A CLASS OF GAMES WITH UNIQUE SOLUTIONS

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# A Class of Games with Unique Solutions

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**Abstract:**

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SUMMARY

In a game with payoff \( M(x,y) = \phi(xy) + \rho(x) + \tau(y) \) played over the unit square (such that \( \rho, \tau \) are continuous and \( \phi \) is analytic and with sufficiently many non-vanishing coefficients in its power series expansion about zero) if either player has a non-step function\(^1\) optimal strategy, the opposing player has a unique optimal strategy. Examples are included which illustrate the fact that games with well-behaved payoffs can have unique solutions\(^2\) which are more or less pathological.

\[ \S 1.\] For any distribution \( f \) (which we may consider as a measure) we may define the spectrum of \( f, \sigma(f) \), as the complement of all open sets of \( f \)-measure zero. The set \( \sigma(f) \) is a closed Borel set, since we may obtain \( \sigma(f) \) by deleting the intervals of \( f \)-measure zero which have rational end points. If one is given a constant, \( v \), strategies \( f \) and \( g \), and functions \( \phi, H, K \), such that

1. \( \phi \) is continuous on the unit square
2. \( H(x) \leq v \), and \( H(x) = v \) on \( \sigma(f) \), \( H \) continuous
3. \( K(y) \geq v \), and \( K(y) = v \) on \( \sigma(g) \), \( K \) continuous

then by setting

\[ M(x,y) = \phi(x,y) - \int \phi df(x) - \int \phi dg(y) + \int \int \phi df(x)dg(y) - v + H(x) + K(y), \]

one obtains the payoff \( M \) of a game which has value \( v \), and \( (f,g) \) as a solution.

\(^1\) By a step function we mean a distribution based on a finite set of points.
\(^2\) By a solution we mean a pair \((f,g)\) consisting of an optimal strategy \( f \) for player I, \( g \) for player II.
For
\[ \int M df(x) = \int \phi df(x) - \int \phi df(x) - \int \phi dfg + \int \phi dfdg \]
\[ - v + v + K(y) = K(y) \geq v , \]
since \( H(x) = v \) on \( \sigma(f) \), and similarly
\[ \int M dg(y) = H(x) \leq v . \]

The representation (4) of the payoff holds trivially in the case of any payoff \( M \), if we select for \( (f,g) \) any solution of the game with payoff \( M \), since we may then set \( v \) equal to the value and
\[ \phi = M , \ H(x) = \int M dg(y) , \ K(y) = \int M df(x) , \]
and obtain (4) as a trivial identity. (4) has, however, some non-trivial consequences if we replace \( \phi(x,y) \) by a function of the product \( xy \).

**Theorem 1:** Let \( f \) and \( g \) be non-step functions, and let \( H, K, \) and \( v \) satisfy (2) and (3) (above). Let \( \phi \) be an analytic function such that
\[ \phi(t) = \sum_{j=0}^{\infty} a_j t^n \] for \( |t| \leq r , \ r > 1 \),
\[ a_j \to 0 \] and \( \sum_{j=1}^{\infty} \frac{1}{n_j} = \infty . \)

Then the game with payoff \( M \) defined by
\[ M(x,y) = \phi(xy) - \int \phi(xy) df(x) - \int \phi(xy) dg(y) + \int \phi(xy) df(x) dg(y) \]
\[ - v + H(x) + K(y) \]
has \( (f,g) \) as its unique solution.

**Proof:** Because of the uniform convergence we have assumed for \( \phi \) we have
\[ p(xy) - \int p(xy) df(x) - \int p(xy) dg(y) + \int \int p(xy) df(x) dg(y) \]

\[ = \sum a_j(x^n - f_{n_j})y^n - x^n g_{n_j} + f_{n_j} g_{n_j} \]

\[ = \sum a_j(x^n - f_{n_j})(y^n - g_{n_j}) \]

where \( f_n \) is the \( n \)-th moment of \( f \). Hence we may write

\[ M(x,y) = \sum a_j(x^n - f_{n_j})(y^n - g_{n_j}) = v + H(x) + K(y) \]

Of course \((f, g)\) is a solution of the game, and we only have to show uniqueness. Let \( f' \) be an optimal strategy for player I. Then

\[ \int Mdf'(x) = \sum a_j(f'_n - f_{n_j})(y^n - g_{n_j}) - v + \int H(x)df'(x) + K(y) \]

\[ = \sum a_j(f'_n - f_{n_j})(y^n - g_{n_j}) - v + v + K(y) \geq v . \]

But \( K(y) = v \) on \( \sigma(g) \) so that

\[ \sum a_j(f'_n - f_{n_j})(y^n - g_{n_j}) \geq 0 \text{ for } y \in \sigma(g) . \]

Actually we must have equality on \( \sigma(g) \), since otherwise there exists a \( y_0 \in \sigma(g) \) such that

\[ \sum a_j(f'_n - f_{n_j})(y^n - g_{n_j}) > 0 , \]

and hence an interval containing \( y_0 \) in which this is true;

however since \( \sum a_j(f'_n - f_{n_j})(y^n - g_{n_j}) \) is non-negative on \( \sigma(g) \), from

\[ \int \sum a_j(f'_n - f_{n_j})(y^n - g_{n_j})dg(y) = \sum a_j(f'_n - f_{n_j})(g_{n_j} - g_{n_j}) = 0 \]
we conclude that this interval is of g-measure zero, hence that 
\( y \notin \sigma(g) \) which is a contradiction. Thus 
\[
\sum a_j (f'_n - f_{n_j}) (y^{n_j} - g_{n_j}) = 0 \quad \text{on } \sigma(g)
\]
and since \( \sigma(g) \) is not a finite set of points the analytic function 
on the left is identically zero, whence 
\[
f'_n = f_{n_j} \quad j = 1, 2, \ldots
\]
However, this implies \( f' = f \), as is shown in [1] say, since 
\[
\sum \frac{1}{n_j} = \infty. \quad \text{A similar argument suffices to show } g \text{ is unique.}
\]
As is evident from the above proof Theorem 1 may be stated in the 
following one-sided form:

Corollary 1: Let \( M, \phi, H, K, v \), satisfy the requirements of 
Theorem 1. Then if either player has a non-step function optimal 
strategy, his opponent has a unique optimal strategy.

\[ \S 2. \] 
Theorem 1 may be simplified to

**Theorem 2:** Let 
\[
M(x,y) = \phi(xy) + \rho(x) + \tau(y)
\]
(\( \phi \) and \( \tau \) are continuous on \([0,1]\) and 
\[
\phi(t) = \sum_{j=0}^{\infty} a_j t^n \quad \text{for } |t| \leq r, \quad r > 1
\]
and \( a_j \neq 0, \quad \sum_{j=1}^{\infty} \frac{1}{n_j} = \infty \) be the payoff of a game 
in which each player has a non-step function optimal strategy. Then 
the optimal strategies are unique.

Proof: Let \( f \) and \( g \) be the non-step function strategies for 
I and II. Then
K(y) = \int Mdf = \int \varphi(xy) df(x) + \int \rho(x) dt(x) + \tau(y) \geq v

H(x) = \int Mdg = \int \varphi(xy) dg(y) + \rho(x) + \int \tau(y) dg(y) \leq v

v = \int H(x) df(x) = \int \rho(xy) df + \int \sigma(y) dg

and H, K and v obviously satisfy (2) and (3). Moreover writing

M(x, y) = (M(x, y) - H(x) - K(y) + v) - v + H(x) + K(y)

and replacing the terms in the parentheses we obtain

M(x, y) = \varphi(xy) + \rho(x) + \tau(y) - \int \varphi(xy) df(x) - \int \rho(x) df(x) - \tau(y)

- \int \varphi(xy) dg(y) - \rho(x) - \int \tau(y) dg(y)

- \int \int \varphi(xy) df(x) dg(y) - \int \rho(x) df(x) - \int \tau(y) dg(y)

- v + H(x) + K(y)

or

M(x, y) = \varphi(xy) - \int \varphi(xy) df(x) - \int \varphi(xy) dg(y)

+ \int \int \varphi(xy) df(x) dg(y) - v + H(x) + K(y)

so that Theorem 1 immediately applies.

Theorem 2 may also be put in a one-sided form

Corollary 2: Let M(x, y) = \varphi(xy) + \rho(x) + \sigma(y) , \rho, \sigma

continuous, \varphi(t) = \sum_{j=1}^{\infty} a_j t^j , for |t| \leq r, r > 1 and a_j \neq 0,

\sum_{j=1}^{\infty} \frac{1}{n_j} = \infty . If either player in the game with payoff M has a non-step function optimal strategy then his opponent has a unique optimal strategy.

3. Examples

The first example is a game with a rational payoff function with a unique solution consisting of distributions with countable spectra. Set
\[ \phi(xy) = \frac{2}{2-xy} - \frac{2}{4-xy} \]

and

\[ f(x) = g(x) = \sum_{n=0}^{\infty} \frac{1}{2} \frac{1}{2^{n+1}} \cos^{2n}(x) \]

Then

\[ \int \phi(xy) df(x) = \sum_{n=0}^{\infty} \frac{1}{2} \frac{1}{2^{n+1}} \left( \frac{2}{2-2^{-n}y} - \frac{2}{4-2^{-n}y} \right) \]

\[ = \sum_{n=0}^{\infty} \left( \frac{1}{2^{n+1} - y} - \frac{1}{2^{n+2} - y} \right) = \frac{1}{2-y} \]

and by symmetry

\[ \int \phi(xy) dg(y) = \frac{1}{2-x} \]

Setting \( H(x) \equiv v \equiv K(y) \), and forming the function \( M \) given by (4) (omitting constants) we obtain

\[ M(x,y) = \frac{2}{2-xy} - \frac{2}{4-xy} - \frac{1}{2-y} - \frac{1}{2-x} \]

as the payoff of a game having \( (f,g) \) as a solution (the strategy \( f = g \) is not a step function in our terminology!), and since \( M \) is of the form required by Theorem 2, the solution is indeed unique.

Our second example is

\[ M(x,y) = e^{xy} - \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} e^x \cos^{2n} - \sum_{n=0}^{\infty} \frac{1}{n!} e^y \sin^{2n-1} \]

which is formed from

\[ \phi(xy) = e^{xy} \]

\[ H(x) \equiv v \equiv K(y) \]

\[ f(x) = \sum_{n=0}^{\infty} \frac{1}{n!} \sin^{2n}(x) \]

\[ g(y) = \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} \cos^{2n}(y) \]
(again omitting constants). The strategies \( f \) and \( g \) have jumps at a dense set of points in \([0,1]\), and are the unique strategies by Theorem 2. We note that the payoff in this example is the sum of two payoffs which have saddle points

\[
e^{xy} \quad \text{and} \quad -\frac{1}{e} \sum \frac{1}{n!} e^y \sin^2 n - \sum \frac{1}{2^{n+1}} e^x \cos^2 n
\]

Reference

[1]. I. Glicksberg and O. Gross, A Class of Games with Unique Density Function Solutions, RM-501
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