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# A DYNAMIC PROCEDURE FOR CALCULATING THE TURBULENT KINETIC ENERGY

B. KNAEPEN, O. DEBLIQUY AND D. CARATI

*Université Libre de Bruxelles, Boulevard du Triomphe, Campus  
Plaine - CP.231, 1050 Brussels, Belgium*

**Abstract.** We propose a dynamic model based on the Germano identities to evaluate the subgrid scale energy  $\langle \bar{k} \rangle$  in LES as a function of the large scale velocity field only. Contrary to traditional transport equation for  $\bar{k}$ , this model does not require any additional equation and provide a very simple first approximation for  $\bar{k}$ .

## 1. Introduction

The aim of LES is to make predictions about turbulent flows which are not accessible by DNS. Therefore, it is important to establish correspondence rules between the physical quantities predicted by LES and their actual measured values. These correspondence rules are also useful when assessing the performance of LES subgrid models through the comparison with resolved DNS, although in that context it may be possible to filter the DNS fields down to the LES scales to produce the desired comparison.

A detailed discussion of how to establish these correspondence rules can be found in (Winckelmans *et al*, 2001). Here we focus our attention to one of most fundamental one: the relation between the total energy (density) of the turbulent fluid and the resolved LES energy density. It can be written as,

$$E = E_R + E_{sgs}, \quad (1)$$

where  $E$  denotes the total energy,  $E_R$  the resolved LES energy and  $E_{sgs}$  is the subgrid-scale energy. This last quantity is traditionally not available in LES and thus needs to be reconstructed to evaluate the total energy density from the LES energy density. As shown in (Winckelmans *et al*, 2001),  $E_{sgs}$  is also required to reconstruct the full Reynolds stress tensor.

In order to calculate  $E_{sgs}$ , a transport equation can be introduced. Although quite effective (Debligny *et al*, 2001), this method has however two side effects. First, to close the transport equation in terms of filtered quantities, one needs further modelling efforts. Second, the resolution of the extra equation increases the computation requirements.

In this paper, we propose a different approach and show how an estimate of the subgrid scale energy  $E_{sgs}$  can be obtained using the Germano identities and a model for the energy spectrum. We also produce some numerical results to illustrate the method.

## 2. Modelling the turbulent kinetic energy

For an incompressible, Boussinesq flow, the Navier-Stokes equations for the LES field  $\bar{u}_i$  read,

$$\partial_t \bar{u}_i + \partial_j (\bar{u}_j \bar{u}_i) = -\partial_i \bar{p} + \nu \Delta \bar{u}_i - \partial_j \bar{\tau}_{ij}. \quad (2)$$

One of the essential difficulty of LES consist in modelling the unknown subgrid-scale stress tensor  $\bar{\tau}_{ij} \equiv \overline{u_i u_j} - \bar{u}_i \bar{u}_j$  which appears in the filtered Navier-Stokes equations. Note that since our numerical code is fully dealiased we have, for consistency, expressed (2) only in terms of filtered quantities, including  $\bar{\tau}_{ij}$ .

Like any second-order symmetric tensor,  $\bar{\tau}_{ij}$  may be decomposed into an isotropic part and a trace-free part:

$$\bar{\tau}_{ij} = \frac{2}{3} \bar{k} \delta_{ij} + \bar{\tau}_{ij}^*, \quad (3)$$

where  $\bar{\tau}_{ij}^* = \bar{\tau}_{ij} - \frac{1}{3} \bar{\tau}_{rr} \delta_{ij}$  and

$$\bar{k} = \frac{1}{2} \bar{\tau}_{rr} \quad (4)$$

$$= \frac{1}{2} (\overline{u_i u_i} - \bar{u}_i \bar{u}_i). \quad (5)$$

Traditionally,  $\bar{k}$  is known in the literature as the turbulent kinetic energy. However, this name might be misleading and one must be careful about the interpretation of  $\bar{k}$ . To be more precise, let us denote by  $u_i$  the non-filtered velocity and by  $u'_i$  the subgrid scale part of  $u_i$ . We then have  $u_i = \bar{u}_i + u'_i$  and we can define three *local* energy densities: 1)  $e = \frac{1}{2} u_i u_i$ ; 2)  $e_R = \frac{1}{2} \bar{u}_i \bar{u}_i$ ; 3)  $e_{sgs} = \frac{1}{2} u'_i u'_i$ . At this stage, it might be tempting to identify the turbulent kinetic energy  $\bar{k}$  with  $\bar{e}_{sgs}$ . However, only their space average are identical. Indeed, locally we have:

$$\bar{k} = \bar{e}_{sgs} + \overline{\bar{u}_i u'_i}. \quad (6)$$

Therefore,  $\bar{k}$  does not represent the *local* contribution of the small scales to the energy budget. To add to this, we also recall that  $\bar{k}$  is in general not even positive definite for arbitrary grid filters (e.g. projectors).

Usually, modelling  $\bar{k}$  is not an issue in the case of incompressible LES since one can define a modified pressure,  $\bar{p}' = \bar{p} + \frac{2}{3}\bar{k}$  which is obtained by enforcing the continuity condition  $\partial_i \bar{u}_i = 0$ . However, this does not apply to the compressible case and as we exposed in the introduction, the knowledge of  $\langle \bar{k} \rangle$  might be very useful for real flow predictions and comparison with DNS.

To proceed, we follow the steps of (Germano *et al*, 1991) and introduce a second coarser filter called the test-filter which we denote by  $\cdot\hat{\cdot}$ . In the sequel all the filters considered will be sharp Fourier cut-offs. Therefore, the "grid+test" filter, denoted  $\widetilde{\cdot\hat{\cdot}}$  is equivalent to the test-filter:  $\widetilde{\cdot\hat{\cdot}} = \hat{\cdot\hat{\cdot}} = \hat{\cdot\hat{\cdot}}$ . The filtered velocity  $\tilde{u}_i$  then satisfies the following equation:

$$\partial_i \tilde{u}_i + \partial_j (\widetilde{\tilde{u}_i \tilde{u}_j}) = -\partial_i \tilde{p} + \nu \Delta \tilde{u}_i - \partial_j \tilde{T}_{ij}, \tag{7}$$

where  $\tilde{T}_{ij} \equiv \widetilde{u_i u_j} - \tilde{u}_i \tilde{u}_j$  is the subgrid-scale stress tensor at the combined grid+test filter level. The Germano identity states that  $\tilde{T}_{ij}$  and  $\tilde{\tau}_{ij}$  are related by,

$$\tilde{L}_{ij} = \tilde{T}_{ij} - \tilde{\tau}_{ij}, \tag{8}$$

where,

$$\tilde{L}_{ij} = \widetilde{\tilde{u}_i \tilde{u}_j} - \tilde{u}_i \tilde{u}_j, \tag{9}$$

is the Leonard tensor.

For any quantity we have the following property,

$$\langle F \rangle = \langle \bar{F} \rangle = \langle \tilde{F} \rangle, \tag{10}$$

where the bracket  $\langle \dots \rangle$  denotes space average:  $\langle F \rangle = \frac{1}{V} \int d^3x F(x)$ . Indeed, for any filter  $G(y)$  ( $\cdot\hat{\cdot}$ ) we have  $\int d^3y G(y) = 1$  so that:

$$\langle \bar{F} \rangle = \frac{1}{V} \int d^3x \bar{F}(x) \tag{11}$$

$$= \frac{1}{V} \int d^3x \int d^3y G(x-y) F(y) \tag{12}$$

$$= \frac{1}{V} \int d^3y F(y) \int d^3z G(z) \tag{13}$$

$$= \frac{1}{V} \int d^3x F(x) = \langle F \rangle \tag{14}$$

where we have used the change of variable:  $z = x - y$ .

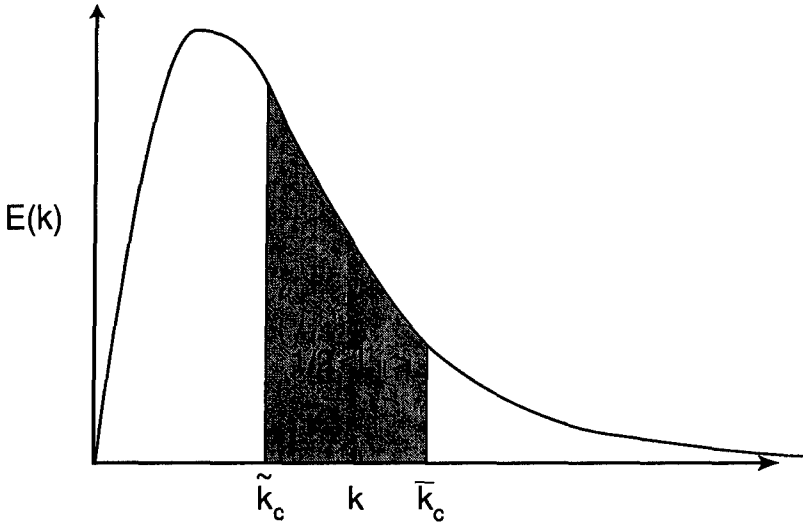


Figure 1. Space average of Leonard tensor.  $\bar{k}_c$  and  $\tilde{k}_c$  denote respectively the cut-offs at grid and grid+test levels.

Using property (10) and the trace of equation (9) we get the relation,

$$\langle \tilde{L}_{ii} \rangle = \langle \bar{u}_i \bar{u}_i \rangle - \langle \tilde{u}_i \tilde{u}_i \rangle, \quad (15)$$

which states that the space average of the Leonard tensor is equal to twice the difference of the energy at grid level and grid+test level. Another way of getting this relation is by taking the trace of (8). Denoting by  $\tilde{K}$  the turbulent kinetic energy density at grid+test level, we obtain,  $\langle \tilde{L}_{ii} \rangle = 2(\langle \tilde{K} \rangle - \langle \bar{k} \rangle)$ , which has of course the same interpretation. Figure 1 illustrates the situation.

The above considerations allow us to write,

$$\langle L_{ii} \rangle = 2 \int_{\tilde{k}_c}^{\bar{k}_c} E(k) dk, \quad (16)$$

where  $\bar{k}_c$  and  $\tilde{k}_c$  denote respectively the cut-offs at grid and grid+test levels. The energy spectra  $E(k)$  is defined so that,  $E = \int_0^\infty E(k) dk$ .

The next step in the analysis is to introduce a model for the energy spectra. To begin with, let us suppose that  $\bar{k}_c$  and  $\tilde{k}_c$  lie in the inertial range which is assumed to be represented by  $E(k) = C_K \epsilon^{2/3} k^{-5/3}$  where  $C_K$  is the Kolmogorov constant and  $\epsilon$  is the global dissipation which is usually not known in LES simulations. We propose here to estimate  $C_K \epsilon^{2/3}$

by substituting  $E(k)$  in (16). After straightforward algebra one gets:

$$C_K \epsilon^{\frac{2}{3}} = \frac{\langle \tilde{L}_{ii} \rangle}{3(\tilde{k}_c^{-\frac{2}{3}} - \bar{k}_c^{-\frac{2}{3}})}. \tag{17}$$

If we now extend the inertial range to infinity (which is the main approximation so far) we finally get a dynamic expression for the mean turbulent kinetic energy:

$$\langle \bar{k} \rangle = \int_{\bar{k}}^{\infty} E(k) dk \tag{18}$$

$$= \frac{\langle \tilde{L}_{ii} \rangle}{2(\frac{\bar{k}_c}{k_c})^{\frac{2}{3}} - 1} \tag{19}$$

$$= \frac{\langle \tilde{L}_{ii} \rangle}{2(\frac{\bar{\Delta}}{\Delta})^{\frac{2}{3}} - 1}, \tag{20}$$

where  $\bar{\Delta}$  and  $\tilde{\Delta}$  denote respectively the width of the grid and grid+test filters. The last equation is written to stress that the model is also suited to the physical formulation approach. Indeed, we only used spectral considerations to establish the model but the final result does not require the spectral formulation. We stress that the expressions (19) and (20) are indeed dynamic since they can be evaluated during the simulation. Usually they do not represent any further computational effort since  $\tilde{L}_{ii}$  is often already required by dynamic eddy-viscosity models.

In the next section we present some results based on our estimate of  $\langle \bar{k} \rangle$ .

### 3. Numerical results

To test model (19), we have build a  $256^3$  DNS database of isotropic decaying turbulence. The initial condition is build according to Rogallo's prescription (Rogallo, 1981) using the spectra of the Comte-Bellot-Corrsin (Comte-Bellot and Corrsin, 1971) experiment at stage 2. The initial random phases of the velocity fields are correlated by time-stepping 100 times the flow and maintaining the spectra constant.

The initial condition of the DNS has then been filtered down to  $32^3$  modes and two kind of LES have been performed. The first one is denoted LES/KOL. It is based on an eddy-viscosity model with Kolmogorov scaling:  $\bar{\tau}_{ij}^* = -2C\epsilon^{\frac{1}{3}}\bar{\Delta}^{\frac{4}{3}}\bar{S}_{ij}$ , where  $\bar{S}_{ij} = \frac{1}{2}(\partial_i\bar{u}_j + \partial_j\bar{u}_i)$ ; the constant  $C$  is evaluated using the dynamic procedure. The second LES run is denoted LES/Transport. It is also based an eddy-viscosity model but this time, the

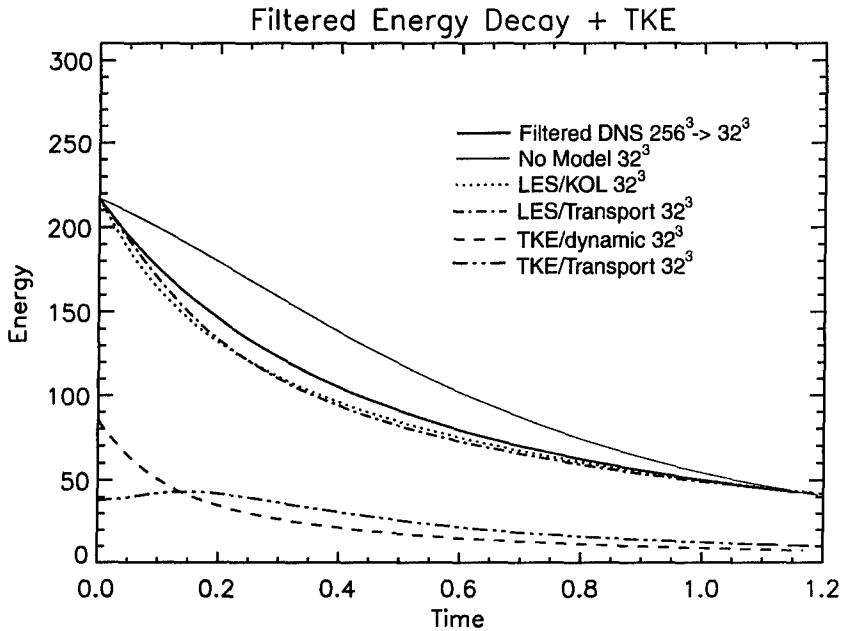


Figure 2. Filtered energy and average turbulent kinetic energy decays.

viscosity is scaled with the turbulent kinetic energy:  $\bar{\tau}_{ij}^* = -2C\bar{k}_+^{\frac{1}{2}}\bar{\Delta}\bar{S}_{ij}$ , where the constant  $C$  is also evaluated using the dynamic procedure and  $k_+ = k$  for  $k > 0$  and  $k = 0$  otherwise. The essential difference is that here  $\bar{k}$  is not evaluated using the diagnostic (19) but is simulated using the transport equation due to Speziale (Speziale, 1991); the details of this LES simulation can be found in (Debliquy *et al*, 2001). For completeness, we have also performed a  $32^3$  unresolved DNS to emphasize the roles of the LES models.

In Figure 2 we plot 1) the energy decay of the  $32^3$  modes resolved by the LES ( $E_R = \langle \epsilon_R \rangle$  in the terminology described above) for the different models used; 2) the subgrid-scale energies predicted by model (19) (dynamic TKE) and by the turbulent kinetic transport equation (transport TKE). From the graph we observe a good agreement between the filtered DNS and the two LES which produce very similar results for the resolved energy. The importance of the subgrid-scale models is stressed by the "No Model curve". The average turbulent kinetic energies predicted by the two LES models are quite different initially but tend to get closer to each other later.

In Figure 3 we plot the decay of the total energy. For the two LES, the curves correspond to the sum of the resolved energies and the average

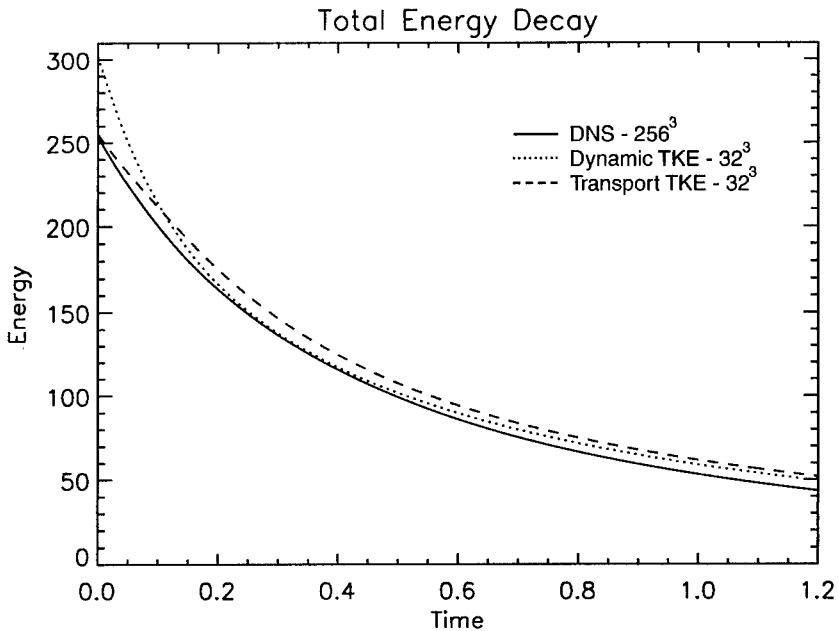


Figure 3. Reconstruction of total DNS energy.

turbulent kinetic energies presented in Figure 2. Initially, model (19) tend to overestimate quite seriously the subgrid scale energy. Later, the behavior is much more satisfactory and even slightly better than than the one of the LES/Transport model. The initial precise match between the DNS and the LES/Transport model is of course natural since the initial condition for  $\bar{k}$  is in that case computed over the DNS. The test for model (19) is thus much more severe. The initial overestimate of  $\bar{k}$  by model (19) may be due to two causes. The first one is the choice of the model for the energy spectrum. Indeed, the resolution of our DNS does not allow a clear inertial range and extending the later to infinity may be a crude oversimplification. The second source of error might come from the dynamic procedure itself. It has been observed already that this procedure takes some time to settle and predict appropriate values. This is probably due to the fact that a filtered DNS field is badly correlated as an initial condition for an LES. Bearing these in mind, the prediction of model (19) are nevertheless satisfactory and show that the expression for  $\langle \bar{k} \rangle$  is a useful first approximation when no DNS is available, and if we want to avoid an additional transport equation for  $\bar{k}$ .



#### 4. Summary and conclusion

In this article we have derived a dynamic expression to estimate the subgrid scale energy  $E_{sgs}$  of a turbulent flow from the resolved LES scales. The knowledge of  $E_{sgs}$  can then be used to reconstruct the complete Reynolds stress and in particular the total energy from the LES field.

We have presented numerical result which indicate that the estimate we propose for  $E_{sgs}$  is satisfactory and rivals the performance of a direct evaluation obtained from a transport equation.

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