putting to zero the coefficients of the variations of the displacements in the integrands of the now simplified variational equation of motion, simplified in that it no longer contains the integration with respect to z. For those of the shell equations that can be solved and satisfied exactly in a given problem, the corresponding coefficients will simply drop out of the simplified variational equation of motion. On the other hand, for those shell equations that cannot be solved exactly, the corresponding coefficients will remain. Subsequent variational approximation may then be performed, which makes the latter equations eventually also satisfied, at least approximately in the variational sense. This procedure will now be demonstrated by the following discussion of axially symmetrical vibration of homogeneous and sandwich cylindrical shells with immovable hinged edges.

The homogeneous cylindrical shell will be considered first, for which Eqs. (14) to (16) yield

$$u'' + \frac{\nu}{a} w' + \frac{h^2}{3a} \psi'' + \omega_s \omega_s' = 0 \tag{17}$$

$$E \frac{2h^3}{3} \left( \psi'' + \frac{1}{a} u'' + \frac{1}{a} \omega_s \omega_s' \right) - 2 \varkappa \mu h ( \psi + w' ) = 0 \tag{18}$$

$$2 \varkappa \mu h ( \psi' + w'' ) - 2Eh \left( u' + \frac{\nu}{a} w + \frac{h^2}{3a} \psi' + \frac{\omega_s^2}{2} \right) \omega_s' - 2Eh \frac{1}{a} \left( \frac{w}{a} + \nu u' + \frac{\nu}{2} \omega_s^2 \right) - 2 \rho h \ddot{w} = 0 \tag{19}$$

where the only rotation component involved is now

$$\omega_s = - \frac{1}{2} ( w' - \psi )$$

The subscript 1 has been dropped from all notations in Eqs. (17) to (19). The shear modulus $\mu$ is now for an istropic material, and the shear coefficient has the usual value of $\pi^2/12$ [14], although it may also be assigned the value of infinity for the purpose of suppressing the transverse shear effect.

From Eqs. (17) and (18) u" is first eliminated. Into the result are substituted

$$w = W\tau(t)\sin\frac{\lambda x}{h}, \qquad \psi = \bar{\Psi}\tau(t)\cos\frac{\lambda x}{h} \tag{20}$$

which satisfy the conditions of zero deflection and moment at the hinged edges $x = 0, \ell$. It is then found that

$$\bar{\Psi} = -\gamma\frac{\lambda}{h}W, \qquad \gamma = \frac{1}{1+2\lambda^2/3\kappa(1-\nu)} \tag{21}$$

In Eqs. (20) and (21) we have $\lambda = n\pi h/\ell$, with $n = 1, 2, 3, \ldots$, designating the number of half-waves in the length $\ell$ of the shell, and $\tau(t)$ is the unknown time function. u is next solved from Eq. (17), into which w and $\psi$ are substituted. Together with the use of the boundary conditions of $u = 0$ at the immovable edges $x = 0, \ell$, we find

$$u = \frac{3\nu+\gamma\lambda^2}{3a}\frac{h}{\lambda}W\tau\left(\cos\frac{\lambda x}{h} - 1 + k\frac{x}{\ell}\right) - \frac{(1+\gamma)^2}{32}\frac{\lambda}{h}W^2\tau^2\sin\frac{2\lambda x}{h} \tag{22}$$

where

$$k = 1-(-1)^n$$

Eqs. (20) to (22) thus satisfy exactly all the boundary conditions and the governing Eqs. (17) and (18). However, it is easily verified that they do not satisfy the remaining governing Eq. (19). Since the left side of Eq. (19) is the coefficient of $\delta w$ in the double integral (actually reduced to a single integral of x in the present plane-strain problem) in the variational equation of motion, and since we now have

$$\delta w = \sin\frac{\lambda x}{h}\,\delta(W\tau)$$

according to Eqs. (20) we may carry out explicitly the integration with respect to x over the length of the shell and put the coefficient of $\delta(W\tau)$ equal to zero. Thus, there results

$$\ddot{\tau} + \alpha_1 \tau + \alpha_2 \tau^2 + \alpha_3 \tau^3 = 0 \tag{23}$$

where

$$\alpha_1 = \frac{\lambda^4}{\rho h^2} \left\{ \frac{1}{3 + \frac{2\lambda^2}{\chi(1-\nu)}} + \frac{1}{\lambda^4} \frac{h^2}{a^2} \left[ 1 - \left( \nu^2 + \frac{\nu\lambda^2}{3 + \frac{2\lambda^2}{\chi(1-\nu)}} \right) \left( 1 - \frac{2k^2}{n^2\pi^2} \right) \right] \right\}$$

$$\alpha_2 = \frac{E\lambda}{\rho a} \frac{k}{\ell} \left[ \left( \nu + \frac{\gamma\lambda^2}{3} \right)(1+\gamma) + \frac{\nu}{4}(1+\gamma)^2 \right] \frac{W}{2h}$$

$$\alpha_3 = \frac{E\lambda^4}{\rho h^2} \left( \frac{1+\gamma}{2} \right)^3 \left( \frac{W}{2h} \right)^2$$

This in essence has completed a second variational approximation, with respect to the second and last remaining space coordinate x, although the approximation applies to only one of the governing differential equations. If needed, the procedure could have also been applied to any other or all of the differential equations and boundary conditions.

To determine the nonlinear frequency it is convenient to carry out a last variational approximation with respect to time, in connection with Eq. (23). The left side of the equation is essentially the coefficient of $\delta(W\tau)$ in the variational equation of motion, which has now been much simplified, since the time integral is the only one remaining. The integration with respect to time may be carried out explicitly by first assuming say, $\tau = \sin\omega t$, and by selecting $t_o = 0$ and $t_1 = 2\pi/\omega$ as the limits of integration. The coefficient of $\delta W$ is finally put equal to zero to yield

$$\omega^2 = \alpha_1 + \frac{3}{4}\alpha_3$$

or

$$\omega^2 = \frac{E\lambda^4}{\rho h^2}\left\{\frac{1}{3+\frac{2\lambda^2}{\chi(1-\nu)}} + \frac{1}{\lambda^4}\frac{h^2}{a^2}\left[1-\left(\nu^2+\frac{\nu\lambda^2}{3+\frac{2\lambda^2}{\chi(1-\nu)}}\right)\left(1-\frac{2k^2}{n^2\pi^2}\right)\right] + \frac{3}{4}\left(\frac{1+\nu}{2}\right)\frac{3}{(W/2h)^2}\right\} \tag{24}$$

It is interesting to note that the approximate nonlinear frequency is independent of $\alpha_2$.

In a similar manner we may investigate the vibration of the sandwich cylindrical shell, to which Eqs. (15) and (16) together with the simplifications in Eqs. (13) are applicable. The equations of motion in terms of the displacements are of the final form

$$u'' + \frac{\nu_2}{a} w' + \frac{h}{a}\left(h_1 + \frac{h_2}{a}\right)\psi'' + w' w'' = 0$$

$$2E_2 h_2\left(h_1+\frac{h_2}{2}\right)\left[\left(h_1+\frac{h_2}{2}\right)\psi'' + \frac{h}{a}\left(u''+w'w''\right)\right] - 2\kappa_1 \mu_1 h_1\left(\psi + w'\right) = 0$$

$$\begin{aligned} &2\kappa_1 \mu_1 h_1\left(\psi' \; w''\right) \\ &\quad + 2E_2 h_2\left[u' + \frac{\nu_2}{a}w + \frac{h}{a}\left(h_1 \; \frac{h_2}{2}\right)\psi' + \frac{1}{2}w'^2\right] w'' \\ &\quad - 2E_2 h_2 \frac{1}{a}\left[\frac{w}{a} + \nu_2 u' + \frac{\nu}{2} w'^2\right] - 2\left(\rho_1 h_1 + \rho_2 h_2\right)\ddot{w} = 0 \end{aligned} \tag{25}$$

The shear coefficient $\chi_1$ may be taken equal to 1 for ordinary sandwich shells [15, 8], although it may also be set equal to infinity for the purpose of suppressing the transverse shear effect.

Eqs. (25) are entirely similar to Eqs. (17) to (19) and the same method of solution is applicable. The results are

$$w = W\tau \sin\frac{\lambda x}{h}$$

$$\psi = -\gamma\frac{\lambda}{h} W\tau\cos\frac{\lambda x}{h}, \qquad \gamma = \frac{1}{1+\frac{r_2 r_h \lambda^2}{\chi_1(1+r_h)}}$$

$$u = \frac{\nu_2 + \gamma\lambda^2/(1+r_h)}{a}\,\frac{h}{\lambda}\,W\tau\left(\cos\frac{\lambda x}{h} - 1 + k\frac{x}{\ell}\right) - \frac{1}{8}\frac{\lambda}{h}W^2\tau^2 \sin\frac{2\lambda x}{h}$$

and

$$\ddot{\tau} + \alpha_1 \tau + \alpha_2 \tau^2 + \alpha_3 \tau^3 = 0$$

where

$$\alpha_1 = \frac{\mu_{z1}}{\rho_1 h^2} \frac{r_z r_h \lambda^4}{1 + r_\rho r_h} \left\{ \frac{1}{1 + r_h + r_z r_h \lambda^2 / \chi_1} + \frac{1}{\lambda^4} \frac{h^2}{a^2} \left[ 1 - \left( \nu_2^2 + \frac{\nu_2 \lambda}{1 + r_h + r_z r_h \lambda^2 / \chi_1} \right) \cdot \right. \right.$$

$$\left. \left. \cdot \left( 1 - \frac{2k^2}{n^2 \pi^2} \right) \right] \right\}$$

$$\alpha_2 = \frac{\mu_{z1}}{\rho_1 a} \frac{r_z r_h \lambda}{1 + r_\rho r_h} \frac{k}{\ell} \left( 3 \nu_2 + \frac{2 \gamma \lambda^2}{1 + r_h} \right) \frac{W}{2h}$$

$$\alpha_3 = \frac{\mu_{z1}}{\rho_1 h^2} \frac{r_z r_h \lambda^4}{1 + r_\rho r_h} \left( \frac{W}{2h} \right)^2$$

$$r_z = E_2 / \mu_{z1}$$

$$r_h = h_2 / h_1$$

$$r_\rho = \rho_2 / \rho_1$$

The nonlinear frequency is again given by the approximate expression

$$\omega^2 = \alpha_1 + \frac{3}{4} \alpha_3$$

which is now

$$\omega^2 = \frac{\mu_{z1}}{\rho_1 h^2} \frac{r_z r_h \lambda^4}{1 + r_\rho r_h} \left\{ \frac{1}{1 + r_h + r_z r_h \lambda^2 / \chi_1} \right.$$

$$\left. + \frac{1}{\lambda^4} \frac{h^2}{a^2} \left[ 1 - \left( \nu_2^2 + \frac{\nu_2 \lambda^2}{1 + r_h + r_z r_h \lambda^2 / \chi_1} \right) \left( 1 - \frac{2k^2}{n^2 \pi^2} \right) \right] + \frac{3}{4} \left( \frac{W}{2h} \right)^2 \right\} \quad (26)$$

Eq. (26) is reducible to the result for the nonlinear frequency of sandwich plates [9] by letting a equal to infinity, and to the result for relatively low linear frequencies of sandwich cylindrical shells [8], by dropping the nonlinear term (3/4) $(W/2h)^2$.

The frequencies of nonlinear symmetrical vibrations of homogeneous and sandwich cylindrical shells have thus been determined solely on the basis of the variation equation of motion, by carrying out variational approximations wherever needed. Further refinements may be made in the analysis and the discussion may readily be extended to non-symmetrical vibrations and to other types of boundary conditions, but the relatively simple cases presented here are clearly sufficient for illustrating the variational procedure that is being proposed.

## Discussion of Results

The results in Eqs. (24) and (26) give the frequencies for the lowest family of axially symmetrical modes of vibration of the cylindrical shells, which are predominately transverse in nature. The effect of thickness shear deformation is associated with the $\varkappa$ - or $\varkappa_1$- terms, and its importance is seen to vary directly with $\lambda^2$. Since putting $\varkappa$ or $\varkappa_1$ equal to infinity is equivalent to the suppression of the shear effect, Eqs. (24) and (26) show that, disregarding nonlinear and curvature effects at this moment, the shear effect by itself becomes negligible if

$$\lambda^2 << \frac{3}{2}(1-\nu) \qquad \text{(homogeneous)}$$

or

$$\lambda^2 << \frac{1+r_h}{r_2 r_h} \qquad \text{(sandwich)}$$

These conditions apply to plates as well as shells. Since $r_2 r_h$ for ordinary sandwich structures is usually of the order of between 10 and 100, the shear effect is much more important for sandwich than for homogeneous plates and shells. In general, the shear effect should be considered for sandwich structures [16].

The $w^2/4h^2$- and $h^2/a^2$- terms in Eqs. (24) and (26) reflect, respectively, the nonlinear and curvature effects, which are completely uncoupled from each other. Terms associated with the coupling between the two effects would be present in the equations, if we had employed more exact expressions of the rotation components. Since it is the approximate expressions of the rotation components in Eqs. (4) which have been used and which are the same as those for plates, the nonlinear terms in Eqs. (24) and (26) are also essentially the same as those for plates. In the case of sandwich plates, the nonlinear effect was discussed before in reference 9, where it was found that the nonlinear effect would overshadow the shear effect if

$$\left(\frac{W}{2h}\right)^2 >> \frac{1}{1+r_h} \qquad (27)$$

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