Reproduced by Armed Services Technical Information Agency

DOCUMENT SERVICE CENTER
KNOTT BUILDING, DAYTON, 2, OHIO

AD -

17504

UNCLASSIFIED
Technical Report No. 4
THE FINITE STURM-LIOUVILLE TRANSFORM

By
A. Cemal Eringen

to
Office of Naval Research
Department of the Navy

Contract N7onr-32909

Department of Mechanics
Illinois Institute of Technology
Technology Center
Chicago, Illinois

1 August 1963
Acknowledgment

The author is deeply indebted to Mr. Thomas King for checking the analysis.
**TABLE OF CONTENTS**

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>Acknowledgment</td>
<td>i</td>
</tr>
<tr>
<td>Table of Contents</td>
<td>ii</td>
</tr>
<tr>
<td>1. Introduction</td>
<td>1</td>
</tr>
<tr>
<td>2. The Sturm-Liouville Expansion</td>
<td>2</td>
</tr>
<tr>
<td>3. Finite Sturm-Liouville Transform</td>
<td>4</td>
</tr>
<tr>
<td>4. Transform of Lf</td>
<td>5</td>
</tr>
<tr>
<td>5. Solution of Some Partial Differential Equations</td>
<td>6</td>
</tr>
<tr>
<td>6. Solutions ( \phi ) and ( \chi ) of Some Special Equations</td>
<td>9</td>
</tr>
<tr>
<td>7. Application</td>
<td>11</td>
</tr>
<tr>
<td>References</td>
<td>17</td>
</tr>
</tbody>
</table>
The Finite Sturm-Liouville Transform

By

A. Cemal Eringen
Illinois Institute of Technology

1. Introduction

It is known that certain partial differential equations can be solved with the use of particular types of definite integrals having appropriate kernels. The choice of kernel depends on the type of boundary value problem. The solution obtained by these transforms is direct in the sense that it contains the boundary values in the solution. This, of course, is lacking in the classical approach. Finite Fourier transforms are of this type [1].

Recently Tranter [2], used a Legendre polynomial as a kernel. Scott [3] following a similar approach used a Jacobi polynomial as a kernel which extends the result of [2].

It is the purpose of the present paper to extend and unify all such special transforms. Thus we employ a kernel which may be determined to suit each particular type of problem. The transform can be employed to solve a wide class of linear second order partial differential equations.

In this paper we deal only with those transforms whose intervals are finite. Thus, the results obtained here are particularly useful for finite domains, or domains which are finite in one direction. An extension to infinite domains and singular cases will be made in a later paper.
2. The Sturm-Liouville Expansion

A second order, linear, homogeneous differential equation containing an arbitrary parameter \( \lambda \) has the general form:

\[
Mv = P_0v, y + P_1v, y + (P_2 + \lambda p_3)v = 0 \tag{1}
\]

\[
a_o \leq y \leq b_o
\]

where \( p_1 = p_1(y) \); \( v \) and \( y \) are the dependent and real independent variables, respectively. Indices after comma represent differentiation, i.e.: \( \phi, y = d\phi/dy \).

Equation (1) can be transformed into the canonical form:

\[
Lu = \lambda u, \quad L = q(x) - \frac{d^2}{dx^2}, \quad a \leq x \leq b \tag{2}
\]

where:

\[
\begin{align*}
\theta(x) &= \theta(x) v, \quad x = \int (p_3/p_o)^{1/2} dy \\
\phi(x) &= \frac{\theta(x)}{\theta(x)} - \phi(x), \quad \theta(x) = (p_3 k^2/p_o)^{1/4} \\
q(x) &= \frac{\theta(x)}{\theta(x)} - \phi(x) \\
k &= \exp\left(\frac{p_1}{p_o}\right) dy
\end{align*}
\]

(3)

It is simpler to work with (2).

We consider two solutions \( \phi(x, \lambda) \) and \( \chi(x, \lambda) \) of (2) satisfying the boundary conditions:

\[
\begin{align*}
\phi(a, \lambda) &= \sin \alpha, & \phi_x(a, \lambda) &= -\cos \alpha \\
\chi(b, \lambda) &= \sin \beta, & \chi_x(b, \lambda) &= -\cos \beta
\end{align*}
\]

Both solutions are unique if \( q(x) \) is real, continuous everywhere in \((a, b)\) and has finite limits for \( x = a \) and \( x = b \).
(see [4], p.6). We can build up \( \phi(x, \lambda) \) and \( \chi(x, \lambda) \) as a linear combination of two independent solutions \( \phi_0(x, \lambda) \) and \( \chi_0(x, \lambda) \) of (2). For let:

\[
\begin{align*}
\omega_0(\lambda) \phi(x, \lambda) &= \phi_0(x, \lambda) [\chi_0(a, \lambda) \cos \alpha + \chi_0, x(a, \lambda) \sin \alpha] \\
- \chi_0(\lambda)(x, \lambda)[\phi_0(a, \lambda) \cos \alpha + \phi_0, x(a, \lambda) \sin \alpha] \\
\omega_0(\lambda) \chi(x, \lambda) &= \phi_0(x, \lambda) [\chi_0(b, \lambda) \cos \beta + \chi_0, x(b, \lambda) \sin \beta] \\
- \chi_0(\lambda)(x, \lambda)[\phi_0(b, \lambda) \cos \beta + \phi_0, x(b, \lambda) \sin \beta] \\
\omega_0(\lambda) &= W(\phi_0, \chi_0) = \phi_0(x, \lambda) \chi_0, x(x, \lambda) - \phi_0, x(x, \lambda) \chi_0(x, \lambda)
\end{align*}
\]

Functions \( \phi(x, \lambda) \) and \( \chi(x, \lambda) \) defined by (5) satisfy (2) and (4). Here \( W(u, v) \) is the Wronskian, and can be shown to be independent of \( x \).

Let \( \lambda_n \) be a root of \( W(\phi, \chi) = 0 \). Then we have:

\[
\omega(\lambda_n) = W(\phi_n, \chi_n) = \phi_n \chi_n, x - \phi_n, x \chi_n = 0
\]

or

\[
\phi_n, x / \phi_n = \chi_n, x / \chi_n
\]

where:

\[
\phi_n = \phi(x, \lambda_n), \quad \text{and} \quad \chi_n = \chi(x, \lambda_n).
\]

Hence the integration gives:

\[
\chi(x, \lambda_n) = k_n \phi(x, \lambda_n)
\]

Consequently \( \phi(x, \lambda_n) \) and \( k_n \phi(x, \lambda_n) \) satisfy all of the boundary conditions (4).

It is also easy to show that \( \phi(x, \lambda_n) \) is an orthogonal set. For, in view of (2), (4) and (6) we have:
\[ (\lambda_m - \lambda_n) \int_a^b \phi_m \phi_n \, dx = \int_a^b (\phi_m L \phi_n - \phi_n L \phi_m) \, dx = W(\phi_m, \phi_n) = 0. \]

\[ (\lambda_m \neq \lambda_n) \]

Hence we can write:
\[
\int_a^b \phi(x, \lambda_n) \phi(x, \lambda_n) \, dx = N^2 \delta_{mn} \tag{7}
\]

where \( \delta_{mn} \) is the Kronecker delta (equal to zero for \( m \neq n \), one for \( m = n \))

It can be shown also that all roots of \( \omega(\lambda) = 0 \) are real and distinct (see [4], pp 11-12).

From (5) and (6) it follows that:
\[
k_n = \frac{\phi_o(b, \lambda_n) \cos \beta + \phi_{o,x}(b, \lambda_n) \sin \beta}{\phi_o(a, \lambda_n) \cos \alpha + \phi_{o,x}(a, \lambda_n) \sin \alpha}
\]
\[
+ \frac{\lambda_{o,b}(b, \lambda_n) \cos \beta + \lambda_{o,x}(b, \lambda_n) \sin \beta}{\lambda_{o,a}(a, \lambda_n) \cos \alpha + \lambda_{o,x}(a, \lambda_n) \sin \alpha} \tag{8}
\]

If now \( f(x) \) is an integrable function over \((a, b)\) and if \( a \neq b \), we have the Sturm-Liouville expansion of \( f(x) \):
\[
f(x) = \sum_{n=0}^{\infty} \left[ k_n / \omega(n, \lambda_n) \right] \phi(x, \lambda_n) \int_a^b \phi(y, \lambda_n) f(y) \, dy \tag{9}
\]


3. Finite Sturm-Liouville Transform

Let \( f(x) \) be a real continuous and integrable function of \( x \), in the interval \((a, b)\). We define the finite Sturm-Liouville transform \( S \{ f \} = F(\lambda_n) \) associated with system (2) and (4) by:
\[
F(\lambda_n) = S \{ f \} = \int_a^b f(y) \psi(y, \lambda_n) \, dy \tag{10}
\]
where:

\[ \psi(y, \lambda_n) = \left[ \frac{k_n}{\omega_n(\lambda_n)} \right]^{1/2} \phi(y, \lambda_n) \]  

(11)

The inversion theorem now follows from (9):

\[ f(x) = \sum_{n=0}^{\infty} \bar{f}(\lambda_n) \psi(x, \lambda_n) \]  

(12)

Here functions \( \psi(y, \lambda_n) \) are orthonormal, i.e.:

\[ \int_{a}^{b} \psi(x, \lambda_n) \psi(x, \lambda_m) \, dx = \delta_{mn} \]  

(13)

This is readily seen from (7) and (11).

Equations (10) and (12) are basic for the solution of some partial differential equations.

4. Transform of \( Lf \)

We are now going to prove that:

\[ S_x \{ Lf \} = S_x \{ q(x)f - \frac{\partial^2 f}{\partial x^2} \} + \lambda_n \bar{f}(\lambda_n, t) \]  

(14)

where:

\[ B_f(\lambda_n, t) = \left[ \frac{k_n}{\omega_n(\lambda_n)} \right]^{1/2} \left\{ -f(b, t)\frac{\cos \beta}{k_n} - f_x(b, t)\frac{\sin \beta}{k_n} + f(a, t)\cos \alpha + f_x(a, t)\sin \alpha \right\} \]  

(15)

is a function depending on the boundary values of \( f(x, t) \), \( f_x(x, t) \) at \( x = a, b \). Subscript \( f \) implies that the function \( B \) contains boundary values of the function \( f \). Here \( S_x \) represents the finite Sturm-Liouville transform taken with respect to \( x \):
To prove (14) we multiply \( Lf \) by \( \phi(x,t) \) and integrate between \((a,b)\), that is, by definition we have:

\[
S\{ Lf \} = \int_a^b [q(x)f(x,t) - \partial^2 f/\partial x^2]\psi(x,\lambda_n) \, dx
\]  

Integrating the second term in the integrand of (17) twice by part we obtain:

\[
S\{ Lf \} = -[f, x \psi - \psi, xf]\bigg|_a^b + \int_a^b f(x,t)L\psi(x,\lambda_n) \, dx
\]

If we now use (4) and (6) in the first term of the right hand side and (2) inside the integrand we obtain (14).

5. Solution of Some Partial Differential Equations

a) Let us consider a parabolic partial differential equation

\[
Mv + p_4 v, t = 0
\]  

where \( Mv \) is defined by (1), \( p_1 = p_1(y,t) \) are known and \( v(y,t) \) is the unknown dependent variable.

The use of transformation (3) reduces (19) to:

\[
u_{xx} + [\lambda - p(x,t)]u + (p_4/p_3)u, t = 0
\]

where:

\[
p(x,t) = q(x,t) - p_4 p_0 \left( k/2p_0 \right)^2 \frac{\partial}{\partial t}(p_0/p_3 \, k^2)
\]

Here \( q(x,t), k(t) \) and \( \lambda(t) \) are given by (3) with \( t \) as a parameter. If \( \psi(x,t,\lambda_n) \) and \( \varphi(x,t,\lambda_n) \) are now obtained as before except with \( q(x) \) replaced by \( p(x,t) \), \( t \) a parameter,
and \( p_4/p_3 \) a function of \( t \) alone, then (20), with the use of the finite Sturm-Liouville transform, can be reduced to an ordinary differential equation in \( t \). For, let \( p_4/p_3 = r(t) \), then application of the transform to (20) gives:

\[
-B_u(\lambda_n, t) + (\lambda - \lambda_n)\overline{u}(\lambda_n, t) + r(t)\overline{u}_t(\lambda_n, t) = 0 \tag{22}
\]

Equation (22) is of first order. The complete solution is:

\[
\overline{u}(\lambda_n, t) = C(\lambda_n)\exp\left(\int_{t_0}^{t} \frac{\lambda - \lambda_n}{x} dt\right) + \left[\exp\left(\int_{t_0}^{t} \frac{\lambda - \lambda_n}{x} dt\right)\right] \cdot 
\]

\[
\int_{t_0}^{t} \frac{B_u}{r} \left[\exp\left(-\int_{t_0}^{t} \frac{\lambda - \lambda_n}{x} dt\right)\right] dt \tag{23}
\]

Now, inversion theorem (12) gives:

\[
u(x, t) = \sum_{n=0}^{\infty} \overline{u}(\lambda_n, t)\psi_n(x, t, \lambda_n) \tag{24}\]

Therefore, the solution of (19) is effected. We must, however, remember that at least one of each pair of boundary values \( u(a, t), u_x(a, t) \) and \( u(b, t), u_x(b, t) \) must be known. By selecting \( a \) and \( \beta \) properly in \( B_u(\lambda_n, t) \) we can make the terms containing the unknown pair zero. Of course any knowledge of the boundary values of \( u \) leading to the complete evaluation of \( B_u \) is sufficient for the solution.

b) A second order linear partial differential equation containing a parameter \( \lambda \) has the general form:

\[
Nv + a_4v_{zz} + a_5v_{zt} + (a_6 + \lambda a_7)v = 0 
\]

\[
Nv = a_1v_{zz} + 2a_2v_{zt} + a_3v_{tt} \tag{25}
\]
where \( a_1 = a_1(z, \tau) \). If the equation is of elliptic type we have:

\[
a_1 a_3 - a_2^2 > 0
\]  

Consider now an elliptic equation (25). Let \( \xi(z, \tau) = \) \( \text{const.} \) and \( \eta(z, \tau) = \text{const.} \), respectively be the solutions of the following differential equations:

\[
\frac{d\xi}{dz} = \zeta_1(z, \tau) \quad \frac{d\eta}{dz} = \zeta_2(z, \tau)
\]

\[
\zeta_1 = \frac{a_2}{a_1} + \frac{a_3}{a_1} \left( 1 + \left( \frac{a_2}{a_1} \right)^2 \right)^{1/2}
\]

If we set:

\[
2y = \xi + \eta \quad 2i\tau = \xi - \eta
\]

equation (25) can be transformed into the canonical form [6]:

\[
v_{yy} + p_1 v_y + (p_2 + \lambda p_3)v + p_4 v_{\tau\tau} + v_{\tau\tau} = 0
\]

where:

\[
p_1 = \frac{2}{\beta} p_y, \quad p_2 = 2a_6/\beta, \quad p_3 = 2a_7/\beta
\]

\[
p_4 = \frac{2}{\beta} p_\tau, \quad \beta = a_1(y_z^2 + t_z^2) + 2a_2(y_{z\tau} + t_{z\tau}) + a_3(y_{\tau}^2 + t_{\tau}^2)
\]

Here operator \( P \) is defined by:

\[
P\phi = N\phi + a_4 \phi_z + a_5 \phi_{\tau}
\]

and \( N\phi \) is given by the second of (25).

Using the transformation (3) with \( p_o = 1 \), (29) becomes:

\[
u_{xx} + (\lambda - p)u + r_1 u_{\tau} + r_2 u_{\tau\tau} = 0
\]
where:

\[ p = q(x,t) - mp_4 \frac{g}{x^4} \frac{d}{dt} \quad r_0 = p_3^{-1} \]
\[ r_1 = mp_4 g + 2mg' \frac{d}{dt} \quad g = (p_3 k^2)^{-1/4} \]
\[ m = \left(\frac{k^2}{p_3}\right)^{3/4} \]  \hfill (33)

In the special case where \( r_1 \) and \( r_0 \) are functions of \( t \) alone, equation (32) can be transformed into an ordinary differential equation by applying the finite Sturm-Liouville transform. Hence:

\[ r_0 \frac{d^2 \bar{u}}{dt^2} + r_1 \frac{d \bar{u}}{dt} + (\lambda - \lambda_n) \bar{u} (\lambda_n, t) = B_u (\lambda_n, t) \]  \hfill (34)

We can now either use transformation (3) with respect to \( t \), with \( r_0, r_1 \) replacing \( p_0, p_1 \) and \( p_2 = 0, p_3 = 1 \) and find the solution of (34) and then invert it or we can solve (34) directly for \( \bar{u} (\lambda_n, t) \) and then (24) gives \( u(x,t) \).

It must be remembered that this method has the advantage that it contains the boundary values of the function within the solution, consequently, the difficulty of satisfying boundary conditions is eliminated. The method is also valid in the cases where the usual technique of separation of variables fail.

6. Solutions \( \phi_0 \) and \( \chi_0 \) of Some Special Equations

Below are given two independent solutions \( \phi_0 \) and \( \chi_0 \) of some special differential equations (D. E.) of mathematical physics. Their canonical forms (2) will be given by only determining \( u, q, \) and \( \lambda \).
a) **Bessel Functions**

\[ v_{yy} + \frac{1}{y} v_y + \left( s^2 - \frac{v^2}{y^2} \right) v = 0 \] (D. E.)

\[ u = x^{1/2} v, \quad y = x, \quad q = (v^2 - \frac{1}{4}) x^{-2}, \quad \lambda = s^2 \]

\[ \phi_o(x, \lambda) = x^{1/2} J_v(xs), \quad \chi_o(x, \lambda) = x^{1/2} Y_v(xs) \]

Interval (a, b) should not contain \( x = 0 \).

b) **Spherical Harmonics**

\[ [(1-v^2) v_{yy}] + [v(v+1) - \frac{2}{1-v^2}] v = 0 \] (D. E.)

\[ u = v \cos^{1/2} x, \quad y = \sin x, \quad q = \frac{\mu^2}{\cos^2 x} - \frac{1}{4} \tan^2 x - \frac{1}{2}, \]

\[ \lambda = v(v+1), \quad \phi_o(x, \lambda) = \cos^{1/2} x \cdot P_v^\mu(\sin x), \]

\[ \chi_o(x, \lambda) = \cos^{1/2} x \cdot Q_v^\mu(\sin x) \]

Interval (a, b) should not contain (-1, 1).

c) **Hermite Polynomials**

\[ v_{yy} - y v_y + n v = 0 \] (D. E.)

\[ u = e^{-x^2/4} v, \quad y = x, \quad q = -\frac{1}{2} + \frac{x^2}{4}, \quad \lambda = n, \]

\[ \phi_o(x, \lambda) = e^{-x^2/4} H_n(x), \quad \chi_o(x, \lambda) = e^{-x^2/4} H_n(x) \]

d) **Tschebyscheff Polynomials**

\[ (1-y^2) v_{yy} - y v_y + n^2 v = 0 \] (D. E.)

\[ u = v, \quad y = \sin x, \quad q = 0, \quad \lambda = n^2 \]

\[ \phi_o(x, \lambda) = T_n(\sin x) = \cos n(\sin^{-1} y), \]

\[ \chi_o(x, \lambda) = U_n(\sin x) = \sin n(\sin^{-1} y) \]
e) Mathieu Functions

\[ u_{xx} + (\lambda - 2n^2 \cos 2x)u = 0 \quad \text{(D. E.)} \]

\[ \phi_0(x, \lambda) = ce_{2n}(x), \ cse_{2n+1}(x), \ se_{2n}(x), \ s_{2n+1}(x) \]

\[ \chi_0(x, \lambda) = cse_{2n}(x), \ cse_{2n+1}(x), \ se_{2n}(x) \]

\[ s_{2n+1}(x) \]

\( (n = 0, 1, 2, \ldots \) except for \( se_{2n} \) only \( n = 1, 2, \ldots \) \)

\( \phi_0(x, \lambda) \) is periodic.

f) Wittaker Functions

\[ v_{yy} + \left( -\frac{1}{4} + \frac{k}{y} + \frac{\frac{4}{y^2} - \mu^2}{y^2} \right)v = 0 \quad \text{(D. E.)} \]

\[ u = (x/2)^{1/2}v, \quad y = x^2/4, \quad q = \frac{16\mu^2 - 17}{4x^2} + \frac{x^2}{16} \]

\[ \phi_0(x, \lambda) = (x/2)^{1/2} W_{k,\mu}(x^2/4), \quad \chi_0(x, \lambda) = (x/2)^{1/2} \]

\[ W_{-k,\mu}(-x^2/4) \quad (\neq) \]

Interval \((a, b)\) should not contain \( x = 0 \).

7. Application

As an illustration we solve a heat conduction problem which may be looked upon as a mathematical model of volcanos. The problem also has application in the exhaust parts of jet engines. As far as I know the problem has not previously been solved.

---

* Here comma does not mean differentiation.
Consider the conical shell enclosed by two coaxial cones having the same apex 0, and two concentric spheres having 0 as their center (Fig. 1).

In polar coordinates the inner and outer conical surfaces are given by $\theta = \theta_0$ and $\theta = \theta_1$, and the end surfaces by $r = r_0$ and $r = r_1$.

**Problem:** Determine the steady temperature distribution within the shell under the general axi-symmetric boundary conditions in temperature. This is the general Dirichlet problem for the domain under consideration. The differential equation and boundary conditions (B. C.) are given below:
(D. E.) \[ \Delta V = r^{-2}(r^2v_r)_r + r^{-2}[(1-y^2)v_y], y = 0 \]
\[ y = \cos\theta, \quad r_o < r < r_1, \quad \theta_o < \theta < \theta_1 \]  

(B. C.) \[ V = V_o(r) \quad \text{for } \theta = \theta_o \]
\[ V = V_1(r) \quad \text{for } \theta = \theta_1 \]  
\[ V = V_3(\theta) \quad \text{for } r = r_o \]
\[ V = V_4(\theta) \quad \text{for } r = r_1 \]  

Here \( V(r, y) \) is the temperature function, \( r \) and \( \theta \) are the polar coordinates.

**Solution:** We can exclude the second term of (D. E.) (35) if we use a finite Sturm-Liouville transform associated with the Legendre equation:

\[ [(1-y^2)v_y], y + v(v+1)v = 0 \]  

In view of (6, b) with \( \mu = 0 \) we find that if we select \( y = \sin x, \ u = v \cos^{1/2} x \) we can transform (37) to canonical form leading to solutions:

\[ \phi_o(x, \lambda) = \cos^{1/2} x \cdot P_v(\sin x), \quad \chi_o(x, \lambda) = \cos^{1/2} x \]
\[ Q_v(\sin x), \quad \lambda = v^{v+1} \]  

where \( P_v \) and \( Q_v \) are Legendre functions of the first and second kind, respectively. In view of (14), when we apply the transform to (35) the second term gives \( v_n(v_n+1) - B_v(x, \lambda_n) \). Now \( B_v \) contains four arbitrary functions which must be specified on the surfaces \( \theta = \theta_o \) and \( \theta = \theta_1 \). The terms containing the derivatives of \( v \) are not given. Thus if we
select \( a = \beta = 0 \) these terms drop out, leaving the terms containing \( v_0(r) \) and \( v_1(r) \), which are given. Hence \( \phi(x,\lambda) \) and \( \omega_0(\lambda) \) of (5) become:

\[
\phi(x,\lambda) = \cos^{1/2}x_0 \cos^{1/2}x [P_v(\sin x)Q_v(\sin x_0)
- P_v(\sin x_0)Q_v(\sin x)] , \quad x_0 = \frac{\pi}{2} - \theta_0 , \quad (39)
\]

\[
x_1 = \frac{\pi}{2} - \theta_1 , \quad \omega_0(\lambda) = 1
\]

By (11) and (8) we have:

\[
\Psi(x,\lambda_n) = \frac{k_n}{\omega_0(\lambda_n)^{1/2}} \phi(x,\lambda_n)
\]

\[
k_n = \frac{\cos^{1/2}x_1 P_v(\sin x_1)}{\cos^{1/2}x_0 P_v(\sin x_0)} \quad (40)
\]

Calculation of \( \omega(\lambda) = \phi \chi_x - \chi \phi_x \) gives

\[
\omega(\lambda) = \cos^{1/2}x_0 \cos^{1/2}x_1 [P_v(\sin x_0)Q_v(\sin x_1)
- P_v(\sin x_1)Q_v(\sin x_0)] \quad (41)
\]

Consequently, the roots \( \lambda_n = \nu_n(\nu_n + 1) \) of \( \omega(\lambda) = 0 \) satisfy the following equation

\[
\frac{P_v(\sin x_1)}{P_v(\sin x_0)} = \frac{Q_v(\sin x_1)}{Q_v(\sin x_0)} \quad (42)
\]

Therefore \( \Psi(x,\lambda_n) \) is completely determined.

After transforming (35) with \( y = \sin x \) and \( u = v \cos^{1/2}x \)
we apply the finite Sturm-Liouville transform. That is, we multiply the equation by \( \Psi(x,\lambda_n) \) and integrate between \( x_0 \) and \( x_1 \). The result is:
\( (r^2 \ddot{V}, r) - v_n(v_n + 1) \ddot{V} = B_V(r, \lambda_n) \)

\[
B_V(r, \lambda_n) = [k_n \lambda_n, \lambda_n]^{1/2} [V_o(r) - k_n^{-1} v_1(r)]
\]

Equation (43) is an ordinary differential equation of Euler type whose solution can be found by variation of parameters. Hence:

\[
\ddot{V}(r, v_n) = C_1(v_n) r^n + C_2(v_n) r^{-v_n - 1} + F(r, v_n)
\]

\[
P(r, v_n) = \int^1_0 \frac{B_V(\rho, \lambda_n)}{1 + 2v_n} (r^n \rho^{-v_n - 1} - r^{-v_n - 1} \rho^v) d\rho
\]

Let the transforms of \( V_3(\theta) \) and \( V_4(\theta) \) be \( \tilde{V}_3(v_n) \) and \( \tilde{V}_4(v_n) \). \( C_1 \) and \( C_2 \) will then be determined from the remaining conditions:

\[
\ddot{V} = \tilde{V}_3(v_n), \text{ for } r = r_0, \quad \theta_0 < \theta < \theta_1
\]

\[
\ddot{V} = \tilde{V}_4(v_n), \text{ for } r = r_1
\]

This gives two linear equations for \( C_1 \) and \( C_2 \) whose solutions are:

\[
C_1(v_n) = \frac{[\tilde{V}_3(v_n) - F(r_0, v_n)] r_1^{-v_n - 1} - [\tilde{V}_4(v_n) - F(r_1, v_n)] r_0^{-v_n - 1}}{r_0^{-v_n - 1} - r_1^{-v_n - 1} - r_1 v_n - r_0^{-v_n - 1}}
\]

\[
C_2(v_n) = \frac{[\tilde{V}_4(v_n) - F(r_1, v_n)] r_0^{-v_n - 1} - [\tilde{V}_3(v_n) - F(r_0, v_n)] r_1^{-v_n - 1}}{r_0^{-v_n - 1} - r_1^{-v_n - 1} - r_1 v_n - r_0^{-v_n - 1}}
\]

Hence, \( \tilde{V}(r, v_n) \) is completely determined. The inversion theorem (12) now gives:

\[
V(r, x) = \sum_n \tilde{V}(r, v_n) \Psi(x, \lambda_n)
\]

where the summation is extended over all roots of (42).
The analysis given above is formal. It can, however, be made rigorous by showing that the solution satisfies both the differential equation and the boundary conditions.

It may be worth while to remark that with the use of the method of separation of variables the solution of the above problem would have been difficult and lengthy, if not impossible.
References


Contributing Personnel:

Dr. A. C. Eringen, Research Associate Professor of Mechanics

Mr. T. King, Staff Computer

Respectfully submitted,

A. C. Eringen

A. C. Eringen

William R. Osgood
Chairman, Department of Mechanics
## Distribution List

### I: Administrative, Reference and Liaison Activities of ONR

<table>
<thead>
<tr>
<th>Role</th>
<th>Details</th>
</tr>
</thead>
</table>
| Chief of Naval Research | Department of the Navy  
Washington 25, D. C. |
Attn: Code 438 (2)  
Code 432 (1)  
Code 466(via Code 108) (1) |
| Director, Naval Research Lab. |  
Washington 25, D. C. |
Attn: Tech. Info., Officer (9)  
Technical Library (1)  
Mechanics Division (2) |

### II: Department of Defense and other interested Government Activities

#### A) GENERAL

<table>
<thead>
<tr>
<th>Role</th>
<th>Details</th>
</tr>
</thead>
</table>
| Research and Development Board  
Department of Defense  
Pentagon Building  
Washington 25, D. C. |  
Attn: Library(Code 3D-1075) (1) |
| Armed Forces Special Weapons Project  
P.O. Box 2610  
Washington, D. C. |  
Attn: Col. G.F. Blunda (2) |
| Joint Task Force 3  
12 St. and Const. Ave., N.W. (Temp. U)  
Washington 25, D. C. |  
Attn: Major B. D. Jones (1) |

#### B) ARMY

<table>
<thead>
<tr>
<th>Role</th>
<th>Details</th>
</tr>
</thead>
</table>
| Chief of Staff  
Department of the Army  
Research and Development Div.  
Washington 25, D. C. |  
Attn: Chief of Res. and Dev. (1) |
| Office of the Chief of Engineers  
Assistant Chief for Works  
Department of the Army  
Bldg. T-7, Gravelly Point  
Washington 25, D. C. |  
Attn: Structural Branch (R. L. Bloor) (1) |
| Engineering Research and Development Laboratory  
Fort Belvoir, Virginia |  
Attn: Structures Branch (1) |
Distribution List (Cont.)

Office of the Chief of Engineers
Asst. Chief for Military
Construction
Department of the Army
Bldg. T-3, Gravelly Point
Washington 25, D. C.
Attn: Structures Branch
(M. F. Carey) (1)
Protective Construction Branch (I.O. Thorley) (1)

Office of the Chief of Engineers
Asst. Chief for Military
Operations
Department of the Army
Bldg T-7, Gravelly Point
Washington 25, D. C.
Attn: Structures Development Branch (W.F. Woollard) (1)

U. S. Army Waterways Experiment Station
P. O. Box 631
Halls Ferry Road
Vicksburg, Mississippi
Attn: Col. H. J. Skidmore (1)

The Commanding General
Sandia Base, P.O.Box 5100
Alburquerque, New Mexico
Attn: Col. Canterbury (1)

Operations Research Officer
Department of the Army
Pt. Lesley J. McNair
Washington 25, D. C.
Attn: Howard Brackney (1)

Office of Chief of Ordnance
Office of Ordnance Research
Department of the Army
The Pentagon Annex No. 2
Washington 25, D. C.
Attn: ORDIB-PS (1)

Ballistics Research Laboratory
Aberdeen Proving Ground
Aberdeen, Maryland
Attn: Dr. C. W. Lampson (1)

(C) NAVY

Chief of Naval Operations
Department of the Navy
Washington 25, D. C.
Attn: OP-31 (1)
OP-363 (1)

Chief of Bureau of Ships
Department of the Navy
Washington 25, D. C.
Attn: Director of Research (2)
Code 423 (1)
Code 442 (1)
Code 421 (1)

Director, David Taylor Model Basin
Department of the Navy
Washington 7, D. C.
Attn: Code 720, Structures Division
Code 740, Hi-Speed Dynamics Div. (1)

Commanding Officer
Underwater Explosion Research Div.
Code 290
Norfolk Naval Shipyard
Portsmouth, Virginia (1)

Commander
Portsmouth Naval Shipyard
Portsmouth, N. H.
Attn: Design Division (1)

Director, Materials Laboratory
New York Naval Shipyard
Brooklyn 1, New York (1)

Chief of Bureau of Ordnance
Department of the Navy
Washington 25, D. C.
Attn: Ad-3, Technical Library (1)
Roc, P.H. Girouard (1)

Naval Ordnance Laboratory
White Oak, Maryland
RFD 1, Silver Spring, Maryland
Attn: Mechanics Division (1)
Explosive Division (1)
Mech. Evaluation Div. (1)
**Distribution List (Cont.)**

(C) **NAVY**
Commander
U.S. Naval Ordnance Test Station
Inyokern, California
Post Office - China Lake, Calif.
Attn: Scientific Officer

Naval Ordnance Test Station
Underwater Ordnance Division
Pasadena, California
Attn: Structures Division

Chief of the Bureau of Aeronautics
Department of the Navy
Washington 25, D. C.
Attn: TD-41, Technical Library

Chief of Bureau of Ships
Department of the Navy
Washington 25, D. C.
Attn: Code P-314
Code C-313

Officer in Charge
Naval Civil Engr. Research and Evaluation Laboratory
Naval Station
Port Hueneme, California

Supervising Officer
U.S. Naval Postgraduate School
Annapolis, Maryland

**Supplementary Distribution List**

<table>
<thead>
<tr>
<th>Name</th>
<th>Address</th>
</tr>
</thead>
<tbody>
<tr>
<td>Professor Lynn Beedle</td>
<td>Fritz Engineering Laboratory, Lehigh University, Bethlehem, Pennsylvania</td>
</tr>
<tr>
<td>Professor R. L. Bisplinghoff</td>
<td>Dept. of Aeronautical Engineering, Massachusetts Institute of Tech., Cambridge 39, Massachusetts</td>
</tr>
<tr>
<td>Professor G. F. Carrier</td>
<td>Graduate Division of Applied Mathematics, Brown University, Providence, R. I.</td>
</tr>
<tr>
<td>Professor R. J. Dolan</td>
<td>Dept. of Theoretical and Applied Mechanics, University Of Illinois, Urbana, Illinois</td>
</tr>
</tbody>
</table>

(D) **AIR FORCES**
Commanding General
U.S. Air Forces
The Pentagon
Washington 25, D. C.
Attn: Res. and Develop. Div.

Deputy Chief of Staff, Operations
Air Targets Division
Headquarters, U.S. Air Forces
Washington 25, D. C.
Attn: AFOIN-T/PV

Flight Research Laboratory
Wright-Patterson Air Force Base
Dayton, Ohio
Attn: Chief, Applied Mechanics Group

(E) **OTHER GOVERNMENT AGENCIES**
U.S. Atomic Energy Commission
Division of Research
Washington, D. C.

Director, National Bureau of Standards
Washington, D. C.
Attn: Dr. W. H. Ramberg

Forest Products Laboratory
Madison 5, Wisconsin
Distribution List (Cont.)

Professor Hans Bleich
Dept. of Civil Engineering
Columbia University
Broadway at 117th St.
New York 27, New York (1)

Professor L. S. Jacobsen
Department of Mechanical Eng.
Stanford University
Stanford, California (1)

Professor B. A. Boley
Dept. of Aeronautical Eng.
Ohio State University
Columbus, Ohio (1)

Dr. W. H. Hoppmann
Department of Applied Mechanics
Johns Hopkins University
Baltimore, Maryland (1)

Professor Lloyd Donnell
Department of Mechanics
Illinois Institute of Tech.
Technology Center
Chicago 16, Illinois (1)

Professor George Lee
Department of Mathematics
Renssalaer Polytechnic Institute
Troy, New York (1)

Professor B. Fried
Dept. of Mechanical Engineering
Washington State College
Pullman, Washington (1)

Professor Glen Murphy, Head
Department of Theoretical and
Applied Mechanics
Iowa State College
Ames, Iowa (1)

Mr. Martin Goland
Midwest Research Institute
4049 Pennsylvania Avenue
Kansas City 2, Missouri (1)

Professor N. M. Newmark
Department of Civil Engineering
University of Illinois
Urbana, Illinois (1)

Dr. J. N. Goodier
School of Engineering
Stanford University
Stanford, California (1)

Professor Jesse Ormondroyd
University of Michigan
Ann Arbor, Michigan (1)

Professor M. Hetenyi
Walter P. Murphy Professor
Northwestern University
Evanston, Illinois (1)

Dr. R. P. Peterson, Director
Applied Physics Division
Sandia Laboratory
Albuquerque, New Mexico (1)

Dr. N. J. Hoff, Head
Department of Aeronautical Engineering and Applied Mechanics
Polytechnic Institute of Brooklyn
99 Livingston Street
Brooklyn 2, New York (1)

Dr. A. Phillips
School of Engineering
Stanford University
Stanford, California (1)

Dr. J. H. Hollomon
General Electric Research Laboratories
1 River Road
Schenectady, New York (1)

Dr. W. Prager, Chairman
Graduate Division of Applied Mathematics
Brown University
Providence 12, R. I. (1)
Distribution List (Cont.)

Dr. S. Raynor
Armour Research Foundation
Illinois Institute of Technology
Chicago 16, Illinois (1)

Professor E. Reissner
Department of Mathematics
Massachusetts Institute of Tech.
Cambridge 39, Massachusetts (1)

Professor M. A. Sadowsky
Illinois Institute of Technology
Technology Center
Chicago 16, Illinois (1)

Professor Paul Lieber
Department of Aeronautical Eng.
Renssalaer Polytechnic Institute
Troy, New York (1)

Professor J. E. Stallmeyer
Talbot Laboratory
Department of Civil Engineering
University of Illinois
Urbana, Illinois (1)

Dr. C. B. Smith
College of Arts and Sciences
Department of Mathematics
Walker Hall
University of Florida
Gainesville, Florida (1)

Professor J. R. Andersen
Towne School of Engineering
University of Pennsylvania
Philadelphia, Pennsylvania (1)

Commander
U.S. Naval Proving Ground
Dahlgren, Virginia (1)

Professor E. Sternberg
Illinois Institute of Technology
Technology Center
Chicago 16, Illinois (1)

Professor F. K. Teichmann
Department of Aeronautical Engineering
New York University
University Heights, Bronx
New York, N. Y. (1)

Professor C. T. Wang
Department of Aeronautical Engineering
New York University
University Heights, Bronx
New York, N. Y. (1)

Project File (2)

Project Staff (5)

For possible future distribution by the University (10)