UNCLASSIFIED

AD 288 931

Reproduced by the

ARMED SERVICES TECHNICAL INFORMATION AGENCY
ARLINGTON HALL STATION
ARLINGTON 12, VIRGINIA

UNCLASSIFIED

D1-82-0203

BOEING SCIENTIFIC RESEARCH LABORATORIES

Improving the Polar Method for Generating a Pair of Normal Random Variables

George Marsaglia

Mathematics Research

September 1962

D1-82-0203

IMPROVING THE POLAR METHOD FOR GENERATING A PAIR OF NORMAL RANDOM VARIABLES

by

George Marsaglia

Mathematical Note No. 271

Mathematics Research Laboratory

BOEING SCIENTIFIC RESEARCH LABORATORIES

September 1962

Box and Muller [1], have pointed out that a pair of independent standard normal random variables may be generated in the form $x = (-2 \ln u_1)^{\frac{1}{2}} \cos 2\pi u_2$, $y = (-2 \ln u_1)^{\frac{1}{2}} \sin 2\pi u_2$, where u_1, u_2 are independent $[0,1]$ random variables. The proof readily follows from the fact that $(-2 \ln u_1)^{\frac{1}{2}}$ is distributed as ρ, and $2\pi u_2$ is distributed as Θ, in the polar representation (ρ, Θ) of (x,y). The method is attractive, but rather slow, requiring sine, cosine, logarithm, and square root subroutines. This note shows how to speed up the procedure and still retain its principal advantage -- ease of programming.

One line of improvement, by way of a suggestion of von Neumann [2], is to generate $\sin \Theta$ and $\cos \Theta$ in the form $u(u^2 + v^2)^{-\frac{1}{2}}, v(u^2 + v^2)^{-\frac{1}{2}}$, where (u,v) are uniformly distributed over the unit circle: $u^2 + v^2 \leq 1$. This avoids the sine and cosine subroutines. The pair (u,v) can be produced, with efficiency $\pi/4$, by conditioning a pair of independent uniform $[-1,1]$ random variables. It seems then that we may avoid the sine and cosine subroutines at the expense of drawing an additional uniform random variable. We may, however, avoid this difficulty by using the following fact: *if* u,v *are uniformly distributed over the unit circle*: $u^2 + v^2 \leq 1$, *then* $u^2 + v^2$ *is uniform* $[0,1]$ *and is independent of* u/v. The proof is elementary and omitted. It follows that $u^2 + v^2$ is independent of $u(u^2 + v^2)^{-\frac{1}{2}}$ and $v(u^2 + v^2)^{-\frac{1}{2}}$, and hence that

$$(1) \qquad \begin{aligned} x &= u[-2 \ln(u^2+v^2)/(u^2+v^2)]^{\frac{1}{2}} \\ y &= v[-2 \ln(u^2+v^2)/(u^2+v^2)]^{\frac{1}{2}} \end{aligned}$$

are independent standard normal random variables, since $[-2 \ln(u^2 + v^2)]^{\frac{1}{2}}$ is distributed as ρ, and $u(u^2 + v^2)^{-\frac{1}{2}}, v(u^2 + v^2)^{-\frac{1}{2}}$ are distributed as $\sin \theta$, $\cos \theta$, and are independent of $u^2 + v^2$.

We then suggest this procedure for generating a pair x,y of independent standard normal random variables: generate pairs (u,v) of independent uniform $[-1,1]$ random variables until one satisfies $u^2 + v^2 \leq 1$, then form x and y according to relations (1). This method is faster than the direct polar coordinate representation and is still very easy to program. It still takes about 5 - 6 times as long as the very fastest methods, but may well serve in situations where ease of programming or limited storage capacity are the primary considerations.

If we try to generalize the above methods - generate a point uniformly over the surface of the unit n-sphere and project the point into space by multiplying by a chi-n variate, we run into the problem of producing a point on the n-sphere. The obvious method, generate a point in an n-cube and reject it if it lies outside the inscribed n-sphere, has efficiency $\pi^{n/2}/2^n\Gamma(\frac{n+2}{2})$, which tends rapidly to zero. If we do not specify which n-sphere we want, however, we can get a satisfactory procedure in the following way: generate independent uniform $[-1,1]$ random variables $v_1, v_2, \ldots, v_n, v_{n+1}$ until

$$(2) \qquad S_n^2 = v_1^2 + \cdots + v_n^2 \leq 1 < v_1^2 + \cdots + v_n^2 + v_{n+1}^2.$$

Then the point $(v_1/S_n, v_2/S_n, \dots, v_n/S_n)$ is uniformly distributed over the surface of the unit n-sphere, since the density function of $(v_1, \dots, v_n)$, given condition (2), is a multiple of $1 - [1 - (z_1^2 + \dots + z_n^2)]^{\frac{1}{2}}$, and is radially symmetric.

We then offer this method for generating a random number of independent, standard normal random variables $x_1, x_2, \dots, x_n$: <u>Generate independent uniform</u> $[-1,1]$ <u>random variables</u> $v_1, \dots, v_n, v_{n+1}$ <u>until</u> $S_n^2 = v_1^2 + \dots + v_n^2 \leq 1 < v_1^2 + \dots + v_n^2 + v_{n+1}^2$, <u>then put</u> $x_i = v_i[R/S_n^2]^{\frac{1}{2}}$, <u>where</u> R <u>has the chi-square-n distribution and is independent of the</u> v's. If $n = 2m$, we may put $R = -2 \ln(u_1 u_2 \cdots u_m)$ and if $n = 2m + 1$, $R = -2 \ln(u_1 u_2 \cdots u_m) + y^2$, where y is normal. We may produce y^2 either as the square of one of the normal variates previously generated, or else in pairs in the form $w_1^2 \ln(w_1^2 + w_2^2)/(w_1^2 + w_2^2)$, $w_2^2 \ln(w_1^2 + w_2^2)/(w_1^2 + w_2^2)$, where w_1, w_2 are uniform $[0,1]$, conditioned by $w_1^2 + w_2^2 \leq 1$.

The expected number of x's produced by this method is about 3.5.

REFERENCES

[1] G. E. P. Box and M. E. Muller, A note on the generation of normal deviates, _Ann. Math. Stat._ 28, (1958), pp. 610-611.

[2] J. von Neumann, Various techniques in connection with random digits, _Monte Carlo Method_, _Nat. Bur. Stand._ Applied Math Series 12, (1951), pp. 36-38.