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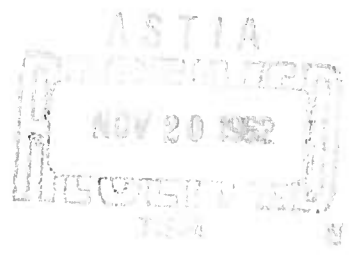
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Elementary Relations Between Uniform  
and Normal Distributions in the Plane



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Mathematics Research

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ELEMENTARY RELATIONS BETWEEN UNIFORM  
AND NORMAL DISTRIBUTIONS IN THE PLANE

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Mathematics Research Laboratory

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## 0. Summary

If  $(U,V)$  is uniformly distributed over the unit circle, then  $U^2 + V^2$  is uniformly distributed over  $[0,1]$  and is independent of  $U/V$ . This simple result leads to an improved method for generating normal random variables via polar coordinates, and also serves to relate normal probability measure and Lebesgue measure.

## 1. Main Paragraph

Let  $(U,V)$  be uniformly distributed over the unit circle:  $\{(u,v): u^2 + v^2 \leq 1\}$ . Then  $U^2 + V^2$  is uniformly distributed over  $[0,1]$  and is independent of  $U/V$ . The proof is simple and omitted. It follows that  $U^2 + V^2$  is independent of  $[1 + (V/U)^2]^{-\frac{1}{2}}$  and of  $[1 + (U/V)^2]^{-\frac{1}{2}} = V(U^2 + V^2)^{-\frac{1}{2}}$ , and hence

$$(1) \quad \begin{aligned} X &= U \left[ \frac{-2 \ln(U^2 + V^2)}{U^2 + V^2} \right]^{\frac{1}{2}} \\ Y &= V \left[ \frac{-2 \ln(U^2 + V^2)}{U^2 + V^2} \right]^{\frac{1}{2}} \end{aligned}$$

are independent standard normal random variables, since  $U(U^2 + V^2)^{-\frac{1}{2}}$  and  $V(U^2 + V^2)^{-\frac{1}{2}}$  are distributed as  $\cos \theta$  and  $\sin \theta$ , and  $[-2 \ln(U^2 + V^2)]^{\frac{1}{2}}$  is distributed as  $\rho$ , in the polar representation  $(\rho, \theta)$  of  $(X, Y)$ .

## 2. Generating Normal Variables

In [1], Box and Muller point out the elementary fact that a pair of normal random variables may be produced in a computer by generating  $\rho$  and  $\theta$ , then putting  $X = \rho \cos \theta$  and  $Y = \rho \sin \theta$ . The obvious method - use two independent uniform  $[0,1]$  random variables  $(U_1, U_2)$  and put  $X = \cos(2\pi U_1)(-2 \ln U_2)^{\frac{1}{2}}$ ,  $Y = \sin(2\pi U_1)(-2 \ln U_2)^{\frac{1}{2}}$ , requires cosine, sine, logarithm, and square root subroutines. This method is not very practical unless some tricks are used to speed it up. One line of improvement is to generate  $\cos \theta$  and  $\sin \theta$  in the form  $U(U^2 + V^2)^{-\frac{1}{2}}$  and  $V(U^2 + V^2)^{-\frac{1}{2}}$ , as suggested by von Neumann [2]. The representation in (1) goes one step further, taking advantage of the fact that  $U^2 + V^2$  is independent of  $U(U^2 + V^2)^{-\frac{1}{2}}$  and of  $V(U^2 + V^2)^{-\frac{1}{2}}$ , is itself uniform  $[0,1]$ , and may be used to form  $\rho$  by way of  $\rho = [-2 \ln(U^2 + V^2)]^{\frac{1}{2}}$ .

Although there are faster methods for generating normal variables, see, e.g., [3] - [5], the method suggested by relations (1) may be suitable for situations where ease of programming is the primary consideration. Furthermore, a slight modification of the procedure may be used to dispose of the problem of handling the tail of the normal distribution in one of the super-fast programs. Let  $r > 0$  be a constant. Then putting

$$(2) \quad \begin{aligned} X &= U \left[ \frac{r^2 - 2 \ln(U^2 + V^2)}{U^2 + V^2} \right]^{\frac{1}{2}} \\ Y &= V \left[ \frac{r^2 - 2 \ln(U^2 + V^2)}{U^2 + V^2} \right]^{\frac{1}{2}} \end{aligned}$$

will produce a normal pair  $(X,Y)$  conditioned by  $X^2 + Y^2 \geq r^2$ . If we want to produce a normal variable  $Z$  conditioned by  $|Z| \geq r$ , we may generate  $X,Y$  according to (2) and put  $Z = X$  if  $|X| \geq r$ . If  $|X| < r$ , test: is  $|Y| \geq r$ ? If yes, put  $Z = Y$ , if no, generate a new pair  $X,Y$  and try again.

### 3. Normal Measure $\rightarrow$ Lebesgue Measure

Finally, expression (1) is convenient for relating normal probability measure and Lebesgue measure in the plane. Let  $T^{-1}$  be the mapping of the unit circle  $C: \{(u,v): 0 < u^2 + v^2 \leq 1\}$  onto the plane,  $R_2$ , of points  $(x,y)$  given by the relations in (1):

$$x = u \left[ \frac{-2 \ln(u^2 + v^2)}{u^2 + v^2} \right]^{\frac{1}{2}}$$

$$y = v \left[ \frac{-2 \ln(u^2 + v^2)}{u^2 + v^2} \right]^{\frac{1}{2}}.$$

Then  $T^{-1}$  is one-to-one, and its inverse,  $T$ , maps points  $(x,y)$  of  $R_2$  into points  $(u,v)$  of  $C$  according to the relations:

$$(3) \quad u = x \left[ \frac{e^{-\frac{1}{2}(x^2+y^2)}}{x^2 + y^2} \right]^{\frac{1}{2}}$$

$$v = y \left[ \frac{e^{-\frac{1}{2}(x^2+y^2)}}{x^2 + y^2} \right]^{\frac{1}{2}}.$$

Let  $\mu$  be standard normal probability measure in the plane, and  $\lambda$  Lebesgue measure. Then

$$\mu(A) = P[(X,Y) \in A] = P[(U,V) \in T(A)] = \frac{\lambda(T(A))}{\pi}.$$

That is, we may find the standard normal probability measure of regions  $A$  by mapping  $A$  into the unit circle according to relations (2), then finding the area of  $T(A)$ .

This type of transformation has been used before, [6]. We have a simpler version here, which may be put in this form: To find the standard normal probability measure of a region  $A$ , map each polar coordinate point  $(\rho, \theta)$  of  $A$  into the polar coordinate point  $(e^{-\frac{1}{2}\rho^2}, \theta)$  of the unit circle, and find  $\frac{1}{\pi}$  times the area of the transformed region. Inverting the mapping  $X = \cos 2\pi U_1(-2 \ln U_2)^{\frac{1}{2}}$ ,  $Y = \sin 2\pi U_1(-2 \ln U_2)^{\frac{1}{2}}$  gives a more suitable method for converting normal to Lebesgue measure, however. The procedure runs as follows: To find the normal probability measure of a region  $A$ , map each polar coordinate point  $(\rho, \theta)$  of  $A$  into the point  $(u, v)$  of the unit square  $\{(u, v): 0 < u < 1, 0 < v < 1\}$  according to the relations

$$u = \theta/2\pi$$

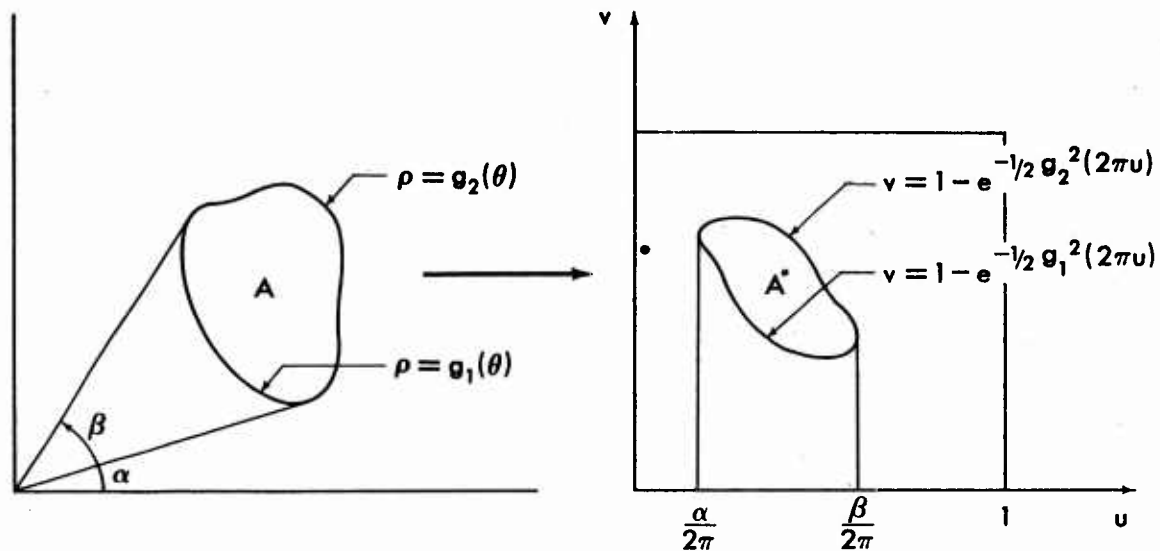
(4)

$$v = 1 - e^{-\frac{1}{2}\rho^2}$$

then find the area of the transformed region. We have found relations (4) to be the most suitable for a general purpose procedure for transforming normal to Lebesgue measure, the advantage being that it is relatively easy to assign an equally spaced set of points  $u_0, u_1, \dots$  and hence use one of the standard numerical integration procedures.



The transformation may be described graphically in this way:



Use of Simpson's rule gives the approximation

$$\mu(A) = \lambda(A^*) \cong \frac{\beta - \alpha}{12\pi n} [t_0 + 4t_1 + 2t_2 + 4t_3 + \dots + 2t_{2n-2} + 4t_{2n-1} + t_{2n}]$$

where  $t_i = e^{-\frac{1}{2}g_1^2(2\pi u_i)} - e^{-\frac{1}{2}g_2^2(2\pi u_i)}$ ,  $u_i = \alpha/2\pi + i\sigma$ , and  $\sigma = (\beta - \alpha)/4\pi n$ .

This elementary procedure makes it possible to write a single computer program which will handle a large number of the commonly encountered regions in the normal probability plane - polygonal regions, ellipses, intersections of ellipses, etc.

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