UNCLASSIFIED

RESEARCH REPORT 5
22 MAY 1961
I.E.R. 172-6

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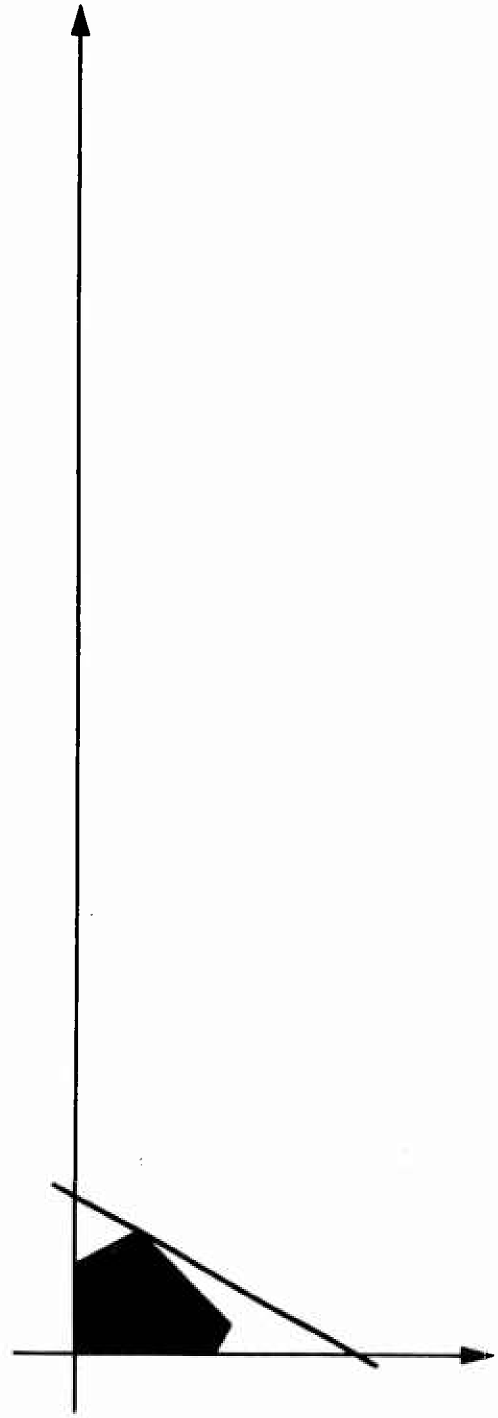

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This research was supported in part by the Office of Naval Research under contract Nonr-222(83) with the University of California. Reproduction in whole or in part, is permitted for any purpose of the United States Government.

SUPPORTS OF A CONVEX FUNCTION

Let C be a real, symmetric, mxm, positive-semi-definite matrix. Let $R^m = \left\{(x_1, \ldots, x_m) \mid x_i \text{ is a real number, } i = 1, \ldots, m\right\}$, and let $K \subset R^m$ be a polyhedral convex cone, i.e., there exists a real mxn matrix A such that $K = \left\{x \mid x \in R^m \text{ and } xA \leq 0\right\}$. Consider the function $\psi : K \rightarrow R$ defined by $\psi(x) = (xCx^T)^{1/2}$ for all $x \in K$. We wish to characterize the set, U, of all supports of ψ, where

(1) $$U = R^m \cap \left\{u \mid x \in K \Rightarrow ux^T \leq (xCx^T)^{1/2}\right\}.$$

Let $R^n_+ = R^n \cap \left\{\pi \mid \pi \geq 0\right\}$ and consider the set

(2) $$V = \left\{v \mid \exists\, x \in R^m,\ \pi \in R^n_+ \text{ and } v = \pi A^T + xC,\ xCx^T \leq 1,\ xA \leq 0\right\}.$$

We shall demonstrate:

<u>THEOREM:</u>

$$U = V.$$

We first show:

<u>LEMMA 1</u>

$$x, y \in R^m \Rightarrow (xCy^T)^2 \leq (xCx^T)(yCy^T).$$

<u>Proof:</u> If $x, y \in R^m$ consider the polynomial $p(\lambda) = \lambda^2 xCx^T + 2\lambda xCy^T + yCy^T = (x+\lambda y)C(x+\lambda y)^T$. Since C is positive-semi-definite, $p(\lambda) \geq 0$ for all real numbers λ, and thus the discriminant of p is non-positive, i.e.,

$$4(xCy^T)^2 - 4(xCx^T)(yCy^T) \leq 0 . \qquad \text{q.e.d.}$$

As an immediate application of Lemma 1 we show:

LEMMA 2

$$V \subset U$$

Proof: Let $v \in V$, then there exist $x \in R^m$, $\pi \in R_+^n$ such that $v = \pi A^T + xC$, $xCx^T \leq 1$. Now if $y \in R^m$, $yA \leq 0$, then $vy^T = yA\pi^T + xCy^T$ and $vy^T \leq xCy^T$, because $yA \leq 0$, $\pi^T \geq 0$ and $yA\pi^T \leq 0$. Thus, $vy^T \leq (xCx^T)^{1/2}(yCy^T)^{1/2}$, by Lemma 1, and $vy^T \leq (yCy^T)^{1/2}$, because $xCx^T \leq 1$. Thus, $v \in U$.

q.e.d.

From the fact that C is positive-semi-definite, it follows that:

LEMMA 3

The set V is convex.

Proof: If $x_k \in R^m$, $\pi_k \in R_+^n$, $x_k A \leq 0$, $u_k = \pi_k A^T + x_k C$, $x_k C x_k^T \leq 1$, $\lambda_k \in R_+$ for $k = 1, 2$ and $\lambda_1 + \lambda_2 = 1$, then: $\lambda_1 u_1 + \lambda_2 u_2 = (\lambda_1 \pi_1 + \lambda_2 \pi_2)A^T + (\lambda_1 x_1 + \lambda_2 x_2)C$, $(\lambda_1 x_1 + \lambda_2 x_2)A \leq 0$, $\lambda_1 x_1 + \lambda_2 x_2 \in R^m$, $\lambda_1 \pi_1 + \lambda_2 \pi_2 \in R_+^n$, and $(\lambda_1 x_1 + \lambda_2 x_2)C(\lambda_1 x_1 + \lambda_2 x_2)^T - 1 \leq$

$$\leq (\lambda_1 x_1 + \lambda_2 x_2)C(\lambda_1 x_1 + \lambda_2 x_2)^T - \lambda_1 x_1 C x_1^T - \lambda_2 x_2 C x_2^T =$$

$$= - \lambda_1 \lambda_2 \left[x_1 C x_1^T - 2x_1 C x_2^T + x_2 C x_2^T \right] =$$

$$= - \lambda_1 \lambda_2 (x_1 - x_2)C(x_1 - x_2)^T \leq 0,$$ because C is positive-semi-definite.

q.e.d.

LEMMA 4

The set V is closed.

Proof: Let $\{w_k\}$ be a sequence with $w_k \epsilon R^m$, $k = 1, 2, \ldots$. We define the (pseudo) norm of w_k, denoted $\left|\{w_k\}\right|$, to be the smallest non-negative integer p such that there exists a k_o and for all $k \geq k_o$, x_k has at most p nonzero components. Now, suppose u is in the closure of V, i.e., there exist sequences $\{u_k\}$, $\{\pi_k\}$ and $\{x_k\}$ such that

(3)
$$\left.\begin{array}{l} \pi_k \epsilon R^n_+, \; x_k \epsilon R^m, \; u_k = \pi_k A^T + x_k C \\ x_k A \leq 0 \quad \text{and} \quad y_k C x_k^T \leq 1, \end{array}\right\} \quad k = 1, 2, \ldots$$

and $\{u_k\}$ converges to u.

Suppose the sequence $\{x_k\}$ is bounded, then we may assume, having taken an appropriate subsequence, that for some $x \epsilon R^m$, $\{x_k\} \to x$ and thus, by (3), $xA \leq 0$ and $xCx^T \leq 1$. Now, $yA \leq 0 \Longrightarrow u_k y^T - x_k C y^T = \pi_k A^T y^T = yA\pi_k^T \leq 0$, all $k \Longrightarrow uy^T - xCy^T \leq 0$. Thus the system,

$$y \epsilon R^m$$

$$yA \leq 0$$

$$(u - xC)y^T > 0$$

has no solution and by the usual feasibility theorem for linear inequalities (see e.q. (4) or (5)) the system:

$$\pi \in R_+^n$$

$$\pi A^T = u - xC$$

has a solution, and thus $u \in V$.

We have just demonstrated that if $\{x_k\}$ is bounded, then $u \in V$. Since $|\{x_k\}| + |\{x_k A\}| \leq m + n$, it is always possible to choose $\{x_k\}$ and $\{\pi_k\}$ satisfying (3) and such that $|\{x_k\}| + |\{x_k A\}|$ is minimal. We shall show next that if $\{x_k\}$, $\{\pi_k\}$ are so chosen, then $\{x_k\}$ is indeed bounded, thus completing the proof. Suppose then that $\{x_k\}$ is not bounded, i.e., $|x_k| = (x_k x_k^T)^{1/2} \to \infty$, and we may assume that $|x_k| > 0$ for all k. Let

$$z_k = \frac{x_k}{|x_k|}, \qquad k = 1, 2, \ldots$$

then $\{z_k\}$ is bounded and we may assume that there is a $z \in R^m$ such that the z_k converge to z and $|z| = 1$. From (3) it follows that $z_k A \leq 0$ and $z_k C z_k^T \leq \frac{1}{|x_k|}$ for all k. Thus, $zA \leq 0$ and $zCz^T \leq 0$. But then, from Lemma 1, $zCy^T = 0$ for all $y \in R^m$, and $zC = 0$. Summarizing:

(4) $$z \in R^m, \; zA \leq 0, \; zC = 0 .$$

Note that if z has a nonzero component, then infinitely many x_k's must have the same component nonzero, this follows from the fact that z is the limit of $\frac{x_k}{|x_k|}$. As a consequence, if $\{\lambda_k\}$ is any sequence of real numbers, then $|\{x_k + \lambda_k z\}| \leq |\{x_k\}|$. If $zA \neq 0$, and a^j, $j = 1, \ldots, n$, denotes the j^{th} column of A, let

$$\lambda_k = \max_{\substack{j \\ za^j < 0}} - \frac{x_k a^j}{za^j} .$$

Then we may replace, in (3), x_k by $x_k + \lambda_k z$ because $\lambda_k za^j + x_k a^j \leq 0$ for all j, and $(x_k + \lambda_k z)A \leq 0$, also $zC = 0$ and thus $(x_k + \lambda_k z)C = x_k C$, $(x_k + \lambda_k z)C(x_k + \lambda_k z)^T = x_k C x_k^T \leq 1$. However each $(x_k + \lambda_k z)A$ has at least one more zero component than $x_k A$, contradicting the minimality of $\left|\left\{x_k\right\}\right| + \left|\left\{x_k A\right\}\right|$. Thus, $zA = 0$ and we may replace, in (3), x_k by $x_k + \lambda_k z$ for an arbitrary sequence $\left\{\lambda_k\right\}$. But $z \neq 0$ and we can define λ_k so that $x_k + \lambda_k z$ has at least one more zero component than x_k has, thus $\left|\left\{x_k + \lambda_k z\right\}\right| < \left|\left\{x_k\right\}\right|$. However, $(x_k + \lambda_k z)A = x_k A$, and $\left|\left\{(x_k + \lambda_k z)A\right\}\right| = \left|\left\{x_k A\right\}\right|$, contradicting the minimality assumption.

q. e. d.

Lastly, we show:

LEMMA 5

$$U \subset V$$

Proof: Suppose $u \notin V$. By Lemmas 3 and 4 V is a closed convex set, hence there is a hyperplane which separates u strongly from V (see [4]). Thus there exist $x \in R^m$ and $\alpha \in R$ such that

$$ux^T > \alpha \geq vx^T \qquad \text{all} \quad v \in V .$$

Now, if $\pi \in R_+^n$ then $v = \pi A^T$ is in V (taking $x = 0$ in the definition of V). Thus $xA\pi^T = \pi A^T x^T \leq \alpha$ for all $\pi \in R_+^n$, and $xA \leq 0$, $x \in K$. Also $v = 0$ is in V, so that $\alpha \geq 0$. If $u \in U$ then $0 \leq \alpha < ux^T \leq (xCx^T)^{1/2}$, thus $xCx^T > 0$ and

$$v = \frac{xC}{(xCx^T)^{1/2}} \quad \epsilon \ V ,$$

consequently,

$$(xCx^T)^{1/2} > \alpha \geq \frac{xCx^T}{(xCx^T)^{1/2}} = (xCx^T)^{1/2}$$

a contradiction. Thus $u \notin U$. q.e.d.

Note: A direct application of Lemmas 2 and 5 yields the theorem stated at the beginning.

REFERENCES

1. Barankin, E. W. and R. Dorfman, On Quadratic Programming, Berkeley: University of California Press, 1958.

2. Dantzig, G. B., "Quadratic Programming: A Variant of the Wolfe-Markowitz Algorithms," Berkeley: Operations Research Center, University of California, Research Report 2, 14 April 1961.

3. Eisenberg, E., "Duality in Homogeneous Programming," to appear in Proceedings Amer. Math. Society.

4. Fenchel, W., Convex Sets, Cones, and Functions, Princeton: Princeton University Lecture Notes, 1953.

5. Gale, D., H. W. Kuhn and A. W. Tucker, "Linear Programming and the Theory of Games," Activity Analysis of Production and Allocation, New York: Cowles Commission Monograph 13, 1951. pp. 317-329.

6. Kuhn, H.W. and A. W. Tucker, "Non-Linear Programming," Proceedings of the Second Berkeley Symposium on Mathematical Statistics and Probability, Berkeley: University of California Press, 1951. pp. 481-492.

7. Markowitz, H., Portfolio Selection -- An Efficient Diversification of Investments, New York: John Wiley and Sons, 1959.

8. Wolfe, P., "A Duality Theorem for Non-Linear Programming," Santa Monica: The RAND Corporation, Report P-2028, 30 June 1960.